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Well quasi-orders and the shuffle closure of finite sets [*][†]

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Abstract

Given a set I of word, the set $L^{\epsilon}_{\vdash_I}$ of all words obtained by the shuffle of (copies of) words of I is naturally provided with a partial order: for u, v in $L^{\epsilon}_{\vdash_I}$, $u \vdash_I^* v$ if and only if v is the shuffle of u and another word of $L^{\epsilon}_{\vdash_I}$. In [3], the authors have opened the problem of the characterization of the finite sets I such that $\vdash_I^*$ is a well quasi-order on $L^{\epsilon}_{\vdash_I}$. In this paper we give an answer in the case when I consists of a single word w.

Keywords: formal languages, well quasi-orders, shuffle

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1 Introduction

A *quasi-order* on a set S is called a *well quasi-order* (*wqo*) if every non-empty subset X of S has at least one minimal element in X but no more than a finite number of (non-equivalent) minimal elements. Well quasi-orders have been widely investigated in the past. We recall the celebrated Higman and Kruskal results [9, 14]. Higman gives a very general theorem on division orders in abstract algebras from which one derives that the *subsequence ordering* in free monoids is a wqo. Kruskal extends Higman's result, proving that certain embeddings on finite trees are well quasi-orders. Some remarkable extensions of the Kruskal theorem are given in [11, 18].

In the last years many papers have been devoted to the application of wqo's to formal language theory [1, 2, 4, 5, 12, 13, 6, 7, 10].

Recently, in the theory of language equations, remarkable results based on wqo's have been obtained by M. Kunc [16]. These results have been culminating in the negative solution of the famous conjecture by Conway stating the regularity of the maximal solutions of the *commutative language equation* $XL = LX$ where L is a finite language of words [15].

In [6], a remarkable class of grammars, called *unitary grammars*, has been introduced in order to study the relationships between the classes of context-free and regular languages. If I is a finite set of words then we can consider the set of productions

$$\{\epsilon \to u \mid u \in I\}$$

and the derivation relation $\Rightarrow_I^*$ of the semi-Thue system associated with I. The language generated by the unitary grammar associated with I is $\mathrm{L}_I^\epsilon = \{w \in A^* \mid \epsilon \Rightarrow_I^* w\}$. Unavoidable sets of words are characterized in terms of the wqo property of the unitary grammars. Precisely it is proved that I is unavoidable if and only if the derivation relation $\Rightarrow_I^*$ is a wqo.

In [8], Haussler investigated the relation $\vdash_I^*$ defined as the transitive and reflexive closure of $\vdash_I$ where, for every pair w, v of words, $v \vdash_I w$ if

$$v = v_1 v_2 \cdots v_{n+1},$$

$$w = v_1 a_1 v_2 a_2 \cdots v_n a_n v_{n+1},$$

where the a_i's are letters, and $a_1 a_2 \cdots a_n \in I$. In particular, a characterization of the wqo property of $\vdash_I^*$ in terms of subsequence unavoidable sets of words was given in [8]. Let $\mathrm{L}_{\vdash_I}^\epsilon$ be the set of all words derived from the empty word by applying $\vdash_I^*$.

A remarkable result proved in [2] states that for any finite set I the derivation relation $\vdash_I^*$ is a wqo on the language L_I^ϵ. It is also proved that, in general, $\Rightarrow_I^*$ is not a wqo on L_I^ϵ and $\vdash_I^*$ is not a wqo on $\mathrm{L}_{\vdash_I}^\epsilon$. In [3] the authors characterize the finite sets I such that $\Rightarrow_I^*$ is a wqo on L_I^ϵ. Moreover, they have left the following problem open: *characterize the finite sets I such that $\vdash_I^*$ is a wqo on $\mathrm{L}_{\vdash_I}^\epsilon$.* In this paper we give an answer in the case when I consists of a single word w.

In this context, it is worth noticing that in [3] the authors prove that $\vdash_{\{w\}}^*$ is not a wqo on $\mathrm{L}_{\vdash_{\{w\}}}^\epsilon$ if $w = abc$. A simple argument allows one to extend the result above in the case that $w = a^i b^j c^h$, $i, j, h \geq 1$. By using Lemma 2.11, this implies that if a word w contains three distinct letters at least, then $\vdash^*_{\{w\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$. Therefore, in order to prove our main result, we can focus our attention to the case where w is a word on the binary alphabet $\{a, b\}$. Let E be the exchange morphism ($E(a) = b$, $E(b) = a$), and let $\tilde{w}$ be the mirror image of w.

Definition 1 *A word w is called* bad *if one of the words w, $\tilde{w}$, $E(w)$ and $E(\tilde{w})$ has a factor of one of the two following forms*

$$a^k b^h \qquad \textit{with } k, h \geq 2 \tag{1}$$

$$a^k b a^l b^m \qquad \textit{with } k > l \geq 1, m \geq 1 \tag{2}$$

A word w is called good *if it is not bad.*

Although it is immediate that a word w is bad if and only if one of the words w, $\tilde{w}$, $E(w)$ and $E(\tilde{w})$ contains a factor of the form a^2b^2 or $a^{k+1}ba^kb$, with $k \geq 1$ it will be useful to consider the definition as above. Morever we observe that, by Lemma 3.1 a word is good if and only if it is a factor of $(ba^n)^\omega$ or $(ab^n)^\omega$ for some $n \geq 0$. The main result of our paper is the following.

Theorem 1.1 *Let w be a word over the alphabet $\{a, b\}$. The derivation relation $\vdash^*_{\{w\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ if and only if w is good.*

We assume the reader to be familiar with the basic theory of combinatorics on words as well as with the theory of well quasi-orders (*see also* [5, 17]). Now let us recall the following theorem which gives a useful characterization of the concept of well quasi-order.

Theorem 1.2 *Let S be a set quasi-ordered by $\leq$. The following conditions are equivalent:*

- *i.* *$\leq$ is a well quasi-order;*
- *ii.* *if $s_1, s_2, \ldots, s_n, \ldots$ is an infinite sequence of elements of S, then there exist integers i, j such that $i < j$ and $s_i \leq s_j$.*

Let $\sigma = (s_i)_{i \geq 1}$ be an infinite sequence of elements of a set S. Then σ is called *good* if it satisfies condition ii of Theorem 1.2 and it is called *bad* otherwise, that is, for all integers i, j such that $i < j$, $s_i \not\leq s_j$. It is worth noting that, by condition ii above, a useful technique to prove that $\leq$ is a wqo on S is to prove that no bad sequence exists in S.

For the sake of clarity, the following well-known notions are briefly recalled. If u is a word over the alphabet A, then, for any $a \in A$, $|u|_a$ denotes the number of occurrences of a in u.

Given a word $v = a_1 \cdots a_k$, with $a_1, \ldots, a_k \in A$, v is said to be a *subsequence (or subword)* of u if there exist words $u_1, \ldots, u_{k+1}$ such that $u = u_1 a_1 \cdots u_k a_k u_{k+1}$.

Given two words u, v over the alphabet A, the symbol $u \sqcup\!\sqcup v$ denotes the set of words obtained by shuffle from u and v, that is the set of all words

$$u_1 v_1 \cdots u_k v_k,$$

where $k \geq 1$ and $u = u_1 \cdots u_k, v = v_1 \cdots v_k$.

2 Bad words

In this section, we prove the "only if" part of Theorem 1.1. We find convenient to split the proof into three sections. In the first two, we prove the claim in the case that w has one of the forms considered in Definition 1.

2.1 Words of form 1

Denote by w a word of the form

$$a^h b^k, \quad \text{with} \;\; h, k \geq 2,$$

and consider the sequence $(S_n)_{n \geq 1}$ of words of A^* defined as: for every $n \geq 1$,

$$S_n = a^h (a^{2h} b^{2k})(aba^{h-1} b^{k-1})^n (a^{2h} b^{2k}) b^k$$

Proposition 2.1 *$(S_n)_{n\geq 1}$ is a bad sequence of $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ with respect to $\vdash^*_{\{w\}}$. In particular $\vdash^*_{\{w\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$.*

In order to prove Proposition 2.1, we prove some technical lemmas. The following lemma is easily proved.

Lemma 2.2 *For every $n \geq 1$, $S_n \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$.*

Now we recall a remarkable characterization of the words of $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$. Let u be a word over $\{a, b\}$. Then we can consider the following integer parameters

$$q^u_a = \lfloor |u|_a / h \rfloor, \quad q^u_b = \lfloor |u|_b / k \rfloor, \quad \text{and}$$

$$r^u_a = |u|_a \bmod h, \quad r^u_b = |u|_b \bmod k.$$

Proposition 2.3 [3] *Let u be a word over the alphabet $A = \{a, b\}$. Then*

$$u \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$$

if and only if the following condition holds: $q^u_a = q^u_b$, $r^u_a = r^u_b$ and, for every prefix p of u, either $q^u_a > q^u_b$ or $q^u_a = q^u_b$ and $r^u_b = 0$.

Now we recall some useful results proved in [3].

Definition 2 *Let $u = a_1 \cdots a_s$ and $v = b_1 \cdots b_t$ be two words over A with $s \leq t$. An* embedding *of u in v is a map $f : \{1, \ldots, s\} \longrightarrow \{1, \ldots, t\}$ such that f is increasing and, for every $i = 1, \ldots, s$, $a_i = b_{f(i)}$.*

It is useful to remark that a word u is a subsequence of v if and only if there exists an embedding of u in v.

Definition 3 *Let $u, v \in A^*$ and let f be an embedding of u in v. Let $v = b_1 \cdots b_t$. Then $\langle v - u \rangle_f$ is the subsequence of v defined as*

$$\langle v - u \rangle_f = b_{i_1} \cdots b_{i_\ell}$$

where $\{i_1, i_2, \ldots, i_\ell\}$ is the increasing sequence of all the integers of $\{1, \ldots, m\}$ not belonging to $\mathrm{Im}(f)$. The word $\langle v - u \rangle_f$ is called the difference of v and u with respect to f.

It is useful to remark that $\langle v - u \rangle_f$ is obtained from v by deleting, one by one, all the letters of u according to f. Moreover, an embedding f of u in v is uniquely determined by two factorizations of u and v of the form

$$u = a_1 a_2 \cdots a_s, \qquad v = v_1 a_1 v_2 a_2 \cdots v_s a_s v_{s+1}$$

with $a_i \in A$, $v_i \in A^*$.

Lemma 2.4 [3] *Let $u, v \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ such that $u \vdash^*_{\{w\}} v$. Then there exists an embedding f of u in v such that*

$$\langle v - u \rangle_f \ \in \ \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}.$$

The following lemma is crucial.

Lemma 2.5 *For every $i, j \geq 1$,*

$$S_i \ \vdash^*_{\{w\}} \ S_j$$

if and only if $i = j$.

Proof. By contradiction, suppose that the claim is false. Hence there exist two positive integers $i < j$ such that $S_i \vdash^*_{\{w\}} S_j$. By Lemma 2.4, there exists an embedding f of S_i into S_j such that

$$\langle S_j - S_i \rangle_f \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}.$$

We divide the proof of the lemma in the following two steps. Let us set

$$P = a^h (a^{2h} b^{2k})(aba^{h-1} b^{k-1})^i,$$

and remark that P is a prefix of S_i and S_j.

Step 1. *Let $Q = a^h(a^{2h}b^{2k})$. The embedding f is the identity on Q.*

Let us first prove that the following condition is true:

$$\exists \ s \in \{1, \ldots, 2k\} \quad \text{with} \quad f(3h + s) = 3h + s. \tag{3}$$

By contradiction, deny. Hence we have $f(3h+2k) = \alpha > 3h+2k$. Moreover we have $\alpha \leq |S_j| - (3k+2h)$ since, otherwise, there would be no room to embed the remaining right part of S_i. Therefore, since $a^h a^{2h} b$ is a prefix of S_i, the prefix $a^h a^{2h}$ of Q must be embedded in a prefix of S_j, that we call T,

$$T = a^h (a^{2h} b^{2k})(aba^{h-1} b^{k-1})^L p,$$

where

$$p \in \{a,\ aba^{h-1}\},$$

with $L \geq 0$. Set $u = \langle T - a^h a^{2h} \rangle_f$. Since $h, k \geq 2$, it is easily checked that $q_a^u < q_b^u$, so contradicting Proposition 2.3. Hence (3) is proved.

