THE AVERAGE SIZE OF GIANT COMPONENTS BETWEEN THE DOUBLE-JUMP

VLADY RAVELOMANANA

ABSTRACT. We study the sizes of connected components according to their excesses during a random graph process built with n vertices. The considered model is the *continuous* one defined in [16]. An ℓ-component is a connected component with ℓ edges more than vertices. ℓ is also called the *excess* of such component. As our main result, we show that when ℓ and $\frac{n}{\ell}$ are both large, the expected number of vertices that ever belong to an ℓ-component is about $12^{1/3}\ell^{1/3}n^{2/3}$. We also obtain limit theorems for the number of creations of ℓ-components.

Random graphs; giant components; double-jump; probabilistic/analytic combinatorics.

1. Introduction

Following Erdős and Rényi's pioneering works around 1960 [9, 10], random graphs have been the subject of intense studies for four decades. Topics on random graphs provide a large and particularly active body of research. We refer to the books of Bollobás [4], of Kolchin [20], and of Janson, Łuczak and Ruciński [18] for excellent treatises related to these subjects.

We consider here labelled graphs on vertex set $V = \{1, 2, \ldots, n\}$ with undirected edges without self-loops or multiple edges. The set of all such graphs is denoted by $\mathcal{G}^n$ and, a random graph is defined by a pair $(\mathcal{G}^n, P)$ where P is a probability distribution over $\mathcal{G}^n$. Let us recall the three popular processes of random graphs in the literature. The first one, $\{\mathbb{G}(n, M)\}_{0\leq M\leq \binom{n}{2}}$, consists of all graphs with vertex set $V = \{1, 2, \ldots, n\}$ having M edges, in which one can randomly pick a graph with the same probability. Thus, with $N = \binom{n}{2}$, we have $0 \leq M \leq N$ and the random graph $\mathbb{G}(n, M)$ has $\binom{N}{M}$ elements with each element occurring with probability $\binom{N}{M}^{-1}$. Secondly, $\{\mathbb{G}(n, p)\}_{0\leq p\leq 1}$, consists of all graphs with the same vertex set $V = \{1, 2, \ldots, n\}$ in which each of the N edges is drawn independently with probability p. The third process, $\{\mathbb{G}(n, t)\}_{0\leq t\leq 1}$ (cf. [15, 16]), may be constructed by letting each edge e, chosen amongst the N possible edges, appear at random time T_e, where T_e are independent random variables uniformly distributed on $[0, 1]$. The random graph $\mathbb{G}(n, t)$ is constructed with all edges e such that $T_e \leq t$. The main difference between $\{\mathbb{G}(n, M)\}_{0\leq M\leq \binom{n}{2}}$ and $\{\mathbb{G}(n, t)\}_{0\leq t\leq 1}$ is that in the first one, edges are added at fixed (slotted) times $1, 2, \ldots, N$ so at any time T we obtain a random graph with n vertices and T edges, whereas in $\{\mathbb{G}(n, t)\}_{0\leq t\leq 1}$ the edges are added at random times. At time $t = 0$, we have a graph with n vertices and 0 edges, and as the time advances all edges e with r.v. T_e

such that $T_e \leq t$ (where t is the current time), are added to the graph until t reaches 1 in which case, one obtains the complete graph K_n.

Following our predecessors [15, 16, 17, 21], let us define the *excess* or the *complexity* of a connected graph as the difference between its number of edges and its number of vertices. Throughout this paper, as the random graph process proceeds, we will often *fix* and study an arbitrary chosen connected component built with $k \leq n$ vertices (where n is the total number of vertices) in the graph. For $\ell \geq -1$, a $(k, k+\ell)$ connected graph is one having k vertices and $k+\ell$ edges, thus its excess is ℓ and we simply called it an *ℓ-component*. A random graph process begins with a set of n isolated vertices. Then, as evolution proceeds, edges are added at random (drawn without replacement) and at first, all components created are trees ((-1)-components), later 0-components (also called *unicyclic* components) will appear and eventually the first ℓ-components are created, with $\ell > 0$. Usually, ℓ-components are called *complex* whenever $\ell > 0$.

As more edges are added, a complex component gradually swallows up some other "simpler" components, and it is worth-noting that with nonzero probability, at least two components can co-exist as the random graph evolves [15, 17]. We denote by $V_n^{(\ell)}$ the number of vertices that at some stage of the random graph process belong to an ℓ-component.

In this paper, we consider the *continuous time* random graph process $\{\mathbb{G}(n,t)\}_{0 \leq t \leq 1}$, and we will study the creation of $(\ell+1)$-components ($\ell \geq 0$). We can observe that there are two manners to create a new $(\ell+1)$-component during the random graph process :
- either by adding an edge inside an existing ℓ-component,
- or by joining with the last added edge a p-component to an $(\ell-p)$-component, with $p \geq 0$.

Following Janson's notations [16], the first transition will be denoted $\ell \to \ell+1$ and the second one $(\ell - p) \oplus p \to \ell + 1$.
We study the random variable $X_n^{(\ell)}$, defined as the number of creations of $(\ell+1)$-components during the evolution of the random graph. As in [15], denote respectively, by $Y_n^{(\ell)}$ and $Z_n^{(\ell)}$ the number of $(\ell+1)$-components created by the two ways described above. More precisely, $Y_n^{(\ell)}$ equals the number of edges added inside an ℓ-component creating an $(\ell+1)$-component, and $Z_n^{(\ell)}$ is the number of bridges added between a p-component and an $(\ell-p)$-component, for all $0 \leq p \leq \ell$ during the evolution of the graph. Thus, by construction $X_n^{(\ell)} = Y_n^{(\ell)} + Z_n^{(\ell)}$.

1.1. **Related works.** In a former paper, Janson [15] obtained limit theorems for the number of *complex components*, i.e., components with more than one cycle, created during the evolution of the graph. In particular, Janson computed the probability that the process never contains more than one complex component is approximately 0.87 (as the number of vertices n tends to infinity). Thus, at least two complex components can co-exist in the random graph and there is not a zero-one law for this process. With the notations of our paper, Janson obtained limit laws for $X_n^{(\ell)}, Y_n^{(\ell)}$ and $Z_n^{(\ell)}$ for $\ell = 1$ (see for instance [15] for precise statements of his results). Using enumerative and analytical methods, Janson, Knuth, Pittel and Łuczak [17] obtained also the exact value $5\pi/18 = 0.872\cdots$ for the limit described above.

In [2, 3], Bender *et al.* studied several properties of labelled graphs. They computed the asymptotic number of connected graphs with k vertices and $k + \ell(k)$ edges for every function $\ell(k)$ as $k \to \infty$. Define a *bridge* or a *cut edge* of a connected component as an edge whose deletion will deconnect the graph. Working in the probability space of connected components, Bender, Canfield and McKay also obtained the asymptotic probability for a random chosen edge to be a bridge. See for instance [3, Section 5].

Speaking about the largest component in $\mathbb{G}(n, M)$, Erdős and Rényi [10] suggested that a "*double jump*" occurs: the largest component changes its size (with respect to the number of vertices n) twice – from $O(\log n)$ to $O_p(n^{2/3})$ – and then from $O_p(n^{2/3})$ to $O(n)$. Note that we use here the notation $X_n = O_p(a_n)$ (e.g. [18, p. 10]): For a r.v. X_n and real positive numbers a_n, we have $X_n = O_p(a_n)$ as $n \to \infty$ if $\forall \delta > 0$ there exist constants c_δ and n_0 s.t. $\mathbb{P}(\|X_n\| \leq c_\delta a_n) > 1 - \delta$, for $n \geq n_0$. In particular, Erdős and Rényi expected that whatever function $M \equiv M(n)$ we choose, the largest component of $\mathbb{G}(n, M)$ can only be either $O(\log n)$ or $O_p(n^{2/3})$ or $O(n)$. In the latter case and for the Bernoulli random graph $\mathbb{G}(n, p)$, for $p = c/n$ with $c > 1$, Barraez, Boucheron and De la Vega [1] have studied precisely the size of the giant component. We refer also to [5] where, among other results, O'Connell has investigated the size of the giant component by means of large deviation principles. Therefore, under the Bernoulli model, it is known that for $p = c/n$ with $c > 1$, the size of the largest connected component, denoted V_n is asymptotically an, where $a > 0$ satisfies $a = 1 - e^{-ac}$ and the sequence V_n/n converges in probability to a. Bender, Canfield and McKay [3] have also determined the probability that a random graph produced under the $\{\mathbb{G}(n, p)\}$ process is connected as well as the asymptotic distribution of the number of edges of such a graph (conditioned on connectedness). Pittel and Wormald [21] presented an alternative *inside-out* approach based on the enumeration of graphs of minimum degree 2. In particular, they obtained the asymptotic number of connected graphs with n vertices and M edges [21, Theorem 3], as well as the joint limiting distribution of the *size* of the 2-*core* (number of vertices of degree at least 2) of the giant component, its *excess* (number of edges minus number of vertices) and the size of its *tree mantle* (number of vertices of the giant component not in its 2-core). Their results hold for the two models of random graphs $\mathbb{G}(n, p)$ and $\mathbb{G}(n, M)$ in the so-called *supercritical case*, i.e., when $n^{1/3}(np - 1) \to \infty$ or $n^{1/3}(2M/n - 1) \to \infty$.

1.2. **Our results.** The kind of problems discussed here are in essence combinatorial. And as already noticed by Janson in [16], combinatorics and probability theory are closely related in such a way that the combination of both approaches can help to study the extremal characteristics of indecomposable structures typified by random graphs.

In order to study the random variables $X_n^{(\ell)}$, $Y_n^{(\ell)}$ and $Z_n^{(\ell)}$, we will use the method of moments (e.g. [18, page 144]). We will investigate the factorial moments $\mathbb{E}(Y_n^{(\ell)})_m$ (resp. $\mathbb{E}(Z_n^{(\ell)})_m$) starting with the simplest cases, viz. the expectations. We will rely $Y_n^{(\ell)}$ and $Z_n^{(\ell)}$ by means of enumerative/analytic tools such as those developed in [11, 17] and in [2, 3]. First, we observe that $(Y_n^{(\ell)})_m$ is the number of ordered m-tuples of edges that are added to ℓ-components (both ends of the edges are in the components) during the evolution of the random graph process. Similarly, $(Z_n^{(\ell)})_m$ is the number of ordered m-tuples of edges that are added between pairs of disjoint complex components to build an $(\ell+1)$-component. As we shall see $(Y_n^{(\ell)})_m$ can be deduced using asymptotic results namely from [2] and [21]. Therefore, our first task is to quantify the number of manners to build an $(\ell+1)$-component arising from the second type of transition.

More precisely, for the *Wright's range*, i.e. for connected components built with k vertices and $k + o(k^{1/3})$ edges (this is the same range as in [29]), we will use the analytical tools associated to the generating functions of Cayley's rooted trees [7], $T(z)$, which plays an important role in the enumerative point of view of the general theory of random graphs (cf. the "giant paper" [17]). Next, for excesses greater then $o(k^{1/3})$, we will use the results of Bender, Canfield and McKay in [2].

