

# Markov solutions for the 3D stochastic Navier–Stokes equations with state dependent noise

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**Abstract:** We construct a Markov family of solutions for the 3D Navier-Stokes equation perturbed by a non degenerate noise. We improve the result of [3] in two directions. We see that in fact not only a transition semigroup but a Markov family of solutions can be constructed. Moreover, we consider a state dependant noise. Another feature of this work is that we greatly simplify the proofs of [3].

**2000 Mathematics Subject Classification AMS:** 35Q10, 60H15, 37L40.

**Key words:** Stochastic Navier–Stokes equations, Markov solutions, invariant measure, ergodicity.

## 1 Introduction

We are concerned with the stochastic Navier–Stokes equations on  $\mathcal{O}$  an open bounded domain of  $\mathbb{R}^3$  with smooth boundary  $\partial\mathcal{O}$ . The unknowns are the velocity  $X(t, \xi)$  and the pressure  $p(t, \xi)$  defined for  $t > 0$  and  $\xi \in \overline{\mathcal{O}}$ :

$$\begin{cases} dX(t, \xi) = [\Delta X(t, \xi) - (X(t, \xi) \cdot \nabla)X(t, \xi)]dt - \nabla p(t, \xi)dt + f(\xi)dt + \Phi(X(t, \cdot))(\xi) dW, \\ \operatorname{div} X(t, \xi) = 0, \end{cases} \quad (1.1)$$

with Dirichlet boundary conditions

$$X(t, \xi) = 0, \quad t > 0, \quad \xi \in \partial\mathcal{O},$$

and supplemented with the initial condition

$$X(0, \xi) = x(\xi), \quad \xi \in \mathcal{O}.$$

We have taken the viscosity equal to 1 since it plays no particular role in this work. The noise is of white noise type and its covariance may depend on the noise through the operator  $\Phi(X)$ .

It is classical that this equation, both in the deterministic and stochastic case, has a global weak solution. Uniqueness of global solutions is a long standing open problem and one of the main challenge of the theory of partial differential equations (see [14] for a survey on these questions). Although an intense research has been performed on this aspect and many new ideas have appeared, this problem seems to be still out of reach.

In the stochastic case, a less difficult problem is the uniqueness in law in the spirit of Stroock and Varadhan [12]. Some progress have recently been obtained on that aspect. In [3], it has been proved that if the noise is additive and sufficiently non degenerate it is possible to construct directly a solution of the associate Kolmogorov

equation. Unfortunately, this solution is not sufficiently smooth and Ito formula cannot be applied. Thus the Stroock and Varadhan program cannot be accomplished and uniqueness in law remains open. However, using other ideas, it has been shown in [3] that it is possible to construct a Markov transition semigroup which is the limit of Galerkin approximations for (1.1). Moreover, this semigroup is associated to weak solutions and thanks to the nondegeneracy of the noise, it is strongly mixing. It follows from [11] that it is exponentially mixing.

Also, in [8], [9], another idea of the book [12] has been used. It has been shown that it is possible to construct a Markov selection from weak solutions to (1.1). Moreover, using the ideas of [3], it is proved that if the noise is sufficiently non degenerate every Markov selection of weak solutions defines a Strong Feller transition semigroup.

In this article, our aim is to improve the result of [3] and to extend it to the case of a state dependent noise. Moreover, we considerably simplify the construction. Indeed, in [3], numerous technical a priori estimates are necessary for a suitable approximation of the Kolmogorov equation introduced thanks to Galerkin approximation. We use the same idea here but show that it is sufficient to estimate the first derivate of their solutions as well as their modulus of continuity in time. This allows to prove compactness of the approximated solution of the Kolmogorov equations. We then notice that in fact this implies the convergence in law of a subsequence of the family of solutions. The limit is clearly a Markov family of solutions. Contrary to [8], [9], this Markov family of solution is obtained in a constructive way.

We also prove irreducibility of the transition semigroup thanks to classical arguments taken from the method to prove support theorems. Ergodicity follows.

## 2 Preliminaries

We set

$$H = \{x \in (L^2(\mathcal{O}))^3 : \operatorname{div} x = 0 \text{ in } \mathcal{O}, x \cdot n = 0 \text{ on } \partial\mathcal{O}\},$$

where  $n$  is the outward normal to  $\partial\mathcal{O}$ , and  $V = (H_0^1(\mathcal{O}))^3 \cap H$ . The norm and inner product in  $H$  will be denoted by  $|\cdot|$  and  $(\cdot, \cdot)$  respectively. We denote by  $\mathcal{L}(H)$  (resp.  $\mathcal{L}_2(H)$ ) the space of linear (resp. Hilbert-Schmidt) operators on  $H$  with norm  $|\cdot|_{\mathcal{L}}$  (resp.  $|\cdot|_{\mathcal{L}_2}$ ). Moreover  $W$  is a cylindrical Wiener process on  $H$  and, for any  $x \in H$ , the operator  $\Phi(x) \in \mathcal{L}(H)$  is Hilbert-Schmidt and such that  $\operatorname{Ker} \phi(x) = \{0\}$ .

We also denote by  $A$  the Stokes operator in  $H$ :

$$A = P\Delta, \quad D(A) = (H^2(\mathcal{O}))^3 \cap (H_0^1(\mathcal{O}))^3 \cap H,$$

where  $P$  is the orthogonal projection of  $(L^2(\mathcal{O}))^3$  onto  $H$  and by  $b$  the operator

$$b(x, y) = -P((x \cdot \nabla)y), \quad b(x) = b(x, x), \quad x, y \in V.$$

Now we can write problem (1.1) in the form

$$\begin{cases} dX(t, x) = (AX(t, x) + b(X(t, x)) + f)dt + \Phi(X(t, x)) dW(t), \\ X(0, x) = x. \end{cases} \quad (2.1)$$

We assume that the forcing term  $f$  is in  $V$ .

Classically, we use fractional powers of the operator  $A$  as well as their domains  $D((-A)^\alpha)$  for  $\alpha \in \mathbb{R}$ . Recall that, thanks to the regularity theory of the Stokes operator,  $D((-A)^\alpha)$  is a closed subspace of the Sobolev space  $(H^{2\alpha}(\mathcal{O}))^3$  and  $|\cdot|_{D((-A)^\alpha)} = |(-A)^\alpha \cdot|$  is equivalent to the usual  $(H^{2\alpha}(\mathcal{O}))^3$  norm.

Let  $0 < \alpha < \beta < \gamma$ . Then the following interpolatory estimate is well known

$$|(-A)^\beta x| \leq c |(-A)^\alpha x|^{\frac{\gamma-\beta}{\gamma-\alpha}} |(-A)^\gamma x|^{\frac{\beta-\alpha}{\gamma-\alpha}}, \quad x \in D((-A)^\gamma). \quad (2.2)$$

Moreover  $(b(x, y), y) = 0$ , whenever the left hand side makes sense. We shall use the following estimates on the bilinear operator  $b(x, y)$  (see [2], [3], [13]):

$$\left| (-A)^{1/2} b(x, y) \right| \leq c |Ax| |Ay|. \quad (2.3)$$

The following result can be proved by classical argument (see [8]).

**Proposition 2.1** *For any  $x \in H$ , there exists a martingale solution of equation (2.1) with trajectories in  $C([0, T]; D((-A)^{-\alpha}))$  and  $L^\infty(0, T; H) \cap L^2(0, T; D((-A)^{1/2})$  for any  $\alpha > 0$  and  $T > 0$ .*

Let  $E$  be any Banach space and  $\varphi : D(A) \rightarrow E$ . For any  $x, h \in D(A)$  we set

$$D\varphi(x) \cdot h = \lim_{s \rightarrow 0} \frac{1}{s} (\varphi(x + sh) - \varphi(x)),$$

provided the limit exists. The limit is intended in  $E$ . The functional space  $C_b(D(A); E)$  is the space of all continuous and bounded mappings from  $D(A)$  (endowed with the graph norm) into  $E$  and, for any  $k \in \mathbb{N}$ ,  $C_k(D(A); E)$  is the space of all continuous mappings from  $D(A)$  into  $E$  such that

$$\|\varphi\|_k := \sup_{x \in D(A)} \frac{|\varphi(x)|}{(|Ax| + 1)^k} < +\infty.$$

