

# ERGODICITY FOR THE WEAKLY DAMPED STOCHASTIC NON-LINEAR SCHRÖDINGER EQUATIONS

ARNAUD DEBUSSCHE    AND    CYRIL ODASSO

ECOLE NORMALE SUPÉRIEURE DE CACHAN, ANTENNE DE BRETAGNE,  
AVENUE ROBERT SCHUMAN, CAMPUS DE KER LANN, 35170 BRUZ (FRANCE).  
AND  
IRMAR, UMR 6625 DU CNRS, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX  
(FRANCE)

**Abstract:** We study a damped stochastic non-linear Schrödinger (NLS) equation driven by an additive noise. It is white in time and smooth in space. Using a coupling method, we establish convergence of the Markov transition semi-group toward a unique invariant probability measure. This kind of method was originally developed to prove exponential mixing for strongly dissipative equations such as the Navier-Stokes equations. We consider here a weakly dissipative equation, the damped nonlinear Schrödinger equation in the one-dimensional cubic case. We prove that the mixing property holds and that the rate of convergence to equilibrium is at least polynomial of any power.

**Key words:** Non-linear Schrödinger equations, Markov transition semi-group, invariant measure, ergodicity, coupling method, Girsanov's formula, expectational Foias–Prodi estimate.

## INTRODUCTION

The non-linear Schrödinger (NLS) equation models the propagation of non-linear dispersive waves in various areas of physics such as hydrodynamics [24], [25], optics, plasma physics, chemical reaction [16]...

When studying the propagation in random media, a noise can be introduced. For instance in [9], [10], the cubic nonlinear Schrödinger equation with additive white noise and damping is introduced. There, the propagation of waves over very long distance is studied. Damping effect cannot be neglected in this case and has to be counterbalanced by amplifiers. The white noise is a model for the description of the randomness in these amplifiers. Such model is valid if the distance between amplifiers is small compared to propagation distance.

Our aim in this work is to study ergodicity for this type of equation. We consider the one-dimensional case with cubic focusing nonlinearity. It has the form

$$(0.1) \quad \begin{cases} du + \alpha u dt - i\Delta u dt - i|u|^2 u dt & = bdW, \\ u(t, x) & = 0, & \text{for } x \in \{0, 1\}, t > 0, \\ u(0, x) & = u_0(x), & \text{for } x \in [0, 1], \end{cases}$$

where  $\alpha > 0$ . The unknown  $u$  is a complex valued process depending on  $x \in [0, 1]$  and  $t \geq 0$ . Dirichlet boundary conditions are considered but we could also use Neumann or periodic boundary condition.

Existence and uniqueness of solutions for (0.1) is not very difficult to prove using straightforward generalization of deterministic arguments. Note that the damping term is necessary to have an invariant measure. Indeed, if  $\alpha = 0$  and  $b \neq 0$  then the  $L^2(0, 1)$  norm grows linearly in time.

The Complex Ginzburg-Landau (CGL) is also a form of dissipative NLS equation. The exponential mixing of the stochastic CGL equation has been established in [14] in a particular case and in [26] in the general case. The method was inspired by the so-called coupling method. This method has been introduced in [3], [14], [19], [20], [21], [23] and [27]. In all these articles, a strongly dissipative stochastic partial differential equation driven by a noise which may be degenerate is considered. Due to the possible degeneracy of the noise Doob Theorem cannot be applied (see [5] for the theory of ergodicity when Doob Theorem can be applied). Indeed, it requires the strong Feller property, which can be proved only when the noise lives in a space of spatially irregular functions, which is clearly not true for a degenerate noise. The main idea is to compensate the degeneracy of the noise by dissipativity arguments, the so-called Foias-Prodi estimates. Roughly speaking, the process can be decomposed into the sum of a strongly dissipative process and another one driven by a non-degenerate noise. The strongly dissipative part is treated as in [4] section 11.5, while the non-degenerate part is treated thanks to probabilistic arguments. The difficulty is of course in the fact that the two parts of the process do not evolve independently so that the two methods have to be used simultaneously. The probabilistic part can be treated either by a generalization of Doob Theorem (see [8], [15], [18]) or by coupling argument (see [19], [20], [21], [23]). Each method has its advantages. The first one allows treating very degenerate noises while the coupling method proves also exponential convergence to equilibrium.

In the case of the NLS equation, it seems hopeless to use Doob Theorem. Indeed, due to the lack of smoothing effect of the deterministic part of equations, only spatially smooth noises can be treated (see [6], [7]). Note that this equation is not strongly dissipative, indeed the eigenvalues of the linear part do not grow to infinity. However, it is known that Foias-Prodi type estimates hold for the deterministic damped NLS equation (see [13]) and we will see that these can be generalized to the stochastic case and it is reasonable to think that the above ideas can be generalized.

In this article, we show that the method based on coupling argument is applicable. However it requires substantial adaptations. For instance, contrary to the strongly dissipative case treated in the above-mentioned articles, we are only able to prove a weaker form of the Foias-Prodi estimates. Indeed, here, we prove that it holds in average and not path-wise. This causes many technical difficulties when trying to use the coupling method. Moreover, another important ingredient in the argument is an exponential estimate on the growth of the solution, which we are unable to prove in our case. This is due to the fact that the Lyapunov structure is more complicated here. It is not a quadratic functional. We only prove polynomial estimate on the growth of the solutions and it results that we can only prove that convergence to equilibrium holds with polynomial speed at any order. Thus, we develop a general result, which gives sufficient conditions for polynomial mixing.

Note also that a crucial step in [21] is the fact that the probability that a solution enters a ball of small radius is controlled precisely. This fact is still true for the

damped NLS equation considered here. However, its proof is more difficult than in the case of the Navier-Stokes equations (see Proposition 2.6 and section 4 hereafter).

The remaining of the article is divided into four parts. First, we give the notations, and state our main result. Its proof is given in section 2. Section 3, 4 and 5 are devoted to the proofs of intermediate results.

## 1. NOTATION AND MAIN RESULT

We set

$$A = -\Delta, \quad D(A) = H_0^1(0, 1) \cap H^2(0, 1)$$

and write problem (0.1) in the form

$$(1.1) \quad du + \alpha u dt + iAu dt - i|u|^2 u dt = bdW,$$

$$(1.2) \quad u(0) = u_0,$$

where  $W$  is a cylindrical Wiener process on  $L^2(0, 1)$  and  $b$  is a linear operator on  $L^2(0, 1)$ .

We denote by  $(\mu_n)_n$  the increasing sequence of eigenvalues of  $A$  and by  $(e_n)_{n \in \mathbb{N}}$  the associated eigenvectors. Also,  $P_N$  and  $Q_N$  are the eigenprojectors onto the space  $Sp(e_k)_{1 \leq k \leq n}$  and onto its complementary space. Recall that for  $s \geq 0$ ,  $D(A^{s/2})$  is a closed subspace of  $H^s(0, 1)$  and that  $\|\cdot\|_s = |A^{s/2} \cdot|_{L^2(0,1)}$  is equivalent to the usual  $H^s(0, 1)$  norm on this space. Moreover

$$D(A^{s/2}) = \left\{ u = \sum_{k \in \mathbb{N}} u_k e_k \in L^2(0, 1) \mid \sum_{n \in \mathbb{N}} \mu_k^s u_k^2 < \infty \right\} \text{ and } \|u\|_s = \sum_{n \in \mathbb{N}} \mu_k^s u_k^2.$$

We denote by  $|\cdot|$ ,  $|\cdot|_p$ ,  $\|\cdot\|$  the norms of  $L^2(0, 1)$ ,  $L^p(0, 1)$ ,  $H_0^1(0, 1)$ .