Now the previous condition obviously implies that, for every $s \leq 3h$, $f(s) = s$. Consequently, if there exists a positive integer s with $1 \leq s \leq 2k$ and $f(3h+s) > 3h+s$, we would have

$$\langle S_j - S_i \rangle_f \in bA^*,$$

which contradicts Proposition 2.3. Hence the embedding f is the identity on Q. $\diamond$

Step 2. *The embedding f is the identity on P.*
By Step 1, it suffices to prove the claim for all indexes $s > |Q|$. Since $h, k \geq 2$, it is easily checked that

$$\forall\ s = 1, \ldots, h+k, \quad f(|Q|+s) = |Q|+s.$$

Indeed, suppose that the condition above does not hold. This implies the existence of a non empty prefix p of $\langle S_j - S_i \rangle_f$ which does not satisfy Proposition 2.3. By iterating the argument above, one completes the proof. $\diamond$

Finally, Step 2 and the fact that Pa^2 is a prefix of S_i implies that

$$f(|P|+1) > |P|+1 \quad \text{or} \quad f(|P|+2) > |P|+2,$$

whence

$$\langle S_j - S_i \rangle_f \in \{ab, b\}A^*,$$

which contradicts Proposition 2.3. Hence the embedding f cannot exist and thus $S_i \not\vdash^*_{\{w\}} S_j$. The proof of the lemma is thus complete. $\square$

Now we are able to prove the announced proposition.
Proof of Proposition 2.1: The claim immediately follows from Lemma 2.2 and Lemma 2.5. $\square$

2.2 Words of form 2

Now denote by w a word of the form

$$a^k b a^\ell b^m, \quad \text{with } k > \ell \geq 1,\ m \geq 1.$$

and consider the sequence $(S_n)_{n\geq 1}$ of words of A^* defined as: for every $n \geq 1$,

$$S_n = a^k b a^\ell a^k b a^\ell b^m (a^k b^{m+1} a^\ell)^n a^k b a^\ell b^m b^m.$$

We prove the following result.

Proposition 2.6 *$(S_n)_{n\geq 1}$ is a bad sequence of $\mathrm{L}^\epsilon_{\vdash_{\{w\}}}$ with respect to $\vdash^*_{\{w\}}$. In particular $\vdash^*_{\{w\}}$ is not a wqo on $\mathrm{L}^\epsilon_{\vdash_{\{w\}}}$.*

The following lemma is easily proved.

Lemma 2.7 *For every $n \geq 1$, $S_n \in \mathrm{L}^\epsilon_{\vdash_{\{w\}}}$.*

Let us define the map $\nu : A^+ \longrightarrow \mathbb{Q} \cup \{\infty\}$, as: for every $u \in A^*$,

$$\nu(u) = \frac{|u|_a}{|u|_b}.$$

The following two lemmas are easily proved by induction on the length of the derivation used to obtain u.

Lemma 2.8 *Let $u \in \mathrm{L}^\epsilon_{\vdash_{\{w\}}}$. For every non empty prefix p of u, we have*

$$\nu(p) \geq \frac{k+\ell}{m+1}.$$

Lemma 2.9 *Let u be a word of $\mathrm{L}^\epsilon_{\vdash_{\{w\}}}$. If $a^\alpha b$ is a prefix of u, then $\alpha \geq k$. If $a^\alpha b^2$ is a prefix of u, then $\alpha \geq 2k$.*

The following lemma is crucial.

Lemma 2.10 *For every $i, j \geq 1$,*

$$S_i \vdash^*_{\{w\}} S_j$$

if and only if $i = j$.

Proof. By contradiction, suppose that the claim is false. Hence there exist two positive integers $i < j$ such that $S_i \vdash^*_{\{w\}} S_j$. By Lemma 2.4, there exists an embedding f of S_i into S_j such that

$$\langle S_j - S_i \rangle_f \in \mathrm{L}^\epsilon_{\vdash_{\{w\}}}.$$

We divide the proof of the lemma in the following two steps. Let us set

$$P = a^k b a^\ell a^k b a^\ell b^m,$$

and remark that P is a prefix of S_i and S_j.

Step 1. *The embedding f is the identity on P.*
Set $Q = a^k b a^\ell a^k b$. We first show that:

$$\exists \ s \in \{1, \dots, \ell\}, \qquad \text{where } \ f(|Q| + s) = |Q| + s. \tag{4}$$

By contradiction, suppose that (4) does not hold. Consequently $f(|P|) > |P|$. Since $a^k b a^\ell b^m b^m$ is a suffix of S_i, $f(|P|) < |P(a^k b^{m+1} a^\ell)^j|$. Since P ends with b and Pa is a prefix of S_i, the prefix P of S_i must be embedded (according to f) in a prefix of S_j, we call T,

$$T = Pa^k (b^{m+1} a^{\ell+k})^\beta b^{m+1},$$

where β is such that $0 \leq \beta < j$. Therefore, the word $\langle T - P \rangle_f$ is a prefix of $\langle S_j - S_i \rangle_f$. On the other hand, an easy computation shows that

$$\nu(\langle T - P \rangle_f) = \frac{|\langle T - P \rangle_f|_a}{|\langle T - P \rangle_f|_b} = \frac{\beta(\ell + k) + k}{(1 + \beta)(m + 1)},$$

and thus

$$\nu(\langle T - P \rangle_f) < \frac{k + \ell}{m + 1},$$

so contradicting Lemma 2.8. Thus condition (4) is proved: it means that f is the identity on Q. Finally this condition implies that f is the identity on P. Indeed, otherwise, $\langle S_j - S_i \rangle_f \ \in a^\alpha b A^*$, with $0 \leq \alpha < l$ which contradicts Lemma 2.9 since $l < k$. $\diamond$

Step 2. *The embedding f is the identity on $P(a^k b^{m+1} a^\ell)^i$.*
By Step 1, it suffices to prove the claim for all indexes $s > |P|$. It is easily checked that, for every $s = 1, \dots, m + 1 + \ell + 2k$,

$$f(|P| + s) = |P| + s.$$

Indeed, otherwise, we would have $\langle S_j - S_i \rangle_f \ \in a^\alpha b^2 A^*$, with $\alpha < 2k$ or $\langle S_j - S_i \rangle_f \ \in a^\alpha b A^*$, with $\alpha < k$, so contradicting Lemma 2.9. By iterating the argument above, one completes the proof. $\diamond$

We have already proved that $S_i = P'R$, $S_j = P'(a^k b^{m+1} a^\ell)^{j-i} R$ where $P' = P(a^k b^{m+1} a^\ell)^i$ and $R = a^k b a^\ell b^m b^m$, and that f is the identity on P'. It follows that $\langle S_j - S_i \rangle_f$ begins with a prefix which is $a^k b^2$ (if $f(|P'| + 1) > f(|P'| + k + m + 1)$) or $a^\alpha b$ where $\alpha < k$ so contradicting Lemma 2.9. Hence the embedding f cannot exist and thus $S_i \not\vdash^*_{\{w\}} S_j$. The proof of the lemma is thus complete. $\square$

Now we are able to prove the announced proposition.
Proof of Proposition 2.6: The claim immediately follows from Lemma 2.7 and Lemma 2.10. $\square$

2.3 Proof of the "only if" part of Theorem 1.1

As pointed out in the previous paragraph, Propositions 2.1 and 2.6 permit to prove that if w is of the forms (1) or (2) of Definition 1, then $\vdash^*_{\{w\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$. This does not suffice to prove the "only if" part of Theorem 1.1. In order to complete the proof, the following lemma (and its symmetric version, say Lemma 2.12) provides a key result: indeed it shows that the property "$\vdash^*_{\{w\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$" is preserved by the factor order.

Lemma 2.11 *Let b be a letter of an alphabet A and let u be a word over A not ending with b. Assume $\vdash^*_{\{u\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{u\}}}$. Then, for every $k \geq 1$, $\vdash^*_{\{ub^k\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{ub^k\}}}$.*

Proof. Let $(w_n)_{n\geq 0}$ be a bad sequence of $\mathrm{L}^{\epsilon}_{\vdash_{\{u\}}}$ with respect to $\vdash^*_{\{u\}}$ and, for every $n \geq 1$, let us denote ℓ_n the positive integer such that

$$\epsilon \vdash^{\ell_n}_{\{u\}} w_n. \tag{5}$$

Since $(w_n)_{n\geq 0}$ is a bad sequence, by using a standard argument, we may choose the sequence $(w_n)_{n\geq 0}$ so that $(\ell_n)_{n\geq 0}$ is a strictly increasing sequence of positive integers. Let k be a positive integer and define the sequence of words $(w_n(b^k)^{\ell_n})_{n\geq 0}$. It is easily checked that, for every $n \geq 1$,

$$\epsilon \vdash^{\ell_n}_{\{ub^k\}} w_n(b^k)^{\ell_n},$$

so that all the words of the sequence defined above belong to the language $\mathrm{L}^{\epsilon}_{\vdash_{\{u\}}}$. Now we prove that this sequence is bad with respect to $\vdash^*_{\{ub^k\}}$. By contradiction, suppose the claim false. Thus there exist positive integers n, m such that

$$w_n(b^k)^{\ell_n} \vdash^*_{\{ub^k\}} w_{n+m}(b^k)^{\ell_{n+m}}. \tag{6}$$

Since, for every $n \geq 1$,

$$|w_n(b^k)^{\ell_n}| = \ell_n k + |w_n| = \ell_n(k + |u|),$$

we have that the length L of the derivation (6) is

$$L = \ell_{n+m} - \ell_n. \tag{7}$$

Now it is useful to do the following remarks. First observe that, since u does not end with the letter b, for every $n \geq 1$, $(b^k)^{\ell_n}$ is the longest power of b which is a suffix of $w_n(b^k)^{\ell_n}$. Second: at each step

$$v \vdash_{\{ub^k\}} v',$$

of the derivation (6), the exponent of the longest power of b which is a suffix of the word v' increases of k at most (with respect to v). Moreover this upper bound can be obtained by performing the insertion of ub^k in the word v only if its suffix b^k is inserted after the last letter of v which is different from b. By the previous remark and by (7), all the insertions of the derivation (6) must be done in this way. This implies that the derivation (6) defines in an obvious way a new one with respect to the relation $\vdash^*_{\{u\}}$ such that

$$w_n \ \vdash^*_{\{u\}} \ w_{n+\ell}.$$

The latter condition contradicts the fact that the sequence of words $(w_n)_{n\geq 0}$ is bad. □

By using a symmetric argument, we can prove the following.

Lemma 2.12 *Let b be a letter of an alphabet A and let u be a word over A not beginning with b. Assume $\vdash^*_{\{u\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{u\}}}$. Then, for every $k \geq 1$, $\vdash^*_{\{b^k u\}}$ is not a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{b^k u\}}}$.*

We are now able to prove the sufficiency of Theorem 1.1.

Theorem 2.13 *If w is a bad word then $\vdash^*_{\{w\}}$ is not a wqo on the language $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$.*

Proof. If w has a factor of the form $a^k b^h$ with $h, k \geq 2$, or $a^k b a^\ell b^m$, with $k > \ell \geq 1$, $m \geq 1$, then the claim is a straightforward consequence of Lemma 2.11, Lemma 2.12, Proposition 2.1, and Proposition 2.6.

In the general case, that is whenever $\tilde{w}$ or $E(w)$ or $E(\tilde{w})$ has a factor of the previous two forms, the proof is similar since the property of wqo is preserved under taking exchange morphism and mirror image of the word w. □

3 Good words

In this section we present the proof of the "if" part of Theorem 1.1. We find convenient to split it into the following seven sections. In the first a characterization of good words and that of the languages of words derivable from a good word are given.

3.1 Form of good words

Lemma 3.1 *A word w is good if and only if $w = \epsilon$ or there exist some integers n, e, i, f such that $w = a^i(ba^n)^e ba^f$ or $w = b^i(ab^n)^e ab^f$, $e \geq 0$, $0 \leq i, f \leq n$, and if $e = 0$ then $n = \max(i, f)$.*

Proof. Clearly if w is a bad word, then w cannot be decomposed as in the lemma.