• As a first result, we obtain Theorem 1.1 which is closely related to the r.v. $Z_n^{(\ell)}$ defined above. We prove that *almost all* $(\ell+1)$-components whose *last added edge* forms a bridge (or a cut edge) between a p-component and an $(\ell-p)$-component, for $0 \leq p \leq \ell$, are built by linking a unicyclic component to an ℓ-component. In fact, we have the following theorem:

Theorem 1.1. *Denote by* $c(r,s)$ *the number of connected graphs with* r *vertices and* s *edges. As* $k, \ell \to \infty$ *and* $\ell \ll k$ *the number of ways,* $c'(k, k+\ell+1)$*, to build an* $(\ell+1)$*-component of order* k *with a distinguished cut edge between a* p*-component and an* $(\ell-p)$*-component,* $p \geq 0$*, satisfies*

$$\begin{aligned} c'(k,k+\ell+1) &= \frac{1}{2}\sum_{p=0}^{\ell}\sum_{t=1}^{k-1}\binom{k}{t} t(k-t)\, c(t,t+p)\ c(k-t,k-t+\ell-p) \\ &= \frac{k^2}{6\ell}\, c(k,k+\ell)\ \left(1+O\left(1/\ell\right)+\nu(\ell,k)\right)\,, \end{aligned} \tag{1}$$

where $\nu(\ell,k)$ *satisfies for* $1 \ll \ell \ll k$

$$\begin{cases} \textbf{(i)} \quad \nu(\ell,k) = O\left(\sqrt{\ell^3/k}\right)\ , \text{ if } \ell = o(k^{1/3}) \\ \textbf{(ii)} \quad \nu(\ell,k) = O\left(\sqrt{\frac{\ell}{k}}\right) + O\left(\frac{\ell^{1/16}}{k^{9/50}}\right)\ , \text{ if } \lim_{k\to\infty}\frac{\ell^3}{k} \neq 0 \text{ and } \ell \ll k\,. \end{cases} \tag{2}$$

Note here that our results differ from those in [3], since we are interested in edges whose additions during the random graph process, increase the complexity of some connected components (whereas in [3] the results are more general but all edges in a given connected component are considered with the same probability).

Note also that Theorem 1.1 will be used to compare the r.v. $Y_n^{(\ell)}$ and $Z_n^{(\ell)}$. We follow the probabilistic methods initiated by Janson and combine them with the enumerative/analytic methods to study the moments of the r.v. $X_n^{(\ell)}, Y_n^{(\ell)}$ and $Z_n^{(\ell)}$ described above, for values of ℓ and n s.t. $\ell, n \to \infty$ but $\ell = o(n)$. More precisely, to obtain the results presented here, methods of the probabilistic random graph process $\{\mathbb{G}(n,t)\}_{0\leq t\leq 1}$, studied in [15, 16], are combined with asymptotic enumeration methods, developed by Wright in [27, 29] and by Bender, Canfield and McKay in [2, 3].

• We turn on the expectations of the size and growth of components according to ℓ and find:

Theorem 1.2. *Let $V_n^{(\ell)}$ be the number of vertices that at some stage of the random graph process belong to an ℓ-component. As $n, \ell \to \infty$, but $\ell = o(n)$, we have*

$$\mathbb{E}(V_n^{(\ell)}) \sim (12\ell)^{1/3}\, n^{2/3}\,. \tag{3}$$

Let $X_n^{(\ell)}$ be the r.v. defined as the number of creations of $(\ell+1)$-components during the evolution of the random graph and denote by $Y_n^{(\ell)}$ (resp. $Z_n^{(\ell)}$) the number of $(\ell+1)$-components created by the transition $\ell \to \ell+1$ (resp. $(\ell-p)\oplus p \to \ell+1$, $\ell \geq p \geq 0$) then as $n, \ell, \frac{n}{\ell} \to \infty$

$$\mathbb{E}(X_n^{(\ell)}) \sim \mathbb{E}(Y_n^{(\ell)}) \sim 1 \ \text{ and } \mathbb{E}(Z_n^{(\ell)}) = O\left(\frac{1}{\ell}\right)\,. \tag{4}$$

• We then obtain for the number of $(\ell+1)$-components, for $1 \ll \ell \ll n$, created during the evolution of the graph:

Theorem 1.3. *Provided that the newly created $(k, k+\ell+1)$ component satisfies $\ell = o(k^{1/3})$ then $Y_n^{(\ell)} \xrightarrow{\mathbf{d}} 1$ and $Z_n^{(\ell)} \xrightarrow{\mathbf{d}} 0$.*

Note that in [17, Section 16, Theorem 9], the authors obtained the asymptotic probability that a random graph of a given configuration evolves to another configuration (see for instance [17, Section 16, Figure 1]). Among other results, they observed the evolution of complex components and proved that the probability that an evolving graph acquires exactly $i \geq 1$ new complex components converges to ${p'}_i$ with ${p'}_1 \approx 0.87266$, ${p'}_2 \approx 0.12120$, ${p'}_3 \approx 0.00598$, ${p'}_4 \approx 0.00015$ (cf. [17, Eq (27.15)]). In other words, the probability that an evolving graph never has more than 4 complex components is strictly greater than 0.999998. Theorem 1.3 confirms this general tendency and we give here an alternative method, connecting the one from generating functions initiated in [27] to those in [15].

1.3. **Outline of the paper.** This paper is organized as follows. The next section gives the enumerative results of this paper (namely the proof of theorem 1.1). In section 3, we compute the expectations of the creations of $(\ell+1)$-component as well as the expected number of vertices that ever belong to such components. Section 4 offers the results about the moments of the random variables $Y_n^{(\ell)}$ and $Z_n^{(\ell)}$. The limit distributions are obtained when studying the factorial moments of these variables.

2. Enumerating complex graphs with distinguished bridge

As mentioned in paragraph 1.2, to investigate $(Z_n^{(\ell)})_m$, i.e., the number of ordered m-tuples of edges added between pairs of complex components to build an $(\ell+1)$-component, we will use tools from enumerative/analytic methods.

The enumeration of connected labelled graphs goes back to Cayley. Denote by $T(z)$ the well-known exponential generating function (EGF) of Cayley's rooted trees [7], we have

$$T(z) = z \exp\left(T(z)\right) = \sum_{n=1}^{\infty} \frac{n^{n-1} z^n}{n!}\,, \tag{5}$$

where the variable z is associated to the labelled vertices. (EIS **A000169**[1]).

Next, Rényi [23] found the EGF W_0 of unicyclic graphs.

$$W_0(z) = -\frac{1}{2} \ln\left(1 - T(z)\right) - \frac{T(z)}{2} - \frac{T(z)^2}{4}\,. \tag{6}$$

More generally, Wright [27] found a recurrence formula satisfied by the EGFs of ℓ-components. Denote by $W_\ell(w, z)$ the bivariate EGF of ℓ-components where the variable w marks the number of edges and the variable z the vertices. Thus, if $c(n, n+\ell)$ is the number of $(n, n+\ell)$ connected graphs with n vertices, we can write

$$W_\ell(w, z) = \sum_n c(n, n+\ell) w^{n+\ell} \frac{z^n}{n!} \tag{7}$$

and Wright's recurrence formula [27] can be stated as follow :

$$\vartheta_w W_{\ell+1} = w\Big(\frac{{\vartheta_z}^2 - \vartheta_z}{2} - \vartheta_w\Big) W_\ell + \frac{w}{2} \left(\sum_{p=-1}^{\ell+1} (\vartheta_z W_p)(\vartheta_z W_{\ell-p}) \right)\,, \tag{8}$$

where we denote by ϑ_w, resp. ϑ_z, the differential operator $w\frac{\partial}{\partial w}$, resp. $z\frac{\partial}{\partial z}$. Thus, the operator ϑ_w corresponds to marking an edge present in a graph. Similarly, ϑ_z corresponds to marking a vertex. The combinatorial pointing operator reflects the distinction of an object among all the others. For the use of pointing and marking, we refer to [13] and for general techniques concerning graphical enumerations we refer to [14]. All these EGFs are given and explained in details in [17]. In terms of coefficients, (8) reads

$$\begin{aligned} (k+\ell+1)\, c(k, k+\ell+1) &= \left(\binom{k}{2} - k - \ell \right) c(k, k+\ell) \\ &+ \frac{1}{2} \sum_{t=1}^{k-1} \sum_{p=-1}^{\ell+1} \binom{k}{t} t(k-t)\, c(t, t+p)\; c(k-t, k-t+\ell-p)\,. \end{aligned} \tag{9}$$

Starting with the differential equation (8), Wright [27, 29] proved that each W_ℓ can be written as :

$$W_\ell(z) = \frac{b_\ell}{(1-T(z))^{3\ell}} - \frac{c_\ell}{(1-T(z))^{3\ell-1}} + \sum_{2 \le s \le 3\ell-2} \frac{\omega_{\ell,s}}{(1-T(z))^s}\,, \quad (\ell \ge 1)\,, \tag{10}$$

where the coefficients (b_ℓ) and (c_ℓ) are rationals and more importantly, the summation is *finite*. (Sequences for ℓ-components are given by EIS **A061540** — EIS **A061544** for respectively $\ell = 1, 2, \cdots, 5$). The $(b_\ell)_{\ell \ge 1}$ are called the Wright's constants of first order (also called

[1]References to EIS correspond to specific entries in [24].

Wright-Louchard-Takács constants, see [25]); $b_1 = \frac{5}{24}$ and for $\ell \geq 1$, b_ℓ is defined recursively by

(11)
$$2(\ell+1)b_{\ell+1} = 3\ell(\ell+1)b_\ell + 3\sum_{p=1}^{l-1} t(\ell-p)b_p b_{l-p}\,.$$

Note that the sequence (c_ℓ) in (10) verifies also the following :

$$2(3\ell+2)c_{\ell+1} = 8(\ell+1)b_{\ell+1} + 3\ell b_\ell + (3\ell+2)(3\ell-1)c_\ell + 6\sum_{p=1}^{l-1} p(3\ell-3p-1)b_t c_{l-p}\,.$$

To study the asymptotic behavior of the coefficients $c(k, k+\ell)$, Wright [29] established that[2]:

(12)
$$\frac{b_\ell}{(1-T(z))^{3\ell}} - \frac{c_\ell}{(1-T(z))^{3\ell-1}} \preceq W_\ell(z) \preceq \frac{b_\ell}{(1-T(z))^{3\ell}}\,,$$

which we shall call *Wright's inequalities.*

We are interested in the number of creation of $(\ell+1)$-components. In this Section, we will study edges added between a p-component and a $(\ell-p)$-component, with $p \geq 0$. Thus, we have to investigate the number of manners to build a component with a distinguished cut edge. The Theorem 1.1 gives an estimate of the number of such combinatorial structures. It will be proved later since its proof involves the decomposition of the Wright's EGFs by means of inverse powers of $(1-T(z))$. In fact, Knuth and Pittel [19] studied combinatorially and analytically the polynomial $t_n(y)$ defined as follows

(13)
$$t_n(y) = n!\,[z^n]\,\frac{1}{\big(1-T(z)\big)^y}\,,$$

which they call *tree polynomial.* The two authors observed that the analysis of these polynomials can be used to study random graphs analytically as shown in [11, 17]. For our purpose, a very similar formula can be defined :

(14)
$$t_{a,n}(y) = n!\,[z^n]\,\frac{T(z)^a}{\big(1-T(z)\big)^y}\,.$$

The lemma below is an application of the saddle point method [6, 12] to study the asymptotic behavior of the coefficients $t_{a,n}(m) = n!\,[z^n]\,T(z)^a(1-T(z))^{-m(n)}$ as m, n tend to infinity but $m \equiv m(n) = o(n)$.