The operator  $b$  is supposed to commute with  $A$ , therefore it is diagonal in the basis  $(e_n)_{n \in \mathbb{N}}$  and we have

$$be_n = b_n e_n.$$

We assume that  $b$  is Hilbert-Schmidt from  $L^2(0, 1)$  with values in  $D(A^{3/2})$ . For any  $s \in [0, 3]$ , we set

$$B_s = |b|_{\mathcal{L}_2(L^2(0,1), D(A^{s/2}))}^2 = \sum_{n=0}^{\infty} \mu_n^s b_n^2.$$

To study ergodic properties, we assume that there exists  $N_*$  such that

$$(1.3) \quad b_n > 0, \text{ for } n \leq N_*.$$

The Hamiltonian plays an important role in the study of the nonlinear Schrödinger equation. It is a conserved quantity in the absence of noise and damping. It is given by

$$\mathcal{H}_*(v) = \frac{1}{2} \|v\|^2 - \frac{1}{4} |v|_4^4, \quad v \in H_0^1(0, 1).$$

In our study, it is the basic tool to derive a priori estimates. Recall that the Gagliardo-Nirenberg inequality gives a constant  $c_0 > 0$  such that

$$|v|_4^4 \leq \frac{1}{4} \|v\|^2 + \frac{c_0}{2} |v|_6^6, \quad v \in H_0^1(0, 1).$$

It follows that, setting

$$\mathcal{H} = \frac{1}{2} \|\cdot\|^2 - \frac{1}{4} |\cdot|_4^4 + c_0 |\cdot|_6^6,$$

we have

$$(1.4) \quad \mathcal{H}(v) \geq \frac{1}{4} \|v\|^2 + \frac{1}{4} |v|_4^4 + \frac{c_0}{2} |v|^6, \quad v \in H_0^1(0, 1).$$

In our computations, we will also use the following quantities which involve the  $k^{\text{th}}$  power of the energy:

$$E_{u,k}(t, s) = \mathcal{H}(u(t))^k + \alpha k \int_s^t \mathcal{H}(u(\sigma))^k d\sigma, \quad t \geq s,$$

when there is no ambiguity we set  $E_{u,k}(t) = E_{u,k}(t, s)$ .

In the following,  $\alpha, B_s$  for  $s \in [0, 3]$  are fixed. All the constants appearing below may depend on them as well as on  $A, b$ .

Well-posedness of equations (1.1), (1.2) is easily proved. Indeed, let  $S(t) = e^{-iAt - \alpha t}$ ,  $t \in \mathbb{R}$ , be the group generated by the linear equation. We look for a mild solution, that is a process  $u$  with paths in  $C(\mathbb{R}^+; H_0^1(0, 1))$  satisfying

$$u(t) = S(t)u_0 + i \int_0^t S(t-s) |u(s)|^2 u(s) ds + \int_0^t S(t-s) b dW(s).$$

Since  $(S(t))_{t \geq 0}$  is a contraction semi-group on  $H_0^1(0, 1)$  and the linear term is locally Lipschitz, local in time existence and uniqueness is straightforward. Note that  $\int_0^t S(t-s) b dW(s)$  lives in  $D(A^{\frac{3}{2}})$ . An a priori estimate is obtained thanks to Ito formula applied to  $\mathcal{H}$  and thanks to (1.4). This use of Ito formula is not rigorous since  $Au$  is not sufficiently smooth. However, an approximation argument can be used to prove rigorously this point. For instance, the initial data can be approximated by a sequence in  $D(A)$  and it is classical that if the initial data is in  $D(A)$  then the solution is continuous with values in  $D(A)$ .

Note that in the following and especially in section 4 and 5, several computations are not rigorous due to the lack of regularity of the solutions. The same approximation argument should be applied.

By classical arguments, the solutions are strong Markov processes. We denote by  $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$  the Markov transition semi-group associated to the solutions of (1.1).

Also, given a Banach space  $E$ , the space  $Lip_b(E)$  consists of all the bounded and Lipschitz real valued functions on  $E$ . Its norm is given by

$$\|\varphi\|_L = \|\varphi\|_\infty + L_\varphi, \quad \varphi \in Lip_b(E),$$

where  $\|\cdot\|_\infty$  is the sup norm and  $L_\varphi$  is the Lipschitz constant of  $\varphi$ . The space of probability measures on  $E$  is denoted by  $\mathcal{P}(E)$ . It can be endowed with the metric defined by the total variation

$$\|\mu\|_{var} = \sup \{ |\mu(\Gamma)| \mid \Gamma \in \mathcal{B}(E) \},$$

where we denote by  $\mathcal{B}(E)$  the set of the Borelian subsets of  $E$ . It is well known that  $\|\cdot\|_{var}$  is the dual norm of  $|\cdot|_\infty$ . We can also use a Wasserstein type metric

$$\|\mu - \lambda\|_W = \sup_{\varphi \in Lip_b(E), \|\varphi\|_L \leq 1} \left| \int_E \varphi(u) d(\mu - \lambda)(u) \right|$$

which is the dual norm of  $\|\cdot\|_L$ . We also use the notation  $\mathcal{D}(Z)$  for the distribution of a random variable  $Z$ .

The aim of this article is to establish the following result

**Theorem 1.1.** *There exists  $N_0$  such that, if (1.3) holds with  $N_* \geq N_0$ , then there exists a unique stationary probability measure  $\nu$  of  $(\mathcal{P}_t)_{t \in \mathbb{R}^+}$  on  $H_0^1(0, 1)$ . Moreover, for any  $p \in \mathbb{N} \setminus \{0\}$ ,  $\nu$  satisfies*

$$(1.5) \quad \int_{H_0^1(0,1)} \|u\|^{2p} d\nu(u) < \infty,$$

and there exists  $C_p > 0$  such that for any  $\mu \in \mathcal{P}(H_0^1(0, 1))$

$$(1.6) \quad \|\mathcal{P}_t^* \mu - \nu\|_W \leq C_p (1+t)^{-p} \left( 1 + \int_{H_0^1(0,1)} \|u\|^2 d\mu(u) \right).$$

**Remark 1.2.** *Note that the existence of a stationary measure is a byproduct of the proof of the mixing property. It could be proved directly by the standard argument involving the Krylov-Boboliubov theorem. However, this would require more a priori estimates on the solutions of the stochastic nonlinear Schrödinger equation.*

**Remark 1.3.** *In many articles and books, a family  $(W_p(\cdot, \cdot))_{p \in [1, \infty)}$  of Wasserstein type metrics is used. Actually, given a polish space  $(E, d_E)$ , these metrics are defined by*

$$W_p(\mu, \lambda) = \inf \left( \int_{E^2} d_E(x, y)^p \mathbb{P}(dx, dy) \right)^{\frac{1}{p}} \quad \text{for any } p \in [1, \infty),$$

where the infimum is taken over all probability measures  $\mathbb{P}$  on  $E^2$  whose marginal laws are  $\lambda$  and  $\mu$ .

Let  $(E, \|\cdot\|_E)$  be a separable Banach space. We set  $d_E(x, y) = \|x - y\|_E \wedge 1$ . Then  $(E, d_E)$  is a polish space. Moreover,  $W_1$  is equivalent to  $\|\cdot\|_W$ . We have chosen not to use the notation  $W_1$  because it might lead to some confusion with the usual notation  $W$  for a Wiener process.

The proof of our result is based on coupling arguments. These arguments have initially been used in the context of stochastic partial differential equations in [19], [20], [21], [23]. The main difficulty here is that the nonlinear Schrödinger equation is not strongly dissipative and several modifications are needed.