Assume now that w is a good word. This means that w has no factor of the form $aabb$, $bbaa$, $a^{n+1}ba^n b$, $ba^n ba^{n+1}$, $b^{n+1}ab^n a$, $ab^n ab^{n+1}$ with $n \geq 1$ an integer.

If $|w|_a = 0$, then $w = \epsilon$ or $w = a^i(ba^n)^e ba^f$ with $i = n = f = 0$. If $|w|_a = 1$, $w = a^p ba^q$ with $\max(p, q) = 1$, that is $w = a^i(ba^n)^e ba^f$ with $i = p$, $f = q$, $n = \max(p, q)$, $e = 0$. Similarly if $|w|_b \leq 1$, w is a good word.

Assume from now on that $|w|_a \geq 2$ and $|w|_b \geq 2$. If both aa and bb are not factors of w, then w is a factor of $(ab)^\omega$ and so $w = a^i(ba^n)^e ba^f$ with $n = 1$.

Let us prove that aa and bb cannot be simultaneously factors of w. Assume the contrary. We have $w = w_1 aaw_2 bbw_3$ (or $w = w_1 bbw_2 aaw_3$ which leads to the same conclusion) for some words w_1, w_2, w_3. Without loss of generality we can assume that aa is not a factor of aw_2 and bb is not a factor of $w_2 b$. This implies that $w_2 = (ba)^m$ for an integer $m \geq 0$. This is not possible since $aabab$ and $aabb$ are not factors of w.

Assume from now on that bb is not a factor of w (the case where aa is not a factor is similar). This implies that $w = a^{i_0} ba^{i_1} ba^{i_2} b \dots ba^{i_p} ba^{i_{p+1}}$ for some integers i_0, i_1, $\dots$, i_{p+1} such that $i_j \neq 0$ for each $j \in \{1, \dots, p\}$. Let j be an integer such that $1 \leq j < j + 1 \leq p$. Since $a^{i_{j+1}+1} ba^{i_{j+1}} b$ and $ba^{i_j} ba^{i_j+1}$ are not factors of w, we have $i_j = i_{j+1}$. Thus set $n = i_1 = \dots = i_p$ and write $w = a^{i_0}(ba^n)^p ba^{i_{p+1}}$. Since $a^{n+1} ba^n b$ and $ba^n ba^{n+1}$ are not factors of w, we have $i_0 \leq n$, $i_{p+1} \leq n$. This ends the proof. $\square$

For X a set of words and n an integer, let $X^{\leq n} = \bigcup_{i=0}^{n} X^i$. Then Lemma 3.1 can be reformulated: the set of good words w is the set

$$\{\epsilon\} \cup \bigcup_{n \geq 0} a^{\leq n}(ba^n)^* ba^{\leq n} \cup \bigcup_{n \geq 0} b^{\leq n}(ab^n)^* ab^{\leq n}.$$

3.2 A fundamental characterization

In this section we prove the next proposition that characterizes words in $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ when w is a good word. The construction which is made in order to prove it also allows us to prove $\vdash^*_{\{w\}}$'s properties (see Lemma 3.3) on some prefixes of elements of $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$.

Proposition 3.2 *Let w be a word over $\{a, b\}$ and let n_w, e_w, i_w, f_w be integers such that $|w|_a \geq 1$, $|w|_b \geq 1$, $w = a^{i_w}(ba^{n_w})^{e_w} ba^{f_w}$, where $0 \leq i_w, f_w \leq n_w$, $e_w \geq 0$ and if $e_w = 0$ then $n_w = \max(i_w, f_w)$.*

A word u belongs to $\in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ if and only if the following conditions are satisfied:

1. $\frac{|u|_a}{|w|_a} = \frac{|u|_b}{|w|_b}$;
2. *for all words p, s, if $u = ps$ then*
   - *2.1)* $|p|_a \geq i_w|p|_b + \max(0, |p|_b - \frac{|u|_b}{|w|_b})(n_w - i_w)$;
   - *2.2)* $|s|_a \geq f_w|s|_b + \max(0, |s|_b - \frac{|u|_b}{|w|_b})(n_w - f_w)$.

In order to prove Conditions 1, 2.1 and 2.2, we now introduce a numbering of the letters which has very good properties (see in particular Lemma 3.3) when the word verifies the three conditions above.

Let w, n_w, e_w, i_w and f_w be as in Proposition 3.2. Let u be a word verifying Condition 1 of Proposition 3.2 and let $x = \frac{|u|_a}{|w|_a} = \frac{|u|_b}{|w|_b}$. We observe that if $u \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ then u is the shuffle of x occurrences of w.

For any $\alpha \in \{a, b\}$, let π_α be the function defined on $\{1, \ldots, |u|_\alpha\}$ as follows: $\pi_\alpha(i)$ is the index of the i^{th} occurrence of the letter α in u.

Example. Let $w = abaaabaa$ and let $u = abaaababaabaaabaabaaaaabaaaabaaa$. We have $x = 4$, $\pi_b(1) = 2$, $\pi_b(2) = 6$, $\pi_b(3) = 8$, $\pi_b(4) = 11$, $\pi_b(5) = 15$, $\pi_b(6) = 18$, $\pi_b(7) = 24$, $\pi_b(8) = 29$.

In order to find x occurrences of w in u, for every $1 \leq i \leq x$, we define the following set of integers:

$$\begin{aligned} P(i) \quad = \quad & \{\pi_a((i-1)i_w + j) \mid 1 \leq j \leq i_w\} \\ & \cup \{\pi_a(xi_w + kxn_w + (i-1)n_w + j) \mid 1 \leq j \leq n_w, 0 \leq k < e_w\} \\ & \cup \{\pi_a(xi_w + e_w x n_w + (i-1)f_w + j) \mid 1 \leq j \leq f_w\} \\ & \cup \{\pi_b(i + kx) \mid 0 \leq k \leq e_w\}. \end{aligned}$$

Note that the idea for introducing the sets $P(i)$ is to try to mark (when $u \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$) some possible occurrences of w as subsequences of u (see also words $u(i)$ below).

Example (continued). We have :

$$\begin{aligned} P(1) &= \{1, 2, 7, 9, 10, 15, 23, 25\}, \\ P(2) &= \{3, 6, 12, 13, 14, 18, 26, 27\}, \\ P(3) &= \{4, 8, 16, 17, 19, 24, 28, 30\}, \\ P(4) &= \{5, 11, 20, 21, 22, 29, 31, 32\}. \end{aligned}$$

The following properties easily follow from the definition of the sets $P(i)$ above:

1. The family $\{P(i)\}_{1 \leq i \leq x}$ is a partition of the set $\{1, \ldots, |u|\}$.
2. For each i with $1 \leq i \leq x$, the set $P(i)$ has exactly $|w|$ elements.

Let i be an integer with $1 \leq i \leq x$. Assume that $P(i) = \{i_1, \ldots, i_{|w|}\}$ with $i_1 < i_2 < \ldots < i_{|w|}$. We denote by $u(i)$ the word $u_{i_1} u_{i_2} \ldots u_{i_{|w|}}$. (In the example, $u(1) = u_1 u_2 u_7 u_9 u_{10} u_{15} u_{23} u_{25} = abaaabaa = w$).

Let us observe that, from an intuitive point of view, it could be useful to consider the word over the alphabet $\{1, \ldots, x\}$ defined as follows: for any $i \in \{1, \ldots, |u|\}$, the i^{th} letter of the word is the integer j such that $i \in P(j)$.

Example (continued). In the first row, we write the word u, while in the second, we write the word defined above:

```
abaaababaabaaabaabaaaaabaaaabaaa
11234213114222133234441312234344.
```

Some useful properties of the previous numbering are proved in the next lemma.

Lemma 3.3 *Let w (resp. u) be a word verifying the hypotheses (resp. Conditions 1 and 2) of Proposition 3.2. Let $x = \frac{|u|_a}{|w|_a}$. Then the following conditions hold:*

1. *For each $1 \leq i \leq x$, $u(i) = w$. Consequently, $u \in \mathrm{L}^{\epsilon}_{\vdash\{w\}}$.*
2. *If p is a prefix of u such that $|p|_a = x(i_w + kn_w)$ with $0 \leq k \leq e_w$, then $p \in \mathrm{L}^{\epsilon}_{\vdash\{p_{w,k},p_{w,k}b\}}$ where $p_{w,k} = a^{i_w}(ba^{n_w})^k$.*

Proof. Let i, $1 \leq i \leq x$. The fact that $u(i) = w$ follows immediately the definition of $u(i)$ (and $P(i)$) and the three following properties :

Property 1. If p is a word such that pb is a prefix of u and $|pb|_b = i$ then $|p|_a \geq i_w|pb|_b = i_w \times i$. This shows that $\pi_a(i_w(i-1)+j) < \pi_b(i)$ for each $1 \leq j \leq i_w$.

Proof of Property 1. By Condition 2.1 of Proposition 3.2, $|p|_a = |pb|_a \geq i_w|pb|_b$.

Property 2. If p and s are the words such that $u = pbs$ and $|pb|_b = e_wx + i$ (that is $|s|_b = x - i$) then $|p|_a \leq x(i_w + e_wn_w) + (i-1)f_w$. This shows that $\pi_b(e_wx+i) < \pi_a(xi_w + e_wxn_w + (i-1)f_w + j)$ for each $1 \leq j \leq i_w$.

Proof of Property 2. By Condition 2.2 of Proposition 3.2, $|s|_a = |bs|_a \geq f_w|bs|_b$. Since $|u|_a = |s|_a + |p|_a$ and $|u|_a = x(i_w + n_we_w + f_w)$, $|p|_a \leq x(i_w + n_we_w) + f_w(x - |bs|_b) = x(i_w + n_we_w) + f_w(i-1)$.

Property 3. If p, v, s are the words such that $u = pbvbs$ with $|pb|_b = i + kx$ with $0 \leq k < e_w$, and $|pbvb|_b = i + (k+1)x$, then $|pb|_a \leq xi_w + (kx+i-1)n_w$ and $xi_w + (kx+i)n_w \leq |pbvb|_a$. This means that $|pb|_b = \pi_b(i+kx) < \pi_a(xi_w + (kx+i-1)n_w + j) < \pi_b(i+(k+1)x)$ for each $1 \leq j \leq n_w$.

Proof of Property 3. First we observe that $|pbvb|_b > x$ and so $\max(0, |pbvb|_b - x) = |pbvb|_b - x$. Hence by Condition 2.1 of Proposition 3.2, $|pbvb|_a \geq i_w|pbvb|_b + (|pbvb|_b - x)(n_w - i_w) = i_wx + n_w(|pbvb|_b - x) = i_wx + n_w(i+kx)$.

Now we observe that $|bvbs|_b \geq x$ (Indeed $|bvbs|_b = |u|_b - |p|_b = x(e_w+1) - (i+kx-1) = x + x(e_w-k-1) + (x-i+1) > x$) and so $\max(0, |bvbs|_b - x) = |bvbs|_b - x$. Hence by Condition 2.2 of Proposition 3.2, $|bvbs|_a \geq f_w|bvbs|_b + (|bvbs|_b - x)(n_w - f_w) = f_wx + n_w(|bvbs|_b - x)$. Since $|u|_a = |p|_a + |bvbs|_a$ and $|u|_a = x(i_w + n_we_w + f_w)$, we have $|p|_a \leq xi_w + n_w(xe_w + x - |bvbs|_b)$. But $(e_w+1)x = |u|_b = |p|_b + |bvbs|_b = i + kx - 1 + |bvbs|_b$, that is $xe_w + x - |bvbs|_b = i + kx - 1$. Thus $|pb|_a = |p|_a \leq xi_w + (kx+i-1)n_w$.

Let us now prove the second part of Lemma 3.3.

First we observe that $xk \leq |p|_b \leq x(k+1)$. Indeed if $|p|_b < xk$, then considering the word s such that $u = ps$, $|s|_b > x(e_w + 1 - k) \geq x$, and by Condition 2.2 of Proposition 3.2, $|s|_a \geq f_w|s|_b + (|s|_b - x)(n_w - f_w) > f_w x(e_w + 1 - k) + x(e_w - k)(n_w - f_w) = x(f_w + (e_w - k)n_w)$, and so $|p|_a = |u|_a - |s|_a = x(i_w + e_w n_w + f_w) - |s|_a < x(i_w + kn_w)$ which contradicts the hypotheses. Moreover if $|p|_b > x(k+1) \geq x$, by Condition 2.1, $|p|_a \geq i_w|p|_b + (|p|_b - x)(n_w - i_w) > i_w x(k+1) + kxn_w - kxi_w = x(i_w + kn_w)$ which also contradicts the hypotheses.