Lemma 2.1. *Let $\rho \equiv \rho(n)$ such that $\rho \to 0$ as $n \to \infty$ but $\rho n \to \infty$, and let a and β be fixed numbers. Then, $t_{a,n}(\rho n + \beta)$ defined in (14) satisfies*

(15)
$$t_{a,n}(\rho n+\beta) = \frac{n!}{2\sqrt{\pi n}}\frac{\exp(nu_0)(1-u_0)^{(1-\beta)}}{{u_0}^n(1-u_0)^{\rho n}}\left(1 + O\Big(\sqrt{\rho}\Big) + O\Big(\frac{1}{\rho^{1/4}n^{1/4}}\Big)\right)$$

where $u_0 = 1 + \frac{\rho}{2} - \sqrt{\rho(1+\frac{\rho}{4})} = 1 - \sqrt{\rho} + \frac{\rho}{2} - \frac{\rho^{3/2}}{8} + O(\rho^2)$.

[2] Remark that if $A(z)$ and $B(z)$ are two formal power series, the notation $A(z) \preceq B(z)$ means that $\forall n$, $[z^n]\,A(z) \leq [z^n]\,B(z)$.

Proof. Cauchy's integral formula gives (if we made the substitution $u = T(z)$ so that $dz = e^{-u}(1-u)du$).

$$\begin{aligned}
t_{a,n}(\rho n+\beta) &= n!\,[z^n]\,\frac{T(z)^a}{\big(1-T(z)\big)^{(\rho n+\beta)}} \\
&= \frac{n!}{2\pi i}\oint \frac{T(z)^a}{\big(1-T(z)\big)^{\rho n+\beta}}\frac{dz}{z^{n+1}} \\
&= \frac{n!}{2\pi i}\oint \frac{e^{nu}\,du}{(1-u)^{\rho n+\beta-1}\,u^{n-a+1}}\,. \qquad (16)
\end{aligned}$$

The power $(\exp(u)/(1-u)^{\rho}/u)^n$ suggests us to use the saddle point method. Let

$$h(u) = u - \ln(u) - \rho\ln(1-u)\,. \qquad (17)$$

We then have

$$t_{a,n}(\rho n+\beta) = \frac{n!}{2\pi i}\oint \frac{(1-u)^{1-\beta}}{u^{1-a}}\exp(nh(u))du\,. \qquad (18)$$

Investigating the roots of $h^{'}(u) = 0$, we find two saddle points, at

$$u_0 = 1+\frac{\rho}{2}-\sqrt{\rho(1+\frac{\rho}{4})} \text{ and } u_1 = 1+\frac{\rho}{2}+\sqrt{\rho(1+\frac{\rho}{4})}\,.$$

We remark that

$$h^{''}(u_0) = 2+3\sqrt{\rho}+O(\rho) \text{ and } h^{''}(u_1) = 2-3\sqrt{\rho}+O(\rho)\,.$$

The main point of the application of the saddle point method here is that $h^{'}(u_0) = 0$ and $h^{''}(u_0) > 0$, hence $nh(u_0\exp(i\theta))$ is approximately $nh(u_0) - n{u_0}^2h^{''}(u_0)\frac{\theta^2}{2}$ in the vicinity of $\theta = 0$. Integrating (18) around a circle passing vertically through $u = u_0$ leads to

$$t_{a,n}(\rho n+\beta) = \frac{n!}{2\pi}\int_{-\pi}^{\pi} u_0^a\,e^{ia\theta}\,(1-u_0e^{i\theta})^{1-\beta}\,\exp(nh(u_0e^{i\theta}))d\theta \qquad (19)$$

where

$$h(u_0e^{i\theta}) = u_0\cos\theta + iu_0\sin\theta - \ln u_0 - i\theta - \rho\ln(1-u_0e^{i\theta})\,. \qquad (20)$$

Let us check that the contribution away from $]-\theta_0,\,\theta_0[$ is bounded away by the integrand at θ_0. Denote by $\mathfrak{Re}(z)$ the real part of z, we have

$$\begin{aligned}
f(\theta) &= \mathfrak{Re}(h(u_0e^{i\theta})) \\
&= u_0\cos\theta - \ln u_0 - \rho\ln(|1-u_0e^{i\theta}|) \\
&= u_0\cos\theta - \ln u_0 - \rho\ln u_0 - \frac{\rho}{2}\ln\big(1+\frac{1}{u_0^2}-\frac{2}{u_0}\cos\theta\big)\,. \qquad (21)
\end{aligned}$$

It comes

$$\begin{aligned}
f^{'}(\theta) &= \frac{d}{d\theta}\big(\mathfrak{Re}(h(u_0e^{i\theta}))\big) \\
&= -u_0\sin\theta - \frac{\frac{\rho}{2}\big(\frac{2}{u_0}\sin\theta\big)^2}{\big(1+\frac{1}{u_0^2}-\frac{2}{u_0}\cos\theta\big)} \qquad (22)
\end{aligned}$$

and $f^{'}(\theta) = 0$ if $\theta = 0$. Also, $f(\theta)$ is a symmetric function of θ and in $[-\pi, -\theta_0] \cup [\theta_0, \pi]$, for a given θ_0, $0 < \theta_0 < \pi$, it takes it maximum value for $\theta = \theta_0$.

Since $|\exp(h(u))| = \exp(\mathfrak{Re}(h(u)))$, for a given θ_0, $\theta_0 < \pi$, when splitting the integral in (19) into three parts, viz. "$\int_{-\pi}^{-\theta_0} + \int_{-\theta_0}^{\theta_0} + \int_{\theta_0}^{\pi}$", we know that it suffices to integrate from $-\theta_0$ to θ_0, for a convenient value of θ_0, because the others can be bounded by the magnitude of the integrand at θ_0.

In fact, we have

$$\begin{aligned} h(u_0 e^{i\theta}) &= h(u_0) + \frac{{u_0}^2 (e^{i\theta}-1)^2}{2!} h^{''}(u_0) + \frac{{u_0}^3 (e^{i\theta}-1)^3}{3!} h^{(3)}(u_0) + \sum_{p\geq 4} \frac{{u_0}^p (e^{i\theta}-1)^p}{p!} h^{(p)}(u_0) \\ &= h(u_0) + \sum_{p\geq 2} \xi_p (e^{i\theta}-1)^p \quad , \end{aligned} \tag{23}$$

where $\xi_p = \frac{{u_0}^p}{p!} h^{(p)}(u_0)$.

The next computations are useful to estimate the error made when replacing $h(u_0 e^{i\theta})$ with an approximation. For $p \geq 2$, we compute $h^{(p)}(u_0) = (p-1)!\Big(\frac{(-1)^p}{{u_0}^p} + \frac{\rho}{(1-u_0)^p}\Big)$, for $p \geq 2$ and for ξ_p, we have

$$\begin{aligned} \xi_p &= \frac{(-1)^p}{p}\Big(1 - \frac{\rho\, {u_0}^p}{(1-u_0)^p}\Big) \\ &= \frac{(-1)^p}{p} + \frac{(-1)^{p+1}}{p} \frac{\rho\,(1+\frac{\rho}{2} - \sqrt{\rho\,(1+\frac{\rho}{4})})^p}{\rho^{\frac{p}{2}}(\sqrt{1+\frac{\rho}{4}} - \frac{\sqrt{\rho}}{2})^p} \, . \end{aligned} \tag{24}$$

Thus, for ρ small enough and $p > 2$, we have

$$|\xi_p| \leq \frac{2^p}{\rho^{\frac{p}{2}-1}} \,, \quad (\rho \to 0,\, p > 2) \, . \tag{25}$$

On the other hand,

$$|e^{i\theta} - 1| = \sqrt{2(1-\cos\theta)} < \theta \,, \ (\theta > 0) \, . \tag{26}$$

Thus, the summation in (23) can be bounded for values of θ and ρ such that $\theta \to 0$, $\rho \to 0$ but $\frac{\theta}{\sqrt{\rho}} \to 0$ and we have

$$|\sum_{p\geq 4} \xi_p (e^{i\theta}-1)^p| \leq \sum_{p\geq 4} |\xi_p \theta^p| \leq \rho \sum_{p\geq 4} \frac{2^p \theta^p}{\rho^{\frac{p}{2}}} = O\Big(\frac{\theta^4}{\rho}\Big) \, . \tag{27}$$

It follows that for $\theta \to 0$, $\rho \to 0$ and $\frac{\theta}{\sqrt{\rho}} \to 0$,

$$\begin{aligned} h(u_0 e^{i\theta}) &= h(u_0) - \frac{1}{2} \frac{u_0}{(1-u_0)^2} (1 + \rho - 2u_0 + {u_0}^2)\theta^2 \\ &+ i\frac{u_0}{6(1-u_0)^3} (1 + \rho + (\rho-3)u_0 + 3{u_0}^2 - {u_0}^3)\theta^3 + O\Big(\frac{\theta^4}{\rho}\Big) \, , \end{aligned} \tag{28}$$

where the term in the big-oh takes into account the terms from $(e^{i\theta}-1)^2$ and $(e^{i\theta}-1)^3$ of (23) which we can neglect since

$$(e^{i\theta}-1)^2 = -\theta^2 - i\theta^3 + O(\theta^4) \text{ and } (e^{i\theta}-1)^3 = -i\theta^3 + \frac{3}{2}\theta^4 + iO(\theta^5)\,.$$

Let

$$\theta_0 = \frac{\rho^{1/8}}{n^{3/8}\tau^{1/2}} \qquad \text{with} \qquad \tau = \frac{u_0(1+\rho-2u_0+{u_0}^2)}{(1-u_0)^2}\,.$$

We can now use the magnitude of the integrand at θ_0 to bound the resulting error. Hence, we can verify our choice of θ_0

$$\begin{aligned} &|u_0^a(1-u_0e^{i\theta_0})^{(1-\beta)}\left(\exp\left(nh(u_0e^{i\theta_0})\right)-nh(u_0)\right)| = \\ &u_0^a|1-u_0e^{i\theta_0}|^{(1-\beta)}\exp\Big(-\frac{n}{2}\tau\,{\theta_0}^2 + O\big(n\frac{{\theta_0}^4}{\rho}\big)\Big) = O\Big(e^{-\frac{\rho^{1/4}n^{1/4}}{2}}\Big)\,. \end{aligned} \tag{29}$$

To estimate $t_{a,n}(\rho\, n+\beta)$, it proves convenient to compute the integral

$$\int_{-\theta_0}^{\theta_0} u_0^a \exp\left(ia\theta\right)(1-u_0e^{i\theta})^{(1-\beta)}\exp\left(nh(u_0e^{i\theta})\right)d\theta\,. \tag{30}$$

