

# Moment estimates for Lévy Processes

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## Abstract

For real Lévy processes  $(X_t)_{t \geq 0}$  having no Brownian component with Blumenthal-Gettoor index  $\beta$ , the estimate  $\mathbb{E} \sup_{s \leq t} |X_s - a_p s|^p \leq C_p t$  for every  $t \in [0, 1]$  and suitable  $a_p \in \mathbb{R}$  has been established by Millar [6] for  $\beta < p \leq 2$  provided  $X_1 \in L^p$ . We derive extensions of these estimates to the cases  $p > 2$  and  $p \leq \beta$ .

*Key words:* Lévy process increment, Lévy measure,  $\alpha$ -stable process, Normal Inverse Gaussian process, tempered stable process, Meixner process.

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## 1 Introduction and results

We investigate the  $L^p$ -norm (or quasi-norm) of the maximum process of real Lévy processes having no Brownian component. A (càdlàg) Lévy process  $X = (X_t)_{t \geq 0}$  is characterized by its so-called local characteristics in the Lévy-Khintchine formula. They depend on the way the "big" jumps are truncated. We will adopt in the following the convention that the truncation occurs at size 1. So that

$$\mathbb{E} e^{iuX_t} = e^{-t\Psi(u)} \text{ with } \Psi(u) = -iua + \frac{1}{2}\sigma^2 u^2 - \int (e^{iux} - 1 - iux \mathbf{1}_{\{|x| \leq 1\}}) d\nu(x) \quad (1.1)$$

where  $u, a \in \mathbb{R}, \sigma^2 \geq 0$  and  $\nu$  is a measure on  $\mathbb{R}$  such that  $\nu(\{0\}) = 0$  and  $\int x^2 \wedge 1 d\nu(x) < +\infty$ . The measure  $\nu$  is called the Lévy measure of  $X$  and the quantities  $(a, \sigma^2, \nu)$  are referred to as the characteristics of  $X$ . One shows that for  $p > 0, \mathbb{E} |X_1|^p < +\infty$  if and only if  $\mathbb{E} |X_t|^p < +\infty$  for every  $t \geq 0$  and this in turn is equivalent to  $\mathbb{E} \sup_{s \leq t} |X_s|^p < +\infty$  for every  $t \geq 0$ . Furthermore,

$$\mathbb{E} |X_1|^p < +\infty \text{ if and only if } \int_{\{|x| > 1\}} |x|^p d\nu(x) < +\infty \quad (1.2)$$

(see [7]). The index  $\beta$  of the process  $X$  introduced in [2] is defined by

$$\beta = \inf\{p > 0 : \int_{\{|x| \leq 1\}} |x|^p d\nu(x) < +\infty\}. \quad (1.3)$$

Necessarily,  $\beta \in [0, 2]$ .

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In the sequel we will assume that  $\sigma^2 = 0$ , *i.e.* that  $X$  has no Brownian component. Then the Lévy-It decomposition of  $X$  reads

$$X_t = at + \int_0^t \int_{\{|x| \leq 1\}} x(\mu - \lambda \otimes \nu)(ds, dx) + \int_0^t \int_{\{|x| > 1\}} x\mu(ds, dx) \quad (1.4)$$

where  $\lambda$  denotes the Lebesgue measure and  $\mu$  is the Poisson random measure on  $\mathbb{R}_+ \times \mathbb{R}$  associated with the jumps of  $X$  by

$$\mu = \sum_{t \geq 0} \varepsilon_{(t, \Delta X_t)} \mathbf{1}_{\{\Delta X_t \neq 0\}},$$

$\Delta X_t = X_t - X_{t-}$ ,  $\Delta X_0 = 0$  (see [4], [7]).

**Theorem 1** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with characteristics  $(a, 0, \nu)$  and index  $\beta$  such that  $\mathbb{E}|X_1|^p < +\infty$  for some  $p \in (\beta, \infty)$  or for  $p = \beta$  provided  $\int_{\{|x| \leq 1\}} |x|^\beta d\nu(x) < +\infty$  and  $\beta > 0$ . Then for every  $t \geq 0$*

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |Y_s|^p &\leq C_p t && \text{if } p < 1, \\ \mathbb{E} \sup_{s \leq t} |X_s - s \mathbb{E} X_1|^p &\leq C_p t && \text{if } 1 \leq p \leq 2 \end{aligned}$$

where  $Y_t = X_t - t(a - \int_{\{|x| \leq 1\}} x d\nu(x))$ . Furthermore, for every  $p > 2$

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t) \quad \text{as } t \rightarrow 0$$

for a finite real constant  $C_p$ .