Let $p(i)$, $1 \leq i \leq x$, be the prefix of $u(i)$ constituted of the letters with index in $P(i) \cap \{1, \ldots, |p|\}$. From $xk \leq |p|_b \leq x(k+1)$, we deduce that the set $\{\pi_b(i + \ell x) \mid 0 \leq l < k\}$ is included in the set $P(i) \cap \{1, \ldots, |p|\} \cap \{\pi_b(j) \mid 1 \leq j \leq |u|_b\}$ which itself is included in the set $\{\pi_b(i + \ell x) \mid 0 \leq l \leq k\}$. Hence $k \leq |p(i)|_b \leq k+1$. Moreover since $|p|_a = x(i_w + kn_w)$, the set $P(i) \cap \{1, \ldots, |p|\} \cap \{\pi_a(j) \mid 1 \leq j \leq |u|_a\}$ equals the union of the sets $\{\pi_a((i-1)i_w + j) \mid 1 \leq j \leq i_w\}$ and $\{\pi_a(xi_w + \ell xn_w + (i-1)n_w + j) \mid 1 \leq j \leq n_w, 0 \leq \ell < k\}$, so that $|p(i)|_a = i_w + kn_w$. Since $u(i) = w$, we deduce that $p(i) \in \{p_{k,w}, p_{k,w}b\}$ and so $p \in \mathrm{L}^{\epsilon}_{\vdash_{\{p_{k,w}, p_{k,w}b\}}}$. □

Proof of Proposition 3.2. The sufficiency of Conditions 1 and 2 is ensured by Lemma 3.3 (1).

It is immediate that Condition (1) is necessary. We prove that it is also the case for Condition 2.1, the proof for Condition 2.2 being similar. Let $u \in \mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$ and let x be the integer such that $\epsilon \vdash^{x}_{\{w\}} u$. If $x = 0$ then $u = \epsilon$ and the claim is trivially verified. Thus suppose $x > 0$.

We have $|u|_a = x|w|_a$ and $|u|_b = x|w|_b$, so that $x = |u|_b/|w|_b = |u|_a/|w|_a$.

Since u is the shuffle of x occurrences of w, any prefix p of u is the shuffle of x prefixes of w: there exist prefixes p_1, …, p_x such that

$$p \in p_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup p_x,$$

Thus

$$|p|_a = \sum_{i=1,\ldots,x} |p_i|_a.$$

Since p_i is a prefix of w, if $|p_i|_b \neq 0$, $|p_i|_a \geq i_w + (p_i|_b - 1)n_w$. Assume without loss of generality that $p_1, \ldots, p_{x'}$ contain at least one b and that $p_{x'+1}, \ldots, p_x$ contain no b. We get

$$|p|_a \geq x' i_w + n_w \sum_{i=1,\ldots,x'} |p_i|_b - x' n_w.$$

But $|p|_b = \sum_{i=1,\ldots,x'} |p_i|_b$. So

$$|p|_a \geq x' i_w + n_w(|p|_b - x') = i_w|p|_b + (|p|_b - x')(n_w - i_w).$$

Since $x' \leq x = |u|_b/|w|_b$, the latter inequality gives

$$|p|_a \geq i_w|p|_b + \max(0, |p|_b - \frac{|u|_b}{|w|_b})(n_w - i_w).$$

The proof is thus complete. ☐

3.3 Some useful wqo's

In this section, we present some useful wqo's. First we recall the following result.

Proposition 3.4 [3] *For any integer $n \geq 0$, if $w \in \{a^nb, ab^n, ba^n, b^na\}$, $\vdash^*_{\{w\}}$ is a wqo on $\mathrm{L}^\epsilon_{\vdash_{\{w\}}} = \mathrm{L}^\epsilon_w$.*

This result allows us to state:

Lemma 3.5 *Let $n \geq 0$ be an integer. Let I be one of the following sets: $\{a^nb, a\}$, $\{a^nb, b\}$, $\{b^na, a\}$, $\{b^na, b\}$, $\{ba^n, a\}$, $\{ba^n, b\}$, $\{ab^n, a\}$, $\{ab^n, b\}$:*

$$\mathrm{L}^\epsilon_{\vdash_I} = \mathrm{L}^\epsilon_I.$$

Proof. Assume $I = \{a^nb, a\}$. It is immediate that $\mathrm{L}^\epsilon_I \subseteq \mathrm{L}^\epsilon_{\vdash_I}$. Let w be a word in $\mathrm{L}^\epsilon_{\vdash_I}$. There exists a word w_1 such that $\epsilon \vdash^*_{\{a^nb\}} w_1 \vdash^*_{\{a\}} w$. By Proposition 3.4, $w_1 \in \mathrm{L}^\epsilon_{a^nb}$, and so $w \in \mathrm{L}^\epsilon_I$.

The proof for the other values of I is similar. ☐

Lemma 3.6 *Let $n \geq 1$ be an integer. The three following assertions are equivalent for a word w:*

1. *$w \in \mathrm{L}^\epsilon_{\vdash_{\{a^nb, a^n\}}}$;*
2. *$|w|_a = 0 \bmod n$, and, for any prefix p of w, $|p|_a \geq n|p|_b$;*
3. *$w \in \mathrm{L}^\epsilon_{\{a^nb, a^n\}}$.*

In particular, $\mathrm{L}^\epsilon_{\vdash_{\{a^nb, a^n\}}} = \mathrm{L}^\epsilon_{\{a^nb, a^n\}}$.

Proof. $3 \Rightarrow 1$ is immediate.

For any word w in $\mathrm{L}^\epsilon_{\vdash_{\{a^nb, a^n\}}}$, obviously $|w|_a = 0 \bmod n$. Moreover w is a prefix of a word in $\mathrm{L}^\epsilon_{\vdash_{\{a^nb\}}}$. Thus $1 \Rightarrow 2$ is a direct consequence of Proposition 3.2. Indeed taking $w = a^nb$, $n_w = n = i_w$, and $e_w = f_w = 0$, Condition 2.1 of Proposition 3.2 says that for any prefix of a word in $\mathrm{L}^\epsilon_{\vdash_{\{a^nb\}}}$, $|p|_a \geq i_w|p|_b = n|p|_b$.

We now prove $2 \Rightarrow 3$ by induction on $|w|_b$. Since $|w|_a = 0 \bmod n$, the result is immediate if $|w|_b = 0$. Assume $|w|_b \geq 1$. Assertion 2 on w implies the existence of an integer $k \geq 0$ and a word w' such that $w = a^ka^nbw'$. Let p be a prefix of a^kw'. If $|p| \leq k$, then $n|p|_b = 0 \leq |p|_a$. If $|p| > k$, $p = a^kp'$ for a prefix p' of w'. Assertion 2 on w implies that $|a^ka^nbp'|_a \geq n|a^ka^nbp'|_b$ that is $|a^kp'|_a \geq n|a^kp'|_b$. Thus a^kw' verifies Assertion 2 and so by inductive hypothesis, $a^nw' \in \mathrm{L}^\epsilon_{\{a^nb, a^n\}}$. It follows that $w \in \mathrm{L}^\epsilon_{\{a^nb, a^n\}}$. ☐

Similarly to Lemma 3.6, one can state that $\mathrm{L}^\epsilon_{\vdash_{\{ba^n, a^n\}}} = \mathrm{L}^\epsilon_{\{ba^n, a^n\}}$ (this needs to exchange prefixes by suffixes), and, exchanging the roles of a and b, $\mathrm{L}^\epsilon_{\vdash_{\{b^na, b^n\}}} = \mathrm{L}^\epsilon_{\{b^na, b^n\}}$ and $\mathrm{L}^\epsilon_{\vdash_{\{ab^n, b^n\}}} = \mathrm{L}^\epsilon_{\{ab^n, b^n\}}$.

Let us recall that:

Theorem 3.7 [1, 2] *For any finite set I, $\vdash_I^*$ is a wqo on L_I^ϵ.*

Hence from this theorem and the previous lemma, we deduce:

Proposition 3.8 *Let $n \geq 0$ be an integer. Let I be one of the following sets: $\{a^n b, a\}$, $\{a^n b, b\}$, $\{b^n a, a\}$, $\{b^n a, b\}$, $\{ba^n, a\}$, $\{ba^n, b\}$, $\{ab^n, a\}$, $\{ab^n, b\}$, $\{a^n b, a^n\}$, $\{ba^n, a^n\}$, $\{b^n a, b^n\}$, $\{ab^n, b^n\}$. The derivation relation $\vdash_I^*$ is a wqo on $\mathrm{L}_{\vdash_I}^\epsilon$.*

3.4 A decomposition tool

Lemma 3.9 *Let $m \geq 1$ be an integer. Any word w over $\{a, b\}$ can be factorized as $w = w_1 w_2 w_3$ with $w_1 \in \mathrm{L}_{\vdash_{\{ba^m, a\}}}^\epsilon$, $w_2 \in \mathrm{L}_{\vdash_{\{ba^m, b\}}}^\epsilon$ and $|w_3|_a < m$.*

Moreover, if w is the shuffle of x occurrences of ba^m and of a word w', then $x \leq |w_1|_b + |w_2|_a / m$.

Proof. We prove the first part of this result by induction on $|w|$. The claim is trivial if $w = \epsilon$. Assume $|w| \geq 1$, so that $w = w'\alpha$ with $\alpha \in \{a, b\}$. By inductive hypothesis, $w' = w_1' w_2' w_3'$ with $w_1' \in \mathrm{L}_{\vdash_{\{ba^m, a\}}}^\epsilon$, $w_2' \in \mathrm{L}_{\vdash_{\{ba^m, b\}}}^\epsilon$ and $|w_3'|_a < m$.

If $\alpha = b$ or if $\alpha = a$ and $|w_3'\alpha|_a < m$, the result is true for w by setting $w_1 = w_1'$, $w_2 = w_2'$ and $w_3 = w_3'\alpha$. Assume now that $\alpha = a$ and $|w_3'\alpha|_a = m$. Two cases have to be considered. If $w_2' \notin \mathrm{L}_{\vdash_{\{ba^m\}}}^\epsilon$, then $w_2' w_3' a \in \mathrm{L}_{\vdash_{\{ba^m, b\}}}^\epsilon$ and thus we can set $w_1 = w_1'$, $w_2 = w_2' w_3' a$ and $w_3' = \epsilon$.

Consider now that $w_2' \in \mathrm{L}_{\vdash_{\{ba^m\}}}^\epsilon$. By replacing w_1' (resp. w_2') by $w_1' w_2'$ (resp. ϵ), we can assume $w_2' = \epsilon$. If w_3' starts with b, then $w_3' a \in \mathrm{L}_{\vdash_{\{ba^m, b\}}}^\epsilon$ and the result is true for w with $w_1 = w_1'$, $w_2 = w_2' w_3'$ and $w_3 = \epsilon$. If w_3' starts with a, $w_3' = ax$ for a word x. The result is true for w with $w_1 = w_1' a$, $w_2 = w_2' = \epsilon$ and $w_3 = x$.

The argument used in the induction above can be used for the proof of the second part of the statement of Lemma 3.9. ☐

3.5 A first inductive result

The aim of this section is to prove the next result which proof is based on the characterization provided by Proposition 3.11.

Proposition 3.10 *Let n, m be two integers such that $n, m \geq 1$ and let w be a word in $a^{\leq n}(ba^n)^* b \cup \{\epsilon\}$ such that $wa^n ba^m$ is a good word. If $\vdash_{\{wa^n, wa^n b\}}^*$ is a wqo on $\mathrm{L}_{\vdash_{\{wa^n, wa^n b\}}}^\epsilon$ then $\vdash_{\{wa^n b, wa^n ba^m\}}^*$ is a wqo on $\mathrm{L}_{\vdash_{\{wa^n b, wa^n ba^m\}}}^\epsilon$.*

Observe that the hypothesis "$wa^n ba^m$ is a good word" means only $1 \leq m \leq n$ when $w \neq \epsilon$.

