A remark on precomposition on $\mathrm{H}^{1/2}(S^1)$ and ε-identifiability of disks in tomography.

M. Dambrine and D. Kateb *

July 7, 2006

Abstract

We consider the inverse conductivity problem with one measurement for the equation $div((\sigma_1 + (\sigma_2 - \sigma_1)\chi_D)\nabla u) = 0$ determining the unknown inclusion D included in Ω. We suppose that Ω is the unit disk of $\mathbb{R}^2$. With the tools of the conformal mappings, of elementary Fourier analysis and also the action of some quasi-conformal mapping on the Sobolev space $\mathrm{H}^{1/2}(S^1)$, we show how to approximate the Dirichlet-to-Neumann map when the original inclusion D is a $\varepsilon-$ approximation of a disk. This enables us to give some uniqueness and stability results.

Keywords: Inverse problem of conductivity, Dirichlet to Neumann map, conformal mapping, Fourier series, precomposition in Sobolev spaces.
AMS classification: 34K29, 42A16, 46E35

1 Introduction.

In this paper, we study the inverse problem of conductivity with one measurement. Given a bounded domain $\Omega \subset \mathbb{R}^2$ with reasonably smooth boundary, an connected open set D compactly contained in Ω, we consider for any $f \in \mathrm{H}^{1/2}(\partial D)$ the problem of recovering the subset D entering the Dirichlet equation

$$P[D,f] \left\{ \begin{array}{rcll} \mathrm{div}\left((\sigma_1 + (\sigma_2 - \sigma_1)\chi_D)\,\nabla u\right) & = & 0, & \text{in } \Omega, \\ u & = & f, & \text{on } \partial\Omega \end{array} \right. \tag{1}$$

from the knowledge of the current flux $g = \sigma_1 \partial_n u$ in $\partial\Omega$ induced by the boundary value $f = u_{|\partial\Omega}$. We will denote $\Lambda_D : \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ the Dirichlet-to-Neumann map which maps the Dirichlet data f onto the corresponding Neumann data $g = \Lambda_D(f) = \sigma_1 \partial_n u$. We can reformulate the inverse problem with one measurement as the determination of D from the Cauchy pair (f, g). We mention here that we do not need the full knowledge of the Dirichlet-to-Neumann map but only one pair of Cauchy data (f, g). For such a problem, we know that the uniqueness question is, in general, an open problem. It has been solved only for the special class of convex polyhedra, disks and balls. For other domains, Fabes, Kang and Seo [6] have studied the global uniqueness and stability within the class of domains which are $\varepsilon-$ perturbations of disks. The main ingredients in the work were layer potential techniques and representation formula for the solution u_D of the problem $P[D, f]$.

*LMAC Université de Technologie de Compiègne BP 20529 60205 Compiègne Cedex - France. Email : Marc.Dambrine@utc.fr, Djalil.Kateb@utc.fr

Our main goal is to revisit the paper [6] with other techniques than boundary integral representations. Throughout our paper, the two-dimensional case will be considered. Instead of layer potential techniques, conformal mappings and Fourier analysis will be another approach to review the two questions of stability and uniqueness within the class of disks and perturbed disks.

Let us illustrate briefly the main steps of our arguments. Since Ω is doubly connected, conformal mappings allows the construction of the conformal transplant function u which is solution of an elliptic problem that is obtained by transporting the original problem $P[D, f]$ by means of a change of variables induced by the conformal mappings. A natural way to study the original Dirichlet-to-Neumann map is to study the transplanted Dirichlet-to-Neumann map; indeed one can show that when the original D is a disk in Ω, then we can give an expression of the new Dirichlet-to-Neumann map by means of Moebius transforms and the classical formula of the Dirichlet-to -Neumann operator related to an concentric annulus. The elementary properties of Moebius transforms allow us to get an uniqueness result within the class of circular inclusions and for some special Dirichlet boundary measurement.

When D is not a disk, things become more difficult. The conformal transplantation furnishes a Dirichlet-to-Neumann operator that is not very convenient to study. Indeed, all the classical tools of perturbation theory by compact operators and Von-Neumann expansion of the inverses are absent and thus we have no way to make our explicit formula more suitable for numerical purposes. However, when the original inclusion D is an $\varepsilon-$perturbation of a disk B, then one can show that we have a reliable expression of the form $\Lambda_D = \Lambda_B + R_\varepsilon$ with a remainder R_ε that is of order ε^α. We show that α depends on the Sobolev regularity of the Dirichlet boundary measurement f and of the regularity of the boundary ∂D. In the conformal transplant, we have to deal with the two conformal mappings that map respectively $\Omega\backslash\overline{D}$ into the annulus and and D on the ball; and the restriction of the maps on the corresponding boundaries will be of great importance in the error estimate. In our context, the diffeomorphism ξ is obtained from a composition of the two boundary correspondence functions. The error estimate is not straighforward, it is a consequence of the hardest problem of estimating $\|h - h\circ\xi\|_{\mathrm{H}^{1/2}(S^1)}$ when $\xi : S^1 \mapsto S^1$ is a $\mathrm{W}^{1,\infty}$ diffeomorphism of the circle and when h is a function that belongs to some Sobolev space $\mathrm{H}^s(S^1)$, $s > \frac{1}{2}$. We were not able to give the best Sobolev exponent s for which the estimate is true. However we give a result for the exponent values $s = 1 + \alpha$ for some $0 < \alpha < 1$. At our best knowledge, the question remains open when h belongs to $\mathrm{H}^s(S^1)$ when $\frac{1}{2} < s < 1$. Our result about the precomposition of Sobolev spaces with quasi-regular diffeomorphisms are in the continuation of the pioneering works (see [3, 4]) where are studied the action of quasi-regular homeomorphisms on the critical Sobolev space $\mathrm{H}^{1/2}(S^1)$.

Let us point out that the resolution of an inverse boundary value problem for harmonic functions arising in electrostatic imaging through conformal mapping techniques has been introduced by Kress and his collaborators. The interested reader can consult the seminal work of Kress and al [1, 7] and our paper in [5].

The paper is organized as follows. In section 2, after introducing some definitions and recalling some preliminary results concerning Moebius conformal mapping, we state the uniqueness results for disks. In section 3, we investigate the continuity properties of the superposition operators on $\mathrm{H}^{1/2}(S^1)$ generated by regular diffeomorphisms of the circle. We then describe the approximation of the Dirichlet-to-Neumann map obtained after a sufficiently small deformation of a disk. In section 4, we prove the main result of uniqueness for disk and the ε identifiability of ε disks. In section 5, we prove the precomposition inequality.

2 Main assumptions and results

We shall assume throughout that Ω is the unit ball of $\mathbb{R}^2$. Let us introduce the notion of small perturbation of disks. Given $\varepsilon \geq 0$, a C^2 domain D is called an $\varepsilon-$ perturbation of a disk if there exists $\delta \in C^2(\partial B)$ with $\|\delta\|_{C^2(\partial B)} < 1$ such that

$$\partial D : x + \varepsilon\delta(x)\nu(x), \quad x \in \partial B$$

where $\nu(x)$ is the outward unit normal to ∂B at x. Denoting $\Omega_0 \subset \Omega$ the set of points at some distance δ_0 from $\partial\Omega$, we will denote by $C[\varepsilon]$ the class of $\varepsilon-$ perturbations of all disks contained in Ω_0 with the radius larger than a fixed number ρ_0 that can be arbitrary small provided than it remains big with respect to ε.
We will assume that the domain D entering in equation (1) is a disk or an $\varepsilon-$ perturbation of a disk $B \subset \Omega_0$. Our main results concern the *identifiability* (the case of a perfect disk) and the approximate identifiability.

In a first time, we deal with the perfect case where $\varepsilon = 0$; we have

Theorem 2.1 *Let D_1 be a disk centered at the origin and of radius R_1. Then there exists a boundary Dirichlet measurement $f(\theta) = \cos\theta$ such that if D_2 is an arbitrary disk contained in Ω and $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$ then $D_1 = D_2$.*

In a second time, we consider the case of perturbed disks. We take the same boundary measurment $f(\theta) = \cos\theta$. We then have the following

Theorem 2.2 *Let $D_0 \in C[\varepsilon]$ and let $\Lambda_{D_0}(f) = g$. Then there exists a positive constant $C > 0$ such that if $D \in C[\varepsilon]$ and $\Lambda_D(f) = g$ on $\partial\Omega$, then*

$$|D\Delta D_0| \leq C\varepsilon^{\alpha}, \tag{2}$$

where $0 < \alpha < 1$ is a constant depending only on the a priori data and where $D\Delta D_0$ denotes the symmetric difference of D and D_0.