The strategy is the following. If the noise is non-degenerate, we observe that starting from different initial data  $u_0^1, u_0^2$ , Girsanov transform can be used to show that there exists a coupling  $(u_1, u_2)$  of the law of the solutions  $u(\cdot, u_0^1), u(\cdot, u_0^2)$  such that, for some time  $T$ ,  $u_1(T) = u_2(T)$  with positive probability. Iterating this argument, exponential convergence to equilibrium follows (see section 1.1 in [26]). Here, as well as in the references above, the noise is assumed to be non-degenerate in the low modes only  $e_k$ ,  $1 \leq k \leq N_*$  and this argument gives a coupling such that  $P_{N_*} u_1(T) = P_{N_*} u_2(T)$  with positive probability. Another ingredient is used. It is based on the observation that if two solutions are such that their low modes have been equal for a long time then they are very close (see section 1.1 in [26]). In the case of a parabolic equation, this is known as Foias-Prodi estimate. This can be generalized to dispersive equations such as the Schrödinger equation considered here. In [13] this has been used to prove a property of asymptotic smoothing in the deterministic case.

The main difference with the result in the parabolic case is that we are not able to prove a path-wise Foias-Prodi estimate, we only prove that this property holds in average. We need to introduce a substantial change in the construction of the coupling. (See Remark 2.12). Moreover, here we only get polynomial convergence

to equilibrium. This comes from the fact that the Lyapunov functional adapted to the nonlinear Schrödinger equation is more complicated, it is not a quadratic functional. We are not able to get exponential estimates on the growth of the solutions.

## 2. PROOF OF THEOREM 1.1

We define  $G$  by

$$D(G) = D(A), \quad Gv = \alpha v + iAv,$$

and set

$$\begin{aligned} X &= P_{N_*} u, \quad Y = Q_{N_*} u, \quad \beta = P_{N_*} W, \quad \eta = Q_{N_*} W, \\ \sigma_l &= P_{N_*} b P_{N_*}, \quad \sigma_h = Q_{N_*} b Q_{N_*}, \\ f(X, Y) &= -i P_{N_*} \left( |X + Y|^2 (X + Y) \right), \\ g(X, Y) &= -i Q_{N_*} \left( |X + Y|^2 (X + Y) \right). \end{aligned}$$

Then the nonlinear Schrödinger equation has the form

$$(2.1) \quad \begin{cases} dX + GX dt + f(X, Y) dt &= \sigma_l d\beta, \\ dY + GY dt + g(X, Y) dt &= \sigma_h d\eta, \\ X(0) = x_0, \quad Y(0) = y_0. \end{cases}$$

Clearly (1.3) states that  $\sigma_l$  is invertible. We set

$$(2.2) \quad \sigma_0 = \|\sigma_l^{-1}\|_{\mathcal{L}(P_{N_*} L^2(0,1))}^{-1} > 0.$$

Given two initial data  $u_0^i = (x_0^i, y_0^i)$ ,  $i = 1, 2$ , we will construct a coupling  $(u_1, u_2) = ((X_1, Y_1), (X_2, Y_2))$  of the laws of the two solutions  $u(\cdot, u_0^i) = (X(\cdot, u_0^i), Y(\cdot, u_0^i))$ ,  $i = 1, 2$ , of (2.1). Recall that  $(u_1, u_2)$  is a coupling of the laws of  $u(\cdot, u_0^i)$ ,  $i = 1, 2$ , if the distribution of  $u_i$  is the distribution of  $u(\cdot, u_0^i)$ .

In fact we are going to construct a coupling  $(V_1, V_2) = ((u_1, W_1), (u_2, W_2))$  of  $\{\mathcal{D}((u(\cdot, u_0^i), W))\}_{i=1,2}$  such that  $X_i = P_{N_*} u_i$ ,  $\eta_i = Q_{N_*} W_i$  satisfy good properties. More precisely, we want that  $X_1 = X_2$  and  $\eta_1 = \eta_2$  for as many trajectories as possible. Clearly, we obtain a coupling of  $\mathcal{D}(u(\cdot, u_0^1))$  and  $\mathcal{D}(u(\cdot, u_0^2))$ . Since the noise may degenerate in the equation for  $Y$ , we are not able to require that  $u_1 = u_2$ . The difference between  $Y_1 = Q_{N_*} u_1$  and  $Y_2 = Q_{N_*} u_2$  will be controlled thanks to a Foias-Prodi estimate. Note that  $W$  is a cylindrical process in  $L^2(0, 1)$  and does not live in  $L^2(0, 1)$ . This is not a problem. Indeed, it is well-known that  $W \in C(\mathbb{R}^+; H^{-1}(0, 1))$  a.s. and we consider its distribution in this space.

We define an integer valued random process  $l_0$  which is particularly convenient when deriving properties of the coupling:

$$l_0(k) = \min \{ l \in \{0, \dots, k\} \mid P_{l,k} \text{ holds} \},$$

where  $\min \emptyset = \infty$  and

$$(P_{l,k}) \left\{ \begin{array}{l} X_1(t) = X_2(t), \quad \eta_1(t) = \eta_2(t), \quad \forall t \in [lT, kT], \\ \mathcal{H}_l \leq d_0, \\ E_{u_i,4}(t, lT) \leq \kappa + 1 + d_0^4 + d_0^8 + B(t - lT), \quad \forall t \in [lT, kT], \quad i = 1, 2, \end{array} \right.$$

where we have set

$$\mathcal{H}_k = \mathcal{H}(u_1(kT)) + \mathcal{H}(u_2(kT)).$$

We say that  $(X_1, X_2)$  are coupled at  $kT$  if  $l_0(k) \leq k$ , in other words if  $l_0(k) \neq \infty$ .

The coupling constructed below will be such that, for any  $q \in \mathbb{N} \setminus \{0\}$ , the following two properties hold

$$(2.3) \quad \left\{ \begin{array}{l} \exists d_0, \kappa, B, T_q > 0 \text{ such that for any } l \leq k, T \geq T_q, \\ \mathbb{P}(l_0(k+1) \neq l \mid l_0(k) = l) \leq \frac{1}{2} (1 + (k-l)T)^{-q}. \end{array} \right.$$

This says that the probability that the trajectories decouple is small. Moreover, the longer they have been coupled, the smaller this probability is.

The second property is that, for any  $R_0, d_0 > 0$ ,

$$(2.4) \quad \left\{ \begin{array}{l} \exists T^*(R_0, d_0) > 0 \text{ and } p_{-1}(d_0) > 0 \text{ such that for any } T \geq T^*(R_0, d_0) \\ \mathbb{P}(l_0(k+1) = k+1 \mid l_0(k) = \infty, \mathcal{H}_k \leq R_0) \geq p_{-1}(d_0). \end{array} \right.$$

In other words, inside a ball, the probability that two trajectories get coupled is bounded below.

The construction can be done by induction. At each step, we construct a probability space  $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$  and a measurable couple of functions  $(\omega_0, u_0^1, u_0^2) \rightarrow (V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$  such that, for any  $(u_0^1, u_0^2)$ ,  $(V_i(\cdot, u_0^1, u_0^2))_{i=1,2}$  is a coupling of  $\{\mathcal{D}((u(\cdot, u_0^i), W))\}_{i=1,2}$  on  $[0, T]$ . Recall that the processes  $(V_i)_{i=1,2}$  live in the space  $C(0, T; H_0^1(0, 1)) \times C(0, T; H^{-1}(0, 1))$ . We first set

$$u_i(0) = u_0^i, \quad W_i(0) = 0, \quad i = 1, 2.$$

Assuming that we have built  $(u_i, W_i)_{i=1,2}$  on  $[0, kT]$ , then we take  $(V_i)_{i=1,2}$  as above independent of  $(u_i, W_i)_{i=1,2}$  on  $[0, kT]$  and set

$$(u_i(kT + t), W_i(kT + t)) = V_i(t, u_1(kT), u_2(kT))$$

for any  $t \in [0, T]$ .