Proposition 3.11 *Let n, m be two integers such that $n, m \geq 1$ and let w be a word in $a^{\leq n}(ba^n)^* b \cup \{\epsilon\}$ such that $wa^n ba^m$ is a good word.*

A word u over $\{a, b\}$ belongs to $\mathrm{L}_{\vdash_{\{wa^n b, wa^n ba^m\}}}^\epsilon$ if and only if $u = u_1 u_2 u_3 u_4$ with

1. $u_1 \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^n\}}}$,

2. $u_2 \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,a\}}}$,

3. $u_3 \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,b\}}}$,

4. $|u_4|_a < m$,

5. $|u_2u_4|_a = 0 \bmod m$,

6. $|u_1|_a(|w|_b + 1) = (|w|_a + n)|u|_b$,

7. $\frac{|u_2|_a+|u_4|_a}{m} - |u_2|_b \leq |u_1| - \frac{|u_1|_a(|w|+n)}{(|w|_a+n)}$.

Proof.

Proof of the "if part". Assume that $u = u_1u_2u_3u_4$ with u_1, u_2, u_3, u_4 verifying Conditions 1 to 7 of the proposition. Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ be the integers (one can verify they are unique) such that:

- any derivation from ϵ to u_1 by $\vdash^*_{\{wa^nb,wa^n\}}$ uses α_1 rewriting steps by $\vdash_{\{wa^nb\}}$ and β_1 steps by $\vdash_{\{wa^n\}}$;
- any derivation from ϵ to u_2 by $\vdash^*_{\{ba^m,a\}}$ uses α_2 rewriting steps by $\vdash_{\{ba^m\}}$ ($\alpha_2 = |u_2|_b$) and β_2 steps by $\vdash_{\{a\}}$ ($\beta_2 = |u_2|_a - m|u_2|_b$);
- any derivation from ϵ to u_3 by $\vdash^*_{\{ba^m,b\}}$ uses α_3 rewriting steps by $\vdash_{\{ba^m\}}$ ($\alpha_3 = |u_3|_a/m$) and β_3 steps by $\vdash_{\{b\}}$.

By hypothesis, $|u_2u_4|_a = 0 \bmod m$: let

$$\beta_2' \quad = \quad |u_2u_4|_a/m - |u_2|_b (= (\beta_2 + |u_4|_a)/m). \tag{8}$$

Let us observe some relations:

- We have $|u_1| = \alpha_1|wa^nb| + \beta_1|wa^n| = \alpha_1 + (\alpha_1 + \beta_1)(|w| + n)$ and $|u_1|_a = \alpha_1|wa^n|_a + \beta_1|wa^n|_a = (\alpha_1 + \beta_1)(|w|_a + n)$. So
$$\alpha_1 \quad = \quad |u_1| - \frac{|u_1|_a(|w| + n)}{|w|_a + n}. \tag{9}$$
- We also have $|u|_b = (\alpha_1 + \beta_1)|w|_b + \alpha_1 + \alpha_2 + \alpha_3 + \beta_3 + |u_4|_b = (\alpha_1 + \beta_1)(|w|_b+1) - \beta_1 + \alpha_2 + \alpha_3 + \beta_3 + |u_4|_b$. Since by hypothesis, $|u_1|_a(|w|_b+1) = (|w|_a + n)|u|_b$, and since $\alpha_1 + \beta_1 = \frac{|u_1|_a}{|w|_a+n}$, we have
$$\beta_1 \quad = \quad \alpha_2 + \alpha_3 + \beta_3 + |u_4|_b. \tag{10}$$

We have defined the integers $\alpha_1, \beta_1, \alpha_2, \beta_2', \alpha_3, \beta_3$ in such a way that:

- u_1 is a shuffle of α_1 words wa^nb and β_1 words wa^n,

- u_2u_4 is a shuffle of α_2 words ba^m, β_2' words a^m and $|u_4|_b$ words b,
- u_3 is a shuffle of α_3 words ba^m and β_3 words b.

Since $\beta_1 = \alpha_2+\alpha_3+\beta_3+|u_4|_b$, the β_1 occurrences of wa^n in u_1 can be associated to the $\alpha_2+\alpha_3$ occurrences of ba^m in u_2u_3 and the $\beta_3+|u_4|_b$ occurrences of b in u_3u_4 in order to obtain $\alpha_2+\alpha_3$ occurrences of wa^nba^m and $\beta_3+|u_4|_b$ occurrences of wa^nb as subwords of u. By Condition 7 and Relations (8) and (9) we have $\beta_2' \leq \alpha_1$. Thus we can associate β_2' occurrences of wa^nb in u_2 with the β_2' occurrences of a^m in u_2u_4 to construct β_2' occurrences of wa^nba^m as subwords in u. So u is the shuffle of $\beta_2'+\alpha_2+\alpha_3$ words wa^nba^m and $(\alpha_1-\beta_2')+\beta_3+|u_4|_b$ words wa^nb and hence $u \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nba^m,wa^nb\}}}$.

Proof of the "only if" part.

Assume $u \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^nba^m\}}}$. Let α and β be the integers (one can verify they are unique) such that any derivation from ϵ to u by $\vdash^*_{\{wa^nb,wa^nba^m\}}$ uses α rewriting steps by $\vdash_{\{wa^nba^m\}}$ and β steps by $\vdash_{\{wa^nb\}}$. An important remark is that $u(a^m)^\beta \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nba^m\}}}$.

We have $|u|_a = \alpha|wa^nba^m|_a + \beta|wa^nb|_a = (\alpha+\beta)(|w|_a+n)+\alpha m$ and $|u|_b = (\alpha+\beta)(|w|_b+1)$. Thus

$$\alpha+\beta \quad = \quad \frac{|u|_b}{|w|_b+1} = \frac{|u|_a}{|w|_a+n} - \frac{\alpha m}{|w|_a+n}. \tag{11}$$

In particular $|u|_b$ is divisible by $|w|_b+1$, and $|u|_a \geq \frac{|u|_b}{|w|_b+1}(|w|_a+n)$. Let u_1 be a prefix of u such that $|u_1|_a = \frac{|u|_b}{|w|_b+1}(|w|_a+n) = (\alpha+\beta)(|w|_a+n)$. By Lemma 3.3(2), since $u(a^m)^\beta$ belongs to $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^nba^m\}}}$, we have $u_1 \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^n\}}}$.

Let s be the word such that $u = u_1s$. By Lemma 3.9, $s = u_2u_3u_4$ with $u_2 \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,a\}}}$, $u_3 \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,b\}}}$ and $|u_4|_a < m$.

Let us observe that $|u_3|_a = 0 \bmod m$ and $|s|_a = |u|_a - |u_1|_a = \alpha m = 0 \bmod m$. Thus $|u_2u_4|_a = |s|_a - |u_3|_a = 0 \bmod m$.

By Condition 2.2 of Proposition 3.2 applied to $n_w = \max(n,m)$ and $f_w = m$, and since $u(a^m)^\beta \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nba^m\}}}$, we have $|u_3u_4(a^m)^\beta|_a \geq m|u_3u_4(a^m)^\beta|_b = m|u_3u_4|_b$, that is,

$$\beta m + |u_3u_4|_a \geq m|u_3u_4|_b = m(|u|_b - |u_1u_2|_b) = m(|u|_b - |u_2|_b - |u_1| + |u_1|_a).$$

The latter inequality can be rewritten as

$$\beta m + |u|_a - |u_1u_2|_a \geq m\left(|u|_b - (|u_1| - |u_1|_a) - |u_2|_b\right),$$

and so

$$|u_2|_a - m|u_2|_b \leq m|u_1| - \left(m|u|_b + (m+1)|u_1|_a - (|u|_a + \beta m)\right).$$

By recalling that $|u|_b = \frac{|u_1|_a(|w|_b+1)}{|w|_a+n}$ and since

$$|u|_a+\beta m = (\alpha+\beta)(|w|_a+n+m) = \frac{|u|_b}{|w|_b+1}(|w|_a+n+m) = \frac{|u_1|_a}{|w|_a+1}(|w|_a+n+m),$$

we have

$$m|u|_b+(m+1)|u_1|_a-(|u|_a+\beta m) = \frac{|u_1|_a}{|w|_a+n}\left(m(|w|_b+1)+(m+1)(|w|_a+n)-(|w|_a+n+m)\right),$$

which gives

$$m|u|_b+(m+1)|u_1|_a-(|u|_a+\beta m) = m\frac{|u_1|_a}{|w|_a+n}(|w|+n).$$

This shows that

$$\frac{|u_2|_a}{m}-|u_2|_b \leq |u_1|-\frac{|u_1|_a}{|w|_a+n}(|w|+n).$$

Now observe that $|u_1|-\frac{|u_1|_a}{|w|_a+n}(|w|+n) = |u_1|-(\alpha+\beta)(|w|+n)$ is an integer, and since $|u_4|_a < m$ and $|u_2u_4|_a = 0 \ mod \ m$, we have $\lceil\frac{|u_2|_a}{m}\rceil = \frac{|u_2|_a+|u_4|_a}{m}$. This implies that

$$\frac{|u_2|_a+|u_4|_a}{m}-|u_2|_b \leq |u_1|-\frac{|u_1|_a}{|w|_a+n}(|w|+n).$$

The proof is thus complete ☐

We are now able to prove Proposition 3.10.

Proof of Proposition 3.10. Let $(u_k)_{k\geq 0}$ be a sequence of words in $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^nba^m\}}}$. By Proposition 3.11, for any $k\geq 0$, there exist words $u_{1,k}$, $u_{2,k}$, $u_{3,k}$ and $u_{4,k}$ such that $u_k = u_{1,k}u_{2,k}u_{3,k}u_{4,k}$ with

- $u_{1,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^n\}}}$,
- $u_{2,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,a\}}}$,
- $u_{3,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,b\}}}$,
- $|u_{4,k}|_a < m$,
- $|u_{2,k}u_{4,k}|_a = 0 \text{ mod } m$,
- $|u_{1,k}|_a(|w|_b+1) = (|w|_a+n)|u_k|_b$,
- $\frac{|u_{2,k}|_a+|u_{4,k}|_a}{m} - |u_{2,k}|_b \leq |u_{1,k}| - \frac{|u_{1,k}|_a(|w|+n)}{(|w|_a+n)}$.

Let us define the following integer sequence $(d_k)_{k\geq 0}$: for every $k\geq 0$,

$$d_k = |u_{1,k}| - \frac{|u_{1,k}|_a(|w|+n)}{(|w|_a+n)} - \left(\frac{|u_{2,k}|_a+|u_{4,k}|_a}{m} - |u_{2,k}|_b\right).$$

By replacing $(u_k)_{k\geq 0}$ with one of its subsequence, we can assume that the sequence $(d_k)_{k\geq 0}$ is non-decreasing.

By hypothesis, $\vdash^*_{\{wa^nb,wa^n\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^n\}}}$, and by Proposition 3.8, $\vdash^*_{\{ba^m,a\}}$ (resp. $\vdash^*_{\{ba^m,b\}}$) is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,a\}}}$ (resp. $\mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,b\}}}$). So still replacing $(u_k)_{k\geq 0}$ by a subsequence, we can assume that, for all $k \geq 0$,

$$u_{1,k} \vdash^*_{\{wa^nb,wa^n\}} u_{1,k+1}, \quad u_{2,k} \vdash^*_{\{ba^m,a\}} u_{2,k+1}, \quad u_{3,k} \vdash^*_{\{ba^m,b\}} u_{3,k+1}.$$

Moreover, since $|u_{4,k}|_a$ is bounded, we can assume that $|u_{4,k}|_a = |u_{4,k+1}|_a$ and since the subsequence ordering is a wqo on A^*, we can assume that $u_{4,k}$ is a subword of $u_{4,k+1}$.