If  $X_1$  is symmetric one observes that  $Y = X$  since the symmetry of  $X_1$  implies  $a = 0$  and the symmetry of  $\nu$  (see [7]). We emphasize that in view of the Kolmogorov criterion for continuous modifications the above bounds are best possible as concerns powers of  $t$ . In case  $p > \beta$  and  $p \leq 2$ , these estimates are due to Millar [6]. However, the Laplace-transform approach in [6] does not work for  $p > 2$ . Our proof is based on the Burkholder-Davis-Gundy inequality.

For the case  $p < \beta$  we need some assumptions on  $X$ . Recall that a measurable function  $\varphi: (0, c] \rightarrow (0, \infty)$  ( $c > 0$ ) is said to be regularly varying at zero with index  $b \in \mathbb{R}$  if, for every  $t > 0$ ,

$$\lim_{x \rightarrow 0} \frac{\varphi(tx)}{\varphi(x)} = t^b.$$

This means that  $\varphi(1/x)$  is regularly varying at infinity with index  $-b$ . Slow variation corresponds to  $b = 0$ .

**Theorem 2** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with characteristics  $(a, 0, \nu)$  and index  $\beta$  such that  $\beta > 0$  and  $\mathbb{E}|X_1|^p < +\infty$  for some  $p \in (0, \beta)$ . Assume that the Lévy measure satisfies*

$$\exists c \in (0, 1], \mathbf{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq \varphi(|x|) \mathbf{1}_{\{0 < |x| \leq c\}} dx \quad (1.5)$$

where  $\varphi: (0, c] \rightarrow (0, \infty)$  is a regularly varying function at zero of index  $-(\beta + 1)$ . Let  $l(x) = x^{\beta+1} \varphi(x)$  and assume that  $l(1/x)$ ,  $x \geq c$  is locally bounded. Let  $\underline{l}(x) = \underline{l}_\beta(x) = l(x^{1/\beta})$ .

(a) Assume  $\beta > 1$ . Then as  $t \rightarrow 0$ , for every  $r \in (\beta, 2]$ ,  $q \in [p \vee 1, \beta)$ ,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\beta} [\underline{l}(t)^{p/r} + \underline{l}(t)^{p/q}]) \quad \text{if } \beta < 2,$$

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\beta} [1 + \underline{l}(t)^{p/q}]) \quad \text{if } \beta = 2.$$

If  $\nu$  is symmetric then this holds for every  $q \in [p, \beta)$ .

(b) Assume  $\beta < 1$ . Then as  $t \rightarrow 0$ , for every  $r \in (\beta, 1]$ ,  $q \in [p, \beta)$

$$\mathbb{E} \sup_{s \leq t} |Y_s|^p = O(t^{p/\beta} [\underline{l}(t)^{p/r} + \underline{l}(t)^{p/q}])$$

where  $Y_t = X_t - t(a - \int_{\{|x| \leq 1\}} x d\nu(x))$ . If  $\nu$  is symmetric this holds for every  $r \in (\beta, 2]$ .

(c) Assume  $\beta = 1$  and  $\nu$  is symmetric. Then as  $t \rightarrow 0$ , for every  $r \in (\beta, 2]$ ,  $q \in [p, \beta)$

$$\mathbb{E} \sup_{s \leq t} |X_s - as|^p = O(t^{p/\beta} [\underline{l}(t)^{p/r} + \underline{l}(t)^{p/q}]).$$

It can be seen from strictly  $\alpha$ -stable Lévy processes where  $\beta = \alpha$  that the above estimates are best possible as concerns powers of  $t$ .

Observe that condition (1.5) is satisfied for a broad class of Lévy processes. It implies that the tail function  $t \mapsto \underline{\nu}(t) := \nu([-t, t]^c)$ ,  $t > 0$  of the Lévy measure is dominated, for  $t \leq c$ , by  $2 \int_t^c \varphi(x) ds + \nu(|x| > c)$ , a regularly varying function at zero with index  $-\beta$ , so that  $\underline{\nu}(t) = O(t\varphi(t))$  as  $t \rightarrow 0$ .

Important special cases are as follows.

**Corollary 1.1** *Assume the situation of Theorem 2 (with  $\nu$  symmetric if  $\beta = 1$ ) and let  $U$  denote any of the processes  $X, Y, (X_t - at)_{t \geq 0}$ .*

(a) *Assume that the slowly varying part  $l$  of  $\varphi$  is decreasing and unbounded on  $(0, c]$  (e.g.  $(-\log x)^a$ ,  $a > 0$ ). Then as  $t \rightarrow 0$ , for every  $\varepsilon \in (0, \beta)$ ,*

$$\mathbb{E} \sup_{s \leq t} |U_s|^p = O(t^{p/\beta} \underline{l}(t)^{p/(\beta-\varepsilon)}).$$