We denote by  $B_b(D(A); \mathbb{R})$  the space of all Borel bounded mappings from  $D(A)$  into  $\mathbb{R}$ . It is also convenient to define the space

$$\mathcal{E} = \left\{ \varphi \in C_b(D(A); \mathbb{R}) \left| \sup_{(x_1, x_2)} \frac{|\varphi(x_2) - \varphi(x_1)|}{|A(x_2 - x_1)|(1 + |Ax_1|^2 + |Ax_2|^2)} < \infty \right. \right\},$$

with the norm

$$\|\varphi\|_{\mathcal{E}} = \|\varphi\|_0 + \sup_{(x_1, x_2)} \frac{|\varphi(x_2) - \varphi(x_1)|}{|A(x_2 - x_1)|(1 + |Ax_1|^2 + |Ax_2|^2)}$$

We introduce the usual Galerkin approximations of equations (2.1). For  $m \in \mathbb{N}$ , we define the projector  $P_m$  onto the first  $m$  eigenvectors of  $A$  and set  $b_m(x) = P_m b(P_m x)$ ,  $\Phi_m(x) = P_m \Phi(x)$ , for  $x \in H$ . Then, we write the following approximations

$$\begin{cases} dX_m(t, x) = (AX_m(t, x) + b_m(X_m(t, x)))dt + \Phi_m(X_m(t, x)) dW(t), \\ X_m(0, x) = P_m x = x_m, \end{cases} \quad (2.4)$$

and

$$\begin{cases} \frac{du_m}{dt} = \frac{1}{2} \operatorname{Tr} [\psi_m(x) D^2 u_m] + (Ax + b_m(x), Du_m), \\ u_m(0) = \varphi, \end{cases} \quad (2.5)$$

where  $\psi_m = \Phi_m \Phi_m^*$ . Equation (2.5) has a unique solution given by

$$u_m(t, x) = P_t^m \varphi(x) = \mathbb{E}[\varphi(X_m(t, x))], \quad \text{for } x \in P_m H. \quad (2.6)$$

We extend  $u_m(t, x)$  to  $H$  by setting  $u_m(t, x) = u_m(t, P_m x)$ . We assume that  $\Phi$  is a  $C^1$  function of  $x$  with values in  $\mathcal{L}(H)$ . Then, if  $\varphi$  is a  $C^1$  function,  $u_m$  is differentiable and its differential can be expressed in terms of  $\eta_m^h = \eta_m^h(t, x)$  the solution of

$$\begin{cases} d\eta_m^h = (A\eta_m^h + b'_m(X_m) \cdot \eta_m^h) dt + \Phi'_m(X_m) \cdot \eta_m^h dW(t), \\ \eta_m^h(0, x) = P_m h, \end{cases} \quad (2.7)$$

with  $b'_m(X_m) \cdot \eta_m^h = b_m(X_m, \eta_m^h) + b_m(\eta_m^h, X_m)$ . Moreover, since we assume that the noise is non degenerate, the differential of  $u_m$  exists even when  $\varphi$  is only continuous thanks to the Bismut–Elworthy–Li formula, see [1] and [7]. Unfortunately, it is impossible to get any estimate on the differential of  $u_m$  by these ideas. Indeed, we are not able to prove an estimate of  $\eta_m^h(t, x)$  uniform in  $m$ . The idea is to introduce the auxiliary Kolmogorov equation

$$\begin{cases} \frac{dv_m}{dt} = \frac{1}{2} \operatorname{Tr} [\psi_m(x) D^2 v_m] + (Ax + b_m(x), Dv_m) - K|Ax|^2 v_m, \\ v_m(0) = \varphi, \end{cases} \quad (2.8)$$

where  $K > 0$  is fixed, which contains a “very negative” potential term. It has a unique solution given by the Feynman-Kac formula

$$v_m(t, x) := S_t^m \varphi(x) = \mathbb{E} \left[ e^{-K \int_0^t |AX_m(s, x)|^2 ds} \varphi(X_m(t, x)) \right]. \quad (2.9)$$

Clearly, the function  $u_m$  can be expressed in terms of the function  $v_m$  by the variation of constants formula:

$$u_m(t, \cdot) = S_t^m \varphi + K \int_0^t S_{t-s}^m (|A \cdot|^2 u_m(s, \cdot)) ds. \quad (2.10)$$

Since the noise is non degenerate, from [6] we know that for any  $\varphi \in C_b(H)$ ,  $S_t^m \varphi$  is differentiable in any direction  $h \in H$  and we have

$$\begin{aligned} & DS_t^m \varphi(x) \cdot h \\ &= \frac{1}{t} \mathbb{E} \left[ e^{-K \int_0^t |AX_m(s, x)|^2 ds} \varphi(X_m(t, x)) \int_0^t (\Phi^{-1}(X_m(s, x)) \eta_m^h(s, x), dW(s)) \right] \\ &+ 2K \mathbb{E} \left[ e^{-K \int_0^t |AX_m(s, x)|^2 ds} \varphi(X_m(t, x)) \int_0^t \left(1 - \frac{s}{t}\right) (AX_m(s, x), A\eta_m^h(s, x)) ds \right]. \end{aligned} \quad (2.11)$$

We are going to prove estimates for the derivatives of  $u_m(t, \cdot)$  through corresponding estimates for  $v_m(t, \cdot)$ . We will see that this is possible provided  $K$  is chosen large enough. This implies some compactness on the sequence  $(u_m)_m$ .

The main assumption in our estimates below is that the covariance operator is at the same time sufficiently smooth and non degenerate. We assume throughout the paper that there exists constants  $M_1 \geq 0$ ,  $r \in (1, 3/2)$  and  $g > 0$  such that

$$\text{Tr} [(-A)^{1+g}\Phi(x)\Phi^*(x)] \leq M_1, \text{ for } x \in H \quad (2.12)$$

and

$$|\Phi^{-1}(x)h| \leq M_1|(-A)^r h|, \text{ for } x \in H, h \in D((-A)^r). \quad (2.13)$$

Note that these two conditions are compatible, for instance if we take  $\Phi(x)$  which is bounded and invertible from  $H$  onto  $D((-A)^\alpha)$ ,  $\alpha > 0$  then conditions (2.12) and (2.13) are satisfied provided

$$\alpha \in (5/4, 3/2),$$

and the norm of  $\Phi(x)$  and its inverse are uniformly bounded.

Since we also work with the differential of the solution with respect to the initial data, we also need some assumption on the derivative of  $\Phi$ . We assume that there exists  $\delta < 3/2$  such that:

$$\text{Tr} [(-A)^2(\Phi'(x) \cdot h)(\Phi'(x) \cdot h)^*] \leq M_1|(-A)^\delta h|^2, \text{ for } x \in H, h \in D((-A)^\delta). \quad (2.14)$$

**Example 2.2** Let  $c > 0$ ,  $\alpha \in (5/4, 3/2)$  and  $\kappa$  a bounded mapping  $H \rightarrow \mathcal{L}_2(H; D(A))$  with bounded continuous derivative. We set

$$\Phi(x) = (-A)^\alpha + c\kappa(x).$$

If  $\text{Ker } \Phi(x) = \{0\}$  for any  $x \in H$ , then  $\Phi$  verifies the previous assumptions. This is the case if  $c$  is small. A first example of such  $\kappa$  is the following. Let  $(\kappa_n)_n$  be a family of functions in  $C_b^1(H)$  with norm bounded by one and  $(\lambda_n)_n \in l^2(\mathbb{N})$ . We denote by  $(\mu_n, e_n)$  the family of eigenvalues and eigenvectors of  $(-A)$ . Then, we set

$$\kappa(x) \cdot h = \sum_n \mu_n^{-\frac{3}{2}} \lambda_n \kappa_n(x) h_n e_n, \quad \text{where } h = \sum_n h_n e_n.$$

Another example is given by

$$(\kappa(x) \cdot h)(\xi) = \int_{O \times O} \mathcal{V}(\xi, \xi', x(\xi')) h(\xi') d\xi',$$

where  $\mathcal{V}$  is a  $C^\infty$  map  $O \times O \times \mathbb{R} \rightarrow \mathbb{R}$  with compact support and free divergence with respect to the first variable (i.e.  $\text{div } \mathcal{V}(\cdot, \xi, r) = 0$ ).