The construction of  $(V_i)_{i=1,2}$  depends on whether  $l_0(k) \leq k$  or  $l_0(k) = \infty$ . The two cases are treated separately in sections 2.5. We first state and prove the Foias-Prodi estimates and give some a priori estimates. We then recall some results on coupling and give a general result implying polynomial mixing. Sections 3, 4 and 5 are devoted to the proof of some results used in the course of the proof.

**2.1. The Foias-Prodi estimates.** We define for any  $(u_1, u_2, r) \in H_0^1(0, 1)$

$$J_*(u_1, u_2, r) = \frac{1}{2} \|r\|^2 - \frac{1}{4} \int_{[0,1]} \left( (|u_1|^2 + |u_2|^2) |r|^2 + (\Re((u_1 + u_2)\bar{r}))^2 \right) dx,$$

where  $\Re(z)$  is the real part of the complex number  $z$ , and

$$J(u_1, u_2, r) = J_*(u_1, u_2, r) + c_1 \left( \sum_{i=1}^2 \mathcal{H}(u_i) \right) |r|^2.$$

We infer from the Sobolev Embedding  $H^1(0, 1) \subset L^\infty(0, 1)$  that there exists  $c > 0$  such that

$$\int_{[0,1]} \left( (|u_1|^2 + |u_2|^2) |r|^2 + (\Re((u_1 + u_2)\bar{r}))^2 \right) dx \leq c(\|u_1\|^2 + \|u_2\|^2) |r|^2.$$

Therefore, by (1.4), there exists  $c_1 > 0$  such that

$$J(u_1, u_2, r) \geq \frac{1}{4} \|r\|^2.$$

We set

$$l(u_1, u_2) = 1 + \sum_{i=1}^2 \mathcal{H}(u_i)^4.$$

For  $N \geq 1$ , given  $u_1, u_2$ , two solutions of (1.1), we define  $J_{FP}^N = J_{FP}^N(u_1, u_2)$  by

$$J_{FP}^N(t) = J(u_1(t), u_2(t), r(t)) \exp \left( 2\alpha t - \frac{\Lambda}{\mu_{N+1}^{\frac{1}{8}}} \int_0^t l(u_1, u_2) ds \right),$$

where  $r = u_1 - u_2$ . The following result will be proved in section 5. It is the Foias-Prodi estimates adapted to the nonlinear Schrödinger equation. It states that two solutions having the same low modes are close. The main difference with similar results in the parabolic case is that we are not able to derive a path-wise estimate. Moreover, we introduce a slight generalization to allow the perturbation of the Wiener process by a drift in the low modes. This generalization is essential in our argument below.

**Proposition 2.1.** *For any  $\kappa_0 > 0$ , there exists  $\Lambda > 0$  depending only on  $\kappa_0$  such that for any  $N \in \mathbb{N} \setminus \{0\}$ , we have the following property:*

*Let  $W_1, W_2$  be two cylindrical Wiener processes and  $h$  be an adapted process with continuous paths in  $P_N L^2(0, 1)$ . Let  $u_1$  be a solution in  $C(0, T; H_0^1(0, 1))$  of*

$$\begin{cases} du_1 + \alpha u_1 dt + iAu_1 dt - i|u_1|^2 u_1 dt &= bdW_1 + hdt, \\ u_1(0) &= u_0^1, \end{cases}$$

*and  $u_2$  be the solution of (1.1), (1.2) for  $u_0 = u_0^2$  and  $W = W_2$ . Let  $\tau$  be a stopping time and assume that*

$$(2.5) \quad P_N u_1 = P_N u_2, \quad Q_N W_1 = Q_N W_2 \text{ on } [0, \tau],$$

*and*

$$(2.6) \quad \|h(t)\|^2 \leq \kappa_0 l(u_1(t), u_2(t))^{3/4} \text{ on } [0, \tau],$$

*then we have*

$$(2.7) \quad \mathbb{E} (J_{FP}^N(u_1, u_2)(t \wedge \tau)) \leq J(u_0^1, u_0^2, r_0), \quad t > 0,$$

*where  $r_0 = u_0^1 - u_0^2$ .*

We deduce a very useful Corollary.

**Corollary 2.2.** *For any  $B, d_0, \kappa_0 > 0$ , there exists  $N_1(B, \kappa_0)$  and  $C^*(d_0)$  such that, with the notations of Proposition 2.1, if (2.5) and (2.6) hold with  $N \geq N_1$ , and for some  $\rho > 0$ ,*

$$(2.8) \quad E_{u_i,4}(t) \leq \rho + 1 + d_0^4 + d_0^8 + Bt \text{ on } [0, \tau], \text{ for } i = 1, 2,$$

then for any  $u_0^1, u_0^2$  such that  $d_0 \geq \sum_{i=1}^2 \mathcal{H}(u_0^i)$  and for any  $a \in \mathbb{R}$ ,

$$\mathbb{P} \left( \|r(T)\| > C^*(d_0) \exp \left( a - \frac{\alpha}{4}T + \rho \right) \text{ and } T \leq \tau \right) \leq \exp \left( -a - \frac{\alpha}{4}T \right).$$

Moreover, there exists  $c > 0$  such that

$$C^*(d_0) \leq cd_0 e^{cd_0^8}.$$

Then, integrating (2.7) in Proposition 2.1 and applying the inequality

$$1 + x \leq C_\delta e^{\delta x} \text{ for any } x \geq 0,$$

we obtain the following result which, in Section 3, ensures that the Novikov condition holds and allows the use of the Girsanov Formula.

**Lemma 2.3.** *For any  $B, d_0, \kappa_0 > 0$  and any  $a \in \mathbb{R}$ , there exists  $N_2(B, \kappa_0, a)$  and  $C^{**}(d_0, B)$  such that, with the notations of Proposition 2.1, if (2.5) and (2.6) hold with  $N \geq N_2$  and (2.8) holds for some  $\rho > 0$ , we obtain that for any  $T$*

$$\begin{aligned} \mathbb{P} \left( \int_T^\tau l(u_1(s), u_2(s)) \|r(s)\|^2 ds > C^{**}(d_0, B) \exp \left( a - \frac{\alpha}{2}T + \rho \right) \text{ and } T \leq \tau \right) \\ \leq \exp \left( -a - \frac{\alpha}{2}T \right). \end{aligned}$$

provided  $d_0 \geq \sum_{i=1}^2 \mathcal{H}(u_0^i)$  holds. Moreover, there exists  $c > 0$  such that

$$C^{**}(d_0, B) \leq C(B)d_0 e^{cd_0^8}.$$

We set

$$(2.9) \quad N_0 = \max(N_1, N_2).$$

**2.2. A priori estimates.** We first give an estimate proven in section 4 on the growth of the solutions of the stochastic nonlinear Schrödinger equation.

**Proposition 2.4.** *Assume that  $u$  is a solution of (1.1), (1.2) associated with a Wiener process  $W$ . Then, for any  $(k, p) \in (\mathbb{N} \setminus \{0\})^2$ , there exists  $C'_k$  and  $K_{k,p}$  depending only on  $k$  and  $p$  such that for any  $\rho > 0$  and  $0 \leq T < \infty$*

$$\begin{aligned} \mathbb{P} \left( \sup_{t \in [0, T]} (E_{u,k}(t) - C'_k t) \geq \mathcal{H}(u_0)^k + \rho (\mathcal{H}(u_0)^{2k} + T) \right) &\leq K_{k,p} \rho^{-p}, \\ \mathbb{P} \left( \sup_{t \in [T, \infty)} (E_{u,k}(t) - C'_k t) \geq \mathcal{H}(u_0)^k + \mathcal{H}(u_0)^{2k} + 1 + \rho \right) &\leq K_{k,p} (\rho + T)^{-p}. \end{aligned}$$

The following result uses the Hamiltonian as a Lyapunov functional and is also proven in section 4.