If we make the substitution $\theta = \frac{t}{\sqrt{n\tau}}$, we have (recall that $\theta_0 = \frac{\rho^{1/8}}{n^{3/8}\tau^{1/2}}$)

$$\frac{u_0^a}{\sqrt{n\tau}}\int_{-\rho^{1/8}n^{1/8}}^{\rho^{1/8}n^{1/8}}\Big(1-u_0e^{\frac{it}{\sqrt{n\tau}}}\Big)^{(1-\beta)}\exp\Big(ia\frac{t}{\sqrt{n\tau}} + nh(u_0e^{\frac{it}{\sqrt{n\tau}}})\Big)dt\,. \tag{31}$$

Since $(1-u_0e^{\frac{it}{\sqrt{n\tau}}})^{(1-\beta)} = (1-u_0)^{(1-\beta)}(1+O(t/\sqrt{n\rho}))$, the integral given in (30) becomes

$$\begin{aligned} \frac{1}{\sqrt{n\tau}}\,u_0^a\int_{-\rho^{1/8}n^{1/8}}^{\rho^{1/8}n^{1/8}}(1-u_0)^{(1-\beta)} \quad \exp\quad &\Big(nh(u_0)-\frac{t^2}{2}+ia\frac{t}{\sqrt{n\tau}}+if_3\frac{t^3}{\sqrt{n\,\rho}}+O\big(\frac{t^4}{n\,\rho}\big)\Big) \\ &\Big(1+O\big(\frac{t}{\sqrt{n\,\rho}}\big)\Big)\,dt \end{aligned}$$

where

$$f_3 = -\frac{\sqrt{\rho}(1+\rho+(\rho-3)u_0+3u_0^2-u_0^3)}{\sqrt{u_0}(1+\rho-2u_0+u_0^2)^{\frac{3}{2}}} = -\frac{\sqrt{2}}{12}-\frac{5}{48}\sqrt{\rho}+O(\rho)\,.$$

Using these approximations, we then obtain

$$\begin{aligned} &u_0^a\frac{(1-u_0)^{(1-\beta)}}{\sqrt{n\tau}}e^{(nh(u_0))} \\ &\times\left[\int_{-\rho^{1/8}n^{1/8}}^{\rho^{1/8}n^{1/8}} e^{-\frac{t^2}{2}}\cos\Big(f_3\frac{t^3}{\sqrt{n\rho}}+a\frac{t}{\sqrt{n\tau}}\Big)\Big(1+O\big(\frac{t}{\sqrt{n\rho}}\big)+O\big(\frac{t^4}{n\rho}\big)\Big)\,dt\right], \end{aligned} \tag{32}$$

since the symmetry of the function leads to the cancellation of the terms with the function sin. Using, $u_0^a = 1+O(\sqrt{\rho})$, $\cos(x) = 1+O(x^2)$ and $\exp(O(x)) = 1+O(x)$ when $x = O(1)$ in (32), we find

$$\frac{(1-u_0)^{(1-\beta)}}{\sqrt{n\tau}}e^{(nh(u_0))}$$

$$
\begin{aligned}
& \times \left[\int_{-\rho^{1/8} n^{1/8}}^{\rho^{1/8} n^{1/8}} e^{-\frac{t^2}{2}} \left(1 + O\Big(\frac{1}{\rho^{1/4}\, n^{1/4}}\Big) + O\Big(\frac{\rho^{1/4}}{n^{3/4}}\Big) \right) \, dt \right] (1 + O(\sqrt{\rho})) \\
= \; & \frac{(1-u_0)^{(1-\beta)}}{\sqrt{n\tau}} e^{(nh(u_0))} \\
& \times \left[\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \left(1 + O\Big(\frac{1}{\rho^{1/4}\, n^{1/4}}\Big) \right) \, dt \; + \; O\Big(e^{-\frac{\rho^{1/4}\, n^{1/4}}{2}} \Big) \right] (1 + O(\sqrt{\rho})) \\
= \; & \frac{\sqrt{2\pi}(1-u_0)^{(1-\beta)} e^{(nh(u_0))}}{\sqrt{n\tau}} \left(1 \, + \, O\Big(\sqrt{\rho}\Big) + O\Big(\frac{1}{\rho^{1/4}\, n^{1/4}}\Big) \, + \, O\Big(e^{-\frac{\rho^{1/4}\, n^{1/4}}{2}} \Big) \right) \\
= \; & \sqrt{\frac{\pi}{n}} (1-u_0)^{(1-\beta)} e^{(nh(u_0))} \left(1 \, + \, O\Big(\sqrt{\rho}\Big) + O\Big(\frac{1}{\rho^{1/4}\, n^{1/4}}\Big) \right) \, .
\end{aligned}
$$

□

Now, we are ready to prove Theorem 1.1. The proof is divided into two parts according to the range of excess. In the first part (**i**), we consider connected graphs with k vertices and $k + o(k^{1/3})$ edges and the methods in use are due to Wright [29]. In the second part (**ii**), we will consider excesses with wider range and the methods are those of Bender-Canfield-McKay [2, 3].

Proof of Theorem 1.1: Part (**i**). In term of EGFs, $c'(k, k+\ell+1)$ represents the coefficient

$$
c'(k, k+\ell+1) = \frac{k!}{2} \left[z^k\right] \sum_{p=0}^{\ell} \Big(\vartheta_z W_p(z) \Big) \Big(\vartheta_z W_{l-p}(z) \Big) \, . \tag{34}
$$

Applying Wright's inequalities , i.e. (12), in (34) yields

$$
L_\ell(z) \preceq \sum_k c'(k, k+\ell+1) \frac{z^k}{k!} \preceq R_\ell(z) \, , \;\; (\ell > 0) \, , \tag{35}
$$

where

$$
R_\ell(z) = \frac{9}{2} \sum_{p=1}^{l-1} \frac{p(\ell-p) b_p b_{l-p} T(z)^2}{\big(1-T(z)\big)^{3\ell+4}} + \frac{3\ell b_\ell T(z)^5}{2(1-T(z))^{3\ell+4}} \tag{36}
$$

and

$$
L_\ell(z) = R_\ell(z) - \left(\sum_{p=1}^{l-1} \frac{3(3p-1)(\ell-p) b_{l-p} c_p T(z)^2}{(1-T(z))^{3\ell+3}} + \frac{(3\ell-1) c_\ell T(z)^5}{2(1-T(z))^{3\ell+3}} \right) \, . \tag{37}
$$

(We used $\vartheta_z T(z) = T(z)/(1-T(z))$.) Our aim is then to show that the difference between the coefficients of the right and left parts of (35), viz. $k! \left[z^k\right] (R_\ell(z) - L_\ell(z))$ can be neglected in comparison to $k! \left[z^k\right] R_\ell(z)$ for $\ell = o(k^{1/3})$. For this purpose, we use lemma 2.1, and the fact that $b_\ell = (\frac{3}{2})^\ell (\ell-1)! d_\ell$ with (d_ℓ) an increasing sequence tending to $\frac{1}{2\pi}$ (cf. [29, eq. (1.4)], [2]).

More precisely, lemma 2.1 tells us that in $R_\ell(z)$, the coefficients of z^k of $T(z)^2/(1-T(z))^{3\ell+4}$ and $T(z)^5/(1-T(z))^{3\ell+4}$ in (36) have the same order of magnitude for $\ell = o(k^{1/3})$. Next, using the definition of Wright's coefficients (11), we find

$$\frac{9}{2}\sum_{p=1}^{\ell-1} p(\ell-p)b_p b_{\ell-p} + \frac{3}{2}\ell b_\ell = 3(\ell+1)(b_{\ell+1} - \frac{3}{2}\ell b_\ell) + \frac{3}{2}\ell b_\ell \,. \tag{38}$$

We then have

$$b_{\ell+1} - \frac{3}{2}\ell b_\ell = (\frac{3}{2})^{\ell+1}\ell!(d_{\ell+1} - d_\ell)$$

where we used $b_\ell = (3/2)^\ell(\ell-1)!d_\ell$ as studied in [29, eq. (1.4)] and in [17]. ¿From the proof given by Meertens in [2, lemma 3.4], we have $0 < d_{\ell+1} - d_\ell = O(1/\ell^2)$. So,

$$\frac{9}{2}\sum_{p=1}^{\ell-1} p(\ell-p)\, b_p\, b_{\ell-p} + \frac{3}{2}\ell\, b_\ell = \Big(\frac{3}{2}\Big)^{\ell+1} \ell!\, d_\ell \Big(1 + O(1/\ell)\Big)\,. \tag{39}$$

On the other hand, the definition (12) of the sequence (c_ℓ) tells us that the summation in (37) satisfies

$$\sum 3(3p-1)(\ell-p)b_{\ell-p}c_p = O(\ell c_\ell)$$

and we know from [29] that $c_\ell = O(\ell b_\ell)$. Finally, lemma 2.1 suggests us to find values of $\ell \equiv \ell(k)$ for which the coefficients of the difference $R_\ell - L_\ell$ satisfy

$$\left[z^k\right]\Big(R_\ell(z) - L_\ell(z)\Big) \ll \left[z^k\right]\Big(R_\ell(z)\Big)\,.$$

It comes $\ell = o(k^{1/3})$ which is the same range as in [29] and in [22] for connected graphs without prefixed (forbidden) configurations, the error terms being of order $O(1/\ell) + O(\sqrt{\ell^3/k})$. After a bit of algebra, we find (replacing $\rho = 3\ell/k$ in the saddle point u_0)

$$\begin{aligned} \frac{3}{2}\,\ell\, b_\ell\, t_{5,3\ell+4} &= \frac{3}{2}\,\ell\, b_\ell \frac{k^{k+3/2\ell+3/2}}{\sqrt{2}(3\ell)^{3\ell/2+3/2}} \exp{(3\ell/2)}\Big(1 + O(\sqrt{\ell^3/k})\Big) \\ &= \frac{1}{\sqrt{48\,\pi\,\ell}}\left(\frac{e}{12l}\right)^{\ell/2} k^{k+3\ell/2+3/2}\Big(1 + O(1/\ell) + O(\sqrt{\ell^3/k})\Big)\,, \end{aligned} \tag{40}$$

which completes the proof of Theorem 1.1 part (**i**). □

Wright showed that the EGFs of all multicyclic components can be expressed in terms of the EGF of Cayley. In order to count the number of ways to label a complex component, one can repeatedly prune it by deleting recursively any vertex of degree 1. The graph obtained after removing all vertices of degree 1 is called a *smooth graph*. The process of removing recursively all vertices of degree 1 is called *smoothing* or *pruning* process [28].