(b) *Assume that  $l$  is increasing on  $(0, c]$  satisfying  $l(0+) = 0$  (e.g.  $(-\log x)^{-a}$ ,  $a > 0, c < 1$ ) and  $\beta \in (0, 2)$ . Then as  $t \rightarrow 0$ , for every  $\varepsilon > 0$ ,*

$$\mathbb{E} \sup_{s \leq t} |U_s|^p = O(t^{p/\beta} \underline{l}(t)^{p/(\beta+\varepsilon)}).$$

The remaining cases  $p = \beta \in (0, 2)$  if  $\beta \neq 1$  and  $p \leq 1$  if  $\beta = 1$  are solved under the assumption that the slowly varying part of the function  $\varphi$  in (1.5) is constant.

**Theorem 3** *Let  $(X_t)_{t \geq 0}$  be a Lévy process with characteristics  $(a, 0, \nu)$  and index  $\beta$  such that  $\beta \in (0, 2)$  and  $\mathbb{E} |X_1|^\beta < +\infty$  if  $\beta \neq 1$  and  $\mathbb{E} |X_1|^p < +\infty$  for some  $p \leq 1$  if  $\beta = 1$ . Assume that the Lévy measure satisfies*

$$\exists c \in (0, 1], \exists C \in (0, \infty), \mathbf{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq \frac{C}{|x|^{\beta+1}} \mathbf{1}_{\{0 < |x| \leq c\}} dx. \quad (1.6)$$

Then as  $t \rightarrow 0$

$$\mathbb{E} \sup_{s \leq t} |X_s|^\beta = O(t(-\log t)) \quad \text{if } \beta >> 1,$$

$$\mathbb{E} \sup_{s \leq t} |Y_s|^\beta = O(t(-\log t)) \quad \text{if } \beta < 1$$

and

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O((t(-\log t))^p) \quad \text{if } \beta = 1, p \leq 1$$

where the process  $Y$  is defined as in Theorem 2.

The above estimates are optimal (see Section 3).

The paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1, 2 and 3. Section 3 contains a collection of examples.

## 2 Proofs

We will extensively use the following compensation formula (see e.g. [4])

$$\mathbb{E} \int_0^t \int f(s, x) \mu(ds, dx) = \mathbb{E} \sum_{s \leq t} f(s, \Delta X_s) \mathbf{1}_{\{\Delta X_s \neq 0\}} = \int_0^t \int f(s, x) d\nu(x) ds$$

where  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$  is a Borel function.

**Proof of Theorem 1.** Since  $\mathbb{E}|X_1|^p < +\infty$  and  $p > \beta$  (or  $p = \beta$  provided  $\int_{\{|x| \leq 1\}} |x|^\beta d\nu(x) < +\infty$  and  $\beta > 0$ ), it follows from (1.2) that

$$\int |x|^p d\nu(x) < +\infty.$$

CASE 1:  $0 < p < 1$ . In this case we have  $\beta < 1$  and hence  $\int_{\{|x| \leq 1\}} |x| d\nu(x) < +\infty$ . Consequently,  $X$  a.s. has finite variation on finite intervals. By (1.4),

$$Y_t = X_t - t \left( a - \int_{\{|x| \leq 1\}} x d\nu(x) \right) = \int_0^t \int x \mu(ds, dx) = \sum_{s \leq t} \Delta X_s$$

so that, using the elementary inequality  $(u + v)^p \leq u^p + v^p$ ,

$$\sup_{s \leq t} |Y_s|^p \leq \left( \sum_{s \leq t} |\Delta X_s| \right)^p \leq \sum_{s \leq t} |\Delta X_s|^p = \int_0^t \int |x|^p \mu(ds, dx).$$

Consequently,

$$\mathbb{E} \sup_{s \leq t} |Y_s|^p \leq t \int |x|^p d\nu(x) \text{ for every } t \geq 0.$$

CASE 2:  $1 \leq p \leq 2$ . Introduce the martingale

$$M_t := X_t - t\mathbb{E} X_1 = X_t - t \left( a + \int_{\{|x| > 1\}} x d\nu(x) \right) = \int_0^t \int x (\mu - \lambda \otimes \nu)(ds, dx).$$

It follows from the Burkholder-Davis-Gundy inequality (see [5]) that

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \mathbb{E} [M]_t^{p/2}$$

for some finite constant  $C$ . Since  $p/2 \leq 1$ , the quadratic variation  $[M]$  of  $M$  satisfies

$$[M]_t^{p/2} = \left( \sum_{s \leq t} |\Delta X_s|^2 \right)^{p/2} \leq \sum_{s \leq t} |\Delta X_s|^p$$

so that

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq Ct \int |x|^p d\nu(x) \text{ for every } t \geq 0.$$