The Sobolev $\mathrm{H}^{1/2}(S^1)$ plays an important role in the proof of stability. We recall that $\mathrm{H}^{1/2}(S^1)$ stands for the Hilbert space of real functions f defined on S^1 (modulo the constants) whose Fourier expansion

$$f(e^{i\theta}) = \sum_{n=-\infty}^{n=\infty} c_n(f)e^{in\theta}$$

where the Fourier coefficients $(c_n(f))$ are such that the sequence $(\sqrt{n}c_n(f))_n$ is square summable. For each f belonging to the Sobolev space $\mathrm{H}^{1/2}(S^1)$, its norm is the weighted l^2 norm $\left(\sum_{n=-\infty}^{\infty} |n||c_n(f)|^2\right)^{1/2}$. We will also write that f belongs to the half Sobolev space if and only if the sequence of its Fourier transform $(c_n(f))$ belongs to $l_2^{1/2}(\mathbb{Z})$.

Let us recall some results about the action or composition (it is en fact a "precomposition") by quasi-symmetric homeomorphisms of the circle S^1. Given an orientation preserving homeomorphism $\xi : S^1 \mapsto S^1$ of the circle, we consider the superposition operator F_ξ generated by ξ defined by

$$F_\xi(f) = f \circ \xi, \ f \in \mathrm{H}^{1/2}(S^1).$$

It is known that F_ξ maps $\mathrm{H}^{1/2}(S^1)$ onto itself if and only if ξ is quasi-symmetric in the sense that we must have the doubling condition

$$\frac{|\xi(2I)|}{|\xi(I)|} \leq K,$$

where $K > 0$ is positive, where I is any interval on S^1 of length less than π and where $2I$ is the interval of S^1 after doubling I but by keeping the same midpoint. Furthermore, we have

$$\|F_\xi\|_{\mathcal{L}(\mathrm{H}^{1/2}(S^1),\mathrm{H}^{1/2}(S^1))} \leq \sqrt{K + \frac{1}{K}}.$$

We recall also that among all quasi-symmetric homeomorphisms of S^1, the Moebius transformations of S^1 act unitarily on $\mathrm{H}^{1/2}(S^1)$.

An important question arises : can we hope to bound the error norm $\|F_\xi(u)-u\|_{\mathrm{H}^{1/2}(S^1)}$ by $C\|u\|_{\mathrm{H}^{1/2}(S^1)}$. The answer is important since it will allow us to estimate the error between the original Dirichlet-to-Neumann operator Λ_D and the *transformed* Λ_B. A first idea is to guess that such an estimate can be possible if ξ is not *far* from a Moebius transform of S^1. However, at this stage of our work, we have to add some regularity assumptions on the target function u. To be more precise, we are only able to prove the following result of continuity for the precomposition by diffeomorphisms.

Theorem 2.3 *Let $0 < \delta < 1$ and let u be a function belonging to $\mathrm{H}^{1+\delta}$. Let ϕ be a regular quasi-regular function that we suppose to be a $\mathrm{W}^{1,\infty}$ diffeomorphism on S^1. Then we have*

$$\|u \circ \phi - u\|_{\mathrm{H}^{1/2}(S^1)} \leq C(\delta')\ \|u\|_{\mathrm{H}^{1+\delta}(S^1)}\ \omega_{\delta'}(\|\phi - I\|_{\mathrm{W}^{1,\infty}(S^1)}) \tag{3}$$

holds for all $\delta' \in (0, \delta)$ where $\omega_{\delta'}$ is the modulus of continuity defined by

$$\omega_{\delta'}(t) = \max(t^{\delta'+1/2}, t^{\delta'}). \tag{4}$$

3 Change of variables and superposition operators.

We suppose that $D \in C[\varepsilon]$ is an $\varepsilon-$ perturbation of a disk. We aim to approximate the Dirichlet-to-Neumann operator Λ_D by Λ_B. In a first time, we wish to transport the original problem $P[D, f]$ by means of conformal transforms and to study the corresponding Dirichlet-to-Neumann operator.

3.1 Change of variables and analysis of the Dirichlet-to-Neumann map

Thanks to the classical mapping theorems, we know that there exists $\rho \in (0, 1)$ and an analytic function Φ^e that maps bijectively $\Omega \setminus \overline{D}$ onto the annulus $\Omega \setminus \overline{B}_\rho$ where B_ρ is the centered disk of radius ρ. If the outer boundaries correspond to each other and if the image of one point on $\partial\Omega$ is prescribed then Φ^e is uniquely determined. Furthermore, thanks to the Riemann's mapping theorem, we know that there exists also a conformal mapping Φ^i that maps bijectively D onto B_ρ. We recall that the restrictions of the conformal maps to the inclusion have the same regularity than ∂D (see [8, 9] for more details).

We denote by $\Psi^i = (\Phi^i)^{-1}$ (respectively $(\Psi^e)^{-1}$) the inverse of Φ^i (respectively Φ^e) and by $\gamma : [0, |\partial D|] \to \partial D$ the parametrization of ∂D in terms of arc-length. We set

$$\phi^i(\theta) = \gamma^{-1}\left(\Psi^i(\rho e^{i\theta})\right), \tag{5}$$

and

$$\phi^e(\theta) = \gamma^{-1}\left(\Psi^e(\rho e^{i\theta})\right). \tag{6}$$

Let $U^e_{B_\rho}$ (respectively $U^i_{B_\rho}$) denote the conformal transplant of $(u_D)_{|\Omega\setminus\overline{D}}$ (respectively $(u_D)_{|D}$). We have

$$u^e = U^e_{B_\rho} \circ \Phi^e$$

and

$$u^i = U^i_{B_\rho} \circ \Phi^i;$$

let us give the explicit form of the elliptic equations satisfied each of the conformal transplants. We have

Proposition 3.1 *Let ξ be the diffeomorphism on ∂B_ρ defined by $\xi = (\phi^e)^{-1} \circ \phi^i$. Then, we have*

$$\begin{aligned}
\Delta U^e_{B_\rho} &= 0 \textit{ in } \Omega \setminus \overline{B}_\rho,\\
U^e_{B_\rho} &= f \circ \Psi^e \textit{ on } \partial\Omega,\\
\Delta U^i_{B_\rho} &= 0 \textit{ in } B_\rho,\\
U^i_{B_\rho} &= U^e_{B_\rho} \circ \xi \textit{ on } \partial B_\rho,\\
\sigma_1\left(\partial_r U^e_{B_\rho} \circ \xi\right)\xi' &= \sigma_2 \partial_r U^i_{B_\rho} \textit{ on } \partial B_\rho.
\end{aligned}$$

Proof of Proposition 3.1: It is elementary and essentially based on the Cauchy-Riemann identities. We left the details to the reader. ■

3.2 Boundary correspondance for ε perturbations of disks.

The important fact to notice now is that if $D \in C[\varepsilon]$, then ξ is a perturbation of the identity. More precisely, one has the following result

Proposition 3.2 *There exists a positive constant $C > 0$ such that*

$$\|\xi - I\|_{W^1_\infty(S^1)} \le C\varepsilon \tag{7}$$

holds for all $D \in C[\varepsilon]$.

Proof of Proposition 3.2: Since $\xi = (\phi^e)^{-1} \circ \phi^i$, the proof is split into two parts: in a first time we estimate the contribution of the interior then in a second time the contribution of the exterior. We claim that

$$\|\phi^i - I\|_{W^1_\infty(S^1)} \le C\varepsilon, \tag{8}$$

and

$$\|\phi^e - I\|_{W^1_\infty(S^1)} \le C\varepsilon. \tag{9}$$

Deducing (7) from the claims (8),(9) is easy and left to the reader. We now prove the claims in the two next sections. ■

3.2.1 Proof of claims (8)-(9).

The simply connected case: claim (8). Without loss of generality, one can assume $\partial\omega$ to be starlike with respect to the origin. We use polar coordinates to write

$$\partial D: \quad z = z(\phi) = r(\phi)e^{i\phi}, \; 0 \leq \phi \leq 2\pi, \tag{10}$$

where r is a given positive regular function of period 2π such that

$$r(\theta) = (R - \delta(\theta)), \; 0 \leq \theta \leq 2\pi \tag{11}$$

δ being a function of period 2π satisfying

$$\sup_{0 \leq \theta < 2\pi} |\delta^{(k)}| < \varepsilon, \; k = 0, 1, 2.$$

From Henrici ([8],[9]), we learn that θ and $\phi^i(\theta)$ are related by the Theodersen's integral equation

$$\phi^i(\theta) - \theta = \mathcal{H}(\log r(\phi(\theta))) \tag{12}$$

where $\mathcal{H}$, the Hilbert transform on the circle, is defined by

$$\mathcal{H}f(\theta) = \frac{1}{2\pi} P.V. \int_0^{2\pi} f(t) \cot\frac{\theta - t}{2}\, dt. \tag{13}$$