Remark 2.2. *Theorem 1.1 tells us that asymptotically almost all* $(\ell+1)$*-components whose situation after smoothing contains a cut edge are built by linking a unicyclic component to another complex component. In fact, (1) reflects simply*

$$c'(k, k+\ell+1) \sim k!\left[z^k\right]\Big(\vartheta_z W_0(z)\Big)\Big(\vartheta_z W_\ell(z)\Big)\,, \qquad 1 \ll \ell \ll k^{1/3}\,. \tag{41}$$

Using the same technics involved in the proof of Theorem 1.1 part (**i**), we obtain a generalization:

Corollary 2.3. *Denote by* $c^r(k, k+\ell+1)$ *the number of manners to build an* $(\ell+1)$*-component of order* k *with a distinguished edge between a* p*-component and an* $(\ell-p)$*-component, with* $p \geq r \geq 0$*. Then, as* $k, \ell \to \infty$*,* $\ell = o(k^{1/3})$ *and for fixed values of* r*, we have*

$$c^r(k, k+\ell+1) \sim k! \left[z^k\right] \Big(\vartheta_z W_r(z)\Big) \Big(\vartheta_z W_{\ell-r}(z)\Big) \, , \qquad 1 \ll \ell \ll k^{1/3} \, . \tag{42}$$

Remark 2.4. *Observe that the value of* $h(u_0)$ *with* h *given by (17) and* u_0 *given in Lemma 2.1 satisfies*

$$h(u_0) = 1 + \left(\frac{1}{2} + \frac{1}{2}\ln\left(\frac{1}{\rho}\right)\right)\rho + \frac{1}{3}\rho^{3/2} - \frac{1}{120}\rho^{5/2} + O\left(\rho^3\right) \, , \quad (\rho \to 0) \, . \tag{43}$$

Thus, for the range $\ell = o(k)$*, it is also possible to obtain an upper-bound of* $c^r(k, k+\ell+1)$ *(for any fixed integer* $r \geq 0$*) by means of the same methods as above and we then get*

$$\begin{aligned} c^r(k, k+\ell+1) \;&\leq\; \frac{1}{\sqrt{48\pi\,\ell}} \Big(e/12\ell\Big)^{\ell/2} k^{k+3\ell/2+3/2} \exp\left(\frac{3^{1/2}\,\ell^{3/2}}{k^{1/2}}\right) \\ &\times\; \left(1 + O\Big(\frac{1}{\ell^{1/4}}\Big) + O\Big(\sqrt{\frac{\ell}{k}}\Big)\right) \, . \end{aligned} \tag{44}$$

Proof of Theorem 1.1: Part (**ii**). The second part of the proof is entirely different and is based upon the results in [2, 3]. We start comparing $c'(k, k+\ell)$ with $(\binom{k}{2} - k - \ell + 1)c(k, k+\ell-1)$ by means of (8) and (9). Using the definition of $c'(k, k+\ell)$, [2, Lemma 4.1 and (4.12)] and denoting $q = k+\ell$, we have

$$\begin{aligned} \frac{c'(k,q)}{q\,c(k,q)} \;&=\; \frac{\sum_{t=1}^{k-1}\binom{k}{t} t\, c(t,t)\,(k-t)\,c(k-t, q-t-1)}{q\binom{\binom{k}{2}}{q}\exp\Big(k\varphi(x) + a(x)\Big)\Big(1 + b(k,\ell)\Big)} + \\ &\quad \frac{1}{2}\sum_{s=1}^{\ell-2}\sum_{t=1}^{k-1} \frac{\binom{k}{t} t\, c(t,t+s)\,(k-t)\,c(k-t, q-t-s-1)}{q\binom{\binom{k}{2}}{q}\exp\Big(k\varphi(x) + a(x)\Big)\Big(1 + b(k,\ell)\Big)} \, , \end{aligned} \tag{45}$$

where $x = q/k = 1 + \ell/k$, φ and a are, respectively, given in [2, (1.12)] and [2, (1.17)], and the error term $b(\cdot, \cdot)$ is given in [2, (1.20a) and (1.20b), Theorem 2]. Thus, we have

$$\frac{\Big(1 + b(k,\ell)\Big)\,c'(k,q)}{q\,c(k,q)} = S_0 + S \, , \tag{46}$$

where (again) S_0 and S are defined in [2, equations (4.2) and (4.3)], i.e., the first and the second summations in the equation (45) above. Hence, the quantity of interest is given by

$$\begin{aligned} \frac{c'(k,q)}{\left(\binom{k}{2} - q + 1\right)c(k,q-1)} &= \frac{S_0 + S}{(1 + b(k,\ell))} \times \frac{q c(k,q)}{\left(\binom{k}{2} - q + 1\right)c(k,q-1)} \\ &= \frac{(S_0 + S)}{(1 + b(k,\ell))} \times \frac{q\binom{\binom{k}{2}}{q}\exp\left(k\varphi(x) + a(x)\right)\,(1 + b(k,\ell))}{\left(\binom{k}{2} - q + 1\right)\binom{\binom{k}{2}}{q-1}\exp\left(k\varphi(x - \frac{1}{k}) + a(x - \frac{1}{k})\right)\,(1 + b(k,\ell-1))} \end{aligned}$$

$$= \frac{S_0 + S}{1 + b(k, \ell - 1)} \times \exp\left(\varphi'(x) - \frac{1}{2k}\varphi''(x) + \frac{1}{6k^2}\varphi^{(3)}(x - \theta_\varphi) + \frac{a'(x)}{k} - \frac{a''(x - \theta_a)}{2k^2}\right), \tag{47}$$

where θ_φ and θ_a are in $(0, \frac{1}{k})$. Taking into account the bounds given in [2, Lemma 3.1], viz., $\frac{\varphi''(x)}{k} = O(1/\ell)$, $\frac{\varphi^{(3)}(x-\theta_\varphi)}{k^2} = O(1/k^2)$, $\frac{a'(x)}{k} = O(1/\sqrt{\ell k})$ and $\frac{a''(x-\theta_a)}{k^2} = O(k^{-1/2}\ell^{-3/2})$, we get

$$\begin{aligned} \frac{c'(k,q)}{\left(\binom{k}{2} - q + 1\right) c(k, q-1)} &= e^{\varphi'(x)} \times (S_0 + S) \times \\ &\left(1 + O\left(\frac{1}{\ell}\right) + O\left(\frac{1}{k^2}\right) + O\left(\frac{1}{\ell^{1/2}k^{1/2}}\right) + O\left(\frac{1}{\ell^{3/2}k^{1/2}}\right) + O\left(\frac{\ell^{1/16}}{k^{9/50}}\right)\right) \\ &= \sqrt{\frac{k}{3\ell}} \times (S_0 + S) \times \\ &\left(1 + O\left(\frac{1}{\ell}\right) + O\left(\frac{1}{\ell^{1/2}k^{1/2}}\right) + O\left(\frac{\ell}{k}\right) + O\left(\frac{\ell^{1/16}}{k^{9/50}}\right)\right). \end{aligned} \tag{48}$$

Now, we can use the approximations of S_0 and S given respectively by [2, equation (4.6c)] and by [2, equation (4.6d)] to get (after a bit of algebra)

$$S_0 + S = \frac{1}{\sqrt{3\ell k}} \left(1 + O\left(\sqrt{\frac{\ell}{k}}\right)\right). \tag{49}$$

The combination of (48) and (49) completes the proof. □

3. Expectations of transitions and size of ℓ-component

When adding an edge in a randomly growing graph, there is a possibility that it joins two vertices of the same component, increasing its excess by 1 (transition $\ell \to \ell + 1$).

Consider an ℓ-component with k vertices. Let $\alpha(\ell; k)$ be the expected number of times that a new edge is added to an ℓ-component of order k (with both ends of the edge in the component). When a new edge is added to an ℓ-component of order k, there are $\binom{n}{k} c(k, k+\ell)$ manners to choose an ℓ-component and $\binom{k}{2} - k - \ell$ ways to choose the new edge. Furthermore, the probability that such possible component is one of $\mathbb{G}(n, t)$ is $t^{k+\ell}(1-t)^{(n-k)k+\binom{k}{2}-k-\ell}$ and with the conditional probability $\frac{dt}{(1-t)}$ that a given edge is added during the interval $(t, t+dt)$ and not earlier, integrating over all times, we obtain (see also [16])

$$\alpha(\ell; k) = \binom{n}{k} c(k, k+\ell) \left(\frac{k^2 - 3k - 2\ell}{2}\right) \int_0^1 t^{k+\ell}(1-t)^{(n-k)k+\binom{k}{2}-k-\ell-1} dt \tag{50}$$

which evaluation leads to

$$\alpha(\ell; k) = (n)_k \frac{(k+\ell)!}{k!} c(k, k+\ell) \frac{(k^2 - 3k - 2\ell)}{2} \frac{(nk - k^2/2 - 3k/2 - \ell - 1)!}{(nk - k^2/2 - k/2)!}. \tag{51}$$

For the second type of transition $(\ell-p)\oplus p \to \ell+1$ (with $0 \le p \le \ell$), let $\beta(\ell-p,p;k_1,k-k_1)$ be the expected number of times an edge is added between an $(\ell-p)$-component of size k_1 and a p-component of size $k-k_1$. Since there are $k_1(k-k_1)$ manners to join two fixed $(\ell-p)$-component and p component of order k_1, respectively $k-k_1$, instead of (50), we have

$$\begin{aligned}\beta(\ell-p,p;k_1,k-k_1) &= \binom{n}{k}\binom{k}{k_1} k_1 c(k_1,k_1+\ell-p)\ (k-k_1)c(k-k_1,k-k_1+p) \\ &\quad \times \int_0^1 t^{k+\ell}(1-t)^{(n-k)k+\binom{k}{2}-k-\ell-1}dt\,.\end{aligned} \tag{52}$$

When summing over p and k_1, we then obtain

$$\binom{n}{k}\underbrace{\left(\frac{1}{2}\sum_{p=0}^{\ell}\sum_{k_1=1}^{k-1}\binom{k}{k_1}\ k_1 c(k_1,k_1+\ell-p)\ (k-k_1)c(k-k_1,k-k_1+p)\right)} \times \int_0^1 t^{k+\ell}(1-t)^{(n-k)k+\binom{k}{2}-k-\ell-1}dt\,, \tag{53}$$

and we recognize that the double-summation represents exactly the coefficient $c'(k,k+\ell+1)$ defined in Theorem 1.1. Therefore, the second kind of transition can be deduced using the first one simply by introducing a factor $O(\frac{1}{\ell})$ as indicated by (1).

Before proving Theorem 1.2, we need several lemmas which are given in the next paragraph.

3.1. **Technical lemmas.** We have the following result which gives bounds of $\alpha(\ell;k)$:

Lemma 3.1. *As $\ell,k,n\to\infty$ but $\ell=o(k)$, we have*

$$\begin{aligned}\alpha(\ell;k) &\le \frac{1}{4}\sqrt{\frac{3}{\pi}}\left(\frac{e}{12\ell}\right)^{\ell/2}\frac{k^{3\ell/2+1/2}}{n^{\ell+1}}\exp\left(-\frac{k^3}{24n^2}+\frac{k^4}{n^3}+\frac{2\ell k}{n}+\frac{\ell^2}{2k}+\frac{3^{1/2}\,\ell^{3/2}}{k^{1/2}}\right) \\ &\quad\times\left(1+O\left(\frac{k}{n}\right)+O\left(\sqrt{\frac{\ell}{k}}\right)+O\left(\frac{1}{\ell^{1/4}}\right)\right), \qquad (k\le n)\,,\end{aligned} \tag{54}$$

and

$$\begin{aligned}\alpha(\ell;k) &\ge \frac{1}{4}\sqrt{\frac{3}{\pi}}\left(\frac{e}{12\ell}\right)^{\ell/2}\frac{k^{3\ell/2+1/2}}{n^{\ell+1}}\exp\left(-\frac{k^3}{24n^2}-\frac{k^4}{n^3}+\frac{\ell k}{2n}-\frac{\ell^3}{6k^2}\right) \\ &\quad\times\left(1+O\left(\frac{k}{n}\right)+O\left(\sqrt{\frac{\ell}{k}}\right)+O\left(\frac{1}{k}\right)\right), \qquad (k\le\frac{n}{2})\,.\end{aligned} \tag{55}$$

Proof. The proof given here are based on the works of Janson in [15, 16]. However, the main difference comes from the fact that our parameter, representing the excess of the sparse components ℓ, is no more fixed as in [16]. Allowing ℓ to grow smoothly with n introduces new difficulties.