CASE 3:  $p > 2$ . One considers again the martingale Lévy process  $M_t = X_t - t\mathbb{E} X_1$ . For  $k \geq 1$  such that  $2^k \leq p$ , introduce the martingales

$$N_t^{(k)} := \int_0^t \int |x|^{2^k} (\mu - \lambda \otimes \nu)(ds, dx) = \sum_{s \leq t} |\Delta X_s|^{2^k} - t \int |x|^{2^k} d\nu(x).$$

Set  $m := \max\{k \geq 1 : 2^k < p\}$ . Again by the Burkholder-Davis-Gundy inequality

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |M_s|^p &\leq C \mathbb{E} [M]_t^{p/2} \\ &= C \mathbb{E} \left( t \int x^2 d\nu(x) + N_t^{(1)} \right)^{p/2} \\ &\leq C \left( t^{p/2} \left( \int x^2 d\nu(x) \right)^{p/2} + \mathbb{E} |N_t^{(1)}|^{p/2} \right) \\ &\leq C (t + \mathbb{E} |N_t^{(1)}|^{p/2}) \end{aligned}$$

for every  $t \in [0, 1]$  where  $C$  is a finite constant that may vary from line to line. Applying successively the Burkholder-Davis-Gundy inequality to the martingales  $N^{(k)}$  and exponents  $p/2^k > 1, 1 \leq k \leq m$ , finally yields

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C(t + \mathbb{E} [N^{(m)}]_t^{p/2^{m+1}}) \quad \text{for every } t \in [0, 1].$$

Using  $p \leq 2^{m+1}$ , one gets

$$[N^{(m)}]_t^{p/2^{m+1}} = \left( \sum_{s \leq t} |\Delta X_s|^{2^{m+1}} \right)^{p/2^{m+1}} \leq \sum_{s \leq t} |\Delta X_s|^p$$

so that

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \left( t + t \int |x|^p d\nu(x) \right) \quad \text{for every } t \in [0, 1].$$

This implies  $\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t)$  as  $t \rightarrow 0$ . □

**Proof of Theorems 2 and 3.** Let  $p \leq \beta$  and fix  $c \in (0, 1]$ . Let  $\nu_1 = \mathbf{1}_{\{|x| \leq c\}} \cdot \nu$  and  $\nu_2 = \mathbf{1}_{\{|x| > c\}} \cdot \nu$ . Construct Lévy processes  $X^{(1)}$  and  $X^{(2)}$  such that  $X \stackrel{d}{=} X^{(1)} + X^{(2)}$  and  $X^{(2)}$  is a compound Poisson process with Lévy measure  $\nu_2$ . Then  $\beta = \beta(X) = \beta(X^{(1)}), \beta(X^{(2)}) = 0$ ,  $\mathbb{E} |X^{(1)}|^q < +\infty$  for every  $q > 0$  and  $\mathbb{E} |X_1^{(2)}|^p < +\infty$ . It follows e.g. from Theorem 1 that for every  $t \geq 0$ ,

$$\mathbb{E} \sup_{s \leq t} |X_s^{(2)}|^p \leq C_p t \quad \text{if } p < 1, \tag{2.1}$$

$$\mathbb{E} \sup_{s \leq t} |X^{(2)} - s \mathbb{E} X_1^{(2)}|^p \leq C_p t \quad \text{if } 1 \leq p \leq 2$$

where  $\mathbb{E} X_1^{(2)} = \int x d\nu_2(x) = \int_{\{|x| > c\}} x d\nu(x)$ .

As concerns  $X^{(1)}$ , consider the martingale

$$Z_t^{(1)} := X_t^{(1)} - t \mathbb{E} X_1^{(1)} = X_t^{(1)} - t \left( a - \int x \mathbf{1}_{\{c < |x| \leq 1\}} d\nu(x) \right) = \int_0^t \int x (\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

where  $\mu_1$  denotes the Poisson random measure associated with the jumps of  $X^{(1)}$ . The starting idea is to part the 'small' and the 'big' jumps of  $X^{(1)}$  in a non homogeneous way with respect to the function  $s \mapsto s^{1/\beta}$ . Indeed one may decompose  $Z^{(1)}$  as follows

$$Z^{(1)} = M + N$$

where

$$M_t := \int_0^t \int x \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} (\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

and

$$N_t := \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} (\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

are martingales. Observe that for every  $q > 0$  and  $t \geq 0$ ,

$$\begin{aligned} \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &= \int |x|^q (|x|^\beta \wedge t) d\nu_1(x) \\ &\leq \int_{\{|x| \leq c\}} |x|^{\beta+q} d\nu(x) < +\infty. \end{aligned}$$

Consequently,

$$N_t = \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\mu_1(s, x) - \psi(t)$$

where  $\psi(t) := \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds$ . Furthermore, for every  $r > \beta$  or  $r = 2$  and  $t \geq 0$

$$\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \leq t \int_{\{|x| \leq c\}} |x|^r d\nu(x) < +\infty. \quad (2.2)$$

In the sequel let  $C$  denote a finite constant that may vary from line to line.