The same author learns us that the Theordersen's integral equation admits exactly one continuous solution $\phi(\theta), \; 0 \leq \theta \leq 2\pi$ under the condition that the ratio

$$\delta = \sup_{0 \leq \phi \leq 2\pi} \left| \frac{r'(\theta)}{r(\theta)} \right| \tag{14}$$

satisfies $\delta < 1$. This condition (14) means that the angle between the outward normal and the radius vectors does not exceed $\arctan \delta < \delta$. It also means that we have to deal with curves $\partial\omega$ that are not too far from a circle. It is referred as the δ condition. In our context, D is not to far from a disk. Some straightforward arguments (primarily due to Montel and Lindelof) show that the boundary correspondence between θ and $\phi^i(\theta)$ is given by

$$\phi^i(\theta) = \theta - \mathcal{H}\delta(\theta) + O(\varepsilon^2). \tag{15}$$

The interested reader will find a geometric proof in ([8]). Hence, if the perturbation δ is in W^2_∞, one can show easily that there exists a positive constant $C > 0$ such that

$$|\phi^i(\theta) - \theta| < C\varepsilon. \tag{16}$$

The doubly connected case: claim (9). As we did in the previous paragraph, we give the asymptotic behavior of $\phi^e - I$ when $\varepsilon \to 0$. The analog of the Theodersen's equations for the doubly connected case is described by the so called Theodersen's and Garrick equations. The boundary correspondence is given by the following result.

Theorem 3.3 *Let O be a doubly connected region conformally equivalent to the annulus $\rho < |w| < 1$. We suppose O bounded by piecewise analytic curves Γ_0 and Γ_1, both starlike with respect to the origin and parametrized as above. If we suppose that*

$$\int_0^{2\pi} (\phi_0(\theta) - \theta)\, d\theta = \int_0^{2\pi} (\phi_1(\theta) - \theta)\, d\theta = 0 \tag{17}$$

then there holds

$$\left\{ \begin{array}{lcl} \phi_0(\theta) - \theta & = & -\mathcal{K}_\rho(\log r_1(\phi_1(\theta))) \\ \phi_1(\theta) - \theta & = & -\mathcal{H}_\rho(\log r_1(\phi_1(\theta))) \end{array} \right. \tag{18}$$

where the operators $\mathcal{K}_\rho : L^2((0,2\pi)) \mapsto L^2((0,2\pi))$ *and* $\mathcal{H}_\rho : L^2((0,2\pi)) \mapsto L^2((0,2\pi))$ *are defined as follows*

$$e^{im\theta} \mapsto \mathcal{H}_\rho(e^{im\theta}) = \left\{ \begin{array}{ll} 0 & , \; m \neq 0 \\ -i\frac{1+\rho^{2n}}{1-\rho^{2n}}, \; m \neq 0, & \end{array} \right. \tag{19}$$

and

$$e^{im\theta} \mapsto \mathcal{K}_\rho(e^{im\theta}) = \left\{ \begin{array}{ll} 0 & , \; m \neq 0 \\ -2i\frac{\rho^n}{1-\rho^{2n}}, \; m \neq 0. & \end{array} \right. \tag{20}$$

Furthermore, the radius ρ *is explicitly given by*

$$\rho = \exp\left(\frac{1}{2\pi} \log \left(\int_0^{2\pi} \log r_1(\phi_1(\theta)) \; d\theta \right) \right). \tag{21}$$

It is straightforward to show that the Theodersen and Garrick equations enables us to get (9) when the radius perturbation $\delta(\theta)$ belongs to $W^{2,\infty}$.

3.3 Approximation of the Dirichlet-to-Neumann map for perturbed disks.

Let D be an fixed $\varepsilon-$ perturbation of a disk B. We want to estimate the correction term $\|\Lambda_D(f) - \Lambda_B(f)\|_{\mathrm{H}^{-1/2}(S^1)}$ with respect to the perturbation factor ε. We have

Theorem 3.4 *Let* D *be a regular domain that is an* ε *perturbation of a disk centered at the origin. Let* α *be a real in* $(0,1)$*. Then there exists a constant* $C > 0$ *such that*

$$\|\Lambda_D(f) - \Lambda_B(f)\|_{\mathrm{H}^{-1/2}(S^1)} \leq C\varepsilon^\alpha \|f\|_{\mathrm{H}^{1+\alpha}(S^1)} \tag{22}$$

holds for the boundary measurement f *belonging to the Sobolev space* $\mathrm{H}^{1+\alpha}(S^1)$.

The proof is lengthy and requires some preliminary results that we will state and prove before.

Computation of the transplanted Dirichlet-to-Neumann map. Our main task is to give the analytic expression of $\Lambda^t_{B_\rho} : \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ defined by

$$\Lambda^t_{B_\rho}(F^e) = \sigma_1 \partial_r U^e_{B_\rho},$$

where we set $F^e = f \circ \Psi^e_{|\partial\Omega} = f \circ \psi^e$. The work will be divided in two parts: in the first one, we give some preliminary results based on the expression of $h = (U^e_{B_\rho})_{|\partial B_\rho}$. This will allow us to get $\Lambda^t_{B_\rho}(F^e)$ and a convenient approximation $\Lambda_{B_\rho}(f)$. A straightforward calculation shows that for $\rho < r < 1$ we have

$$\begin{aligned} U^e(re^{i\theta}) \;\; = \;\; & \frac{\ln r}{\ln \rho}(c_0(h) - c_0(F^e)) + c_0(F^e) + \sum_{n\neq 0} \frac{1}{1-\rho^{2|n|}} \left[r^{|n|} - \frac{\rho^{2|n|}}{r^{|n|}} \right] c_n(F^e) e^{in\theta} \\ & + \sum_{n \neq 0} \frac{\rho^{|n|}}{\rho^{2|n|}-1} \left[r^{|n|} - \frac{1}{r^{|n|}} \right] c_n(h) e^{in\theta}; \end{aligned}$$

hence after identification of the Fourier coefficients, it comes that

$$\left\{ \begin{array}{ll} c_0\left(\Lambda^t_{B_\rho}(F^e)\right) & = \dfrac{c_0(h) - c_(F^e)}{\ln\rho} = 0, \\ c_n\left(\Lambda^t_{B_\rho}(F^e)\right) & = \sigma_1 \dfrac{|n|}{1-\rho^{2|n|}}\left[(1+\rho^{2|n|})c_n(F^e) - 2\rho^{|n|}c_n(h)\right],\ n \neq 0. \end{array} \right. \tag{23}$$

We see that the knowledge of h determines uniquely the Dirichlet-to-Neumann operator Λ_{B_ρ}. It is then useful to get some informations about h. First, we have

Proposition 3.5 *We have*

$$\begin{aligned} \sum_{n\neq 0} |n| c_n(h\circ\xi) e^{in\theta} + 2\xi'(\theta)\frac{\sigma_1}{\sigma_2}\sum_{n\neq 0}\frac{1+\rho^{2|n|}}{1-\rho^{2|n|}}|n|c_n(h)e^{in\xi(\theta)} \\ = 2\xi'(\theta)\frac{\sigma_1}{\sigma_2}\sum_{n\neq 0}\frac{\rho^{|n|}}{1-\rho^{2|n|}}|n|c_n(F^e)e^{in\xi(\theta)}. \end{aligned} \tag{24}$$

Proof of Proposition 3.5: Since U^i is solution of a Dirichlet problem in the disk B_ρ, we obtain

$$\partial_r U^i(\rho e^{i\theta}) = \frac{1}{\rho}\sum_{n\neq 0}|n|c_n(h\circ\xi)e^{in\theta},$$

and thanks to the jump condition satisfied by the normal derivatives $\partial_r U^i_{|\partial B_\rho}$ and $\partial_r U^e_{|\partial B_\rho}$, we deduce equation (24). ■

To solve (24) is a difficult task since the explicit expression of h is hard to manipulate; however when the perturbation factor ε is very small, one can give a suitable approximation. Before entering in the details, we have to give some qualitative properties of h.

A regularity result. We first prove the following tangential regularity result for solution of the conductivity problem.