For $1 \leq k \leq n$ and $\ell = o(n)$, the value of the integral in (50) is

$$\frac{(nk-k^2/2-3k/2-\ell-1)!}{(nk-k^2/2-k/2)!} = k^{-k-\ell-1}(n-k/2)^{-k-\ell-1} \times \left(1+O\left(\frac{k}{n}\right)+O\left(\frac{\ell}{n}\right)+O\left(\frac{\ell^2}{k\,n}\right)\right). \tag{56}$$

We have,

$$\begin{aligned}\frac{(n)_k}{(n-k/2)^k} &= \exp\Big(\sum_{i=1}^{k-1}\ln\left(1-\frac{i}{n}\right)-k\ln\left(1-\frac{k}{2n}\right)\Big)\\ &\leq \exp\Big(\sum_{i=1}^{k-1}\ln\left(1-\frac{i}{n}\right)+\frac{k^2}{2n}+\frac{k^3}{8n^2}+\frac{k^4}{n^3}\Big)\\ &\leq \exp\Big(-\frac{k^3}{24n^2}+\frac{k^4}{n^3}\Big)\left(1+O\left(\frac{k}{n}\right)\right),\end{aligned} \tag{57}$$

and assuming that $1 \leq k \leq \frac{n}{2}$ we find the following lower-bound

$$\frac{(n)_k}{(n-k/2)^k} \geq \exp\Big(-\frac{k^3}{24n^2}-\frac{k^4}{n^3}\Big)\left(1+O\left(\frac{k}{n}\right)\right), \qquad (1 \leq k \leq \frac{n}{2}). \tag{58}$$

(We used $\ln(1-x) \geq -x-x^2/2-4x^3\,, x \in [0,\,1/2]$.)
Obviously $\binom{k}{2}-k-\ell \leq \frac{k^2}{2}$ and

$$\binom{k}{2}-k-\ell = \frac{k^2}{2}\left(1+O(1/k)+O(\ell/k^2)\right). \tag{59}$$

Thus, combining (56), (57), (58) and (59) in (51), we infer that

$$\begin{aligned}\alpha(\ell;k) &\leq \frac{1}{2}\frac{(k+\ell)!}{k!}\frac{c(k,k+\ell)}{(n-k/2)^{\ell+1}k^{k+\ell-1}}\exp\Big(-\frac{k^3}{24n^2}+\frac{k^4}{n^3}\Big)\\ &\times \left(1+O\left(\frac{k}{n}\right)+O\left(\frac{\ell^2}{kn}\right)+O\left(\frac{\ell}{n}\right)\right) \qquad (1 \leq k \leq n) \qquad \text{and}\\ \alpha(\ell;k) &\geq \frac{1}{2}\frac{(k+\ell)!}{k!}\frac{c(k,k+\ell)}{(n-k/2)^{\ell+1}k^{k+\ell-1}}\exp\Big(-\frac{k^3}{24n^2}-\frac{k^4}{n^3}\Big)\\ &\times \left(1+O\left(\frac{k}{n}\right)+O\left(\frac{\ell^2}{kn}\right)+O\left(\frac{1}{k}\right)+O\left(\frac{\ell}{k^2}\right)+O\left(\frac{\ell}{n}\right)\right) \qquad (1 \leq k \leq \frac{n}{2}).\end{aligned} \tag{60}$$

Also, we simply get (using $-2x \leq \ln(1-x) \leq -x\,, x \in [0,\,1/2]$)

$$\exp\left(-\frac{2\ell k}{n}\right) \leq \frac{(n-k/2)^{\ell+1}}{n^{\ell+1}} \leq \exp\left(-\frac{\ell k}{2n}\right). \tag{61}$$

Taylor expansions lead to

$$\frac{\ell^2}{2k}-\frac{\ell^3}{6k^2}+O\left(\frac{\ell}{k}\right) \leq \ln\left(\frac{(k+\ell)!}{k^\ell k!}\right) \leq \frac{\ell^2}{2k}+O\left(\frac{\ell}{k}\right). \tag{62}$$

Now, by Wright's inequality and by means of (15), we can get an upper-bound of the quantity $c(k, k+\ell)$ above. After a bit of algebra, we then have

$$\begin{aligned} c(k,k+\ell) &\leq \frac{1}{2}\sqrt{\frac{3}{\pi}}\left(\frac{e}{12\ell}\right)^{\frac{\ell}{2}} k^{k+3/2\ell-1/2}\exp\left(\frac{3^{1/2}\ell^{3/2}}{k^{1/2}}\right) \\ &\times \left(1+O\left(\frac{1}{\ell^{1/4}}\right)+O\left(\sqrt{\frac{\ell}{k}}\right)\right). \end{aligned} \tag{63}$$

Using the result [21, Theorem 3] (whenever $\ell/k \to 0$ but $(k+\ell)\exp(-2(k+\ell)/k) \to \infty$), one can get a lower-bound of the same quantity, viz.

$$c(k,k+\ell) \geq \frac{1}{2}\sqrt{\frac{3}{\pi}}\left(\frac{e}{12\ell}\right)^{\frac{\ell}{2}} k^{k+3/2\ell-1/2}\exp\left(-\frac{\ell^2}{2\,k}\right)\left(1+O\left(\sqrt{\frac{\ell}{k}}\right)\right). \tag{64}$$

Combining the above inequalities, we find the bounds of the quantity defined by $\alpha(\ell;k)$. □

Since $\mathbb{E}(Y_n^{(\ell)}) = \sum_{1\leq k\leq n}\alpha(\ell;k)$, the bounds given in lemma 3.1 suggest us to consider the asymptotic behavior of sums of the form

$$\sum_k k^a \exp\left(-\frac{k^3}{24n^2}+c_1\frac{k^4}{n^3}+c_2\frac{2\ell k}{n}+c_3\frac{\ell^2}{2k}+c_4\frac{\ell^3}{k^2}+c_5\frac{\ell^{3/2}}{k^{1/2}}\right) \tag{65}$$

where $a = \frac{3\ell+1}{2}$, $\ell \equiv \ell(n)$, as $n \to \infty$ and the c_i, $1 \leq i \leq 5$ are absolute constants.

Now, our plan is to show

$$\mathbb{E}(Y_n^{(\ell)}) = \sum_{k=1}^{n}\alpha(\ell;k) \sim \sum_{k=\omega(n)}^{n/2}\alpha(\ell;k)$$

where

$$\omega(n) = \frac{\ell}{10}\ln\left(\frac{n}{\ell}\right).$$

In the second part of the summation, the values of k satisfy $\ell \ll \omega(n) \leq k \leq n/2$. So, we can use the bounds given in lemma 3.1. Therefore, we have to prove that the sums $\sum_{k=1}^{\omega(n)-1}\alpha(\ell;k)$ and $\sum_{k=n/2+1}^{n}\alpha(\ell;k)$ can be neglected. In these directions, we have the following lemma:

Lemma 3.2. *As $1 \ll \ell \ll n$, set $\omega(n) = \frac{\ell}{10}\ln\left(\frac{n}{\ell}\right)$. We have,*

$$\sum_{k=1}^{\omega(n)}\alpha(\ell;k) = o\left(\frac{1}{\sqrt{n}}\right). \tag{66}$$

Proof. There is a constant A s.t. $c(k,k+\ell) \leq (A/\ell)^{\ell/2}k^{k+3\ell/2-1/2}$ for every k and ℓ (see for instance [4]). Using this and with similar bounds to those given during the proof of lemma 3.1, we successively get (for $k \leq \omega(n)$ and $\ell = o(n)$):

$$\alpha(\ell;k) \leq (n)_k\frac{(k+\ell)!}{k!}\left(\frac{A}{\ell}\right)^{\ell/2} k^{k+3\ell/2+3/2}\,\frac{(nk-k^2/2-3k/2-\ell-1)!}{(nk-k^2/2-k/2)!} \tag{67}$$

and since $\frac{(k+\ell)!}{k!} \leq \ell^\ell \exp{(2k)}$

$$\alpha(\ell;k) \leq (n)_k \exp{(2k)} \,(A\ell)^{\ell/2} \;\; k^{k+3\ell/2+3/2} \,\frac{(nk-k^2/2-3k/2-\ell-1)!}{(nk-k^2/2-k/2)!}\,. \tag{68}$$

In the considered ranges, we have

$$\frac{(nk-k^2/2-3k/2-\ell-1)!}{(nk-k^2/2-k/2)!} \leq 2\,\frac{\exp\left(\frac{4\ell^2}{kn}\right)}{k^{k+\ell+1}(n-k/2)^{k+\ell+1}}\,. \tag{69}$$

and (since $k \leq n$)

$$\frac{1}{(n-k/2)^{\ell+1}} = \frac{1}{n^{\ell+1}}\,\exp\Big(-(\ell+1)\ln(1-k/2n)\Big) \leq \frac{\exp\left(\frac{(\ell+1)k}{n}\right)}{n^{\ell+1}} \leq \frac{\exp{(k)}}{n^{\ell+1}}\,. \tag{70}$$

Combining the above inequalities with (57), we get

$$\begin{aligned}\alpha(\ell;k) &\leq 2\,\frac{(A\ell)^{\ell/2}}{n^{\ell+1}}\,k^{\ell/2+1/2}\,\exp\left(-\frac{k^3}{24n^2}+\frac{k^4}{n^3}+3k+\frac{4\ell^2}{kn}\right) \\ &\leq 2\,\frac{(A\ell)^{\ell/2}}{n^{\ell+1}}\,k^{\ell/2+1/2}\,\exp{(5k+4\ell)}\,. \end{aligned} \tag{71}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{\omega(n)}\alpha(\ell;k) &\leq 2\,\frac{(Ae^8\ell)^{\ell/2}}{n^{\ell+1}}\sum_{k=1}^{\omega(n)}k^{\ell/2+1/2}\exp{(5k)} \leq 2\,\frac{(Ae^8\ell)^{\ell/2}}{n^{\ell+1}}\,\exp{(5\,\omega(n))}\sum_{k=1}^{\omega(n)}k^{\ell/2+1/2} \\ &\leq O\left(\frac{1}{\sqrt{n}}\,\frac{\ln\left(\frac{n}{\ell}\right)^{3/2}}{\sqrt{\left(\frac{n}{\ell}\right)}}\,\left(\frac{\tilde{A}\ln\left(\frac{n}{\ell}\right)}{\left(\frac{n}{\ell}\right)}\right)^{\ell/2}\right)\,. \end{aligned} \tag{72}$$