**Lemma 2.5.** *There exists  $C_k > 0$  such that for any  $k \in \mathbb{N} \setminus \{0\}$ , for any  $t \in \mathbb{R}^+$  and for any stopping time  $\tau$*

$$\begin{cases} \mathbb{E} \left( \mathcal{H}(u(t, u_0))^k \right) &\leq \mathcal{H}(u_0)^k e^{-\alpha k t} + \frac{C_k}{2}, \\ \mathbb{E} \left( \mathcal{H}(u(\tau, u_0))^k 1_{\tau < \infty} \right) &\leq \mathcal{H}(u_0)^k + C_k \mathbb{E}(\tau 1_{\tau < \infty}). \end{cases}$$

The following result states that we control the probability of entering a small ball.

**Proposition 2.6.** *For any  $R_0, R_1 > 0$ , there exists  $T_{-1}(R_0, R_1) \geq 0$  and  $\pi_{-1}(R_1) > 0$  such that*

$$\mathbb{P}(\mathcal{H}(u(t, u_0^1)) + \mathcal{H}(u(t, u_0^2)) \leq R_1) \geq \pi_{-1}(R_1),$$

provided  $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) \leq R_0$  and  $t \geq T_{-1}(R_0, R_1)$ .

### 2.3. Basic properties of couplings.

Let  $(\mu_1, \mu_2)$  be two distributions on a same space  $(E, \mathcal{E})$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $(Z_1, Z_2)$  be two random variables  $(\Omega, \mathcal{F}) \rightarrow (E, \mathcal{E})$ . Recall that  $(Z_1, Z_2)$  is a coupling of  $(\mu_1, \mu_2)$  if  $\mu_i = \mathcal{D}(Z_i)$  for  $i = 1, 2$  and that we have denoted by  $\mathcal{D}(Z_i)$  the law of the random variable  $Z_i$ .

Let  $\mu, \mu_1$  and  $\mu_2$  be three probability measures on a space  $(E, \mathcal{E})$  such that  $\mu_1$  and  $\mu_2$  are absolutely continuous with respect to  $\mu$ . We set

$$d(\mu_1 \wedge \mu_2) = \left( \frac{d\mu_1}{d\mu} \wedge \frac{d\mu_2}{d\mu} \right) d\mu.$$

This definition does not depend on the choice of  $\mu$  and we have

$$\|\mu_1 - \mu_2\|_{var} = \frac{1}{2} \int_E \left| \frac{d\mu_1}{d\mu} - \frac{d\mu_2}{d\mu} \right| d\mu.$$

Remark that if  $\mu_1$  is absolutely continuous with respect to  $\mu_2$ , we have

$$(2.10) \quad \|\mu_1 - \mu_2\|_{var} \leq \frac{1}{2} \sqrt{\int \left( \frac{d\mu_1}{d\mu_2} \right)^2 d\mu_2 - 1}.$$

Next result is a fundamental result in the coupling methods, the proof is given for instance in the Appendix of [26].

**Lemma 2.7.** *Let  $(\mu_1, \mu_2)$  be two probability measures on  $(E, \mathcal{E})$ . Then*

$$\|\mu_1 - \mu_2\|_{var} = \min \mathbb{P}(Z_1 \neq Z_2).$$

*The minimum is taken over all couplings  $(Z_1, Z_2)$  of  $(\mu_1, \mu_2)$ . There exists a coupling which reaches the minimum value. It is called a maximal coupling and has the following property:*

$$\mathbb{P}(Z_1 = Z_2, Z_1 \in \Gamma) = (\mu_1 \wedge \mu_2)(\Gamma) \text{ for any } \Gamma \in \mathcal{E}.$$

Next result is a refinement of Lemma 2.7 used in [23] (see also Proposition 1.7 in [26]).

**Proposition 2.8.** *Let  $E$  and  $F$  be two Polish spaces,  $f_0 : E \rightarrow F$  be a measurable map and  $(\mu_1, \mu_2)$  be two probability measures on  $E$ . We set*

$$\nu_i = f_0^* \mu_i, \quad i = 1, 2.$$

*Then there exist a coupling  $(V_1, V_2)$  of  $(\mu_1, \mu_2)$  such that  $(f_0(V_1), f_0(V_2))$  is a maximal coupling of  $(\nu_1, \nu_2)$ .*

#### 2.4. Sufficient conditions for polynomial mixing.

We now state and prove a general result which allows to reduce the proof of polynomial convergence to equilibrium to the verification of some conditions. This result is a polynomial version of Theorem 1.8 of subsection 1.3 in [26] which gives sufficient conditions for exponential mixing.

We are concerned with  $v(\cdot, (u_0, W_0)) = (u(\cdot, u_0), W(\cdot, W_0))$  a couple of strongly Markov processes defined on Polish spaces  $(E, d_E)$  and  $(F, d_F)$ . We denote by  $(\mathcal{P}_t)_{t \in I}$  the Markov transition semigroup associated to  $u$ , where  $I = \mathbb{R}^+$  or  $T\mathbb{N} = \{kT, k \in \mathbb{N}\}$ . We are also given a real valued function  $\mathcal{H}$  defined on  $E$ .

We consider for any couple of initial conditions  $(v_0^1, v_0^2)$  a coupling  $(v_1, v_2)$  of  $\{\mathcal{D}(v(\cdot, v_0^1)), \mathcal{D}(v(\cdot, v_0^2))\}$ . We write  $v_i = (u_i, W_i)$ . Let  $l_0 : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$  be a random integer valued process which has the following properties

$$(2.11) \quad \left\{ \begin{array}{l} l_0(k+1) = l \text{ implies } l_0(k) = l, \text{ for any } l \leq k, \\ l_0(k) \in \{0, 1, 2, \dots, k\} \cup \{\infty\}, \\ l_0(k) \text{ depends only of } v_1|_{[0, kT]} \text{ and } v_2|_{[0, kT]}, \\ l_0(k) = k \text{ implies } \mathcal{H}_k \leq d_0, \end{array} \right.$$

where

$$\mathcal{H}_k = \mathcal{H}(u_1(kT)) + \mathcal{H}(u_2(kT)), \quad \mathcal{H} : E \rightarrow \mathbb{R}^+,$$

and  $d_0 > 0$ .