Lemma 3.6 *Let u be the solution of the problem P[D,f]* (1) *where ∂D is assumed to be of class $\mathcal{C}^2$. Then $u_{|\partial D}$ belongs to $\mathrm{H}^{1+\delta}(\partial D, \mathbb{R})$ for some $\delta \in (0,1)$.*

Proof of Lemma 3.6: From classical methods, the problem $P[D,f]$ has a unique solution in the variational space $\mathrm{H}^1(\Omega)$ with a trace $u_{|\partial\Omega} = f \in \mathrm{H}^{1/2}(\partial\Omega)$ and a normal derivative $\partial_n u_{|\partial\Omega} := g \in \mathrm{H}^{-1/2}(\partial\Omega)$. For all $x \in \partial D$, we use the classical representation formulae for harmonic functions with the help of the single layer and double layer potential: since u is harmonic in $\Omega \setminus D$ and in D we have

$$\begin{aligned} \frac{1}{2}u^+(x) &= \int_{\partial\Omega}\partial_n G(x,y)f(y)ds(y) - \int_{\partial D}\partial_n G(x,y)u^+(y)ds(y) \\ &\quad - \int_{\partial\Omega} G(x,y)g(y)ds(y) + \int_{\partial D} G(x,y)\partial_n u^+(y)ds(y), \\ \frac{1}{2}u^-(x) &= \int_{\partial D}\partial_n G(x,y)u^-(y)ds(y) - \int_{\partial D} G(x,y)\partial_n u^-(y)ds(y) \end{aligned}$$

where G is the Newtonian potential and where the normal n to ∂D is oriented to the exterior. Using the jump conditions $[u] = [\sigma \partial_n u] = 0$ across the interface ∂D, we check that $u = u^+ = u^-$ solves the integral equation

$$\begin{aligned}\frac{1}{2}u(x) \quad &+ \quad \frac{\sigma_2 - \sigma_1}{\sigma_1 + \sigma_2} \int_{\partial D} \partial_n G(x,y) u(y) ds(y) \\ &= \quad \frac{\sigma_1}{\sigma_1 + \sigma_2} \left[\int_{\partial \Omega} \partial_n G(x,y) f(y) ds(y) - \int_{\partial \Omega} G(x,y) g(y) ds(y) \right].\end{aligned}$$

Since $\partial D \cap \partial \Omega = \emptyset$, the right hand side of this equation is of class $\mathcal{C}^\infty$. From the fact that the boundary ∂D is of class $\mathcal{C}^2$, the double layer potential on ∂D maps $\mathrm{H}^s(\partial D)$ into $\mathrm{H}^{s+1}(\partial D)$ for and hence is compact as operator from $\mathrm{H}^s(\partial D)$ into itself. We conclude thanks to the Fredholm alternative. ■

The perturbation argument. Let $T_\xi : \mathrm{H}^{1+\alpha}(\partial B_\rho) \to \mathrm{H}^{-1/2}(\partial B_\rho)$ the operator defined by

$$h \mapsto T_\xi(h)(\theta) = \sum_{n \neq 0} |n| c_n(h \circ \xi) e^{in\theta} + \xi'(\theta) \frac{\sigma_1}{\sigma_2} \sum_{n \neq 0} \frac{1 + \rho^{2|n|}}{1 - \rho^{2|n|}} c_n(h) e^{in\xi(\theta)},$$

and $T : \mathrm{H}^{1+\alpha}(\partial B_\rho) \to \mathrm{H}^{-1/2}(\partial B_\rho)$ the operator defined by

$$h \mapsto T(h)(\theta) = \sum_{n \neq 0} |n| c_n(h) e^{in\theta} + \frac{\sigma_1}{\sigma_2} \sum_{n \neq 0} \frac{1 + \rho^{2|n|}}{1 - \rho^{2|n|}} c_n(h) e^{in\theta}.$$

We set $DT_\xi = T_\xi - T$. We have

Proposition 3.7 *There exists a constant $C > 0$ such that*

$$\|T_\xi(u) - T(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C \|\xi - I\|_{W^1_\infty} \|u\|_{\mathrm{H}^{1+\alpha}(\partial B_\rho)} \tag{25}$$

holds for all u belonging to $\mathrm{H}^{1+\alpha}(\partial B_\rho)$ with $0 < \alpha < 1$.

Proof of Proposition 3.7: We decompose $DT_\xi(u) = T_1(u) + T_2(u) + T_3(u)$ where

$$\begin{aligned} T_1(u)(\theta) &= \sum_{n \neq 0} |n| c_n(u \circ \xi - u) e^{in\theta}, \\ T_2(u)(\theta) &= -\frac{\sigma_1}{\sigma_2} \left(\xi'(\theta) - 1 \right) \sum_{n \neq 0} |n| \frac{1 + \rho^{2|n|}}{1 - \rho^{2|n|}} c_n(u) e^{in\xi(\theta)}, \\ T_3(u)(\theta) &= -\frac{\sigma_1}{\sigma_2} \sum_{n \neq 0} |n| \frac{1 + \rho^{2|n|}}{1 - \rho^{2|n|}} c_n(u) \left[e^{in\theta} - e^{in\xi(\theta)} \right]. \end{aligned}$$

We begin to estimate $\|T_2(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)}$, we have

$$\|T_2(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C(\sigma_1, \sigma_2) \|\xi' - 1\|^2_\infty \|g_\xi\|_{\mathrm{H}^{-1/2}(\partial B_\rho)},$$

where $g_\xi = g_1 \circ \xi + g_2 \circ \xi$ with

$$g_1(\theta) = \sum_{n \neq 0} |n| c_n(u) e^{in\theta}$$

and

$$g_2(\theta) = 2\sum_{n\neq 0} |n| \frac{\rho^{2|n|}}{1-\rho^{2|n|}} c_n(u) e^{in\theta}.$$

While the estimation of $\|g_2 \circ \xi\|_{\mathrm{H}^{-1/2}(\partial B_\rho)}$ is straightforward

$$\|g_2 \circ \xi\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C(\delta_0)\|u\|_{\mathrm{H}^{1/2}(\partial B_\rho)},$$

the estimation of $\|g_2 \circ \xi\|_{\mathrm{H}^{-1/2}(\partial B_\rho)}$ is a little bit harder. We first observe that, since u belongs to $\mathrm{H}^s(\partial\Omega)$, $s > 1$, we can define $\mathcal{H}u' = (\mathcal{H}s)'$ where $\mathcal{H}$ is the Hilbert transform on the circle. It then comes that

$$g_1 \circ \xi = \mathcal{H}u' \circ \xi = (\mathcal{H}u)' \circ \xi,$$

and then that

$$\begin{aligned}\|g_1 \circ \xi\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C\|(\mathcal{H}u)' \circ \xi\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} &\leq C\|(\mathcal{H}u)'\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \\ &\leq C\|\mathcal{H}u\|_{\mathrm{H}^{1/2}(\partial B_\rho)} \leq C\|u\|_{\mathrm{H}^{1/2}(\partial B_\rho)}.\end{aligned}$$

In the last inequality, we used the fact that the Hilbert transform corresponds to an unimodular multiplier and then is an isometry on $\mathrm{H}^{1/2}$. Gathering the estimates on $g_1 \circ \xi$ and $g_1 \circ \xi$, we get

$$\|T_2(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C(\sigma_1, \sigma_2, \delta_0)\|\xi' - 1\|_\infty^2 \|u\|_{\mathrm{H}^{1/2}(\partial B_\rho)},$$

It remains to estimate $\|T_3(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)}$, we have

$$\begin{aligned}\|T_3(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C(\sigma_1, \sigma_2) \| \sum_{n\neq 0} |n| c_n(u) \left(e^{in\xi(\theta)} - e^{in\theta}\right) \|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \\ + \| \sum_{n\neq 0} |n| \frac{\rho^{2|n|}}{1-\rho^{2|n|}} c_n(u) \left(e^{in\xi(\theta)} - e^{in\theta}\right) \|_{\mathrm{H}^{-1/2}(\partial B_\rho)}.\end{aligned}$$

We focus on the first part of the sum. We have

$$\begin{aligned}\sum_{n\neq 0} |n| c_n(u) \left(e^{in\xi(\theta)} - e^{in\theta}\right) &= (\mathcal{H}u)' \circ \xi - \mathcal{H}u' \\ &= (\mathcal{H}u' \circ \xi)\xi' + (1-\xi')(\mathcal{H}u') \circ \xi - (\mathcal{H}u)' \circ \xi - (\mathcal{H}u)' \\ &= ((\mathcal{H}u) \circ \xi - \mathcal{H}u)' + (1-\xi')(\mathcal{H}u') \circ \xi.\end{aligned}$$

Hence,

$$\begin{aligned}&\| \sum_{n\neq 0} |n| c_n(u)(e^{in\xi(\theta)} - e^{in\theta})\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \\ &\qquad \leq \|(\mathcal{H}u) \circ \xi - \mathcal{H}u\|_{\mathrm{H}^{1/2}(\partial B_\rho)} + \|\xi' - 1\|_\infty \|u'\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \\ &\qquad \leq \| \sum_{n\neq 0} c_n(\mathcal{H}u)(e^{in\xi(\theta)} - e^{in\theta})\|_{\mathrm{H}^{1/2}(\partial B_\rho)} + \|\xi' - 1\|_\infty \|u\|_{\mathrm{H}^{1/2}(\partial B_\rho)}\end{aligned}$$