($\tilde{A}$ is some constant.) □

For summation of the form described in (65), we have the following approximation

Lemma 3.3. *Let $c > 0$ and c_1, c_2, c_3, c_4 be fixed constants. If $\ell, n \to \infty$ but $n \gg \ell$ and $a \equiv a(\ell) = \Theta(\ell)$ then we have*

$$\begin{aligned}\sum_{k=1}^{n/c}\exp{(\phi(k,n,\ell))} &\stackrel{def}{=} \sum_{k=1}^{n/c}k^a\exp\left(-\frac{k^3}{24n^2}+c_1\frac{k^4}{n^3}+c_2\frac{\ell k}{n}+c_3\frac{\ell^2}{k}+c_4\frac{\ell^3}{k^2}+c_5\frac{\ell^{3/2}}{k^{1/2}}\right) \\ &\sim 2^{a+1}3^{(a-2)/3}\Gamma\left(\frac{a+1}{3}\right)n^{2(a+1)/3}\,. \end{aligned} \tag{73}$$

Proof. We have

$$\sum_{k=1}^{n/c}\exp{(\phi(k,n,\ell))} \sim \int_1^{n/c}\exp{(\phi(t,n,\ell))}dt\,. \tag{74}$$

If we denote by I_n the integral, we have after substituting t for $2n^{2/3}e^z$:

$$I_n = 2^{a+1} n^{\frac{2(a+1)}{3}} \int_{-\frac{2}{3}\ln n - \ln 2}^{\frac{1}{3}\ln n - \ln(2c)} \exp(H(z))dz \,. \tag{75}$$

where

$$H(z) = (a+1)\,z - \frac{e^{3\,z}}{3} + 16\,\frac{c_1 e^{4\,z}}{n^{1/3}} + \frac{c_2 \ell\, e^{-z}}{2n^{5/3}} + \frac{c_3 \ell^2 e^{-z}}{2n^{2/3}} + \frac{c_4 \ell^3 e^{-2\,z}}{4n^{4/3}} + \frac{c_5 \ell^{3/2} e^{-z/2}}{2^{1/2} n^{1/3}} \,. \tag{76}$$

Also, we have

$$H'(z) = a + 1 - e^{3\,z} + 64\,\frac{c_1 e^{4\,z}}{n^{1/3}} - \frac{c_2 \ell\, e^{-z}}{2n^{5/3}} - \frac{c_3 \ell^2 e^{-z}}{2n^{2/3}} - \frac{c_4 \ell^3 e^{-2\,z}}{2n^{4/3}} - \frac{c_5 \ell^{3/2} e^{-z/2}}{2^{3/2} n^{1/3}} \tag{77}$$

and more generally (for $q > 1$)

$$\begin{aligned} H^{(q)}(z) &= -3^{q-1} e^{3\,z} + 4^{q+2}\,\frac{c_1 e^{4\,z}}{n^{1/3}} + (-1)^q \frac{c_2 \ell\, e^{-z}}{2\,n^{5/3}} + (-1)^q \frac{c_3 \ell^2 e^{-z}}{2\,n^{2/3}} \\ &+ (-2)^{q-2} \frac{c_4 \ell^3 e^{-2\,z}}{n^{4/3}} + \left(-\frac{1}{2}\right)^q \frac{c_5 \ell^{3/2} e^{-z/2}}{2^{1/2} n^{1/3}} \,. \end{aligned} \tag{78}$$

Let z_0 be the solution of $H^{'}(z) = 0$. By hypothesis, $a = \Theta(\ell)$ so that a is large. Therefore, z_0 is located near $\frac{1}{3}\ln(a+1)$. We can proceed by an iterative method (see [6, Chapter 2]) to get a full asymptotic expansion of z_0. For our present purpose the first few terms of such expansion suffice. If we let $x_0 = \exp(z_0)$, solving $H^{'}(z_0) = 0$ we obtain

$${x_0}^3 = (a+1) + O\left(\frac{\ell^{4/3}}{n^{1/3}}\right) . \tag{79}$$

We also have

$$H(z_0) = \frac{a+1}{3}\ln(a+1) - \frac{a+1}{3} + O\left(\frac{\ell^{4/3}}{n^{1/3}}\right) \tag{80}$$

and

$$H^{''}(z_0) = -3(a+1) + O\left(\frac{\ell^{4/3}}{n^{1/3}}\right) . \tag{81}$$

That is $H^{''}(z_0) < 0$. At this stage, we can consider $\exp\left(H^{''}(z_0)\frac{(z-z_0)^2}{2}\right)$ as the main factor of the integrand. We refer here to the book of De Bruijn [6, §4.4 and §6.8] for more discussions about asymptotic estimates on integrals of the forms "$\int x^a\, e^{\text{Polynomial}(x)}$" and we infer that

$$\int_{-\frac{2}{3}\ln n - \ln 2}^{\frac{1}{3}\ln n - \ln(2c)} \exp(H(z))dz \sim \sqrt{-\frac{2\pi}{H^{''}(z_0)}} \exp\left(H(z_0)\right) . \tag{82}$$

Using the Stirling formula for Gamma function, i.e., $\Gamma(t+1) \sim \sqrt{2\pi t}\frac{t^t}{e^t}$ and since $z_0 \sim \frac{1}{3}\ln(a+1)$, $H(z_0) \sim \frac{(a+1)}{3}(\ln(a+1) - 1)$ and $H^{''}(z_0) \sim -3(a+1)$, we can see that (82) leads to (73) which is similar to the formula already obtained by Janson in [16]. $\square$

3.2. **Proof of Theorem 1.2.** Using lemmas 3.1, 3.2, 3.3 (namely with $a = \frac{3\ell+1}{2}$) and theorem 1.1, after nice cancellations, we find the results announced in the theorem.

Now, let us describe briefly how to proceed. Using the upper-bound given in (54) valid for $1 \ll \ell k \le n$ and lemma 3.2, we obtain

$$\begin{aligned} \mathbb{E}(Y_n^{(\ell)}) &\le o\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{4}\sqrt{3}\pi \left(\frac{e}{12\ell}\right)^{\ell/2} 2^{3\ell/2+3/2}\, 3^{3\ell/2-1/2}\Gamma\left(\frac{\ell}{2}+\frac{1}{2}\right)(1+o(1)) \\ &\le 1 + o(1)\,. \end{aligned} \tag{83}$$

Next, using the lower-bound (55) and summing only for $\omega(n) \ll k \ll n/2$, we get (using lemma 3.2) $\mathbb{E}(Y_n^{(\ell)}) \ge 1 + o(1)$. We find $\mathbb{E}(Z_n^{(\ell)}) \sim \frac{1}{3\ell}$ by Theorem 1.1 and also

$$\mathbb{E}(V_n^{(\ell)}) \sim \left(\sum_{k=\omega(n)}^{n} k\alpha(\ell-1;k)\right)\left(1 + O\left(\frac{1}{\ell}\right)\right) \sim 12^{1/3}\ell^{1/3}n^{2/3}\,. \tag{84}$$

4. **Higher moments**

To simplify computations, we consider in the rest of the paper only $(k, k+o(k^{1/3}))$ connected graphs.

4.1. **Adding edges to an ℓ-component: higher moments.** As already said, proves given here follow (humbly) the works of Janson in [15, 16] but in our work we allow the parameter ℓ to grow relatively with n. Turning to higher moments, we observe that $\mathbb{E}(Y_n^{(\ell)})_m$ is the number of m-tuples of edges added to a $i-$th ℓ-component of order k_i during the evolution of the random graph process.

There are $\binom{n}{k_1\dots k_m}\prod_i c(k_i, k_i+\ell)$ manners to choose an ℓ-component having respectively $k_1, \dots, k_m$ vertices. There are $\prod_i \left(\binom{k_i}{2} - k_i - \ell\right)$ ways to choose the new edge. Furthermore, the probability that such possible component is one of $\{\mathbb{G}(n,t)\}_{0\le t\le 1}$ is

$$\prod_i t_i^{k_i+\ell}(1-t_i)^{(n-\sum k_j)k_i + \binom{k_i}{2} - k_i - \ell}\prod_{i<j}(1 - t_i \vee t_j)^{k_ik_j}$$

and with the conditional probability $\frac{dt_i}{(1-t_i)}$ that a given edge is added during the interval $(t_i, t_i + dt_i)$ and not earlier, integrating over all times, i.e. $t_i \in [0,\, 1]$ and summing over k_i, we obtain

$$\begin{aligned} \mathbb{E}(Y_n^{(\ell)})_m = &\sum_{k_1=1}^{n}\cdots\sum_{k_m=1}^{n}\int_0^1\cdots\int_0^1 (n)_{k_*}\prod_i \frac{c(k_i,k_i+\ell)}{k_i!}\left(\binom{k_i}{2} - k_i - \ell\right) \\ & t_i^{k_i+\ell}(1-t_i)^{(n-k_i)k_j + \binom{k_i}{2} - k_i - \ell - 1}\prod_{i<j}(1 - t_i\vee t_j)^{k_ik_j}\, dt_1\dots dt_m \end{aligned} \tag{85}$$

where $k_* = \sum_i k_i$. We remark here that

$$c(k_i, k_i+\ell) = 0 \text{ for } k_i = 1, 2, \cdots \lceil(3+\sqrt{9+8\ell})/2\rceil - 1\,.$$

Rewriting the integrand in (85) as a function of k_i and t_i, viz.

$$\varphi_n(k_i,t_i) \equiv \varphi_n(k_1,\cdots,k_m,t_1,\cdots,t_m)\,,$$

with $\varphi_n(k_i,t_i)=0$ if $\exists j\in[1,m]$ s.t. $k_j\le\lceil(3+\sqrt{9+8\ell})/2\rceil-1$ or $k_j>n$ or $t_j\notin(0,1)$ and substituting $k_i=\lceil x_i n^{2/3}\rceil$ and $t_i=n^{-1}+u_i n^{-4/3}$, we have

$$\begin{aligned}\mathbb{E}({Y_n}^{(\ell)})_m &= \int_0^{n^{1/3}}\cdots\int_0^{n^{1/3}}\int_{-n^{1/3}}^{n^{4/3}-n^{1/3}}\cdots\int_{-n^{1/3}}^{n^{4/3}-n^{1/3}} \\ &\quad\times\ \varphi_n\left(\lceil x_i n^{2/3}\rceil, \frac{1}{n}+\frac{u_i}{n^{4/3}}\right)\frac{du_i\cdots du_m\,dx_i\cdots dx_m}{n^{2m/3}} \\ &= \int_0^{\infty}\cdots\int_0^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Psi_n^{(m)}(x_i,u_i)\,du_i\cdots du_m\,dx_i\cdots dx_m\,, \qquad (86)\end{aligned}$$

where

$$\Psi_n^{(m)}(x_i,u_i)\equiv\Psi_n^{(m)}(x_1,\cdots,x_m,\,u_1,\cdots,u_m)=\frac{\varphi_n\left(\lceil x_i n^{2/3}\rceil,\frac{1}{n}+\frac{u_i}{n^{4/3}}\right)}{n^{2m/3}}\,. \qquad (87)$$