Before stating our main result, we recall the following definition

**Definition 2.3** Let  $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)_{x \in D(A)}$  be a family of probability spaces and  $(X(\cdot, x))_{x \in D(A)}$  be a family of random processes on  $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)_{x \in D(A)}$ . We denote by  $(\mathcal{F}_x^t)_t$  the filtration generated by  $X(\cdot, x)$  and by  $\mathcal{P}_x$  the law of  $X(t, x)$  under  $\mathbb{P}_x$ . The family  $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x, X(\cdot, x))_{x \in D(A)}$  is a Markov family if the following conditions hold:

(i) For any  $x \in D(A)$ ,  $t \geq 0$ , we have

$$\mathbb{P}_x(X(t, x) \in D(A)) = 1,$$

(ii) the map  $x \rightarrow \mathcal{P}_x$  is measurable and for any  $x \in D(A)$ ,  $t_0, \dots, t_n \geq 0$ ,  $A_0, \dots, A_n$  borelian subsets of  $D(A)$ , we have

$$\mathbb{P}_x(X(t + \cdot) \in \mathcal{A} \mid \mathcal{F}_x^t) = \mathcal{P}_{X(t, x)}(\mathcal{A}),$$

where  $\mathcal{A} = \{y \in (H)^{\mathbb{R}^+} \mid y(t_0) \in A_0, \dots, y(t_n) \in A_n\}$ .

The Markov transition semigroup  $(P_t)_{t \geq 0}$  associated to the family is then defined by

$$P_t \varphi(x) = \mathbb{E}_x(\varphi(X(t, x))), \quad x \in D(A), t \geq 0.$$

for  $\varphi \in B_b(D(A); \mathbb{R})$ .

The main result of the paper is the following. The proof is given in the case  $f = 0$ , the generalization to a general  $f \in V$  is easy.

**Theorem 2.4** *There exists a Markov family of martingale solutions  $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x, X(\cdot, x))_{x \in D(A)}$  of the stochastic Navier-Stokes equations (2.1). Furthermore, the transition semigroup  $(P_t)_{t \geq 0}$  is stochastically continuous.*

We also study ergodic properties and prove the following result.

**Theorem 2.5** *There exists a Markov process  $X(\cdot, \nu)$  on a probability space  $(\Omega_\nu, \mathcal{F}_\nu, \mathbb{P}_\nu)$  which is a martingale stationary solution of the stochastic Navier-Stokes equations (2.1). The law  $\nu$  of  $X(t, \nu)$  is the unique invariant measure on  $D(A)$  of the transition semigroup  $(P_t)_{t \geq 0}$ . Moreover*

- (i) the invariant measure  $\nu$  is ergodic,
- (ii) the law  $\mathcal{P}_\nu$  of  $X(\cdot, \nu)$  is given by

$$\mathcal{P}_\nu(\mathcal{A}) = \int_{D(A)} \mathcal{P}_x(\mathcal{A}) \nu(dx),$$

for  $\mathcal{A} = \{y \in (H)^{\mathbb{R}^+} \mid y(t_0) \in A_0, \dots, y(t_n) \in A_n\}$  with  $t_0, \dots, t_n \geq 0$  and  $A_0, \dots, A_n$  borelian subsets of  $D(A)$ ,

- (iii) the transition semigroup  $(P_t)_{t \geq 0}$  is strong Feller, irreducible, and therefore strongly mixing.

**Remark 2.6** *In fact, we prove that there exists a subsequence  $(m_k)_k$  such that, for any  $x \in D(A)$ ,  $X_{m_k}(\cdot, x) \rightarrow X(\cdot, x)$  in law. Moreover  $X_{m_k}(\cdot, \nu_{m_k})$  the unique stationary solution of (2.4) converges to  $X(\cdot, \nu)$ . Thus, the solutions  $(X(\cdot, x))_x$  and  $X(\cdot, \nu)$  are obtained in a constructive way.*

### 3 A priori estimates

For any predictable process  $X$  with values in  $H$ , we set:

$$Z_X(t) = \int_0^t e^{(t-s)A} \Phi(X(s)) dW(s). \quad (3.1)$$

We have the following estimates on  $Z_X$  which will be useful in the sequel.

**Proposition 3.1** *For any  $T \geq 0$ ,  $\varepsilon < g/2$  and any  $m \geq 1$ , there exists a constant  $c(\varepsilon, m, T)$  such that, for any predictable process  $X$  with values in  $H$ ,  $Z_X$  has continuous paths with values in  $D((-A)^{1+\varepsilon})$  and*

$$\mathbb{E} \left( \sup_{t \in [0, T]} |(-A)^{1+\varepsilon} Z_X(t)|^{2m} \right) \leq c(\varepsilon, m, T). \quad (3.2)$$

Moreover, for any  $\beta < \min\{g/2 - \varepsilon, 1/2\}$ , there exists a constant  $c(\varepsilon, \beta, m, T)$  such that for  $t_1, t_2 \in [0, T]$ ,

$$\mathbb{E}(|(-A)^{1+\varepsilon}(Z_X(t_1) - Z_X(t_2))|^{2m}) \leq c(\varepsilon, \beta, m, T)|t_1 - t_2|^{2\beta m}. \quad (3.3)$$

**Proof:** The proof uses the factorization method (see [4, Section 5.3]). We write

$$Z_X(t) = \int_0^t (t-s)^{\alpha-1} e^{A(t-s)} Y(s) ds$$

with  $\alpha$  to be chosen below and

$$Y(s) = \frac{\sin \pi \alpha}{\alpha} \int_0^s e^{A(t-s)} (s-\sigma)^{-\alpha} \Phi(X(\sigma)) dW(\sigma).$$

Using Burkholder-Davies-Gundy inequality, we deduce for  $m \in \mathbb{N}$ :

$$\begin{aligned} & \mathbb{E} (|(-A)^{1+\varepsilon} Y(s)|^{2m}) \\ & \leq c \mathbb{E} \left( \left( \int_0^s |(-A)^{1+\varepsilon} e^{A(s-\sigma)} (s-\sigma)^{-\alpha} \Phi(X(\sigma))|_{\mathcal{L}_2}^2 d\sigma \right)^m \right) \\ & \leq c \mathbb{E} \left( \left( \int_0^s (s-\sigma)^{-2\alpha} |(-A)^{1/2+g/2} \Phi(X(\sigma))|_{\mathcal{L}_2}^2 |(-A)^{1/2+\varepsilon-g/2} e^{A(s-\sigma)}|_{\mathcal{L}}^2 d\sigma \right)^m \right) \\ & \leq c M_1 \left( \int_0^s (s-\sigma)^{-(1+2\varepsilon-g)^+ - 2\alpha} d\sigma \right)^m, \end{aligned}$$

thanks to well-known smoothing properties of the semigroup  $(e^{At})_{t \geq 0}$  and assumption (2.12). This is a finite quantity provided  $\alpha < \min\{g/2 - \varepsilon, 1/2\}$ . Moreover, it is a bounded function of  $s \in [0, T]$ . It follows easily that  $Y \in L^{2m}(\Omega \times [0, T]; D((-A)^{1+\varepsilon}))$ . Proposition 3.1 follows now from Proposition A.1.1 of [5].  $\square$

The proof of the following estimate is the same as the proof of Lemma 3.1 in [3].

**Lemma 3.2** *There exists  $c > 0$  such that, for any  $m \in \mathbb{N}$ ,  $t \in [0, T]$  and  $x \in D(A)$ ,*

$$e^{-c \int_0^t |AX_m(s, x)|^2 ds} |AX_m(t, x)|^2 \leq 2|Ax|^2 + c \sup_{s \in [0, T]} |AZ_{X_m}(s)|^2. \quad (3.4)$$

**Lemma 3.3** For any  $\gamma \in (\delta - 1/2, 1]$ , there exists  $c_\gamma > 0$  such that for any  $m \in \mathbb{N}$ ,  $t \in [0, T]$  and any  $x, h \in D(A)$  we have

$$\begin{aligned} & \mathbb{E} \left( e^{-c_\gamma \int_0^t |AX_m(s, x)|^2 ds} |(-A)^\gamma \eta_m^h(t, x)|^2 + \int_0^t e^{-c_\gamma \int_0^s |AX_m(\tau, x)|^2 d\tau} |(-A)^{\gamma+1/2} \eta_m^h(s, x)|^2 ds \right) \\ & \leq e^{c_\gamma t} |(-A)^\gamma h|^2, \end{aligned} \quad (3.5)$$