Suppose an instant that u is a trigonometric polynomial of degree d. Thanks to the composition Theorem 2.3, for $\delta \in (0,1)$, we have

$$\begin{aligned}\| \sum_{n\neq 0} c_n(\mathcal{H}u)(e^{in\xi(\theta)} - e^{in\theta})\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} &\leq \sum_{|n|\leq d} |c_n(u)| \|e^{in\xi(\theta)} - e^{in\theta}\|_{\mathrm{H}^{1/2}(\partial B_\rho)} \\ &\leq C(\delta)\ \|u\|_{\mathrm{H}^{1+\delta}(\partial B_\rho)}\ \omega_\delta(\|\phi - I\|_{\mathrm{W}^{1,\infty}(S^1)}),\end{aligned}$$

where $\omega_{2\delta}$ is the modulus of continuity defined by (4). Then, there exists a constant $C(\sigma_1, \sigma_2, \delta_0)$ such that

$$\|T_3(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C\omega_\delta(\|\xi - Id\|_{\mathrm{W}^{1,\infty}(\partial B_\rho)})\|u\|_{\mathrm{H}^{1+\delta}(\partial B_\rho)}.$$

The estimation of $T_1(u)$ obeys to the same computation : we have by following the same lines

$$\|T_1(u)\|_{\mathrm{H}^{-1/2}(\partial B_\rho)} \leq C\omega_\delta(\|\xi - Id\|_{\mathrm{W}^{1,\infty}(\partial B_\rho)})\|u\|_{\mathrm{H}^{1+\delta}(\partial B_\rho)}$$

end this ends our proof. ■

Proof of Theorem 3.4: Let us return to the equations satisfied by h. Set

$$b(F^e, \rho, \xi, \sigma_1, \sigma_2)(\theta) = 2\xi'(\theta)\frac{\sigma_1}{\sigma_2}\sum_{n\neq 0}\frac{\rho^{|n|}}{1-\rho^{2|n|}}|n|c_n(F^e)e^{in\xi(\theta)},$$

and suppose that the perturbation parameter ε is sufficiently small such that

$$\|(DT_\xi)T^{-1}\| < 1.$$

It follows that

$$\begin{aligned} h(\theta) &= T_\xi^{-1}b(F^e, \rho, \xi, \sigma_1, \sigma_2) \\ &= T^{-1}\left(I + DT_\xi T^{-1}\right)^{-1}b(F^e, \rho, \xi, \sigma_1, \sigma_2) \\ &= T^{-1}b(F^e, \rho, \xi, \sigma_1, \sigma_2) - T^{-1}\left(I + DT_\xi T^{-1}\right)^{-1}DT_\xi T^{-1}b(F^e, \rho, \xi, \sigma_1, \sigma_2). \end{aligned}$$

An easy calculation of T^{-1} shows that

$$h(\theta) = 2\frac{\sigma_1}{\sigma_2}\sum_{n\neq 0}\frac{\dfrac{\rho^{|n|}}{1-\rho^{2|n|}}}{\rho^{|n|} + 2\dfrac{\sigma_1}{\sigma_2}\dfrac{1+\rho^{2|n|}}{1-\rho^{2|n|}}}c_n(f)e^{in\theta} + \delta u(\theta),$$

where $\delta u \in \mathcal{C}^\infty$ is such that for all $s > 0$ $\|\delta u\|_{\mathrm{H}^m(S^1)} \leq C\|\xi - I\|_{\mathrm{W}^1_\infty(S^1)}\|f\|_{\mathrm{H}^{1/2}}$ for all integers $m \in \mathbb{N}$. Plugging this expression of h in formula (23) and setting $\mu = \frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_1}$ it follows that

$$\left\{\begin{array}{ll} c_0\left(\Lambda^t_{B_\rho}(F^e)\right) & = \dfrac{c_0(h) - c_(F^e)}{\ln\rho} = 0, \\ c_n\left(\Lambda^t_{B_\rho}(F^e)\right) & = |n|\sigma_1\left(\dfrac{1+\mu\rho^{2|n|}}{1-\mu\rho^{2|n|}}c_n(f) + \ c_n(\delta h)\right) n \neq 0. \end{array}\right. \tag{26}$$

where δh is a function such that $\|\delta h\|_{H^1/2(S^1)} \leq C\varepsilon$.

We are ready now to finish the proof. First of all, we know from the Cauchy-Riemann identities that

$$\Lambda^t_{B_\rho}(F^e) = (g \circ \psi^e)\,(\psi^e)'$$

and using the same arguments that we developed above, one can easily show that there exists a positive constant $C > 0$ depending on ρ_0 and δ_0 such that

$$\|\Lambda^t_{B_\rho}(F^e) - \Lambda_D(f)\|_{\mathrm{H}^{-1/2}(S)} \leq C\varepsilon^\alpha\|f\|_{\mathrm{H}^{1+\alpha}}. \tag{27}$$

Hence we get

$$\begin{aligned}
\|\Lambda_B(f)-\Lambda_D(f)\|_{\mathrm{H}^{-1/2}(S^1)} &= \|\Lambda_B(f)-\Lambda^t_{B_\rho}(F^e)+\Lambda^t_{B_\rho}(f)-\Lambda_D(f)\|_{\mathrm{H}^{-1/2}(S^1)},\\
&\leq \|\Lambda_B(f)-\Lambda^t_{B_\rho}(F^e)+C\varepsilon^\alpha\|f\|_{\mathrm{H}^{1+\alpha}(S^1)},\\
&\leq \|\Lambda_B(f)-\Lambda_{B_\rho}(f)\|_{\mathrm{H}^{-1/2}(S^1)}+C\|\left(\mathcal{H}(\delta h)\right)'\|_{\mathrm{H}^{-1/2}(S^1)},\\
&\quad +C\varepsilon^\alpha\|f\|_{\mathrm{H}^{1+\alpha}(S^1)},\\
&\leq \|\Lambda_B(f)-\Lambda_{B_\rho}(f)\|_{\mathrm{H}^{-1/2}(S^1)}+C\varepsilon^\alpha\|f\|_{\mathrm{H}^{1+\alpha}(S^1)}.
\end{aligned}$$

Recall that the Fourier coefficients of $\Lambda_{B_\rho}(f)$ are given by

$$c_n(\Lambda_{B_\rho}(f))=|n|\sigma_1\frac{1+\mu\rho^{2|n|}}{1-\mu\rho^{2|n|}}c_n(f).$$

Hence after denoting ρ_1 the radius of the disk B, we get

$$\Lambda_D(f)-\Lambda_{B_\rho}(f)=\sigma_1\sum_{n\neq 0}|n|\left(\frac{1+\mu\rho^{2|n|}}{1-\mu\rho^{2|n|}}-\frac{1+\mu\rho_1^{2|n|}}{1-\mu\rho_1^{2|n|}}\right)c_n(f)e^{in\theta} \tag{28}$$

and this implies that we can find a constant $C>0$ depending on δ_0,ρ_0 and on the conductivities $\sigma_i,\ i=1,2$ such that

$$\|\Lambda_D(f)-\Lambda_{B_\rho}(f)\|_{\mathrm{H}^{-1/2}(S^1)}\leq C|\rho-\rho_1|\|f\|_{\mathrm{H}^{1/2}(S^1)}; \tag{29}$$

we conclude thanks to the fact that $|\rho_1-\rho_2|\leq C\varepsilon$. This ends the proof of our theorem. ■

4 Proof of the main theorems

We subdivide the section in two parts : in the first one, we focus on the case where the inclusions are disks. In the second part, the inclusions belong to $C[\varepsilon]$.

4.1 Identifiability for disks : proof of Theorem 2.1

For the case where the domains are disks, we can always suppose that D_1 is centered at the origin.