We now give four conditions on the coupling. The first condition states that when  $(v_1, v_2)$  have been coupled for a long time then the probability that  $(u_1, u_2)$  are close is high. It will be a consequence of the Foias-Prodi estimate.

$$(2.12) \quad \left\{ \begin{array}{l} \text{There exist } c_0 \text{ and } q > 0 \text{ such that for any } t \in [lT, kT] \cap I \\ \mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_0 (t - lT)^{-q} \text{ and } l_0(k) \leq l\right) \leq c_0 (t - lT)^{-q}, \end{array} \right.$$

The next two properties are exactly (2.3) and (2.4).

$$(2.13) \quad \left\{ \begin{array}{l} \exists (p_k)_{k \in \mathbb{N}}, c_1 > 0, q_0 > 1 + q \text{ such that,} \\ \mathbb{P}(l_0(k+1) = l \mid l_0(k) = l) \geq p_{k-l}, \text{ for any } l \leq k, \\ 1 - p_k \leq c_1 ((k+1)T)^{-q_0}, p_k > 0 \text{ for any } k \in \mathbb{N}. \end{array} \right.$$

$$(2.14) \quad \left\{ \begin{array}{l} \text{There exist } p_{-1} > 0, R_0 > 0 \text{ such that} \\ \mathbb{P}(l_0(k+1) = k+1 \mid l_0(k) = \infty, \mathcal{H}_k \leq R_0) \geq p_{-1}. \end{array} \right.$$

The last ingredient is the so-called Lyapunov structure and follows from Lemma 2.5. It allows the control of the probability to enter the ball of radius  $R_0$ . It states that there exist  $K_1$  and  $K'$  constants such that for any initial data  $v_0$  and any

stopping time  $\tau'$  taking values in  $\{kT, k \in \mathbb{N}\} \cup \{\infty\}$

$$(2.15) \quad \begin{cases} \mathbb{E}\mathcal{H}(v(t, v_0)) & \leq e^{-\alpha t}\mathcal{H}(v_0) + \frac{K_1}{2}, \quad t \geq 0, \\ \mathbb{E}(\mathcal{H}(v(\tau', v_0))1_{\tau' < \infty}) & \leq K'(\mathcal{H}(v_0) + \mathbb{E}(\tau'1_{\tau' < \infty})). \end{cases}$$

The process  $V = (v_1, v_2)$  is said to be  $l_0$ -Markov if the laws of  $V(kT + \cdot)$  and of  $l_0(k + \cdot) - k$  on  $\{l_0(k) \in \{k, \infty\}\}$  conditioned by  $\mathcal{F}_{kT}$  only depend on  $V(kT)$  and are equal to the laws of  $V(\cdot, V(kT))$  and  $l_0$ , respectively.

In this article, we construct a coupling  $(u_i, W_i)_{i=1,2}$  of two solutions which is  $l_0$ -Markov but not Markov. We could modify the construction so that it is Markov at discrete times  $T\mathbb{N} = \{kT, k \in \mathbb{N}\}$ . However, it does not seem to be possible to modify the coupling to be Markov at any times. As shown below, the following result implies Theorem 1.1. Its proof is given in section 3.

**Theorem 2.9.** *Assume that for any  $(u_0^1, W_0^1), (u_0^2, W_0^2)$  there exists a coupling  $V = (v_1, v_2)$  of the laws of  $(u(\cdot, u_0^1), W(\cdot, W_0^1))$  and  $(u(\cdot, u_0^2), W(\cdot, W_0^2))$  which is  $l_0$ -Markov and satisfies (2.11), (2.12), (2.13), (2.14) and (2.15) with  $R_0 > 4K_1$  and  $R_0 \geq d_0$ . Then there exists  $c_4 > 0$  such that, for any  $\varphi \in Lip_b(E)$  and any  $u_0^1, u_0^2 \in E$ ,*

$$(2.16) \quad |\mathbb{E}\varphi(u(t, u_0^1)) - \mathbb{E}\varphi(u(t, u_0^2))| \leq c_4(1+t)^{-q} \|\varphi\|_L(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)).$$

**Corollary 2.10.** *Under the same assumptions as Theorem 2.9, there exists a unique stationary probability measure  $\nu$  of  $(\mathcal{P}_t)_{t \in I}$  on  $E$ . It satisfies,*

$$(2.17) \quad \int_E \mathcal{H}(u) d\nu(u) \leq \frac{K_1}{2}.$$

Moreover for any  $\mu \in \mathcal{P}(E)$

$$(2.18) \quad \|\mathcal{P}_t^* \mu - \nu\|_W \leq 2c_4(1+t)^{-q} \left(1 + \int_E \mathcal{H}(u) d\mu(u)\right).$$

To prove Theorem 2.9, we first note that it is sufficient to prove that, for any initial data  $u_0^1$  and  $u_0^2$ , the coupling satisfies

$$(2.19) \quad \mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_3(1+t)^{-q}\right) \leq c_3(1+t)^{-q}(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2))$$

where, as above,  $v_i = (u_i, W_i)$ . Indeed we have, since  $(u_1, u_2)$  is a coupling of  $\{\mathcal{D}(u(\cdot, u_0^1)), \mathcal{D}(u(\cdot, u_0^2))\}$ ,

$$\begin{aligned} |\mathbb{E}\varphi(u(t, u_0^1)) - \mathbb{E}\varphi(u(t, u_0^2))| &= |\mathbb{E}\varphi(u_1(t)) - \mathbb{E}\varphi(u_2(t))| \\ &\leq L_\varphi c_3(1+t)^{-q} + 2\|\varphi\|_\infty \mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_3(1+t)^{-q}\right) \\ &\leq L_\varphi c_3(1+t)^{-q} + 2\|\varphi\|_\infty c_3(1+t)^{-q}(1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)) \end{aligned}$$

so that (2.16) follows. The existence and uniqueness of a stationary measure is then straightforward. Moreover, (2.18) is an easy consequence of (2.16) and (2.17) follows from (2.15).

**2.5. Construction of the coupling.** We first state the following result.

**Proposition 2.11.** *There exists a measurable map*

$$\Phi : C((0, T); P_{N_*} H_0^1(0, 1)) \times C((0, T); Q_{N_*} H^{-1}(0, 1)) \times H_0^1(0, 1) \rightarrow C((0, T); Q_{N_*} H_0^1(0, 1)),$$

such that for any  $(u, W)$  solution of (1.1) and (1.2)

$$Y = \Phi(X, \eta, u_0) \quad \text{on } [0, T], \quad \text{where } X = P_{N_*} u, \quad Y = Q_{N_*} u, \quad \eta = Q_{N_*} W.$$

Moreover  $\Phi$  is a non-anticipative functions of  $(X, \eta)$ .

To prove this result, we note that the equation

$$y(t) = S(t)y_0 - \int_0^t S(t-s)g(x(s), y(s))ds + \int_0^t S(t-s)dz(s),$$

can be solved by a fixed point argument in  $C(0, T; H_0^1(0, 1))$  for any deterministic functions  $x \in C(0, T; P_{N_*} H_0^1(0, 1))$  and  $z \in C(0, T; D(A^{\frac{3}{2}}))$ . The last term is defined thanks to an integration by part. Clearly  $y = \Psi(x, z, y_0)$  for a measurable function  $\Psi$ . Thus  $Y = \Psi(X, \sigma_h \eta, Q_{N_*} u_0)$ . We set  $\Phi(x, \tilde{z}, u_0) = \Psi(x, \sigma_h \tilde{z}, Q_{N_*} u_0)$  for  $\tilde{z}$  such that  $\sigma_h \tilde{z} \in C(0, T; D(A^{\frac{3}{2}}))$  and 0 otherwise. It is clear that  $\Phi$  is not anticipative.

As already explained, the coupling  $(u_1, u_2)$  is constructed by induction and we start by constructing a coupling for two solutions  $u(\cdot, u_0^i)$ ,  $i = 1, 2$  on an interval  $[0, T]$ . In fact, we construct three different couplings. At time  $kT$ , we choose between these depending on whether  $l_0(k) = \infty$  and  $\mathcal{H}(u_1(kT)) + \mathcal{H}(u_2(kT)) \leq R_0$  (case a) or  $l_0(k) \leq k$  (case b). In this latter case,  $P_{N_*} u_1(kT) = P_{N_*} u_2(kT)$ . In the third case,  $l_0(k) = \infty$  and  $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) > R_0$ , we choose the trivial coupling.

**Case a:**  $l_0(k) = \infty$  and  $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) \leq R_0$ . We construct a coupling such that (2.4) holds.