**Proof.** We use Ito formula to obtain

$$\begin{aligned} & d \left[ e^{-c_\gamma \int_0^t |AX(s)|^2 ds} |(-A)^\gamma \eta(t)|^2 + 2 \int_0^t e^{-c_\gamma \int_0^s |AX(\tau)|^2 d\tau} |(-A)^{\gamma+1/2} \eta(s)|^2 ds \right] \\ & = e^{-c_\gamma \int_0^t |AX(s)|^2 ds} \left[ \text{Tr} \left[ (-A)^{2\gamma} (\Phi'(X(t)) \cdot \eta(t)) (\Phi'(X(t)) \cdot \eta(t))^* \right] dt \right. \\ & \quad + 2 \left( (-A)^{2\gamma} \eta(t), (\Phi'(X(t)) \cdot \eta(t)) dW(t) \right) + 2 \left( b'(X(t)) \cdot \eta(t), (-A)^{2\gamma} \eta(t) \right) dt \\ & \quad \left. - c_\gamma |AX(t)|^2 |(-A)^\gamma \eta(t)|^2 dt \right]. \end{aligned}$$

We have written for simplicity  $\eta_m^h(t) = \eta(t)$  and  $X(t) = X_m(t, x)$ . Using (2.3), Poincaré inequality and (2.2), we obtain

$$\begin{aligned} \left( b'(X(t)), \eta(t), (-A)^{2\gamma} \eta(t) \right) & \leq c |AX(t)| |A\eta(t)| |(-A)^{2\gamma-1/2} \eta(t)| \\ & \leq c |AX(t)| |(-A)^\gamma \eta(t)| |(-A)^{\gamma+1/2} \eta(t)| \\ & \leq c |AX(t)|^2 |(-A)^\gamma \eta(t)|^2 + \frac{1}{4} |(-A)^{\gamma+1/2} \eta(t)|^2. \end{aligned}$$

Moreover, since  $\delta < \gamma + 1/2$  in (2.14), we have

$$\begin{aligned} & \text{Tr} \left[ (-A)^{2\gamma} (\Phi'(X(t)) \cdot \eta(t)) (\Phi'(X(t)) \cdot \eta(t))^* \right] \\ & \leq c \text{Tr} \left[ (-A)^2 (\Phi'(X(t)) \cdot \eta(t)) (\Phi'(X(t)) \cdot \eta(t))^* \right] \\ & \leq c |(-A)^\delta \eta(t)|^2 \\ & \leq c |(-A)^\gamma \eta(t)|^2 + \frac{1}{2} |(-A)^{\gamma+1/2} \eta(t)|^2. \end{aligned}$$

It follows that, for  $c_\gamma$  sufficiently large,

$$\begin{aligned} & d \left[ e^{-c_\gamma \int_0^t |AX(s)|^2 ds} |(-A)^\gamma \eta(t)|^2 + \int_0^t e^{-c_\gamma \int_0^s |AX(\tau)|^2 d\tau} |(-A)^{\gamma+1/2} \eta(s)|^2 ds \right] \\ & \leq 2e^{-c_\gamma \int_0^t |AX(s)|^2 ds} \left[ \left( (-A)^{2\gamma} \eta(t), (\Phi'(X(t)) \cdot \eta(t)) dW(t) \right) + c |(-A)^\gamma \eta(t)|^2 dt \right]. \end{aligned}$$

We deduce the result by taking the expectation and integrating.  $\square$

We now get bounds on the Feynman-Kac semigroup  $S_t^m$ .

**Lemma 3.4** For any  $1 \geq \gamma > \max\{\delta - 1/2, r - 1/2\}$ , where  $r \in (1, 3/2)$  is defined in (2.13) and  $k \in \mathbb{N}$ , if  $K$  is sufficiently large there exists  $c(\gamma, k) > 0$  such that for any  $\varphi \in C_k(D(A); \mathbb{R})$

$$\|(-A)^{-\gamma} DS_t^m \varphi\|_k \leq c(\gamma)(t^{-1/2-(r-\gamma)} + 1) \|\varphi\|_k, \quad t > 0,$$

for all  $m \in \mathbb{N}$ .

**Proof.** Let  $h \in H$ . We write (2.11) as  $DS_t^m \varphi(x) \cdot h = I_1 + I_2$  and estimate separately the two terms. We again write for simplicity  $\eta_m^h(t) = \eta(t)$  and  $X(t) = X_m(t, x)$ . Concerning  $I_1$  we have, using the Hölder inequality,

$$\begin{aligned} I_1 &\leq \frac{1}{t} \|\varphi\|_k \mathbb{E} \left[ e^{-K \int_0^t |AX(s)|^2 ds} (1 + |AX(t)|)^k \left( \int_0^t (\Phi^{-1}(X(s))\eta(s), dW(s)) \right) \right] \\ &\leq \frac{1}{t} \|\varphi\|_k \left[ \mathbb{E} \left( e^{-K \int_0^t |AX(s)|^2 ds} (1 + |AX(t)|)^{2k} \right) \right]^{1/2} \\ &\quad \times \left[ \mathbb{E} \left( e^{-K \int_0^t |AX(s)|^2 ds} \left( \int_0^t (\Phi^{-1}(X(s))\eta(s), dW(s)) \right)^2 \right) \right]^{1/2}. \end{aligned}$$

Choosing  $K$  sufficiently large, the first factor is easily majorized by  $c(1 + |Ax|)^{2k}$  thanks to Lemma 3.2 and Proposition 3.1. To estimate the second factor we proceed as in [3] and set

$$\xi(t) = e^{-\frac{K}{2} \int_0^t |AX(s)|^2 ds} \int_0^t (\Phi^{-1}(X(s))\eta(s), dW(s))$$

and use Ito formula to compute  $\mathbb{E}(\xi^2(t))$ . We obtain

$$\mathbb{E}(\xi^2(t)) \leq \mathbb{E} \left[ \int_0^t e^{-K \int_0^s |AX_m(\tau, x)|^2 d\tau} |\Phi^{-1}(X(s))\eta(s)|^2 ds \right].$$

Recalling assumption (2.13) and the interpolatory estimate (2.2) we find

$$|\Phi^{-1}(X(s))x|^2 \leq M_1 |(-A)^r x|^2 \leq c |(-A)^\gamma x|^{2(1-2(r-\gamma))} |(-A)^{\gamma+1/2} x|^{4(r-\gamma)}.$$

Consequently, by Hölder inequality and Lemma 3.3, we get

$$\mathbb{E}(\xi(t)^2) \leq ct^{1-2(r-\gamma)} |(-A)^\gamma h|^2,$$

provided  $K$  is sufficiently large. Thus

$$I_1 \leq ct^{-1/2-(r-\gamma)} |(-A)^\gamma h| (|Ax| + 1)^k.$$

Finally, since  $\gamma \geq 1/2$ , the following estimate easily follows from Hölder inequality and Lemmas 3.2, 3.3:

$$I_2 \leq c \|\varphi\|_k (1 + |Ax|)^k |(-A)^\gamma h|.$$

Consequently, if  $K$  is sufficiently large, we find

$$|DS_t^m \varphi(x) \cdot h| \leq c \|\varphi\|_k (1 + |Ax|)^k (1 + t^{-1/2-(r-\gamma)}) |(-A)^\gamma h|.$$

The conclusion follows.  $\square$

We are now ready to get uniform estimates on the approximated solutions to the Kolmogorov equation.

**Proposition 3.5** *If  $\varphi \in C_b(D(A); \mathbb{R})$ , then  $u_m(t) \in C_b(D(A); \mathbb{R})$  and, for any  $1 \geq \gamma > \max\{\delta - 1/2, r - 1/2\}$ ,  $(-A)^{-\gamma} Du_m \in C_2(D(A); \mathbb{R})$  for all  $t > 0$ ,  $m \in \mathbb{N}$ . Moreover, we have*

$$\|u_m(t)\|_0 \leq \|\varphi\|_0$$

and

$$\|(-A)^{-\gamma} Du_m(t)\|_2 \leq c(\gamma)(1 + t^{-1/2-(r-\gamma)})\|\varphi\|_0, \quad t > 0.$$

**Proof.** The first estimate follows from (2.6). By (2.10) and Lemma 3.4, it follows

$$\begin{aligned} \|(-A)^{-\gamma} Du_m(t)\|_2 &\leq c(1 + t^{-1/2-(r-\gamma)})\|\varphi\|_2 \\ &\quad + \int_0^t c(1 + (t-s)^{-1/2-(r-\gamma)})\| |Ax|^2 u_m(s) \|_2 ds. \end{aligned}$$