Proof of Theorem 2.1: We use conformal mappings that maps a non concentric disk D_2 into a disk centered at the origin. We know that this can be done by means of the Moebius transform

$$w(z)=\frac{z-b}{1-\bar{b}z}\text{ with }|b|<1.$$

In [2, 8, 9], the interested reader will find all the details about the properties on such transforms . The radius of the transformed disk is denoted by R_2. If $z=re^{i\theta}$, then $w(z)$ writes $\rho(r,\theta)e^{i\phi(r,\theta)}$ where

$$\rho^2(r,\theta)=\frac{r^2+|b|^2-r(\bar{b}e^{i\theta}+be^{-i\theta})}{1+|b|^2-r(\bar{b}e^{i\theta}+be^{-i\theta})},$$

and where

$$\phi(r,\theta)=\arctan\frac{r\sin\theta-\Im br^2+r[\sin\theta(\Im b^2-\Re b^2)+2\Re b\Im b\cos\theta]}{2\cos\theta-\Re br^2-\Re b+r[(\Re b^2-\Im b^2)\cos\theta-2\Re b\Im b\sin\theta]}.$$

Here $\Re b$ and $\Im b$ denote the real and imaginary part of the complex b. A straightforward computation shows that

$$\partial_r \rho(1,\theta) = \frac{1-|b|^2}{1+|b|^2 - (\overline{b}e^{i\theta} + be^{-i\theta})} \text{ and } \partial_r \phi(1,\theta) = 0.$$

From the chain rule of differentiation, we get:

$$\Lambda_{D_2}(f) = \partial_r \rho(1,\theta) \Lambda_C^{R_2}(f \circ \phi^{-1}),$$

where $\Lambda_C^{R_2}$ is the Dirichlet-to-Neumann map for the transformed and concentric problem. It follows that

$$\Lambda_{D_1}(f) = \Lambda_{D_2}(f) \Rightarrow \Lambda_{D_1}(f) = \frac{1-|b|^2}{1+|b|^2 - (\overline{b}e^{i\theta} + b^{-i\theta})} \Lambda_C^{R_2}(f \circ \phi^{-1})$$

and the key for solving the problem is to use the Dirichlet-to-Neumann map for concentric disk. We have:

$$\frac{1+\mu R_1^2}{1-\mu R_1^2} \cos\theta = \frac{1-|b|^2}{1+|b|^2 - (\overline{b}e^{i\theta} + b^{-i\theta})} \sum_{k \neq 0} |k| \frac{1+\mu R_2^{2|k|}}{1-\mu R_2^{2|k|}} c_k(f \circ \phi^{-1}) e^{ik\phi(\theta)},$$

or equivalently:

$$\frac{1+\mu R_1^2}{1-\mu R_1^2} \left[(1+|b|^2)\cos(\theta) - \Re(b) - \frac{1}{2}(\overline{b}e^{2i\theta} + be^{-2i\theta}) \right]$$
$$= (1-|b|^2) \sum_{k \neq 0} |k| \frac{1+\mu R_2^{2|k|}}{1-\mu R_2^{2|k|}} c_k(f \circ \phi^{-1}) e^{ik\phi(\theta)}$$

Replacing θ by $\phi^{-1}(\theta)$, we then get that $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$ implies that the 2π periodic function

$$F(\theta) = (1-|b|^2) \sum_{k \neq 0} |k| \frac{1+\mu R_2^{2|k|}}{1-\mu R_2^{2|k|}} c_k(f \circ \phi^{-1}) e^{ik\theta} - \frac{1+\mu R_1^2}{1-\mu R_1^2}$$
$$\left[(1+|b|^2)\cos(\phi^{-1}(\theta)) - \Re(b) - \frac{1}{2}(\overline{b}e^{2i\phi^{-1}(\theta)} + be^{-2i\phi^{-1}(\theta)}) \right]$$

satisfies $c_k(F) = 0$ for all $k \in \mathbb{Z}$.
We tackle the computation of these Fourier coefficients. First of all, we need to compute $c_k(f \circ \phi^{-1})$ and $c_k(e^{2i\phi^{-1}})$. A straightforward computation shows that:

$$c_k(f \circ \phi^{-1}) = \begin{cases} \Re(b) \text{ if } k = 0, \\ \frac{1}{2}(1-|b|^2)(-b)^{k-1} \text{ else;} \end{cases}$$

and that

$$c_k(e^{i\phi^{-1}}) = \begin{cases} b \text{ if } k = 0, \\ (1-|b|^2)(-\overline{b})^{k-1} \text{ if } k > 0, \\ 0 \text{ else.} \end{cases}$$

Since

$$e^{i\phi^{-1}(\theta)} = b + (1-|b|^2) \frac{e^{i\theta}}{1+\overline{b}e^{i\theta}},$$

we deduce that

$$e^{2i\phi^{-1}(\theta)} = b^2 + 2b(1-|b|^2)\frac{e^{i\theta}}{1+\overline{b}e^{i\theta}} + (1-|b|^2)^2\frac{e^{2i\theta}}{(1+\overline{b}e^{i\theta})^2},$$

and then

$$c_k(e^{2i\phi^{-1}(\theta)}) = \begin{cases} b^2 \text{ if } k=0, \\ (-\overline{b})^{k-2}(1-|b|^2)\left[k-1-(k+1)|b|^2\right] \text{ if } k>0, \\ 0 \text{ else.} \end{cases}$$

Hence, the Fourier coefficients $(c_k(F))_k$ are given by the following formulae

$$c_0(F) = 0 \text{ and } c_1(F) = \mu(1-|b|^2)^2\frac{R_2^2-R_1^2}{(1-\mu R_2^2)(1-\mu R_1^2)}$$

and for $k \geq 2$:

$$\begin{aligned} c_k(F) = & \frac{(-1)^{k-1}(1-|b|^2)}{2}\Big[k\frac{1+\mu R_2^{2|k|}}{1-\mu R_2^{2|k|}}(1-|b|^2)b^{k-1} - \frac{1+\mu R_1^2}{1-\mu R_1^2}(1+|b|^2)b^{k-1} \\ & - \frac{1+\mu R_1^2}{1-\mu R_1^2}\left[k-1-|b|^2(k+1)\right]\overline{b}^{k-1}\Big], \\ = & \frac{(-1)^{k-1}(1-|b|^2)}{2}\left[b^{k-1}2\mu k\frac{(R_2^{2|k|}-R_1^2)(1-|b|^2)}{(1-\mu R_2^{2|k|})(1-\mu R_1^2)} + (\overline{b}^{k-1}-b^{k-1})(1+|b|^2)\frac{1+\mu R_1^2}{1-\mu R_1^2}\right]. \end{aligned}$$

Let us show that $c_k(F) = 0,\ \forall k \geq 0$ implies $D_1 = D_2$. First of all, the condition $c_1(F) = 0$ implies that $R_1 = R_2$. It remains to show that the Moebius transform is, in fact, the identity. Equivalently, we need to prove that $b = 0$.

Let us assume that we have $b \neq 0$; the condition $c_k(F) = 0,\ \forall k \geq 2$ would imply that $\overline{b}/b$ is real and then that $\overline{b} = \pm b$. From the identity

$$2\mu\frac{(R_1^{2|k|})(1-|b|^2)}{(1-\mu R_2^{2|k|})(1-\mu R_1^2)} + \left[\left(\frac{\overline{b}}{b}\right)^{k-1} - 1\right](1+|b|^2)\frac{1+\mu R_1^2}{1-\mu R_1^2} = 0,$$

we would get for $k = 2p+1,\ p \geq 1$

$$\frac{2\mu(R_2^{4p+2}-R_1^2)(1-|b|^2)}{(1-\mu R_2^{4p+2})(1-\mu R_1^2)} = 0$$

and then $|b| = 1$; this is impossible since the Moebius transform requires $|b| < 1$. Hence $b = 0$ is the only possibility. Gathering the two identities $R_1 = R_2$ and $b = 0$, it then comes that $D_1 = D_2$. ■

4.2 Proof of theorem 2.2 : the approximate identifiability for approximate disks

In this section, we need an intermediary lemma showing the relations between the norm of the superposition operators generated by a diffeomorphism ξ and its inverse ξ^{-1}. Its proof can found in ([4]); we have

Lemma 4.1 *Let $T_\xi : \mathrm{H}^{1/2}(S^1) \to \mathrm{H}^{1/2}(S^1)$ be the composition operator defined by $T_\xi(f) = f \circ \xi$. Then, if T_ξ is bounded on $\mathrm{H}^{1/2}(S^1)$, then so is $T_{\xi^{-1}}$. Furthermore, we have*

$$\|T_\xi\|_{\mathcal{L}(\mathrm{H}^{1/2}(S^1),\mathrm{H}^{1/2}(S^1))} = \|T_{\xi^{-1}}\|_{\mathcal{L}(\mathrm{H}^{1/2}(S^1),\mathrm{H}^{1/2}(S^1))}.$$

Proof of Theorem 2.2: If Ψ_1 is the conformal mapping which maps $\Omega \setminus \overline{D}_1$ onto an annulus $A(1, R_1^\varepsilon) = \Omega \setminus B(0, R_1^\varepsilon)$, it then comes from the results of the preceding section that

$$\|\Lambda_{B_1^\varepsilon}(f \circ \varphi_{1,\varepsilon}) - \Lambda_{D_1}(f)\|_{\mathrm{H}^{-1/2}} \le C\varepsilon^\alpha, \; 0 < \alpha < 1$$

where $\varphi_{1,\varepsilon} = \Psi_1^{-1}{}_{|\partial\Omega}$ and where B_1^ε denotes the disk $B(0, R_1^\varepsilon)$.