We shall now investigate the integrand in (85). For this purpose, we consider each term of the products in this integrand and we assume that $x_i n^{2/3}$ are integers. In the following, for each factor, we use the substitutions $k_i=x_i n^{2/3}$ and $t_i=n^{-1}+u_i n^{-4/3}$ as done above. We then have (denoting $x_*=\sum x_i$)

$$(n)_{k_*}=n^{k_*}\exp\left(-\frac{{x_*}^2}{2}n^{1/3}-\frac{{x_*}^3}{6}+O\Big(\frac{x_*}{n^{1/3}}(1+{x_*}^3)\Big)\right)\,, \qquad (88)$$

$$\begin{aligned}\binom{k_i}{2}-k_i-\ell &= \frac{{k_i}^2}{2}\Big(1+O\Big(\frac{1}{k_i}\Big)\Big) \\ &= \frac{x_i^2}{2}n^{4/3}\Big(1+O\Big(\frac{1}{x_i n^{2/3}}\Big)\Big)\,. \qquad (89)\end{aligned}$$

Using Stirling's formula and asymptotic formulae for $c(k_i,k_i+\ell)$ (see for instance [3, 29]), it yields

$$\begin{aligned}\frac{c(k_i,k_i+\ell)}{k_i!} &= \sqrt{\frac{3}{2}}d\,\exp\left(\frac{\ell}{2}+k_i\right){k_i}^{\frac{3}{2}\ell-1}\frac{1}{(12\,\ell)^{\ell/2}}\left(1+O\Big(\frac{1}{\ell}\Big)+O\Big(\frac{1}{k_i}\Big)+O\Big(\frac{\ell^{3/2}}{k_i^{1/2}}\Big)\right) \\ &= \sqrt{\frac{3}{2}}d\,\exp\left(\frac{\ell}{2}+x_i n^{2/3}\right){x_i}^{\frac{3}{2}\ell-1}n^{\ell-\frac{2}{3}}\frac{1}{(12\,\ell)^{\ell/2}} \\ &\quad\times\left(1+O\Big(\frac{1}{\ell}\Big)+O\Big(\frac{1}{x_i n^{2/3}}\Big)+O\Big(\frac{\ell^{3/2}}{{x_i}^{1/2}n^{1/3}}\Big)\right)\,. \qquad (90)\end{aligned}$$

(We use formulae $c(k_i,k_i+\ell)$ for $l=o(k_i^{1/3})$ and $d=\frac{1}{2\pi}$ as described in [3].) Also, after the same substitutions

$$\begin{aligned}{t_i}^{k_i+l} &= \frac{1}{n^{k_i+\ell}}\Big(1+\frac{u_i}{n^{1/3}}\Big)^{k_i+\ell} \\ &= \frac{1}{n^\ell n^{x_i n^{2/3}}}\exp\left(\frac{\ell u_i}{n^{1/3}}-\frac{\ell u_i}{2n^{2/3}}+x_i u_i n^{1/3}-\frac{x_i u_i^2}{2}+O\Big(\frac{\ell u_i^3}{n}\Big)+O\Big(\frac{x_i u_i}{n^{1/3}}\Big)\right)\,, \qquad (91)\end{aligned}$$

and

$$(1-t_i \vee t_j)^{k_ik_j} = \exp\left(-k_ik_j(t_i \vee t_j) + \frac{1}{2}k_ik_j(t_i \vee t_j)^2 + O(k_ik_j)(t_i \vee t_j)^3\right)$$

$$= \exp\left(-x_ix_jn^{1/3} - x_ix_j(u_i \vee u_j) + O\Big(\frac{x_ix_j}{n^{2/3}}\Big)\right). \quad (92)$$

Finally,

$$(1-t_i)^{(n-k_*)k_i+\binom{k_i}{2}-k_i-\ell-1} = \exp\Big(- x_in^{2/3} - (u_i + x_*)x_in^{1/3} + u_ix_ix_* - \frac{x_i^2}{2}\left(n^{1/3}+u_i\right) + \frac{3}{2}\frac{x_i}{n^{1/3}}\left(1+\frac{u_i}{n^{1/3}}\right) + \frac{\ell}{n} + \frac{\ell u_i}{n^{4/3}} + \frac{1}{n} + \frac{u_i}{n^{4/3}} + O\Big(\frac{x_i}{n^{1/3}}\Big)\Big). \quad (93)$$

Using the equations (88) – (93) given above, the integrand in (86) reads

$$\Psi_n^{(m)}(x_i,u_i) = A_m \exp(B_m)(1+\varepsilon). \quad (94)$$

A bit of algebra gives A_m and B_m

$$A_m = \left(\sqrt{\frac{3}{2}}\right)^m d^m \prod_{i=1}^{m} \frac{x_i^{3/2\ell+1}}{2}, \quad (95)$$

$$B_m = \frac{ml}{2}\Big(1-\ln(12\ell)\Big) - \frac{{x_*}^3}{6} - \sum_{i=1}^{m}\frac{x_iu_i^2}{2} + x_*\sum_{i=1}^{m}x_iu_i - \frac{1}{2}\sum_{1\le i,\, j\le m} x_ix_j(u_i \vee u_j). \quad (96)$$

The ε in (94) regroups all the big-Ohs produced by (88) – (93). In particular, if (x_i), (u_i) and $(1/x_i)$ are fixed, as $n\to\infty$, we have

$$\Psi_n^{(m)}(x_i,u_i) = A_m \exp(B_m)(1+o(1)). \quad (97)$$

So, if $x_i>0$, $u_i \in (-\infty,\infty)$ fixed, without restricting each $x_in^{2/3}$ to be an integer, we get

$$\Psi_n^{(m)}(x_i,u_i) \to \left(\sqrt{\frac{3}{2}}\right)^m d^m \frac{\exp\left(\frac{m\ell}{2}\right)}{(12\ell)^{\frac{m\ell}{2}}}\left(\prod_{i=1}^{m}\frac{x_i^{3/2\ell+1}}{2}\right) \times \exp\left(-\frac{{x_*}^3}{6} - \sum_{i=1}^{m}\frac{x_iu_i^2}{2} + x_*\sum_{i=1}^{m}x_iu_i - \frac{1}{2}\sum_{1\le i,\, j\le m}x_ix_j(u_i\vee u_j)\right) \quad (98)$$

as $n\to\infty$. Next, we use the estimate

$$\prod_{i<j}(1-t_i\vee t_j)^{k_ik_j} \le \prod_{i\ne j}(1-t_i)^{k_ik_j/2},$$

to state that, there is a constant C_1 such that

$$\Psi_n^{(m)} \le C_1(n)_{k_*}\prod_i \frac{c(k_i,k_i+\ell)}{k_i!}\left(\frac{k_i^2-3k_i}{2}-\ell-1\right){t_i}^{k_i}(1-t_i)^{k_i(n-2-k_*/2)}. \quad (99)$$

Then, using the bounds given in [15, eq. (2.12) – (2.18)] with (88) – (93), we get (the C_i below are constants)

$$\begin{aligned}\Psi_n^{(m)} \;\leq\;& g_m(x_i,u_i) = C_2 \exp\left(-\delta {x_*}^3\right) \prod_i x_i^{3\ell/2+1} \exp\left(-\delta x_i u_i^2\right)\\ &+\; C_3 \exp\left(-\delta {x_*}^3\right) \prod_i x_i^{3\ell/2+1} \exp\left(-\delta x_i u_i\right)\\ &+\; C_4 \exp\left(-\delta {x_*}^3\right) \prod_i \frac{1}{(1+u_i^2)}\,,\end{aligned} \tag{100}$$

valid for all n, x_i, u_i. Since $\int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty g_m(x_i,u_i) dx_1 \cdots dx_m du_1 \cdots du_m < \infty$, (86), (98) and the use of Lebesgue dominated convergence yield

$$\mathbb{E}(Y_n^{(\ell)})_m \sim \left(\sqrt{\frac{3}{8}}\frac{d\,\exp\left(\frac{\ell}{2}\right)}{(12\ell)^{\frac{\ell}{2}}}\right)^m a_m^{(\ell)}\,, \tag{101}$$

where

$$\begin{aligned} a_m^{(\ell)} \;=\;& \int_0^\infty \cdots \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \left(\prod_{i=1}^m x_i^{3\ell/2+1}\right)\\ &\times\; \exp\Big(-\frac{1}{6}{x_*}^3 - \frac{1}{2}\sum_{i=1}^m x_i u_i^2 + x_* \sum_{i=1}^m x_i u_i\Big)\\ &\times\; \exp\Big(-\frac{1}{2}\sum_{1\le i,j\le m} x_i x_j (u_i \vee u_j)\Big) dx_1 \cdots dx_m du_1 \cdots du_m\\ \leq\;& \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^m x_i^{3\ell/2+1}\right) \exp\left(-\frac{1}{24}\left(\sum_{i=1}^m x_i\right)^3\right) dx_1 \cdots dx_m\\ \leq\;& \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^m x_i^{3\ell/2+1}\right) \exp\left(-\frac{1}{24}\sum_{i=1}^m x_i^3\right) dx_1 \cdots dx_m\\ \leq\;& \left(\frac{4}{3}\, 2^\ell 6^{\ell/2} 3^{2/3} \Gamma\left(\ell/2+2/3\right)\right)^m.\end{aligned} \tag{102}$$

Using this latter inequality with (101), we get that

$$\mathbb{E}(Y_n^{(\ell)})_m \to 0 \qquad (m>0, \quad n,\, \ell \to \infty)\,. \tag{103}$$

4.2. **Joining two complex components: higher moments.** We observe that $(Z_n^{(\ell)})_m$ is the number of m-tuples of edges added between a p-component and, resp., a $(\ell-p)$-component of order k_i and, resp., k_j. By Theorem 1.1, we find

$$c'(k_i, k_i+\ell+1) = \frac{1}{3\ell}\Big(k_i^2/2 - 3k_i/2 - \ell\Big)\, c(k_i, k_i+\ell)\Big(1 + O(1/\ell) + O(\ell^{3/2}/k_i^{1/2})\Big) \tag{104}$$

which means that we can obtain expressions for $\mathbb{E}(Z_n^{(\ell)})_m$ by simply introducing a factor $1/3\ell$ in (85). Therefore,

$$\mathbb{E}(Z_n^{(\ell)})_m \to 0, \qquad (m > 0, \quad n,\, \ell \to \infty)\,. \tag{105}$$

5. **Conclusion**

In this paper, we have studied the growths of complexity of connected components in an evolving graph. We have shown, using a combination of the methods from [15] and the theory of generating functions, how one can quantify asymptotically properties of such components growths. Amongst other things, we study complex components that increase their complexity by receiving new edges and/or by merging other complex components. As $\ell \to \infty$, our results show that whenever the second case occurs, almost all times, it is a unicyclic component that is swallowed by the considered ℓ-component. Our other result states that as $1 \ll \ell \ll n$, the expected number of vertices that ever belong to an ℓ-component is about $(12\ell)^{1/3}\, n^{2/3}$.

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E-mail address: `vlad@lipn.univ-paris13.fr`

Vlady RAVELOMANANA, LIPN – UMR 7030, Institut Galilée –Université de Paris-Nord, 99, Avenue J. B. Clément. F 93430 Villetaneuse, France.