Clearly  $\|\varphi\|_2 \leq \|\varphi\|_0$  and  $\| |Ax|^2 u_m(s) \|_2 \leq \|u_m(s)\|_0 \leq \|\varphi\|_0$ . The result follows.  $\square$

**Proposition 3.6** *Let  $\varphi \in \mathcal{E}$ . Then for any  $\beta < \min\{g/2, 1/2\}$ , there exists  $c(\beta)$  such that for any  $t_1, t_2 > 0$ ,  $m \in \mathbb{N}$  and  $x \in D(A)$  we have*

$$|u_m(t_1, x) - u_m(t_2, x)| \leq c\|\varphi\|_{\mathcal{E}} (|Ax| + 1)^6 (|t_1 - t_2|^\beta + |A(e^{t_1 A} - e^{t_2 A})x|).$$

**Proof.** By (2.10), we have for  $t_1 < t_2$

$$\begin{aligned} u_m(t_1, x) - u_m(t_2, x) &= (S_{t_1}^m - S_{t_2}^m) \varphi(x) + K \int_0^{t_1} (S_{t_1-s}^m - S_{t_2-s}^m) (|Ax|^2 u_m(s)) (x) ds \\ &\quad + K \int_{t_1}^{t_2} S_{t_2-s}^m (|Ax|^2 u_m(s)) (x) ds \\ &= T_1 + T_2 + T_3. \end{aligned}$$

For the first term we use the decomposition, with  $X(t) = X_m(t, x)$ ,

$$\begin{aligned} |T_1| &= \left| \mathbb{E} \left( \left( e^{-K \int_0^{t_1} |AX(s)|^2 ds} - e^{-K \int_0^{t_2} |AX(s)|^2 ds} \right) \varphi(X(t_1)) \right) \right| \\ &\quad + \left| \mathbb{E} \left( e^{-K \int_0^{t_2} |AX(s)|^2 ds} (\varphi(X(t_1)) - \varphi(X(t_2))) \right) \right| \\ &\leq K \|\varphi\|_{\mathcal{E}} \mathbb{E} \left( \int_{t_1}^{t_2} |AX(s)|^2 e^{-K \int_0^s |AX(\tau)|^2 d\tau} ds \right) \\ &\quad + \|\varphi\|_{\mathcal{E}} \mathbb{E} \left( e^{-K \int_0^{t_2} |AX(s)|^2 ds} \left( 1 + \sup_{t \in [0, T]} |AX(t)|^2 \right) |A(X(t_1) - X(t_2))| \right). \end{aligned}$$

Thanks to Lemma 3.2, we majorize the first term by  $c\|\varphi\|_0(|Ax|^2 + 1)|t_1 - t_2|$ . For the second term, we write

$$\begin{aligned} X(t_1) - X(t_2) &= (e^{t_1 A} - e^{t_2 A})x + Z_X(t_1) - Z_X(t_2) \\ &\quad + \int_0^{t_1} e^{A(t_1-s)} b(X(s)) ds - \int_0^{t_2} e^{A(t_2-s)} b(X(s)) ds. \end{aligned}$$

By (2.3), and classical property of  $(e^{At})_{t>0}$ , for any  $\lambda \in (0, 1/2)$  we have the following estimate:

$$\begin{aligned} & \mathbb{E} \left( e^{-K \int_0^{t_2} |AX(s)|^2 ds} \left| A \left( \int_0^{t_1} e^{A(t_1-s)} b(X(s)) ds - \int_0^{t_2} e^{A(t_2-s)} b(X(s)) ds \right) \right| \right) \\ & \leq \mathbb{E} \left[ e^{-K \int_0^{t_2} |AX(s)|^2 ds} \left( \int_{t_1}^{t_2} \left| (-A)^{1/2} e^{A(t_2-s)} \right|_{\mathcal{L}} \left| (-A)^{1/2} b(X(s)) \right| ds \right. \right. \\ & \quad \left. \left. + \int_0^{t_1} \left| (-A)^{1/2} \left( e^{A(t_1-s)} - e^{A(t_2-s)} \right) \right|_{\mathcal{L}} \left| (-A)^{1/2} b(X(s)) \right| ds \right) \right] \\ & \leq c_\lambda \mathbb{E} \left( e^{-K \int_0^{t_2} |AX(s)|^2 ds} \sup_{s \in [0, t_2]} |AX_m(s, x)|^2 \right) |t_1 - t_2|^{1/2-\lambda} \end{aligned}$$

Therefore, with Proposition 3.1 and Lemma 3.2, we obtain for any  $\beta < \min\{g/2, 1/2\}$

$$|T_1| \leq c \|\varphi\|_{\mathcal{E}} (|Ax|^4 + 1) (|t_1 - t_2|^\beta + |A(e^{t_1 A} - e^{t_2 A})x|).$$

Similarly, we have

$$\begin{aligned} (S_{t_1-s}^m - S_{t_2-s}^m) (|Ax|^2 u_m(s)) & \leq c (\|u_m\|_0 + \|(-A)^{-1} D u_m(s)\|_2) (|Ax|^6 + 1) \\ & \quad \times (|t_1 - t_2|^\beta + |A(e^{t_1 A} - e^{t_2 A})x|). \end{aligned}$$

Thus, thanks to Proposition 3.5,

$$|T_2| \leq c \|\varphi\|_{\mathcal{E}} (|Ax|^6 + 1) (|t_1 - t_2|^\beta + |A(e^{t_1 A} - e^{t_2 A})x|).$$

Finally, the last term  $T_3$  is easy to treat and majorized by  $c \|\varphi\|_0 (|Ax|^2 + 1) |t_1 - t_2|$ . Gathering these estimates yields the result.  $\square$

## 4 Proof of Theorem 2.4

For  $\varphi \in \mathcal{E}$ , let  $(u_m)_{m \in \mathbb{N}}$  be the sequence of solutions of the approximated Kolmogorov equations. Thanks to the a priori estimates derived in the previous section, we are now able to show that  $(u_m)_{m \in \mathbb{N}}$  has a convergent subsequence.

We set  $K_R = \{x \in D(A) : |Ax| \leq R\}$ . For  $\gamma < 1$ , it is a compact subset of  $D((-A)^\gamma)$ .

**Lemma 4.1** *Assume that  $\varphi \in \mathcal{E}$ , then there exists a subsequence  $(u_{m_k})_{k \in \mathbb{N}}$  of  $(u_m)$  and a function  $u$  bounded on  $[0, T] \times D(A)$ , such that*

(i)  $u \in C_b([0, T] \times D(A))$  and for any  $\delta > 0$ ,  $R > 0$

$$\lim_{k \rightarrow \infty} u_{m_k}(t, x) = u(t, x) \quad \text{uniformly on } [\delta, T] \times K_R.$$

(ii) For any  $x \in D(A)$ ,  $u(\cdot, x)$  is continuous on  $[0, T]$ .

(iii) For any  $1 \geq \gamma > \max\{\delta - 1/2, r - 1/2\}$ ,  $\delta > 0$ ,  $R \geq 0$ ,  $\beta < \min\{g/2, 1/2\}$ , there exists  $c(\gamma, \beta, \delta, R, T, \varphi)$  such that for  $x, y \in K_R$ ,  $t, s > \delta$ ,

$$|u(t, x) - u(s, y)| \leq c(\gamma, \beta, \delta, R, T, \varphi) (|(-A)^\gamma (x - y)| + |t - s|^\beta)$$

(iv) For any  $t \in [0, T]$ ,  $u(t, \cdot) \in \mathcal{E}$ .

(v)  $u(0, \cdot) = \varphi$ .