Concerning D_2 which is the ε-perturbation of a non concentric disk B_2, we begin to transform it via Ψ_2 the Moebius transform that maps B_2 onto a concentric disk of radius R_2. It is obvious that $\tilde{D}_2 = \Psi_2(D_2)$ is a slight perturbation of the concentric disk $\tilde{B}_2 = \Psi_2(B_2)$.

The chain rule derivative gives

$$\Lambda_{D_2}(f) = \frac{1 - |b_2|^2}{1 + |b_2|^2 - \overline{b}_2 e^{i\theta} - b_2 e^{-i\theta}} \Lambda_{\tilde{D}_2}(f \circ \varphi_2),$$

where $\varphi_2 = \Psi_2^{-1}{}_{|\partial\Omega}$. Thanks to our preceding results, we know that if $\Omega\setminus\overline{B(0, R_2^\varepsilon)}$ is the conformal transform of $\Omega\setminus\overline{\tilde{D}_2}$ then

$$\|\Lambda_{B(0,R_2^\varepsilon)}(f \circ \varphi_{1,\varepsilon}) - \Lambda_{\tilde{D}_2}(f)\|_{\mathrm{H}^{-1/2}} \le C\varepsilon^\alpha, \; 0 < \alpha < 1$$

and the assumption $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$ implies that

$$\begin{aligned}
\|\Lambda_{B(0,R_1^\varepsilon)}(f) - &\frac{1 - |b_2|^2}{1 + |b_2|^2 - \overline{b_2} e^{i\theta} - b_2 e^{-i\theta}} \Lambda_{B(0,R_2^\varepsilon)}(f \circ \phi_2)\|_{\mathrm{H}^{-1/2}(S^1)} \\
&\le \|\Lambda_{D_1}(f) - \Lambda_{B(0,R_1^\varepsilon)}(f)\|_{\mathrm{H}-1/2(S^1)} \\
&+ \sup_{\theta\in[0,2\pi]} \left| \frac{1 - |b_2|^2}{1 + |b_2|^2 - \overline{b_2} e^{i\theta} - b_2 e^{-i\theta}} \right| \|\Lambda_{\tilde{D}_2}(f \circ \phi_2) - \Lambda_{B(0,R_2^\varepsilon)}(f \circ \phi_2)\|_{\mathrm{H}-1/2(S^1)} \\
&\le C\varepsilon^\alpha
\end{aligned}$$

where $C = C(\delta_0)$ is a constant depending only on δ_0 and where $0 < \alpha < 1$. Following the same lines for the proof of Theorem 2.1, we show that $\Lambda_{D_1}(f) = \Lambda_{D_2}(f)$ implies that

$$\|G\|_{\mathrm{H}^{-1/2}(S^1)} \le C\varepsilon^\alpha$$

where G is the $2\pi-$ periodic function defined by

$$\begin{aligned}
G(\theta) =&(1 - |b_2|^2) \sum_{n \neq 0} |k| \frac{1 + \mu(R_2^\varepsilon)^{2|k|}}{1 - \mu(R_2^\varepsilon)^{2|k|}} c_k(f \circ \phi^{-1}) e^{ik\phi(\theta)} \\
&- \frac{1 + \mu(R_1^\varepsilon)^2}{1 - \mu(R_1^\varepsilon)^2} \left((1 + |b|^2) \cos\theta - \frac{1}{2} \overline{b_2} e^{2i\theta} - \frac{1}{2} b_2 e^{-2i\theta} - \Re(b_2) \right).
\end{aligned}$$

We set $F = G \circ \phi^{-1}$; from Lemma 4.1, we deduce that

$$\Lambda_{D_1}(f) = \Lambda_{D_2}(f) \implies \|F\|_{\mathrm{H}^{-1/2}(S^1)} \le C\varepsilon^\alpha,$$

or equivalently

$$\sum_{n\neq 0} \frac{|c_k(F)|^2}{|k|} \leq C\varepsilon^{2\alpha}. \tag{30}$$

We claim that this last inequality implies the two following inequalities

$$|b_2|^2 \leq C\varepsilon^2 \text{ and } |R_1^\varepsilon - R_2^\varepsilon| \leq C\varepsilon^{2\alpha}.$$

Indeed, we write

$$c_k(F) = \alpha_k + \beta_k + \gamma_k,$$

where for $k \geq 2$

$$\begin{aligned}
\alpha_k &= (-1)^{k-1}(1-|b_2|^2)^2 \mu b^{k-1} k \frac{(R_2^\varepsilon)^{2k} - (R_1\varepsilon)^{2k}}{(1-\mu(R_1^\varepsilon)^2)(1-\mu(R_2^\varepsilon)^{2k})}, \\
\beta_k &= \overline{b}^{k-1}(1+|b_2|^2)^2 \frac{1+\mu(R_1^\varepsilon)^2}{1-\mu(R_1^\varepsilon)^2},
\end{aligned}$$

and where $\gamma_k = -\overline{\beta}_k$. When $k = 1$, we have

$$c_1(F) = \mu(1-|b_2|^2)^2 \frac{(R_2\varepsilon)^2 - (R_1^\varepsilon)^2}{(1-\mu R_1^2)(1-\mu R_2^2)}.$$

From (30), we get

$$|c_1(F)| \leq C\varepsilon^\alpha \text{ and } \sum_{|k|\geq 2} \frac{|c_k(f)|^2}{|k|} \leq C\varepsilon^{2\alpha}.$$

As we got for the case of disks, the condition $|c_1(F)| \leq C\varepsilon^\alpha$ implies

$$|(R_1^\varepsilon)^2 - (R_2^\varepsilon)^2|(1-|b_2|^2)^2 \leq C\varepsilon^{2\alpha}$$

where the constant C depends on δ_0. Since $(1-|b_2|^2)^2 > \delta_1$, we then get $|R_2^\varepsilon - R_1^\varepsilon| \leq C\varepsilon^\alpha$ This means that the radii of the two disks are very close.

Let us prove that

$$\sum_{|k|\geq 2} \frac{|c_k(f)|^2}{|k|} \leq C\varepsilon^2 \implies |b_2| \leq C\varepsilon^{2\alpha}.$$

Since

$$c_k(F) = (-1)^k b_2^{k-1} k \left[(1-|b_2|^2)^2 \frac{(R_2^\varepsilon)^{2k} - (R_1^\varepsilon)^2}{(1-\mu(R_1^\varepsilon)^2)(1-\mu(R_2^\varepsilon)^{2k})} + \left(\left(\frac{\overline{b_2}}{b_2}\right)^{k-1} - 1 \right) \frac{1+\mu(R_1^\varepsilon)^2}{k(1-\mu R_1^2)} \right]$$

we also get

$$\sum_{k\geq 2} k|b_2|^{2(k-1)} \delta_k \leq \varepsilon^{2\alpha}, \tag{31}$$

where we set

$$\delta_k = \left| (1-|b_2|^2)^2 \frac{(R_2^\varepsilon)^{2k} - (R_1^\varepsilon)^2}{(1-\mu(R_1^\varepsilon)^2)(1-\mu(R_2^\varepsilon)^{2k})} + \left(\left(\frac{\overline{b_2}}{b_2}\right)^{k-1} - 1 \right) \frac{1+\mu(R_1^\varepsilon)^2}{k(1-\mu(R_1^\varepsilon)^2)} \right|.$$

Let us check that $\delta_2 > 0$. We argue by contradiction and assume the converse: if $\delta_2 = 0$ then we would have

$$(1-|b_2|^2)^2 \frac{(R_2^\varepsilon)^4 - (R_1^\varepsilon)^2}{(1-\mu(R_1^\varepsilon)^2)(1-\mu(R_2^\varepsilon)^4)} + \left(\frac{\overline{b_2}}{b_2} - 1\right) \frac{1+\mu(R_1^\varepsilon)^2}{k(1-\mu(R_1^\varepsilon)^2)} = 0.$$

Taking the imaginary part of b_2, we would get $\overline{b_2} = \pm(b_2)^{-1}$ and then $(R_2^\varepsilon)^4 = (R_1^\varepsilon)^2$ which is impossible since we have $|R_1^\varepsilon - R_2^\varepsilon| \leq C\varepsilon^{2\alpha}$ with $R_1^\varepsilon, R_2^\varepsilon > \rho_0$. Hence, from (31), we deduce

$$|b_2|^2 \leq C\varepsilon^{2\alpha} \tag{32}$$

where $C > 0$ is a positive constant that depends on δ_0 and ρ_0.
Let us sum up our conditions :

- we have $|(R_1^\varepsilon)^2 - (R_\varepsilon^2|) \leq C\varepsilon^\alpha$; this means that

  $$\big||B_1^\varepsilon| - |B_2^\varepsilon t|\big| \leq C\varepsilon^{2\alpha}$$

  with a constant $C > 0$ depending on δ_0 and ρ_0.