The previous arguments imply the existence, for any $k \geq 0$, of words $v_{1,k}$, $v_{2,k}$, $v_{3,k}$, $v_{4,k}$ such that

$$u_{i,k+1} \in u_{i,k} \shuffle v_{i,k}, v_{1,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^n\}}}, v_{2,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,a\}}}, v_{3,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^m,b\}}}, |v_{4,k}|_a = 0.$$

The equality $|v_{2,k}v_{4,k}|_a = 0 \bmod m$ easily follows from $|u_{2,k}u_{4,k}|_a = 0 \bmod m$ and $|u_{2,k+1}u_{4,k+1}|_a = 0 \bmod m$. We have $|v_{1,k}|_a = |u_{1,k+1}|_a - |u_{1,k}|_a$ and, taking $v_k = v_{1,k}v_{2,k}v_{3,k}v_{4,k}$, $|v_k|_b = |u_{k+1}|_b - |u_k|_b$. Since $|u_{1,j}|_a(|w|_b + 1) = (|w|_a + n)|u_j|_b$ for $j \in \{k, k+1\}$, we can deduce that $|v_{1,k}|_a(|w|_b + 1) = (|w|_a + n)|v_k|_b$. By the fact that the sequence $(d_k)_{k\geq 0}$ is non-decreasing, we have

$$\frac{|v_{2,k}|_a + |v_{4,k}|_a}{m} - |v_{2,k}|_b \leq |v_{1,k}| - \frac{|v_{1,k}|_a(|w| + n)}{(|w|_a + n)}.$$

Now, by applying Proposition 3.11 to the words v_k, we have $v_k \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^nba^m\}}}$. Since, for all $k \geq 0$, $u_{k+1} \in u_k \shuffle v_k$, the latter condition gives $u_k \vdash^*_{\{wa^nb,wa^nba^m\}} u_{k+1}$. Therefore $\vdash^*_{\{wa^nb,wa^nba^m\}}$is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^nba^m\}}}$. $\square$

3.6 A second inductive result

The aim of this section is to prove the next result which proof is based on the characterization provided by Proposition 3.13.

Proposition 3.12 *Let $n \geq 1$ be an integer and let w be a word in $a^{\leq n}(ba^n)^*$. If $\vdash^*_{\{wb,wba^n\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wb,wba^n\}}}$ then $\vdash^*_{\{wba^n,wba^nb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^n,wba^nb\}}}$.*

Proposition 3.13 *Let $n \geq 1$ be an integer and let $w \in a^{\leq n}(ba^n)^*$. A word u belongs to $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^n,wba^nb\}}}$ if and only if $u = u_1u_2u_3u_4u_5u_6$ with*[1]*:*

1. *$u_1b^{|u_2|_b} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wb,wba^n\}}}$,*
2. *$|u_1u_2|_b(|w|_a + n) = |u|_a(|w|_b + 1)$,*
3. *$u_2u_3 = \epsilon$ or $|u_2u_3|_a = n$,*
4. *$|u_4|_a < n$,*
5. *$u_5 \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,b\}}}$,*

---

[1] the value of $\overline{\delta}_{u_2u_3,\epsilon}$ is 0 if $u_2u_3 = \epsilon$ and 1 otherwise

6. $u_6 \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^n b, a\}}}$,
7. $|u_3|_b \leq \frac{1}{n}\left[|u_1|_a - \frac{|u_1u_2|_b}{|w|_b+1}|w|_a\right]$,
8. $|u_5|_b - \frac{|u_5|_a}{n} + |u_3u_4|_b \leq \frac{1}{n}\left[|u_1|_a - \frac{|u_1u_2|_b}{|w|_b+1}|w|_a\right] + \overline{\delta}_{u_2u_3,\epsilon}$,
9. $\frac{|u|_a - |u_1|_a}{n} \geq |u_2|_b + \overline{\delta}_{u_2u_3,\epsilon}$.

Proof.

Proof of the "if" part. Assume first that u can be factorized in the product of six words satisfying the properties of the proposition. Let α_1, β_1, α_5, β_5, α_6, β_6 be the integers (one can verify they are unique) such that:

- any derivation from ϵ to $u_1 b^{|u_2|_b}$ by $\vdash^*_{\{wba^n, wb\}}$ uses α_1 rewriting steps by $\vdash_{\{wba^n\}}$ and β_1 steps by $\vdash_{\{wb\}}$;
- any derivation from ϵ to u_5 by $\vdash^*_{\{a^n b, b\}}$ uses α_5 rewriting steps by $\vdash_{\{a^n b\}}$ ($\alpha_5 = |u_5|_a/n$) and β_5 steps by $\vdash_{\{b\}}$ ($\beta_5 = |u_5|_b - \alpha_5$);
- any derivation from ϵ to u_6 by $\vdash^*_{\{a^n b, a\}}$ uses α_6 rewriting steps by $\vdash_{\{a^n b\}}$ ($\alpha_6 = |u_6|_b$) and β_6 steps by $\vdash_{\{a\}}$ ($\beta_6 = |u_6|_a - n\alpha_6$).

Let us observe some relations:

- We have $|u_1|_a = \alpha_1 |wba^n|_a + \beta_1 |wb|_a = n\alpha_1 + (\alpha_1 + \beta_1)|w|_a$ and $|u_1u_2|_b = (\alpha_1 + \beta_1)(|w|_b + 1)$. So we have

  $$\alpha_1 \quad = \quad \frac{1}{n}\left[|u_1|_a - \frac{|u_1u_2|_b}{|w|_b+1}|w|_a\right]. \qquad (12)$$

  Thus Properties 7 and 8 can be rephrased $|u_3|_b \leq \alpha_1$ and $\beta_5 + |u_3u_4|_b \leq \alpha_1 + \overline{\delta}_{u_2u_3,\epsilon}$ respectively.

- We also have $|u|_a = \alpha_1(|w|_a + n) + \beta_1|w|_a + |u_2u_3u_4u_6|_a + n\alpha_5 = (\alpha_1 + \beta_1)(|w|_a + n) - \beta_1 n + |u_2u_3u_4u_6|_a + n\alpha_5$. Thus from Property 2 and the equality $|u_1u_2|_b = (\alpha_1 + \beta_1)(|w|_b + 1)$, we have:

  $$\beta_1 n \quad = \quad |u_2u_3u_4u_6|_a + n\alpha_5. \qquad (13)$$

We first consider the case where $u_2u_3 = \epsilon$. The previous equality shows that $|u_4u_6|_a$ is a multiple of n. Moreover the β_1 occurrences of wb in u_1 can be associated to the $\alpha_5 + \alpha_6$ occurrences of $a^n b$ in u_5u_6 and to the $|u_4u_6|_a/n - \alpha_6$ remaining occurrences of a in u_4u_6 to form $\alpha_5 + \alpha_6$ occurrences of wba^nb and $(|u_4u_6|_a - n\alpha_6)/n$ occurrences of wba^n. We have seen as a consequence of Relation (12), that $\beta_5 + |u_4|_b \leq \alpha_1$. Thus $\beta_5 + |u_4|_b$ occurrences of wba^n in u_1 can be associated to some corresponding b in u_4u_5 to form some occurrences of wba^nb in u. Finally we have shown that u is the shuffle of $\alpha_5 + \alpha_6 + \beta_5 + |u_4|_b$ of wba^nb and $(|u_4u_6|_a - n\alpha_6)/n + \alpha_1 - (\beta_5 + |u_4|_b)$ occurrences of wba^n.

We now consider the case where $u_2u_3 \neq \epsilon$. We start exploiting Property 9 : $\frac{|u|_a - |u_1|_a}{n} \geq |u_2|_b + 1$. We already know that $|u_1u_2|_b = (\alpha_1 + \beta_1)(|w|_b + 1)$, so by Property 2, $|u|_a = (\alpha_1 + \beta_1)(|w|_a + n)$. Moreover $|u_1|_a = (\alpha_1 + \beta_1)|w|_a + \alpha_1 n = |u|_a - \beta_1 n$. Thus Property 9 can be rewritten $\beta_1 \geq |u_2|_b + 1$. This means that at least one occurrence of the β_1 occurrences of wb in $u_1b^{|u_2|_b}$ is completely included as a subword in u_1. There exists a subword x_1 of u_1 such that $x_1b^{|u_2|_b} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wb, wba^nb\}}}$, $|x_1|_b = |u_1|_b - |wb|_b$, $|x_1|_a = |u_1|_a - |w|_a$. Let $u'_1 = x_1b^{|u_2|_b}$, $u'_2 = u'_3 = \epsilon$.

If $|u_4|_b \neq 0$, let x_4 be a subword of u_4 with $|x_4|_a = |u_4|_a$, $|x_4|_b = |u_4|_b - 1$ and let $u'_4 = b^{|u_3|_b}x_4$, $u'_5 = u_5$, $u'_6 = u_6$. If $|u_4|_b = 0$, let $u'_4 = b^{|u_3|_b}u_4$. If $|u_4|_b = 0$ and $|u_5|_b - \frac{|u_5|_a}{n} \neq 0$, let u'_5 be the subword of u_5 obtained by erasing the first occurrence of b in u_5 and let $u'_6 = u_6$. If $|u_4|_b = 0$ and $|u_5|_b - \frac{|u_5|_a}{n} = 0$, let $u'_5 = u_5$, $u'_6 = u_6$. Finally let $u' = u'_1u'_2u'_3u'_4u'_5u'_6$.

By the previous construction, the word u is the shuffle of u' and one of the two words wba^n or wba^nb (constituted with a subword wb in u_1, the $|u_2u_3|_a = n$ occurrences of a in u_2u_3, and possibly a b occurring in u_4u_5). We now verify that the words u', u'_1, u'_2, u'_3, u'_4, u'_5, u'_6 satisfy Properties 1 to 9 of the Proposition. We have already said that $u'_1b^{|u'_2|_b} = u'_1 \in \mathrm{L}^{\epsilon}_{\vdash_{\{wb, wba^n\}}}$. We have $|u'_1u'_2|_b = |u_1u_2|_b - (|w|_b + 1)$ and $|u'|_a = |u|_a - (|w|_a + n)$ which gives $|u'_1u'_2|_b(|w|_a + n) = |u'|_a(|w|_b + 1)$. The verification (left to the reader) of Properties 3 to 7 and 9 are immediate.

Let us prove Property 8.

Let $X = |u_5|_b - \frac{|u_5|_a}{n} + |u_3u_4|_b$, $Y = \frac{1}{n}\left[|u_1|_a - \frac{|u_1u_2|_b}{|w|_b+1}|w|_a\right]$, $X' = |u'_5|_b - \frac{|u'_5|_a}{n} + |u'_3u'_4|_b$, $Y' = \frac{1}{n}\left[|u'_1|_a - \frac{|u'_1u'_2|_b}{|w|_b+1}|w|_a\right]$. By Property 9 for u, we have $X \leq Y + 1$ and we want to prove that $X' \leq Y'$. As a consequence of the definition of the words u'_i, it is easily seen that

$$X = X' + 1 \text{ or } X = X'.$$

Moreover, one can easily verify that the last equality occur only if

$$|u_4|_b = |u_5|_b - \frac{|u_5|_a}{n} = 0,$$

which gives

$$X = |u_3|_b.$$

On the other hand, since $|u_1|_a = |u'_1|_a + |w|_a$ and $|u_1u_2|_b = |u'_1u'_2|_b + (|w|_b + 1)$, we have

$$Y = Y'.$$

By the latter equality, $X = X' + 1$ immediately gives $X' \leq Y'$, while, if $X = X'$, by Property 7, $X \leq Y$, that is $X' \leq Y'$.

Thus the words u', u'_1, u'_2, u'_3, u'_4, u'_5, u'_6 satisfy Properties 1 to 9 of the Proposition with $u'_2u'_3 = \epsilon$. By the previous case, $u' \in \mathrm{L}^{\epsilon}_{\vdash_{\{wba^n, wba^nb\}}}$ and so $u \in \mathrm{L}^{\epsilon}_{\vdash_{\{wba^n, wba^nb\}}}$.

Proof of the " only if" part. Let us first note that, by definition of w, there exists an integer i_w between 0 and n such that $wba^nb = a^{i_w}b(a^nb)^{|w|_b+1}$.

Assume u belongs to $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^n, wba^nb\}}}$. There exist unique integers α and β such that any derivation from ϵ to u by $\vdash^*_{\{wba^n, wba^nb\}}$ uses α derivation steps by $\vdash_{\{wba^nb\}}$ and β derivation steps by $\vdash_{\{wba^n\}}$. We have:

$$|u|_a = (\alpha+\beta)(|w|_a + n), \text{ and}$$

$$|u|_b = (\alpha+\beta)(|w|_b + 1) + \alpha.$$

In particular, $|u|_a$ is divisible by $|w|_a + n$ and $|u|_b \geq \frac{|u|_a(|w|_b+1)}{|w|_a+n}$.