**Proof.** Let  $R > 0$ ,  $\delta > 0$  and  $t, s \in [\delta, T]$ ,  $x, y \in K_R$ . Then by Proposition 3.5 and 3.6 it follows that, for  $\beta < \min\{g/2, 1/2\}$ ,

$$|u_m(t, x) - u_m(s, y)| \leq c(\delta, T) \|\varphi\|_{\mathcal{E}} (|(-A)^\gamma (x - y)| + |t - s|^\beta) \quad (4.1)$$

From the the Ascoli–Arzelà theorem and a diagonal extraction argument, we deduce that we can construct a subsequence such that

$$u_{m_k}(t, x) \rightarrow u(t, x),$$

uniformly in  $[\delta, T] \times K_R$  for any  $\delta > 0$ ,  $R > 0$ . So that (i) follows. Moreover, taking the limit in (4.1), we deduce (iii). We define  $u(0, \cdot) = \varphi$ . We can take the limit  $m_k \rightarrow \infty$  in Proposition 3.6 with  $t_1 = t > 0$  and  $t_2 = 0$  and obtain for  $x \in K_R$

$$|u(t, x) - \varphi(x)| \leq c(\varphi, R) (t^\beta + |A(e^{tA} - I)x|)$$

This proves (ii) thanks to the strong continuity of  $(e^{At})_{t \geq 0}$ . Finally, (iv) is an obvious consequence of Proposition 3.5.  $\square$

**Remark 4.2** *We could prove that  $u$  is differentiable. This follows from a priori estimate on the modulus of continuity of  $Du_m$  and Ascoli–Arzelà theorem. This further a priori estimate is rather technical but its proof does not require new ideas.*

Assume that  $\varphi \in C_b^1(D(A); \mathbb{R})$ , the limit  $u$  of the subsequence  $(u_{m_k})$  constructed above may depend on the choice of  $(m_k)$ . Therefore, at this point it is not clear that this is any helpful to construct a transition semigroup  $P_t \varphi$  for  $\varphi \in B_b(D(A); \mathbb{R})$ . To avoid this problem, we shall use the fact that we know the existence of a martingale solution and of a stationary solution of (2.1). The stationary solution will provide a candidate for the invariant measure  $P_t$ .

In order to emphasize the dependence on the initial datum, we shall denote by  $u_m^\varphi$  the solution of (2.5).

To prove Theorem 2.4 we need further a priori estimates on  $X_m(t, x)$ . The following Lemma is proved exactly as Lemma 7.4 in [3].

**Lemma 4.3** *For any  $\delta \in (\frac{1}{2}, 1 + g]$ , there exists a constant  $c(\delta) > 0$  such that for any  $x \in H$ ,  $m \in \mathbb{N}$ , and  $t \in [0, T]$ :*

$$(i) \quad \mathbb{E}(|X_m(t, x)|^2) + \mathbb{E} \int_0^t |(-A)^{1/2} X_m(s, x)|^2 ds \leq |x|^2 + t \operatorname{Tr} Q.$$

$$(ii) \mathbb{E} \int_0^T \frac{|(-A)^{\frac{\delta+1}{2}} X_m(s, x)|^2}{(1 + |(-A)^{\frac{\delta}{2}} X_m(s, x)|^2)^{\gamma_\delta}} ds \leq c(\delta), \text{ with } \gamma_\delta = \frac{2}{2\delta-1} \text{ if } \delta \leq 1 \text{ and } \gamma_\delta = \frac{2\delta+1}{2\delta-1} \text{ if } \delta > 1.$$

It is well known that Lemma 4.3 can be used to prove that the family of laws  $(\mathcal{L}(X_m(\cdot, x)))_{m \in \mathbb{N}}$  is tight in  $L^2(0, T; D((-A)^{s/2}))$  for  $s < 1$  and in  $C([0, T]; D((-A)^{-\alpha})$  for  $\alpha > 0$ . Thus, by the Prokhorov theorem, it has a weakly convergent subsequence  $(\mathcal{L}(X_{m_k}(\cdot, x)))_{k \in \mathbb{N}}$ . We denote by  $\nu_x$  its limit. By the Skohorod theorem there exists a stochastic process  $X(\cdot, x)$  on a probability space  $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$  which belongs to  $L^2(0, T; D((-A)^{s/2}))$  for  $s < 1$  and in  $C([0, T]; D((-A)^{-\alpha})$  for  $\alpha > 0$ , satisfying (2.1) and such that for any  $x \in D(A)$

$$X_{m_k}(\cdot, x) \rightarrow X(\cdot, x), \quad \mathbb{P}_x \text{ a.s.}, \quad (4.2)$$

in  $L^2(0, T; D((-A)^{s/2}))$  and in  $C([0, T]; D((-A)^{-\alpha})$ .

Note that it is not straightforward to build a family of solution  $(X(t, x), (\Omega_x, \mathcal{F}_x, \mathbb{P}_x))_x$  which is Markov. Indeed, the sequence  $(m_k)_{k \in \mathbb{N}}$  in (4.2) may depend on  $x$ . The key idea is to use the following Lemma which states that the sequence  $(m_k)_k$  in Lemma 4.1 can be chosen independently of  $\varphi$ .

**Lemma 4.4** *There exists a sequence  $(m_k)_{k \in \mathbb{N}}$  such that for any  $\varphi$  in  $\mathcal{E}$  we have*

$$u_{m_k}^\varphi(t, x) \rightarrow u^\varphi(t, x), \quad \text{uniformly in } [\delta, T] \times K_R \text{ for any } \delta > 0, R > 0.$$

**Proof.** The proof is the same as for Lemma 7.5 in [3], we indicate the main ideas. Let  $D$  be a dense countable set of  $D(A)$ . It follows from a diagonal extraction argument that there exists a sequence  $(m_k)_k$  such that (4.2) holds for any  $x \in D$ . Then, it follows from Lemma 4.1-i) that, for any  $\varphi$  in  $\mathcal{E}$  and any subsequence of  $(u_{m_k}^\varphi)$ , we can extract a subsequence which converges to a continuous map  $u^\varphi$ . Moreover, it follows from Lemma 4.3-ii) that

$$X_{m_k}(\cdot, x) \rightarrow X(\cdot, x) \quad \text{in } D(A), \quad d\mathbb{P} \times dt \text{ a.s.}, \quad \text{for any } x \in D.$$

It follows that  $\mathbb{E}_x[\varphi(X(t, x))]$  is defined  $dt$  a.s. for  $x \in D$  and taking the limit in (2.6)

$$u^\varphi(t, x) = \mathbb{E}_x[\varphi(X(t, x))] \quad dt \text{ a.s.}, \quad \text{for any } x \in D. \quad (4.3)$$

Therefore, any two accumulation points of  $(u_{m_k}^\varphi)$  coincide on  $D$ . By continuity, there is only one and the whole sequence  $(u_{m_k}^\varphi)$  converges to  $u^\varphi$ . This ends the proof.  $\square$

We now fix the sequence  $(m_k)_{k \in \mathbb{N}}$  and define for  $\varphi \in \mathcal{E}$ :

$$P_t \varphi(x) = u^\varphi(t, x), \quad t \in [0, T], \quad x \in D(A).$$

As in [3], it is easily deduced that  $P_t^* \delta_x$  defines a unique probability measure on  $D(A)$ . Therefore we can define  $P_t \varphi(x)$  for  $\varphi \in B_b(D(A); \mathbb{R})$ .

Then, for any  $x \in D(A)$ , we build a martingale solution  $X(\cdot, x)$  by extracting a subsequence  $(m_k^x)_k$  of  $(m_k)_k$  such that (4.2) holds. It follows that

$$P_t \varphi(x) = \mathbb{E}_x[\varphi(X(t, x))], \quad x \in D(A), \quad t \in [0, T], \quad (4.4)$$

provided  $\varphi \in C_b(D((-A)^{-\alpha}); \mathbb{R}) \cap \mathcal{E}$ . It easily checked that (4.4) remains true for  $\varphi$  uniformly continuous in  $D((-A)^{-\alpha})$ . Thus  $P_t^* \delta_x$  - seen as a probability measure on  $D((-A)^{-\alpha})$  - is the law of  $X(t, x)$ . Since  $P_t^* \delta_x$  is a probability measure on  $D(A)$ , i) of Definition 2.3 follows. Moreover (4.4) remains true for  $\varphi \in B_b(D(A); \mathbb{R})$ .

Recall that for any subsequence of  $(m_k)_k$  and any  $x \in D(A)$ , we have a subsequence  $(m_k^x)_k$  and a martingale solution  $X(\cdot, x)$  such that  $X_{m_k^x}(\cdot, x) \rightarrow X(\cdot, x)$  in law in  $C(0, T; D((-A)^{-\alpha}))$ .

To end the proof of Theorem 2.4, we prove below the following result.