In this case, we consider  $u_0^1, u_0^2$  such that  $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) \leq R_0$ . The construction of the coupling is done in two steps. We set

$$\mu_i = \mathcal{D}((u(\cdot, u_0^i), W)), \quad \text{on } [0, T_1], \quad i = 1, 2.$$

Step 1:

We first prove that, for any  $d_0 > 0$ , there exist  $T_1(d_0) > 0$ ,  $R_1 = R_1(d_0) > 0$  and a coupling  $(\tilde{V}_i(\cdot, u_0^1, u_0^2))_{i=1,2}$  of  $(\mu_1, \mu_2)$  such that for any  $(u_0^1, u_0^2)$  satisfying  $\sum_{i=1}^2 \mathcal{H}(u_0^i) \leq R_1$  we have

$$(2.20) \quad \mathbb{P} \left( \tilde{X}_1(T_1, u_0^1, u_0^2) = \tilde{X}_2(T_1, u_0^1, u_0^2), \sum_{i=1}^2 \mathcal{H}(\tilde{u}_i(T_1, u_0^1, u_0^2)) \leq d_0 \right) \geq \frac{1}{2},$$

where

$$\tilde{V}_i(\cdot, u_0^1, u_0^2) = (\tilde{u}_i(\cdot, u_0^1, u_0^2), \tilde{W}_i(\cdot, u_0^1, u_0^2)), \quad \tilde{X}_i(\cdot, u_0^1, u_0^2) = P_{N_*} \tilde{u}_i(\cdot, u_0^1, u_0^2), \quad i = 1, 2.$$

To construct  $\tilde{V}_i$  such that (2.20) holds, we take  $R_1, T_1 > 0$  and we set

$$\begin{aligned} E &= C((0, T); H_0^1(0, 1)) \times C((0, T); H^{-1}(0, 1)), \\ F &= C((0, T); P_{N_*} H_0^1(0, 1)) \times C((0, T); Q_{N_*} H^{-1}(0, 1)), \\ f_0(u, W) &= (P_{N_*} u, Q_{N_*} W) = (X, \eta), \\ \hat{\mu}_1 &= \mathcal{D} \left( (u(\cdot, u_0^1) + \frac{T_1 - \cdot}{T_1} P_{N_*} (u_0^2 - u_0^1), W) \right) \text{ on } [0, T_1], \\ \nu_i &= f_0^* \mu_i, \quad \hat{\nu}_1 = f_0^* \hat{\mu}_1. \end{aligned}$$

We apply Proposition 2.8 to  $(E, F, f_0, (\hat{\mu}_1, \mu_2))$  and obtain  $(\hat{V}_1(\cdot, u_0^1, u_0^2), \tilde{V}_2(\cdot, u_0^1, u_0^2))$  a coupling of  $(\hat{\mu}_1, \mu_2)$ . Moreover, setting

$$(\tilde{X}_2, \tilde{\eta}_2) = f_0(\tilde{V}_2(\cdot, u_0^1, u_0^2)), \quad (\hat{X}_1, \eta_1) = f_0(\hat{V}_1(\cdot, u_0^1, u_0^2)),$$

$((\tilde{X}_2, \tilde{\eta}_2), (\hat{X}_1, \eta_1))$  is a maximal coupling of  $(\hat{\nu}_1, \nu_2)$ .

Finally, we set

$$\tilde{V}_1 = \left( \hat{u}_1 - \frac{T_1 - \cdot}{T_1} P_{N_*} (u_0^2 - u_0^1), W_1 \right) \text{ on } [0, T_1], \text{ where } \hat{V}_1 = (\hat{u}_1, W_1).$$

We also write

$$\beta_1 = P_{N_*} W_1, \quad \tilde{V}_1 = (\tilde{u}_1, W_1), \quad \tilde{V}_2 = (\tilde{u}_2, W_2).$$

To prove (2.20) we first remark that since  $\hat{u}_1(T_1) = \tilde{u}_1(T_1)$  and  $\hat{X}_1 = P_{N_*} \hat{u}_1$ ,  $\tilde{X}_i = P_{N_*} \tilde{u}_i$ , then

$$(2.21) \quad \begin{aligned} &\mathbb{P} \left( \tilde{X}_1(T_1) = \tilde{X}_2(T_1) \text{ and } \sum_{i=1}^2 \mathcal{H}(\tilde{u}_i(T_1))^6 \leq \kappa'(\rho, T_1, R_1) \right) \\ &\geq \mathbb{P} \left( \hat{X}_1 = \tilde{X}_2 \text{ on } [0, T_1] \text{ and } \sum_{i=1}^2 E_{\tilde{u}_i, 6}(t) \leq \kappa'(\rho, t, R_1) \text{ on } [0, T_1] \right), \end{aligned}$$

where

$$\kappa'(\rho, t, R_1) = 2(R_1^6 + C_6' t + \rho(R_1^{12} + t)), \quad t > 0,$$

$\rho$  to be chosen below.

Let us consider  $\bar{X}_1$  the unique solution of

$$(2.22) \quad \begin{cases} d\bar{X}_1 + G\bar{X}_1 dt - \delta(t) + 1_{t \leq \tau} f(\bar{X}_1 - \hat{\delta}, \Phi(\bar{X}_1 - \hat{\delta}, \eta_1, u_0^1)) dt = \sigma_1 d\beta_1, \\ \bar{X}_1(0) = x_0^2, \end{cases}$$

where  $\delta(t) = \left( \frac{T_1 - t}{T_1} - \frac{1}{T_1} G \right) P_{N_*} (u_0^2 - u_0^1)$ ,  $\hat{\delta}(t) = \frac{T_1 - t}{T_1} P_{N_*} (u_0^2 - u_0^1)$ , and  $\tau = \tau_1 \wedge \tau_2$  where

$$\begin{cases} \tau_1 = \inf \left\{ t \in [0, T_1] \mid E_{\bar{X}_1 - \hat{\delta} + \Phi(\bar{X}_1 - \hat{\delta}, \eta_1, u_0^1), 6}(t) > \kappa'(\rho, t, R_1) \right\}, \\ \tau_2 = \inf \left\{ t \in [0, T_1] \mid E_{\bar{X}_1 + \Phi(\bar{X}_1, \eta_1, u_0^2), 6}(t) > \kappa'(\rho, t, R_1) \right\}. \end{cases}$$

Clearly,  $\bar{X}_1 = \hat{X}_1 = P_{N_*} \tilde{u}_1 + \hat{\delta}$  on  $[0, \tau]$ . We denote by  $\lambda_1$  the distribution of  $(\bar{X}_1, \eta_1)$  under the probability  $\mathbb{P}$ . We set  $\tilde{\beta}_1(t) = \beta_1(t) + \int_0^t d(s) ds$  where

$$(2.23) \quad d(t) = \delta(t) + 1_{t \leq \tau} \sigma_1^{-1} \begin{pmatrix} f(\bar{X}_1(t) - \hat{\delta}(t), \Phi(\bar{X}_1(t) - \hat{\delta}(t), \eta_1, u_0^1)(t)) \\ -f(\bar{X}_1(t), \Phi(\bar{X}_1(t), \eta_1, u_0^2)(t)) \end{pmatrix}.$$

Then  $\bar{X}_1$  is a solution of

$$(2.24) \quad \begin{cases} d\bar{X}_1 + G\bar{X}_1 dt + 1_{t \leq \tau} f(\bar{X}_1, \Phi(\bar{X}_1, \eta_1, u_0^2)) dt = \sigma_1 d\tilde{\beta}_1, \\ \bar{X}_1(0) = x_0^2. \end{cases}$$

It is not difficult to see that since  $\sigma_l$  is bounded below and by the definition of  $\tau$ , the Novikov condition is satisfied:

$$\mathbb{E} \left( \exp \left( \int_0^T |d(t)|^2 dt \right) \right) < \infty$$

and the Girsanov formula can be applied. Then we set

$$d\tilde{\mathbb{P}} = \exp \left( \int_0^T d(s) dW(s) - \frac{1}{2} \int_0^T |d(s)|^2 dt \right) d\mathbb{P}$$

and deduce that  $\tilde{\mathbb{P}}$  is a probability under which  $(\tilde{\beta}_1, \eta_1)$  is a cylindrical Wiener process. We denote by  $\lambda_2$  the law of  $(\bar{X}_1, \eta_1)$  under  $\tilde{\mathbb{P}}$ .