We first claim that for every  $t \geq 0, r \in (\beta, 2] \cap [1, 2]$  and for  $r = 2$ ,

$$\mathbb{E} \sup_{s \leq t} |M_s| \leq C \left( \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r}. \quad (2.3)$$

In fact, it follows from the Burkholder-Davis-Gundy inequality and from  $p/r \leq 1, r/2 \leq 1$  that

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |M_s|^p &\leq \left( \mathbb{E} \sup_{s \leq t} |M_s|^r \right)^{p/r} \\ &\leq C \left( \mathbb{E} [M]_t^{r/2} \right)^{p/r} \\ &= C \left( \mathbb{E} \left( \sum_{s \leq t} |\Delta X_s^{(1)}|^2 \mathbf{1}_{\{|\Delta X_s^{(1)}| \leq s^{1/\beta}\}} \right)^{r/2} \right)^{p/r} \\ &\leq C \left( \mathbb{E} \sum_{s \leq t} |\Delta X_s^{(1)}|^r \mathbf{1}_{\{|\Delta X_s^{(1)}| \leq s^{1/\beta}\}} \right)^{p/r} \\ &= C \left( \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r}. \end{aligned}$$

Exactly as for  $M$ , one gets for every  $t \geq 0$  and every  $q \in [p, 2] \cap [1, 2]$  that

$$\mathbb{E} \sup_{s \leq t} |N_s|^p \leq C \left( \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q}. \quad (2.4)$$

If  $\nu$  is symmetric then (2.4) holds for every  $q \in [p, 2]$  (which of course provides additional information in case  $p < 1$  only). Indeed,  $\psi = 0$  by the symmetry of  $\nu$  so that

$$N_t = \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\mu_1(s, x)$$

and for  $q \in [p, 1]$

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} \left| \int_0^s \int x \mathbf{1}_{\{|x| > u^{1/\beta}\}} \mu_1(du, dx) \right|^p &\leq \left( \mathbb{E} \sup_{s \leq t} \left| \int_0^t \int x \mathbf{1}_{\{|x| > u^{1/\beta}\}} \mu_1(du, dx) \right|^q \right)^{p/q} \\
&\leq \left( \mathbb{E} \sum_{s \leq t} \left| \Delta X_s^{(1)} \right|^q \mathbf{1}_{\{|\Delta X_s^{(1)}| > s^{1/\beta}\}} \right)^{p/q} \\
&= \left( \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q}.
\end{aligned} \tag{2.5}$$

In the case  $\beta < 1$  we consider the process

$$\begin{aligned}
Y_t^{(1)} &:= Z_t^{(1)} + t \int x d\nu_1(x) = X_t^{(1)} - t \left( a - \int_{\{|x| \leq 1\}} x d\nu(x) \right) \\
&= M_t + N_t + t \int x d\nu_1(x) \\
&= \int_0^t \int x \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} \mu_1(ds, dx) + \int_0^1 \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} \mu_1(ds, dx).
\end{aligned}$$

Exactly as in (2.5) one shows that for  $t \geq 0$  and  $r \in (\beta, 1]$

$$\mathbb{E} \sup_{s \leq t} \left| \int_0^s \int x \mathbf{1}_{\{|x| \leq u^{1/\beta}\}} \mu_1(du, dx) \right|^p \leq \left( \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r}. \tag{2.6}$$

Combining (2.1) and (2.3) - (2.6) we obtain the following estimates. Let

$$Z_t = X_t - t \left( a - \int x \mathbf{1}_{\{c < |x| \leq 1\}} d\nu(x) \right).$$

CASE 1:  $\beta \geq 1$  and  $p < 1$ . Then for every  $t \geq 0, r \in (\beta, 2] \cup \{2\}, q \in [1, 2]$ ,

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |Z_s|^p &\leq C \left( t + \left( \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\
&\quad \left. + \left( \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right).
\end{aligned} \tag{2.7}$$

If  $\nu$  is symmetric (2.7) is even valid for every  $q \in [p, 2]$ .