Let p be a prefix of w such that $|p|_b = \frac{|u|_a(|w|_b+1)}{|w|_a+n} (= (\alpha+\beta)(|w|_b+1))$, and let s be the word such that $u = ps$. Since $i_w \leq n$, the $(\alpha+\beta)^{\text{th}}$ occurrence of the letter b is preceded by at least $(\alpha+\beta)i_w$ occurrences of the letter a. Let u_1 be the longest prefix of p such that $|u_1|_a \geq (\alpha+\beta)i_w$ and $|u_1|_a - (\alpha+\beta)i_w \bmod n = 0$, and let u_2 be the word such that $p = u_1u_2$: by construction $u_2 = \epsilon$, or, u_2 begins with the letter a and $0 < |u_2|_a < n$. Observe $|u|_a - (\alpha+\beta)i_w = 0 \bmod n$. So we can consider the shortest prefix u_3 of s such that $|u_2u_3|_a = 0 \bmod n$. We observe that if $u_2 = \epsilon$ then $u_3 = \epsilon$, and otherwise $u_3 \neq \epsilon$ and $|u_2u_3|_a = n$.

By Lemma 3.9, there exist words u_4, u_5, u_6 such that $\tilde{s} = \tilde{u}_6\tilde{u}_5\tilde{u}_4$ with $\tilde{u}_6 \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^n,a\}}}$, $\tilde{u}_5 \in \mathrm{L}^{\epsilon}_{\vdash_{\{ba^n,b\}}}$ and $|\tilde{u}_4|_a < n$. Thus $s = u_4u_5u_6$, $|u_4|_a < n$, $u_5 \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,b\}}}$, $u_6 \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,a\}}}$.

Up to now, we have constructed words $u_1, \ldots, u_6$ verifying required Properties 2 to 6. We have $|u_1|_a \bmod n = |u|_a \bmod n = (\alpha+\beta)i_w \bmod n$, $|u_2u_3|_a = 0 \bmod n$ and $|u_5|_a = 0 \bmod n$: thus $|u_4u_6|_a = 0 \bmod n$. We now concentrate our efforts on Properties 1 and 7 to 9. The word ub^β belongs to $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^nb\}}}$ and $|ub^\beta| = (\alpha+\beta)|wba^nb|$. Let us recall that $wba^nb = a^{i_w}b(a^nb)^{|w|_b+1}$. Condition 2.1 of Proposition 3.2 shows that, taking $x = \alpha+\beta = \frac{|ub^\beta|}{|wba^nb|}$, $|p|_a \geq i_wx + n(|p|_b - x)$. But $|p|_a = |ub^\beta|_a - |s|_a = x|wba^nb|_a - |s|_a = x(i_w + (|w|_b+1)n) - |s|_a = xi_w + n|p|_b - |s|_a$. Thus $\frac{|s|_a}{n} \leq x$.

By Proposition 3.2 and Lemma 3.3, we know that ub^β is the shuffle of the $(\alpha+\beta)$ words $(ub^\beta)(i)$ $(1 \leq i \leq \alpha+\beta)$ defined just before Lemma 3.3. Let us recall that $(ub^\beta)(i)$ is the subword of ub^β constituted by the letters in position in $P(i)$. Let $p(i)$ be the subword of p constituted by the letters in position in $P(i) \cap \{1, \ldots, |p|\}$, and let $s(i)$ be the words such that $(ub^\beta)(i) = p(i)s(i)$.

The proof is divided into the following two cases according to the value of $|s|_a \bmod n = |u_3|_a$.

Case $|s|_a = 0 \bmod n$. In particular $u_2 = u_3 = \epsilon$. In this case, Properties 7 and 9 are trivially satisfied.

Let $y = \frac{|s|_a}{n}$. By the construction of the $(ub^\beta)(i)$'s (and in particular of the values of elements of $P(i)$) we have that:

- $p(i) = wba^n$, $s(i) = b$, for $1 \leq i \leq x - y$,
- $p(i) = wb$, $s(i) = a^nb$, for $x - y + 1 \leq i \leq x$.

This implies $p = u_1 b^{|u_2|_b} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wba^n, wb\}}}$ and $sb^{\beta} \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^n b, b\}}}$. In particular we have Property 1.

There exist unique integers α_5 and β_5 such that any derivation from ϵ to u_5 by $\vdash^*_{\{a^n b, b\}}$ uses α_5 derivation steps by $\vdash_{\{a^n b\}}$ and β_5 derivation steps by $\vdash_{\{b\}}$, and there exist unique integers α_6 and β_6 such that any derivation from ϵ to u_6 by $\vdash^*_{\{a^n b, a\}}$ uses α_6 derivation steps by $\vdash_{\{a^n b\}}$ and β_6 derivation steps by $\vdash_{\{a\}}$. In particular, we have $\beta_5 = |u_5|_b - \frac{|u_5|_a}{n}$.

Let us prove that $\beta_5 + |u_4|_b \leq x - y$. By Lemma 3.9, the value of $\alpha_5 + \alpha_6$ is the greatest number z such that $u_4 u_5 u_6$ can be viewed as the shuffle of z occurrences of $a^n b$ with some occurrences of a and some occurrences of b. Due to the fact that $sb^{\beta} = u_4 u_5 u_6 b^{\beta}$ is the shuffle of y occurrences of $a^n b$ and $(x-y)$ occurrences of b, we get $y \leq \alpha_5 + \alpha_6 + \beta$. It follows: $x = |sb^{\beta}|_b = |u_4 u_5 u_6 b^{\beta}|_b = |u_4|_b + \alpha_5 + \beta_5 + \alpha_6 + \beta \geq |u_4|_b + \beta_5 + y$. So $x - y \geq \beta_5 + |u_4|_b$.

Since $p = u_1$, p is the shuffle of $x - y$ occurrences of wba^n and y occurrences of wb. We have $|p|_a = (x-y)(|w|_a + n) + y|w|_a = x|w|_a + n(x-y)$ and $|p|_b = (x-y)|wb|_b + y|wb|_b = x(|w|_b + 1)$. Thus $n(x-y) = |p|_a - \frac{|p|_b |w|_a}{|w|_b + 1}$. Since $u_2 = u_3 = \epsilon$, $p = u_1 u_2$, $\beta_5 + |u_4|_b \leq x - y$ and $\beta_5 = |u_5|_b - \frac{|u_5|_a}{n}$, we have

$$|u_5|_b - \frac{|u_5|_a}{n} + |u_3 u_4|_b \leq \frac{1}{n}\left[|u_1|_a - \frac{|u_1 u_2|_b}{|w|_b + 1}|w|_a\right].$$

Hence Property 8 is proved.

Case $|s|_a \neq 0 \bmod n$. We still have $\alpha + \beta = x \geq \frac{|s|_a}{n}$. Let $y = \lfloor \frac{|s|_a}{n} \rfloor$: $0 \leq y < x$. By construction of the $(ub^{\beta})(i)$'s,

- $p(i) = wba^n$, $s(i) = b$, for $1 \leq i \leq x - y - 1$;
- $p(x-y) = wba^r$, $s(x-y) = a^{n-r}b$ for an integer r, $1 \leq r < n$;
- $p(i) = wb$, $s(i) = a^n b$ for $x - y + 1 \leq i \leq x$.

It follows that $|u_2|_a = r$ and $u_1 b^{|u_2|_b} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wba^n, wb\}}}$. Hence we have proved Property 1.

Let us recall that $s = u_3 u_4 u_5 u_6$ and sb^{β} is the shuffle of the x words $s(i)$. Since $b^{|u_3|_b} u_4 u_5 u_6 b^{\beta}$ is the shuffle of y occurrences of $a^n b$ and $(x-y)$ occurrences of b, by using an argument similar to that of the previous case, we have that $|u_5|_b - \frac{|u_5|_a}{n} + |u_3 u_4|_b \leq x - y$.

Here p is the shuffle of $x - y - 1$ occurrences of wba^n, one occurrence of wba^r and y occurrences of wb. Thus $|u_1 u_2|_a = |p|_a = (x - y - 1)(|w|_a + n) + (|w|_a + r) + y|w|_a$ with $r = |u_2|_a$. So $|u_1|_a = x|w|_a + (x-y)n - n$. Since $x = |u_1 u_2|_b / (|w|_b + 1)$, we get $n(x-y) = |u_1|_a - \frac{|u_1 u_2|_b |w|_a}{|w|_b + 1} + n$. And so, we have Property 8:

$$|u_5|_b - \frac{|u_5|_a}{n} + |u_3 u_4|_b \leq \frac{1}{n}\left[|u_1|_a - \frac{|u_1 u_2|_b}{|w|_b + 1}|w|_a\right] + 1.$$

By construction of the words $s(i)$'s, for all i such that $x - y + 1 \leq i \leq x$, the occurrences of the letter a in $s(i)$ appear in ub^{β} after the occurrences of the letter a in $s(x-y)$. More precisely, for an integer $i \geq x - y + 1$, if the letter a occurs in ub^{β} at two positions j and k with $j \in P(x-y) \cap \{|p|+1, \ldots, |u|\}$, and $k \in P(i) \cap \{|p|+1, \ldots, |u|\}$, then $j < k$. On the other hand, by definition of u_3, the last letter of u_3 is a. Hence for any $i \geq x - y$, each letter b in $s(i)$ cannot occur in u_3, so that $|u_3|_b < x - y$. Therefore, we have

$$|u_3|_b < \frac{1}{n}\left[|u_1|_a - \frac{|u_1u_2|_b}{|w|_b + 1}|w|_a\right] + 1,$$

and Property 7 is proved.

By construction, u_2 starts with the letter a. It follows that u_1 contains all the b's occurring in the $p(i)$'s for $1 \leq i \leq x-y$, and those occurring in the prefix w of the $p(i)$'s for $x - y + 1 \leq i \leq x$, that is, $|u_1|_b \geq (x-y)|wb|_b + y|w|_b = x|wb|_b - y = |u_1u_2|_b - y$ and, hence, $y \geq |u_2|_b$. But $|u_1|_a = x|w|_a + (x-y-1)n = x(|w|_a + n) - (y+1)n = |u|_a - (y+1)n$. Consequently, we have Property 9:

$$\frac{|u|_a - |u_1|_a}{n} \geq |u_2|_b + 1 = |u_2|_b + \overline{\delta}_{u_2u_3,\epsilon}.$$

$\square$

Proof of Proposition 3.12. The proof follows the same scheme of that of Proposition 3.10 but the arguments used here are more technical.

Let $(u_k)_{k\geq 0}$ be a sequence of words in $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^n, wba^nb\}}}$. By Proposition 3.13, for any $k \geq 0$, there exist six words $u_{1,k}$, …, $u_{6,k}$ such that $u_k = u_{1,k} \ldots u_{6,k}$ with

- $u_{1,k}b^{|u_{2,k}|_b} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wb, wba^n\}}}$,
- $|u_{1,k}u_{2,k}|_b(|w|_a + n) = |u_k|_a(|w|_b + 1)$,
- $u_{2,k}u_{3,k} = \epsilon$ or $|u_{2,k}u_{3,k}|_a = n$,
- $|u_{4,k}|_a < n$,
- $u_{5,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^nb, b\}}}$,
- $u_{6,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^nb, a\}}}$,
- $|u_{3,k}|_b \leq \frac{1}{n}\left[|u_{1,k}|_a - \frac{|u_{1,k}u_{2,k}|_b}{|w|_b+1}|w|_a\right]$,
- $|u_{5,k}|_b - \frac{|u_{5,k}|_a}{n} + |u_{3,k}u_{4,k}|_b \leq \frac{1}{n}\left[|u_{1,k}|_a - \frac{|u_{1,k}u_{2,k}|_b}{|w|_b+1}|w|_a\right] + \overline{\delta}_{u_{2,k}u_{3,k},\epsilon}$,
- $\frac{|u_k|_a - |u_{1,k}|_a}{n} \geq |u_{2,k}|_b + \overline{\delta}_{u_{2,k}u_{3,k},\epsilon}$.