**Lemma 4.5** *Let  $X(\cdot, x)$  be a limit process of a subsequence of  $(X_{m_k}(\cdot, x))_{m_k}$ . Then, for any  $(n, N) \in \mathbb{N}^2$ ,  $t_1, \dots, t_n \geq 0$  and  $(f_k)_{k=0}^n \in C_c^\infty(P_N H)$  (i.e.  $f_k(x) = f_k(P_N x)$ ), we have*

$$\begin{aligned} \mathbb{E}_x (f_0(X(0, x))f_1(X(t_1, x)), \dots, f_n(X(t_1 + \dots + t_n, x))) \\ = f_0(x)P_{t_1} \left[ f_1 P_{t_2} \left( f_2 P_{t_3} (f_3 \dots) \right) \right] (x). \end{aligned} \quad (4.5)$$

By classical arguments, uniqueness in law of the limit process follows from (4.5). Combining uniqueness of the limit and compactness of the sequence  $(X_{m_k}(\cdot, x))_k$ , we obtain that  $X_{m_k}(\cdot, x) \rightarrow X(\cdot, x)$  in law in  $C(0, T; D((-A)^{-\alpha}))$ . It follows that the map  $x \rightarrow \mathcal{P}_x$  defined in Definition 2.3 depends measurably on  $x$ , and that  $\mathcal{P}_{X(t, x)}(\mathcal{A})$  is a random variable for any  $\mathcal{A}$  as in Definition 2.3 (ii).

We set, for  $t_1, \dots, t_k, s_1, \dots, s_n \geq 0$  and  $A_0, \dots, A_k, A'_0, \dots, A'_n \in \mathcal{B}(D(A))$ ,

$$\begin{cases} \mathcal{A} &= \{X(0) \in A_0, \dots, X(t_1 + \dots + t_k) \in A_k\}, \\ \mathcal{A}' &= \{X(0) \in A'_0, \dots, X(s_1 + \dots + s_n) \in A'_n\}. \end{cases}$$

Since (4.5) is easily extended to any  $(f_0, \dots, f_n) \in (B_b(D(A); \mathbb{R}))^{n+1}$ , we deduce

$$\mathcal{P}_x(\mathcal{A}) = 1_{A_0}(x)P_{t_1} \left[ 1_{A_1} P_{t_2} \left( 1_{A_2} P_{t_3} (1_{A_3} \dots) \right) \right] (x).$$

Applying successively (4.5) to  $(1_{A'_0}, \dots, 1_{A'_n}, x \mapsto \mathcal{P}_x(\mathcal{A}'))$ ,  $s_1, \dots, s_n, t - (s_1 + \dots + s_n)$  and to  $(1_{A'_0}, \dots, 1_{A'_n}, 1_{A_0}, \dots, 1_{A_k})$ ,  $s_1, \dots, s_n, t - (s_1 + \dots + s_n), t_1, \dots, t_n$ , we obtain

$$\mathbb{P}_x(X(\cdot, x) \in \mathcal{A}', X(t + \cdot, x) \in \mathcal{A}) = \mathbb{E}_x(1_{\mathcal{A}'}(X(\cdot, x))\mathcal{P}_{X(t, x)}(\mathcal{A})). \quad (4.6)$$

provided  $t \geq s_1 + \dots + s_n$ . This yields (ii) of Definition 2.3.

**Proof of Lemma 4.5.** We set  $f(x_0, \dots, x_n) = f_0(x_0) \dots f_n(x_n)$ . It follows from the weak Markov property that

$$\mathbb{E}_x^{m_k^x} (f(X_{m_k^x}(0, x), \dots, X_{m_k^x}(t_1 + \dots + t_n, x))) = f_0(x)P_{t_1}^{m_k^x} \left[ f_1 P_{t_2}^{m_k^x} \left( f_2 P_{t_3}^{m_k^x} (f_3 \dots) \right) \right] (x). \quad (4.7)$$

Moreover, the convergence in  $C(0, T; D((-A)^{-\alpha}))$  gives

$$\mathbb{E}_x (f(X(0, x), \dots, X(t_1 + \dots + t_n, x))) = \lim \mathbb{E}_x^{m_k^x} (f(X_{m_k^x}(0, x), \dots)). \quad (4.8)$$

Therefore it remains to prove that

$$f_0(x)P_{t_1}^{m_k} \left[ f_1 P_{t_2}^{m_k} \left( f_2 P_{t_3}^{m_k} (f_3 \dots) \right) \right] (x) \rightarrow f_0(x)P_{t_1} \left[ f_1 P_{t_2} \left( f_2 P_{t_3} \dots \right) \right] (x), \quad (4.9)$$

uniformly on  $D(A)$ .

We prove (4.9) by induction on  $n \in \mathbb{N}$ . For  $n = 0$ , this is trivial. For  $n = 1$ , this is Lemma 4.4. Assume that (4.9) is true for  $n \in \mathbb{N}$ . We set

$$\begin{cases} I_m(x) &= f_0(x)P_{t_1}^m \left[ f_1 P_{t_2}^m \left( f_2 P_{t_3}^m (f_3 \dots) \right) \right] (x) - f_0(x)P_{t_1} \left[ f_1 P_{t_2} \left( f_2 \dots \right) \right] (x), \\ J_m(x) &= f_1(x)P_{t_2}^m \left[ f_2 P_{t_3}^m \left( f_3 P_{t_4}^m (f_4 \dots) \right) \right] (x) - f_1(x)P_{t_2} \left[ f_2 P_{t_3} \left( f_3 \dots \right) \right] (x). \end{cases}$$

Remark that

$$I_{m_k}(x) = f_0(x) \left( P_{t_1}^{m_k} J_{m_k}(x) + (P_{t_1}^{m_k} - P_{t_1}) \left[ f_1 P_{t_2} \left( f_2 P_{t_3} \dots \right) \right] (x) \right). \quad (4.10)$$

By the induction argument,  $J_{m_k}(x) \rightarrow 0$  uniformly. Hence

$$\|f_0 P_{t_1}^{m_k} J_{m_k}\|_0 \leq \|f_0\|_0 \|J_{m_k}\|_0 \rightarrow 0.$$

Moreover, since  $f_0 = 0$  out of a bounded set, then it follows from Lemma 4.4

$$f_0(x)(P_{t_1}^{m_k} - P_{t_1}) \left[ f_1 P_{t_2} \left( f_2 P_{t_3} \dots \right) \right] (x) \rightarrow 0,$$

uniformly on  $D(A)$ , which yields (4.10).  $\square$

## 5 Proof of Theorem 2.5

Now we observe that, since the noise is nondegenerate, then  $P_t^{m_k}$  has a unique invariant measure  $\nu_{m_k}$ . Moreover, we have the following result proved as in [3] (see Lemma 7.6).

**Lemma 5.1** *There exists a constant  $C_1$  such that for any  $k \in \mathbb{N}$*

$$\int_H \left[ |(-A)^{1/2} x|^2 + |Ax|^{2/3} + |(-A)^{1+\frac{g}{2}} x|^{\frac{1+2g}{10+8g}} \right] \nu_{m_k}(dx) < C_1.$$

It follows that the sequence  $(\nu_{m_k})_{k \in \mathbb{N}}$  is tight on  $D(A)$  and there exists a subsequence, which we still denote by  $(\nu_{m_k})_{k \in \mathbb{N}}$ , and a measure  $\nu$  on  $D(A)$  such that  $\nu_{m_k}$  converges weakly to  $\nu$ . Moreover  $\nu(D((-A)^{1+\frac{g}{2}})) = 1$ .

Let us take  $\varphi \in \mathcal{E}$ . It follows from the invariance of  $\nu_m$  and the convergence properties of the approximations of  $P_t \varphi$  that for any  $t \geq 0$

$$\int_H P_t \varphi(x) \nu(dx) = \int_H \varphi(x) \nu(dx). \quad (5.1)$$

Therefore  $\nu$  is an invariant measure. The strong Feller property is a consequence of Proposition 3.5. Hence, by Doob Theorem, the strong mixing property is a consequence of the irreducibility. This latter property is implied by the following Lemma. Its proof significantly differs from the additive case treated in [3].