CASE 2:  $\beta \geq 1$  and  $p \geq 1$ . Then for every  $t \geq 0, r \in (\beta, 2] \cup \{2\}, q \in [p, 2]$ ,

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |X_s - s \mathbb{E} X_1|^p &\leq C \left( t + \left( \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\
&\quad \left. + \left( \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right).
\end{aligned} \tag{2.8}$$

CASE 3:  $\beta < 1$ . Then for every  $t \geq 0, r \in (\beta, 1], q \in [p, 1]$

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |Y_s|^p &\leq C \left( t + \left( \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\
&\quad \left. + \left( \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right).
\end{aligned} \tag{2.9}$$

If  $\nu$  is symmetric then  $Y = Z = (X_t - at)_{t \geq 0}$  and (2.9) is valid for every  $r \in (\beta, 2], q \in [p, 2]$ .

Now we deduce Theorem 2. Assume  $p \in (0, \beta)$  and (1.5). The constant  $c$  in the above decomposition of  $X$  is specified by the constant from (1.5). Then one just needs to investigate the integrals appearing in the right hand side of the inequalities (2.7) - (2.10). One observes that Theorem 1.5.11 in [1] yields for  $r > \beta$ ,

$$\int |x|^r \varphi(|x|) \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} dx \sim \frac{2}{r - \beta} s^{\frac{r}{\beta} - 1} l(s^{1/\beta}) \text{ as } s \rightarrow 0$$

which in turn implies that for small  $t$ ,

$$\begin{aligned} \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds &\leq \int_0^t \int |x|^r \varphi(|x|) \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} dx ds \\ &\sim \frac{2\beta}{(r - \beta)r} t^{r/\beta} l(t^{1/\beta}) \text{ as } t \rightarrow 0. \end{aligned} \quad (2.10)$$

Similarly, for  $0 < q < \beta$ ,

$$\begin{aligned} \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &\leq \int_0^t \int |x|^q \varphi(|x|) \mathbf{1}_{\{|x| > s^{1/\beta}\}} dx ds \\ &\sim \frac{2\beta}{(\beta - q)q} t^{q/\beta} l(t^{1/\beta}) \text{ as } t \rightarrow 0. \end{aligned} \quad (2.11)$$

Using (2.2) for the case  $\beta = 2$  and  $t + t^p = o(t^{p/\beta} l(t)^\alpha)$  as  $t \rightarrow 0, \alpha > 0$ , for the case  $\beta > 1$  one derives Theorem 2.

As for Theorem 3, one just needs a suitable choice of  $q$  in (2.7) - (2.9). Note that by (1.6) for every  $\beta \in (0, 2)$  and  $t \leq c^\beta$ ,

$$\begin{aligned} \int_0^t \int |x|^\beta \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &= \int_0^t \int |x|^\beta \mathbf{1}_{\{c \geq |x| > s^{1/\beta}\}} d\nu(x) ds \\ &\leq C \int_0^t \int |x|^{-1} \mathbf{1}_{\{c \geq |x| > s^{1/\beta}\}} dx ds \\ &= Ct(-\log t) \end{aligned}$$

so that  $q = \beta$  is the right choice. (This choice of  $q$  is optimal.) Since by (2.10), for  $r \in (\beta, 2] (\neq \emptyset)$ ,

$$\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds = O(t^{r/\beta})$$

the assertions follow from (2.7) - (2.9). □

### 3 Examples

Let  $K_\nu$  denote the modified Bessel function of the third kind and index  $\nu$  given by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty u^{\nu-1} \exp\left(-\frac{z}{2}\left(u + \frac{1}{u}\right)\right) du, \quad z > 0.$$

• *The  $\Gamma$ -process* is a subordinator (increasing Lévy process) whose distribution  $\mathbb{P}_{X_t}$  at time  $t > 0$  is a  $\Gamma(1, t)$ -distribution

$$\mathbb{P}_{X_t}(dx) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} \mathbf{1}_{(0, \infty)}(x) dx.$$

The characteristics are given by

$$\nu(dx) = \frac{1}{x} e^{-x} \mathbf{1}_{(0,\infty)}(x) dx$$

and  $a = \int_0^1 x d\nu(x) = 1 - e^{-1}$  so that  $\beta = 0$  and  $Y = X$ . It follows from Theorem 1 that

$$\mathbb{E} \sup_{s \leq t} X_s^p = \mathbb{E} X_t^p = O(t)$$

for every  $p > 0$ . This is clearly the true rate since

$$\mathbb{E} X_t^p = \frac{\Gamma(p+t)}{\Gamma(t+1)} t \sim \Gamma(p)t \text{ as } t \rightarrow 0.$$

- The  $\alpha$ -stable Lévy Processes indexed by  $\alpha \in (0, 2)$  have Lévy measure

$$\nu(dx) = \left( \frac{C_1}{x^{\alpha+1}} \mathbf{1}_{(0,\infty)}(x) + \frac{C_2}{|x|^{\alpha+1}} \mathbf{1}_{(-\infty,0)}(x) \right) dx$$

with  $C_i \geq 0, C_1 + C_2 > 0$  so that  $\mathbb{E} |X_1|^p < +\infty$  for  $p \in (0, \alpha), \mathbb{E} |X_1|^\alpha = \infty$  and  $\beta = \alpha$ . It follows from Theorems 2 and 3 that for  $p \in (0, \alpha)$ ,