Now let us define the following three sequences of integers: for every $k \geq 0$,

$$d_{1,k} = \frac{1}{n}\left[|u_{1,k}|_a - \frac{|u_{1,k}u_{2,k}|_b}{|w|_b + 1}|w|_a\right] - |u_{3,k}|_b,$$

$$d_{2,k} = \frac{1}{n}\left[|u_{1,k}|_a - \frac{|u_{1,k}u_{2,k}|_b}{|w|_b + 1}|w|_a\right] + \overline{\delta}_{u_{2,k}u_{3,k},\epsilon} - \left(|u_{5,k}|_b - \frac{|u_{5,k}|_a}{n} + |u_{3,k}u_{4,k}|_b\right),$$

$$d_{3,k} = \frac{|u|_a - |u_{1,k}|_a}{n} - \left(|u_{2,k}|_b + \overline{\delta}_{u_{2,k}u_{3,k},\epsilon}\right).$$

By hypothesis, $\vdash^*_{\{wba^n,wb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^n,wb\}}}$, and by Proposition 3.8, $\vdash^*_{\{a^nb,b\}}$ (resp. $\vdash^*_{\{a^nb,a\}}$) is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,b\}}}$ (resp. $\mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,a\}}}$).

By the fact that the subsequence ordering is a wqo on A^* and by taking a suitable subsequence of $(u_k)_{k\geq 0}$, we can assume that, for all $k \geq 0$, the following conditions are satisfied:

- $u_{1,k} \vdash^*_{\{wba^n,wb\}} u_{1,k+1}$,
- $u_{i,k}$ is a subword of $u_{i,k+1}$, for $i = 2, 3, 4$,
- $|u_{i,k}|_a = |u_{i,k+1}|_a$, for $i = 2, 3, 4$,
- $u_{5,k} \vdash^*_{\{a^nb,b\}} u_{5,k+1}$,
- $u_{6,k} \vdash^*_{\{a^nb,a\}} u_{6,k+1}$,
- $d_{i,k}$ is non-decreasing for $i = 1, 2, 3$.

We have $|u_{2,k}u_{3,k}|_a = |u_{2,k+1}u_{3,k+1}|_a$ and so $\overline{\delta}_{u_{2,k}u_{3,k},\epsilon} = \overline{\delta}_{u_{2,k+1}u_{3,k+1},\epsilon}$.

From the previous conditions, for any $k \geq 0$, we can easily deduce the existence of words $v_{1,k}, v_{2,k}, v_{3,k}, v_{4,k}, v_{5,k}, v_{6,k}$, such that

$$u_{i,k+1} \in u_{i,k} \sqcup\!\sqcup\, v_{i,k}, \quad v_{1,k}b^{|v_{2,k}|_b} \in \mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^n\}}}, \quad |v_{i,k}|_a = 0,$$

for $i = 2, 3, 4$ and

$$v_{5,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,b\}}}, \quad v_{6,k} \in \mathrm{L}^{\epsilon}_{\vdash_{\{a^n,a\}}}.$$

Let $v'_{1,k} = v_{1,k}b^{|v_{2,k}|_b}$, $v'_{2,k} = \epsilon$, $v'_{3,k} = \epsilon$, $v'_{4,k} = b^{|v_{3,k}|_b}v_{4,k}$, $v'_{5,k} = v_{5,k}$ and $v'_{6,k} = v_{6,k}$.

By using an argument similar to that of the proof of Proposition 3.10, we can deduce that, for all $k \geq 0$, the words $v_k = v_{1,k} \ldots v_{6,k} = v'_{1,k} \ldots v'_{6,k}$ satisfy all the properties of Proposition 3.13, and therefore $v'_k \in \mathrm{L}^{\epsilon}_{\vdash_{\{wba^n,wba^nb\}}}$. This implies that, for all $k \geq 0$, $v_k \in \mathrm{L}^{\epsilon}_{\vdash_{\{wba^n,wba^nb\}}}$. Since, for all $k \geq 0$, $u_{k+1} \in u_k \sqcup\!\sqcup\, v_k$, the latter implies that $u_k \vdash^*_{\{wba^n,wba^nb\}} u_{k+1}$, that is $\vdash^*_{\{wba^n,wba^nb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wba^n,wba^nb\}}}$. □

3.7 Proof of the "if" part of Theorem 1.1

From the results of the previous section we can deduce:

Theorem 3.14 *For any integers* $n, m \geq 1$*, and for any word* w *in* $a^{\leq n}(ba^n)^*b \cup \{\epsilon\}$ *such that* wa^nba^m *is a good word, one has:*

1. $\vdash^*_{\{wa^n,wa^nb\}}$ *is a wqo on* $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^n,wa^nb\}}}$*;*
2. $\vdash^*_{\{wa^nb,wa^nba^m\}}$ *is a wqo on* $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^nb,wa^nba^m\}}}$*.*

Proof. We act by induction on $|w|_b$.

When $|w|_b = 0$, $w = \epsilon$ and we know by Proposition 3.8 that $\vdash^*_{\{a^n,a^nb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{a^n,a^nb\}}}$. By Proposition 3.10, we deduce that $\vdash^*_{\{a^nb,a^nba^m\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{a^nb,a^nba^m\}}}$.

Assume now $|w|_b \geq 1$. Then $w = a^hb$ with $0 \leq h \leq n$ or $w = w'a^nb$ with $w' \in a^{\leq n}(ba^n)^*b$. If $w = b$, then by Proposition 3.8, $\vdash^*_{\{b,ba^n\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{b,ba^n\}}}$. In the other cases, by inductive hypothesis, $\vdash^*_{\{w,wa^n\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w,wa^n\}}}$. So in all cases by Proposition 3.12, $\vdash^*_{\{wa^n,wa^nb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^n,wa^nb\}}}$, and by Proposition 3.10, we deduce that $\vdash^*_{\{wa^nb,wa^nba^m\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{wa^n,wa^nba^m\}}}$. ☐

Corollary 3.15 *Let* $n \geq 1$ *be an integer. For any word* w *in* $a^{\leq n}(ba^n)^*ba^{\leq n}$*,* $\vdash^*_{\{w\}}$ *is a wqo on* $\mathrm{L}^{\epsilon}_{\vdash_{\{w\}}}$*.*

Proof. The result is immediate if $|w|_b = 0$. Assume from now on $|w|_b > 0$.

First we consider the case where w ends with b. Two cases are possible: $w = a^mb$ with $1 \leq m \leq n$ or $w = w'ba^nb$ with w' in $a^{\leq n}(ba^n)^*$. If $w = a^mb$, the result is stated in Proposition 3.4.

Assume $w = w'ba^nb$. By Theorem 3.14, we know that $\vdash^*_{\{w'ba^n,w'ba^nb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^n,w'ba^nb\}}}$. Let $(u_k)_{k\geq 0}$ be a sequence of words in $\mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^nb\}}}$. Since $\mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^nb\}}} \subseteq \mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^n,w'ba^nb\}}}$, $u_k \in \mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^n,w'ba^nb\}}}$ and so we can replace the sequence $(u_k)_{k\geq 0}$ by a subsequence such that $u_k \vdash^*_{\{w'ba^n,w'ba^nb\}} u_{k+1}$ for each $k \geq 0$. For any k this means there exists a word v_k in $\mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^n,w'ba^nb\}}}$ such that $u_{k+1} \in u_k \sqcup\!\sqcup\, v_k$. The word v_k is the shuffle of α_k occurrences of $w'ba^n$ and β_k occurrences of $w'ba^nb$, and the words u_k and u_{k+1} are the shuffle of γ_k and γ_{k+1} occurrences of $w'ba^nb$ respectively. From $|v_k|_a = |u_{k+1}|_a - |u_k|_a$ and $|v_k|_b = |u_{k+1}|_b - |u_k|_b$, we deduce respectively $\alpha_k + \beta_k = \gamma_{k+1} - \gamma_k$ and $(\gamma_{k+1} - \gamma_k)|w'ba^nb|_b = (\alpha_k + \beta_k)|w'ba^nb|_b - \alpha_k$ which imply $\alpha_k = 0$, that is, $v_k \in \mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^nb\}}}$. Hence $u_k \vdash^*_{\{w'ba^nb\}} u_{k+1}$, so that $\vdash^*_{\{w'ba^nb\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w'ba^nb\}}}$.

Now we consider the case where w ends with a so that $w = w'ba^m$ with $w' \in a^{\leq n}(ba^n)^* \cup \{\epsilon\}$ and $n \geq m \geq 1$. By Theorem 3.14(2), $\vdash^*_{\{w'b,w'ba^m\}}$ is a wqo on $\mathrm{L}^{\epsilon}_{\vdash_{\{w'b,w'ba^m\}}}$. The proof ends as in the previous case. ☐

We are now able to prove the "if" part of Theorem 1.1.

Proof of the "if" part of Theorem 1.1. Assume w is a word such that w, $\tilde{w}$, $E(w)$ and $E(\tilde{w})$ have no factor of the two possible forms 1 and 2 of Definition 1. By Lemma 3.1, we know that

$$w \in \{\epsilon\} \cup \bigcup_{n \geq 0} a^{\leq n}(ba^n)^* ba^{\leq n} \cup \bigcup_{n \geq 0} b^{\leq n}(ab^n)^* ab^{\leq n}.$$

The result is trivial if $|w|_a = 0$ or $|w|_b = 0$ and stated by Corollary 3.15 if $w \in a^{\leq n}(ba^n)^* ba^{\leq n}$ with $n \geq 1$. The case $w \in b^{\leq n}(ab^n)^* ab^{\leq n}$ with $n \geq 1$ is treated as the previous case by exchanging the role of a and b. □

References

[1] F. D'Alessandro and S. Varricchio. On Well Quasi-orders On Languages. In *Proceedings of DLT 2003 (Szeged, Hungary)*, volume 2710 of *Lecture Notes in Computer Science*, pages 230–241. Springer-Verlag, Berlin, 2003.

[2] F. D'Alessandro and S. Varricchio. Well quasi-orders and context-free grammars. *Theoretical Computer Science*, 327(3):255–268, 2004.

[3] F. D'Alessandro and S. Varricchio. Well quasi-orders, unavoidable sets, and derivation systems. *RAIRO Theoretical Informatics and Applications*, to appear.

[4] A. de Luca and S. Varricchio. Well quasi-orders and regular languages. *Acta Informatica*, 31:539–557, 1994.

[5] A. de Luca and S. Varricchio. *Finiteness and regularity in semigroups and formal languages.* EATCS Monographs on Theoretical Computer Science. Springer, Berlin, 1999.

[6] A. Ehrenfeucht, D. Haussler, and G. Rozenberg. On regularity of context-free languages. *Theoretical Computer Science*, 27:311–332, 1983.

[7] T. Harju and L. Ilie. On quasi orders of words and the confluence property. *Theoretical Computer Science*, 200:205–224, 1998.

[8] D. Haussler. Another generalization of Higman's well quasi-order result on Σ^*. *Discrete Mathematics*, 57:237–243, 1985.

[9] G. H. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.*, 3:326–336, 1952.

[10] L. Ilie and A. Salomaa. On well quasi orders of free monoids. *Theoretical Computer Science*, 204:131–152, 1998.

[11] B. Intrigila and S. Varricchio. On the generalization of Higman and Kruskal's theorems to regular languages and rational trees. *Acta Informatica*, 36:817–835, 2000.

[12] M. Ito, L. Kari, and G. Thierrin. Shuffle and scattered deletion closure of languages. *Theoretical Computer Science*, 245(1):115–133, 2000.

[13] M. Jantzen. Extending regular expressions with iterated shuffle. *Theoretical Computer Science*, 38:223–247, 1985.

[14] J. Kruskal. The theory of well-quasi-ordering: a frequently discovered concept. *J. Combin. Theory, Ser. A*, 13:297–305, 1972.

[15] M. Kunc. The power of commuting with finite sets of words. In *Proceedings of STACS 2005 (Stuttgart, Germany)*, volume 3404 of *Lecture Notes in Computer Science*, pages 569–580. Springer-Verlag, Berlin, 2005.

[16] M. Kunc. Regular solutions of language inequalities and well quasi-orders. *Theoretical Computer Science*, 348(2-3):277–293, 2005.

[17] Lothaire. *Combinatorics on words*, volume 17 of *Series Encyclopedia of Mathematics and its Applications.* Addison-Wesley, Reading, Mass., 1983.

[18] L. Puel. Using unavoidable sets of trees to generalize Kruskal's theorem. *J. Symbolic Comput.*, 8(4):335–382, 1989.