**Lemma 5.2** *Let  $x_0 \in D(A)$ ,  $\varepsilon > 0$  and  $\varphi \in \mathcal{E}$  be such that  $\varphi(x) = 1$  for  $x$  in  $B_{D(A)}(x_0, \varepsilon)$ , the ball in  $D(A)$  of center  $x_0$  and radius  $\varepsilon$ . Then for any  $t > 0$  and  $x \in D(A)$  we have  $P_t \varphi(x) > 0$ .*

**Proof.** It is classical that, since  $x \in D(A)$ , there exists  $T^* > 0$  and  $\bar{x} \in C([0, T^*]; D(A))$  such that

$$\bar{x}(t) = e^{tA}x + \int_0^t e^{(t-s)A}b(\bar{x}(s))ds, \quad t \in [0, T^*].$$

This follows from a fixed point argument. Moreover, it is not difficult to see that  $\bar{x} \in L^2(0, T^*; D((-A)^{3/2}))$ , so that  $\bar{x}(t) \in D((-A)^{3/2})$  a.e. and we may change  $T^*$  so that  $\bar{x}(T^*) \in D((-A)^{3/2})$ . Moreover, we can assume that  $x_0 \in D((-A)^{3/2})$ . Then we set  $\bar{g} = 0$  on  $[0, T^*]$  and define  $\bar{x}, \bar{g}$  on  $[T^*, T]$  by

$$\bar{x}(t) = \frac{T-t}{T-T^*} \bar{x}(T^*) + \frac{t-T^*}{T-T^*} x_0, \quad \bar{g}(t) = \frac{d\bar{x}}{dt} - A\bar{x} - b(\bar{x}).$$

We also define

$$R = 2 \sup_{t \in [0, T]} |A\bar{x}(t)|, \quad b_R(x) = \vartheta \left( \frac{|Ax|}{R} \right) b(x),$$

where  $\vartheta \in C_0^\infty(\mathbb{R})$  takes its values in  $[0, 1]$ , is equal to 1 on  $[0, 1]$  and vanishes on  $[2, \infty)$ . Then, for  $n \in \mathbb{N}$ , we write

$$\Delta t = T/n, \quad t_k = k\Delta t, \quad \text{for } k \in \mathbb{N}, \quad \text{and } \dot{W}_n(t) = \frac{P_n W(t_k) - P_n W(t_{k-1})}{\Delta t}, \quad t \in [t_k, t_{k+1}).$$

It is not difficult to see that the equation

$$\begin{cases} dX^{n,R} = (AX^{n,R} + b_R(X^{n,R}) + P_n \bar{g} - \Phi_n(e^{A(t-t_{k-1})} X^{n,R}(t_{k-1})) \dot{W}_n) dt \\ \quad + \Phi(X^{n,R}) dW, \quad t \in [t_k, t_{k+1}), \\ X^{n,R}(0) = x \end{cases}, \quad (5.2)$$

has a unique solution  $X^{n,R}$  in  $C([0, T]; D(A))$ . We set

$$G^n(t) = \Phi(X^{n,R}(t))^{-1} \left( P_n \bar{g}(t) - \Phi_n(e^{A(t-t_{k-1})} X^{n,R}(t_{k-1})) \dot{W}_n(t) \right), \quad t \in [t_k, t_{k+1}).$$

By Girsanov Theorem  $\widetilde{W}_n(t) = W(t) - \int_0^t G^n(s) ds$  defines a cylindrical Wiener process for the probability measure

$$d\mathbb{P}_n = \exp \left[ \int_0^T (G^n(s), dW(s)) - \frac{1}{2} \int_0^T |G^n(s)|^2 ds \right] d\mathbb{P}.$$

It follows that the laws of  $X^{n,R}$  and  $X^R$  are equivalent, where  $X^R$  is the solution of

$$\begin{cases} dX^R = (AX^R + b_R(X^R)) dt + \Phi(X^R) dW, \quad t \in [t_k, t_{k+1}), \\ X^R(0) = x. \end{cases} \quad (5.3)$$

Moreover, we have, for a subsequence  $m_k$ ,

$$\begin{aligned} P_t \varphi(x) &= \lim_{m_k \rightarrow \infty} \mathbb{E}(\varphi(X_{m_k}(t, x))) \\ &\geq \lim_{m_k \rightarrow \infty} \mathbb{E}(\varphi(X_{m_k}(t, x)) \mathbf{1}_{\tau_{m_k}^R \geq T}) \\ &\geq \lim_{m_k \rightarrow \infty} \mathbb{E}(\varphi(X_{m_k}^R(t, x)) \mathbf{1}_{\tilde{\tau}_{m_k}^R \geq T}) \end{aligned}$$

where  $\tau_{m_k}^R = \inf\{t \in [0, T], |AX_{m_k}(t, x)| \geq R\}$ ,  $\tilde{\tau}_{m_k}^R = \inf\{t \in [0, T], |AX_{m_k}^R(t, x)| \geq R\}$ , and  $X_{m_k}^R$  is the solution of the Galerkin approximation where  $b$  has been replaced by  $b_R$ . Since, it easy to check that

$$\lim_{m_k \rightarrow \infty} \mathbb{E}(\varphi(X_{m_k}^R(t, x)) \mathbf{1}_{\tilde{\tau}_{m_k}^R \geq T}) = \mathbb{E}(\varphi(X^R(t, x)) \mathbf{1}_{\tau^R \geq T})$$

where  $\tau^R = \inf\{t \in [0, T], |AX^R(t, x)| \geq R\}$ , we deduce that it is sufficient to prove that  $\mathbb{E}(\varphi(X^R(t, x)) \mathbf{1}_{\tau^R \geq T}) > 0$  and, since the laws of  $X^{n,R}$  and  $X^R$  are equivalent, that  $\mathbb{E}(\varphi(X^{n,R}(t, x)) \mathbf{1}_{\tau^{n,R} \geq T}) > 0$  with  $\tau^{n,R} = \inf\{t \in [0, T], |AX^{n,R}(t, x)| \geq R\}$ . We prove below that  $X^{n,R}$  converges to  $\bar{x}$  in  $L^2(\Omega; C([0, T]; D(A)))$  as  $n \rightarrow \infty$ . Since  $\varphi(\bar{x}(T)) = 1$ , the claim follows.

To prove that  $X^{n,R}$  converges to  $\bar{x}$ , we first observe that  $X^{n,R}$  is uniformly bounded with respect to  $n$  in  $L^2(\Omega; C([0, T]; D((-A)^\gamma)))$  for  $\gamma < 1 + g/2$ . This is proved thanks to the integral form of (5.2) and similar argument as in Proposition 3.1. This enables to control the difference between  $\Phi_n(e^{A(t-t_{k-1})} X^{n,R}(t_{k-1}))$  and  $\Phi(X^{n,R}(t))$ , for  $t \in [t_k, t_{k+1}]$ .

Then, we write the difference between the integral equation satisfied by  $X^{n,R}$  and  $\bar{x}$  and see that it is sufficient to prove that  $\int_0^t e^{A(t-s)} \Phi_n(e^{A(s-s_{k-1})} X^{n,R}(s_{k-1})) \dot{W}_n ds - \int_0^t e^{A(t-s)} \Phi(X^{n,R}(s)) dW(s)$  goes to zero in  $L^2(\Omega; C([0, T]; D(A)))$ , where  $s_{k_1} = t_{k-1}$  for  $s \in [t_k, t_{k+1}]$ . This latter point is not difficult to prove.  $\square$

Let  $(X_{m_k}(\cdot, \nu_{m_k}))_k$  be the sequence of stationary solutions. Proceeding as in the end of section 4, we remark that to prove the convergence, it is sufficient to establish uniqueness in law of the limit of subsequence of  $(X_{m_k}(\cdot, \nu_{m_k}))_k$ . Remark that Theorem 2.5-(ii) implies such uniqueness.

So, to conclude Theorem 2.5, it remains to establish (ii). By classical arguments, it follows from Lemma 4.5 that it is sufficient to establish that

$$\begin{aligned} &\mathbb{E}_\nu(f_0(X(0, \nu)) \dots f_n(X(t_1 + \dots + t_n, \nu))) \\ &= \int_{D(A)} f_0(x) P_{t_1} \left[ f_1 P_{t_2} \left( f_2 P_{t_3} (f_3 \dots) \right) \right] (x) \nu(dx). \end{aligned} \quad (5.4)$$

for any  $(t_1, \dots, t_n)$  and  $(f_0, \dots, f_n)$  as in Lemma 4.5.

Remark that

$$\begin{aligned} &\mathbb{E}_{\nu_{m_k}}(f_0(X_{m_k}(0, \nu_{m_k})) \dots f_n(X_{m_k}(t_1 + \dots + t_n, \nu_{m_k}))) \\ &= \int_{D(A)} f_0(x) P_{t_1}^{m_k} \left[ f_1 P_{t_2}^{m_k} \left( f_2 P_{t_3}^{m_k} (f_3 \dots) \right) \right] (x) \nu_{m_k}(dx). \end{aligned}$$

The convergence of the right-hand-side comes from the convergence in law in  $C(0, T; D((-A)^{-\alpha}))$ . Applying (4.9), we can take the limit in the left-hand-side and then conclude.

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