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t^{p/\alpha}) & \text{if } \alpha > 1, \\ \mathbb{E} \sup_{s \leq t} |Y_s|^p &= O(t^{p/\alpha}) & \text{if } \alpha < 1, \\ \mathbb{E} \sup_{s \leq t} |X_s|^p &= O((t(-\log t))^p) & \text{if } \alpha = 1. \end{aligned}$$

Here Theorem 3 gives the true rate provided  $X$  is not strictly stable. In fact, if  $\alpha = 1$  the scaling property in this case says that  $X_t \stackrel{d}{=} tX_1 + Ct \log t$  for some real constant  $C \neq 0$  (see [7], p.87) so that for  $p < 1$

$$\mathbb{E} |X_t|^p = t^p \mathbb{E} |X_1 + C \log t|^p \sim C^p t^p |\log t|^p \quad \text{as } t \rightarrow 0.$$

Now assume that  $X$  is *strictly*  $\alpha$ -stable. If  $\alpha < 1$ , then  $a = \int_{|x| \leq 1} x d\nu(x)$  and thus  $Y = X$  and if  $\alpha = 1$ , then  $\nu$  is symmetric (see [7]). Consequently, by Theorem 2, for every  $\alpha \in (0, 2), p \in (0, \alpha)$ ,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\alpha}).$$

In this case Theorem 2 provides the true rate since the self-similarity property of strictly stable Lévy processes implies

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = t^{p/\alpha} \mathbb{E} \sup_{s \leq 1} |X_s|^p.$$

- *Tempered stable processes* are subordinators with Lévy measure

$$\nu(dx) = \frac{2^\alpha \cdot \alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} \exp\left(-\frac{1}{2} \gamma^{1/\alpha} x\right) \mathbf{1}_{(0,\infty)}(x) dx$$

and first characteristic  $a = \int_0^1 x d\nu(x), \alpha \in (0, 1), \gamma > 0$  (see [8]) so that  $\beta = \alpha, Y = X$  and  $\mathbb{E} X_1^p < +\infty$  for every  $p > 0$ . The distribution of  $X_t$  is not generally known. It follows from Theorems 1,2 and 3 that

$$\begin{aligned} \mathbb{E} X_t^p &= O(t) & \text{if } p > \alpha, \\ \mathbb{E} X_t^p &= O(t^{p/\alpha}) & \text{if } p < \alpha \\ \mathbb{E} X_t^\alpha &= O(t(-\log t)) & \text{if } p = \alpha. \end{aligned}$$

For  $\alpha = 1/2$ , the process reduces to the *inverse Gaussian process* whose distribution  $\mathbb{P}_{X_t}$  at time  $t > 0$  is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{1}{2}\left(\frac{t}{\sqrt{x}} - \gamma\sqrt{x}\right)^2\right) \mathbf{1}_{(0,\infty)}(x) dx.$$

In this case all rates are the true rates. In fact, for  $p > 0$ ,

$$\begin{aligned} \mathbb{E}X_t^p &= \frac{t}{\sqrt{2\pi}} e^{t\gamma} \int_0^\infty x^{p-3/2} \exp\left(-\frac{1}{2}\left(\frac{t}{x} + \gamma^2 x\right)\right) dx \\ &= \frac{t}{\sqrt{2\pi}} e^{t\gamma} \left(\frac{1}{\gamma}\right)^{p-3/2} t^{p-1/2} \int_0^\infty y^{p-3/2} \exp\left(-\frac{t\gamma}{2}\left(\frac{1}{y} + \gamma^2 y\right)\right) dy \\ &= \frac{2}{\sqrt{2\pi}} \gamma^{p-3/2} t^{p+1/2} e^{t\gamma} K_{p-1/2}(t\gamma) \end{aligned}$$

and, as  $z \rightarrow 0$ ,

$$\begin{aligned} K_{p-1/2}(z) &\sim \frac{C_p}{z^{p-1/2}} && \text{if } p > \frac{1}{2}, \\ K_{p-1/2}(z) &\sim \frac{C_p}{z^{1/2-p}} && \text{if } p < \frac{1}{2} \\ K_0(z) &\sim |\log z| \end{aligned}$$

where  $C_p = 2^{p-3/2}\Gamma(p-1/2)$  if  $p > 1/2$  and  $C_p = 2^{-p-1/2}\Gamma(\frac{1}{2}-p)$  if  $p < 1/2$ .