- We have $|b_2| \leq C\varepsilon^\alpha$, this means that the center b_2 of B_2 is near the origin 0; a straightforward calculus gives

  $$|B_2 \Delta B_2^\varepsilon| \leq C\varepsilon^\alpha.$$

Since $|D_2 \Delta B_2| \leq C\varepsilon$, we then get

$$|D_2 \Delta B_2^\varepsilon| \leq \varepsilon^\alpha$$

It then follows that

$$|D_2 \Delta B_1^\varepsilon| \leq \varepsilon^\alpha$$

and then that

$$|D_2 \Delta D_1| \leq \varepsilon^\alpha.$$

This ends the proof of the result. ■

5 Proof of the precomposition theorem.

The main tool for proving Theorem 2.3 is the following lemma. It provides the behaviour of the precomposition on the Fourier basis functions.

Lemma 5.1 *Assume ϕ is a $\mathrm{W}^{1,\infty}$ diffeomorphism on S^1. Then, for all $\delta \in (0, 1/2)$*

$$\|e^{in\phi(\theta)} - e^{in\theta}\|_{\mathrm{H}^{1/2}(S^1)} \leq C(\delta)\ n^{1+2\delta}\ \omega_{2\delta}(\|\phi - I\|_{\mathrm{W}^{1,\infty}(S^1)}) \tag{33}$$

where $\omega_{2\delta}$ is the modulus of continuity defined by (4).

Proof of Lemma 5.1: The intrinsic definition of the $\mathrm{H}^{1/2}(S^1)$ gives

$$\|e^{in\phi(\theta)} - e^{in\theta}\|_{\mathrm{H}^{1/2}(S^1)} = \frac{1}{4\pi} \iint_{S^1 \times S^1} \frac{|e^{in\phi(\theta)} - e^{in\theta} - e^{in\phi(\alpha)} + e^{in\alpha}|^2}{|e^{i\theta} - e^{i\alpha}|^2} d\theta d\alpha.$$

Noting that

$$|e^{in\phi(\theta)} - e^{in\theta} - e^{in\phi(\alpha)} + e^{in\alpha}|^2 \leq 2\big[|e^{in(\phi-Id)(\theta)} - e^{in(\phi-Id)(\alpha)}|^2 + |e^{in(\phi-Id)(\alpha)} - 1|^2 |e^{in\theta} - e^{in\alpha}|^2\big]$$

we write $\|e^{in\phi(\theta} - e^{in\theta}\|_{\mathrm{H}^{1/2}(S^1)} \leq I_1 + I_2$ where

$$I_1 = \frac{1}{2\pi} \iint_{S^1 \times S^1} |e^{in(\phi-Id)(\alpha)} - 1|^2 \frac{|e^{in\theta} - e^{in\alpha}|^2}{|e^{i\theta} - e^{i\alpha}|^2} d\theta d\alpha,$$

and

$$I_2 = \frac{1}{2\pi} \iint_{S^1 \times S^1} \frac{|e^{in(\phi-Id)(\theta)} - e^{in(\phi-Id)(\alpha)}|^2}{|e^{i\theta} - e^{i\alpha}|^2} d\theta d\alpha.$$

Concerning I_1, we follow an idea of Bourgain, Brezis and Mironescu ([3]) and first integrate with respect to θ. Thanks to Parseval's formula, we get for all $\delta \in (0,1)$

$$I_1 = n \int_{S^1} |e^{in(\phi-Id)(\alpha)} - 1|^2 d\alpha \leq C\ n^{1+2\delta}\ \|\phi - Id\|_\infty^{2\delta}.$$

To estimate I_2, we introduce a non-negative parameter l and split I_2 into $I_2^d + I_2^r$ where

$$I_2^d = \frac{1}{2\pi} \iint_{|\theta-\alpha|<l} \left(\frac{\sin \frac{n}{2} [(\phi - Id)(\theta) - (\phi - Id)(\alpha)]}{\sin \frac{\theta-\alpha}{2}} \right)^2 d\theta d\alpha.$$

$$I_2^r = \frac{1}{2\pi} \iint_{|\theta-\alpha|\geq l} \left(\frac{\sin \frac{n}{2} [(\phi - Id)(\theta) - (\phi - Id)(\alpha)]}{\sin \frac{\theta-\alpha}{2}} \right)^2 d\theta d\alpha.$$

Since for all $\delta' \in (0,2)$ and for small enough l, one has

$$\left(\frac{\sin \frac{n}{2} [(\phi - Id)(\theta) - (\phi - Id)(\alpha)]}{\sin \frac{\theta-\alpha}{2}} \right)^2 \leq Cn^{\delta'} \|\phi' - 1\|_\infty^{\delta'-2}.$$

One checks that for $\delta' \in (1,2)$, there is a constant $C(\delta')$ such that

$$I_2^d \leq C(\delta')\ n^{\delta'}\ \|\phi' - 1\|_\infty^{\delta'}.$$

Concerning I_2^r, we get easily for all $\delta'' \in (0,1)$

$$I_2^r \leq C\ n^{2\delta''}\ \|\phi' - 1\|_\infty^{2\delta''}.$$

Summing up the estimates for I_1 and I_2^d and I_2^r, we get the stated result (33). ■

Let us notice that the exponent $1 + 2\delta$ can hardly by reduced since its main part I_1 is deduced from the Parseval's equality. Another remark is that the constant $C(\delta)$ blows up when $\delta \to 0$. We can now prove Theorem 2.3.

Proof of Theorem 2.3: In a first step, we assume that u is a trigonometric polynomial namely

$$u(\theta) = \sum_{|k|\leq n} c_k(u) e^{ik\theta}.$$

We fix δ' in $(0,\delta)$ and set $\alpha = \delta/\delta' - 1 > 0$. Then we have

$$\begin{aligned}\|u\circ\phi - u\|_{\mathrm{H}^{1/2}(S^1)} &\le \sum_{|k|\le n} |c_k(u)|\ \|e^{ik\phi(\theta)} - e^{ik\theta}\|_{\mathrm{H}^{1/2}(S^1)}\\ &\le C(\delta')\ \omega_{\delta'}(\|\phi - Id\|_{\mathrm{W}^{1,\infty}(S^1)}) \sum_{|k|\le n} k^{1/2+\delta'}|c_k(u)|.\end{aligned}$$

Writing

$$\frac{1}{2} + \delta' = -\frac{1}{2} - \alpha\delta' + 1 + \delta'(1+\alpha) = -\frac{1}{2} - \alpha\delta' + 1 + \delta,$$

we get

$$\sum_{|k|\le n} k^{1/2+\delta'}\ |c_k(u)|\ \le \left(\sum_{n\ne 0} |k|^{-1-2\alpha\delta'}\right)^{1/2} \left(\sum_{n\ne 0} |k|^{2+2(1+\alpha)\delta'}\ |c_k(u)|^2\right)^{1/2}.$$

Thanks to the Fatou's property that is satisfied by the Sobolev spaces ([10]), we extend the result to the most general Sobolev space $\mathrm{H}^{1+\delta}(S^1)$. ■

References

[1] I. Akduman, and R. Kress. *Electrostatic imaging via conformal mapping.* Inverse Problems 18 (2002), pp. 1659-1672.

[2] L. Ahlfors. *Möbius transformations in several dimensions.* School of Mathematics, University of Minnesota (1981).

[3] J. Bourgain, H. Brézis, and P. Mironescu. $\mathrm{H}^{1/2}$ *maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation.* Publ. Math. Inst. Hautes Études Sci., 99 (2004), pp. 1-115.

[4] G. Bourdaud, and W. Sickel . *Homeomorphisms which act on Besov spaces*, Function Spaces, Differential Operators and Nonlinear Analysis. Proceedings of Paseky na Jizerou, September 3-9, 1995. Editor: J. Rákosník.

[5] M. Dambrine, and D. Kateb. *Conformal mapping and inverse conductivity problem with one measurement.* to appear in ESAIM : Control Optimisation and Calculus of Variations.

[6] E. Fabes, H. Kang, and J.K. Seo. *Inverse conductivity problem with one measurement: Error estimates and approximate identification for perturbed disks.* SIAM J. Math. Anal, 30 (1999), pp. 699–720.

[7] H. Haddar, and R. Kress. *Conformal mappings and inverse boundary value problems.* Inverse Problems 21 (2005), pp. 935-953.

[8] P. Henrici. *Applied and computational Complex Analysis Vol. 1.* John Wiley & Sons (1974).

[9] P. Henrici. *Applied and computational Complex Analysis Vol. 3.* John Wiley & Sons (1986).

[10] H. Triebel. *Theory of Function Spaces.* Birkhäuser, Basel, Boston, Stuttgart, 1983.