• *The Normal Inverse Gaussian (NIG)* process was introduced by Barndorff-Nielsen and has been used in financial modeling (see [8]). The NIG process is a Lévy process with characteristics  $(a, 0, \nu)$  where

$$\begin{aligned} \nu(dx) &= \frac{\delta\alpha}{\pi} \frac{\exp(\gamma x) K_1(\alpha|x|)}{|x|} dx, \\ a &= \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\gamma x) K_1(\alpha x) dx, \end{aligned}$$

$\alpha > 0$ ,  $\gamma \in (-\alpha, \alpha)$ ,  $\delta > 0$ . Since  $K_1(|z|) \sim |z|^{-1}$  as  $z \rightarrow 0$ , the Lévy density behaves like  $\delta\pi^{-1}|x|^{-2}$  as  $x \rightarrow 0$  so that (1.6) is satisfied with  $\beta = 1$ . One also checks that  $\mathbb{E}|X_1|^p < +\infty$  for every  $p > 0$ . It follows from Theorems 1 and 3 that, as  $t \rightarrow 0$

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t) && \text{if } p > 1, \\ \mathbb{E} \sup_{s \leq t} |X_s|^p &= O((t(-\log t))^p) && \text{if } p \leq 1. \end{aligned}$$

If  $\gamma = 0$ , then  $\nu$  is symmetric and by Theorem 2,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^p) \quad \text{if } p < 1.$$

The distribution  $\mathbb{P}_{X_t}$  at time  $t > 0$  is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t\delta\alpha}{\pi} \exp(t\delta\sqrt{\alpha^2 - \gamma^2} + \gamma x) \frac{K_1(\alpha\sqrt{t^2\delta^2 + x^2})}{\sqrt{t^2\delta^2 + x^2}} dx$$

so that Theorem 3 gives the true rate for  $p = \beta = 1$  in the symmetric case. In fact, assuming  $\gamma = 0$ , we get as  $t \rightarrow 0$

$$\mathbb{E}|X_t| = \frac{2t\delta\alpha}{\pi} e^{t\delta\alpha} \int_0^\infty \frac{x K_1(\alpha\sqrt{t^2\delta^2 + x^2})}{\sqrt{t^2\delta^2 + x^2}} dx$$

$$\begin{aligned}
&= \frac{2t\delta\alpha}{\pi} e^{t\delta\alpha} \int_{t\delta}^{\infty} K_1(\alpha y) dy \\
&\sim \frac{2\delta}{\pi} t \int_{t\delta}^1 \frac{1}{y} dy \\
&\sim \frac{2\delta}{\pi} t(-\log(t)).
\end{aligned}$$

• *Hyperbolic Lévy motions* have been applied to option pricing in finance (see [3]). These processes are Lévy processes whose distribution  $\mathbb{P}_{X_1}$  at time  $t = 1$  is a symmetric (centered) hyperbolic distribution

$$\mathbb{P}_{X_1}(dx) = C \exp(-\delta\sqrt{1+(x/\gamma)^2})dx, \quad \gamma, \delta > 0.$$

Hyperbolic Lévy processes have characteristics  $(0, 0, \nu)$  and satisfy  $\mathbb{E}|X_1|^p < +\infty$  for every  $p > 0$ . In particular, they are martingales. Their (rather involved) symmetric Lévy measure has a Lebesgue density that behaves like  $Cx^{-2}$  as  $x \rightarrow 0$  so that (1.6) is satisfied with  $\beta = 1$ . Consequently, by Theorems 1,2 and 3, as  $t \rightarrow 0$

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t) \quad \text{if } p > 1, \\
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t^p) \quad \text{if } p < 1, \\
\mathbb{E} \sup_{s \leq t} |X_s| &= O(t(-\log t)) \quad \text{if } p = 1.
\end{aligned}$$

• *Meixner processes* are Lévy processes without Brownian component and with Lévy measure given by

$$\nu(dx) = \frac{\delta e^{\gamma x}}{x \sinh(\pi x)} dx, \quad \delta > 0, \quad \gamma \in (-\pi, \pi)$$

(see [8]). The density behaves like  $\delta/\pi x^2$  as  $x \rightarrow 0$  so that (1.6) is satisfied with  $\beta = 1$ . Using (1.2) one observes that  $\mathbb{E}|X_1|^p < +\infty$  for every  $p > 0$ . It follows from Theorems 1 and 3 that

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t) \quad \text{if } p > 1, \\
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O((t(-\log t))^p) \quad \text{if } p \leq 1.
\end{aligned}$$

If  $\gamma = 0$ , then  $\nu$  is symmetric and hence Theorem 2 yields

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^p) \quad \text{if } p < 1.$$

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