We prove below that

$$(2.25) \quad \begin{aligned} & \mathbb{P} \left( \hat{X}_1(t) \neq \tilde{X}_2(t) \text{ or } \sum_{i=1}^2 E_{\tilde{u}_i, 6}(t) > \kappa'(\rho, t, R_1) \text{ for some } t < T_1 \right) \\ & \leq \|\lambda_1 - \lambda_2\|_{var} + \mathbb{P} \left( E_{\tilde{u}_1, 6}(\tau) \geq \frac{1}{2} \kappa'(\rho, \tau, R_1) \right) + \mathbb{P} \left( E_{\tilde{u}_2, 6}(\tau) \geq \frac{1}{2} \kappa'(\rho, \tau, R_1) \right) \end{aligned}$$

We choose

$$\rho = 8K_{6,1}$$

in the definition of  $\kappa'(\rho, t, R_1)$  and deduce from Proposition 2.4 that

$$(2.26) \quad \mathbb{P} \left( E_{\tilde{u}_1, 6}(\tau) \geq \frac{1}{2} \kappa'(\rho, \tau, R_1) \right) + \mathbb{P} \left( E_{\tilde{u}_2, 6}(\tau) \geq \frac{1}{2} \kappa'(\rho, \tau, R_1) \right) \leq \frac{1}{4}.$$

Moreover using (2.10), we obtain

$$\|\lambda_1 - \lambda_2\|_{var} \leq \frac{1}{2} \sqrt{\mathbb{E} \exp \left( c \int_0^T |d(s)|^2 dt \right)} - 1,$$

and then, for  $T_1, R_1$  sufficiently small,

$$\|\lambda_1 - \lambda_2\|_{var} \leq 2(R_1 (T_1 + 1) (1 + R_1^2) + \kappa'(\rho, T_1, R_1)).$$

We choose

$$T_1 = R_1,$$

and deduce

$$(2.27) \quad \|\lambda_1 - \lambda_2\|_{var} \leq cR_1 (1 + R_1^{11}).$$

Taking into account (2.21), (2.25), (2.26) and (2.27), we can choose  $R_1^0 > 0$  sufficiently small such that for any  $R_1 \leq R_1^0$

$$(2.28) \quad \mathbb{P} \left( \tilde{X}_1(T_1) = \tilde{X}_2(T_1) \text{ and } \sum_{i=1}^2 \mathcal{H}(\tilde{u}_i(T_1))^6 \leq \kappa'(\rho, R_1, R_1) \right) \geq \frac{1}{2}.$$

Remark that there exists  $R_1(d_0) \in (0, R_1^0)$  such that  $R_1 \leq R_1(d_0)$  implies

$$\left\{ \sum_{i=1}^2 \mathcal{H}(\tilde{u}_i(T_1))^6 \leq \kappa'(\rho, R_1, R_1) \right\} \subset \left\{ \sum_{i=1}^2 \mathcal{H}(\tilde{u}_i(T_1)) \leq d_0 \right\},$$

so that (2.20) follows.

It remains to prove (2.25). We write

$$\begin{aligned} & \mathbb{P} \left( \hat{X}_1(t) \neq \tilde{X}_2(t) \text{ or } \sum_{i=1}^2 E_{\tilde{u}_i,6}(t) > \kappa'(\rho, t, R_1) \text{ for some } t < T_1 \right) \\ &= \mathbb{P} \left( \hat{X}_1|_{[0,\tau]} \neq \tilde{X}_2|_{[0,\tau]} \text{ or } \sum_{i=1}^2 E_{\tilde{u}_i,6}(\tau) = \kappa'(\rho, \tau, R_1) \right) \\ &\leq \mathbb{P} \left( \hat{X}_1|_{[0,\tau]} \neq \tilde{X}_2|_{[0,\tau]} \right) + \mathbb{P} \left( E_{\tilde{u}_1,6}(\tau) \geq \frac{1}{2}\kappa'(\rho, \tau, R_1) \right) \\ &\quad + \mathbb{P} \left( E_{\tilde{u}_2,6}(\tau) \geq \frac{1}{2}\kappa'(\rho, \tau, R_1) \right). \end{aligned}$$

Let  $\bar{X}_2$  is the solution of equation (2.24) where  $\beta_1$  is replaced by  $\beta_2 = P_{N_*}W_2$  then, with the probability  $\mathbb{P}$ ,  $\bar{X}_2$  has the same law as  $\bar{X}_1$  under the probability  $\tilde{\mathbb{P}}$  and

$$\mathbb{P} \left( P_{N_*}\hat{u}_1|_{[0,\tau]} \neq P_{N_*}\tilde{u}_2|_{[0,\tau]} \right) \leq \mathbb{P}(\bar{X}_1 \neq \bar{X}_2).$$

Thus, (2.25) would follow if  $((\bar{X}_1, \eta_1), (\bar{X}_2, \eta_2))$  was a maximal coupling of  $(\lambda_1, \lambda_2)$  (here, we have set  $\eta_2 = Q_{N_*}W_2$ ). However, we only know that  $((\hat{X}_1, \eta_1), (\tilde{X}_2, \tilde{\eta}_2))$  is a maximal coupling of  $(\nu_1, \nu_2)$ . It is not difficult to remedy this problem. Indeed, the above result holds for any coupling of  $(\nu_1, \nu_2)$ . Thus, instead of  $((\hat{X}_1, \eta_1), (\tilde{X}_2, \tilde{\eta}_2))$ , we choose another coupling such that the processes constructed as  $((\bar{X}_1, \eta_1), (\bar{X}_2, \eta_2))$  above is a maximal coupling of  $(\lambda_1, \lambda_2)$ . Then, the right hand side is equal to the right hand side of (2.25) while, by Lemma 2.7, the left hand side is larger than the left hand side of (2.25).

Step 2: Construction of the coupling under the assumptions of case a.

Thanks to Proposition 2.6, we know that there exists  $T_{-1}(R_0, R_1) > 0$  and  $\pi_{-1}(R_1) > 0$  such that

$$(2.29) \quad \mathbb{P} \left( \sum_{i=1}^2 \mathcal{H}(u(\theta, u_0^i)) \leq R_1 \right) \geq \pi_{-1}(R_1),$$

provided  $\sum_{i=1}^2 \mathcal{H}(u_0^i) \leq R_0$  and  $\theta \geq T_{-1}(R_0, R_1)$ .

We set  $T^*(R_0, d_0) = T_{-1}(R_0, R_1(d_0)) + T_1(d_0)$  and assume that  $T \geq T^*(R_0, d_0)$ . We also write  $\theta = T - T_1$ . Then on  $[0, \theta]$ , we take the trivial coupling which we denote by  $(V'_1, V'_2)$ . Finally, we consider  $(\tilde{V}_1, \tilde{V}_2)$  as above independent of  $(V'_1, V'_2)$  and we set

$$V_i^a(t, u_0^1, u_0^2) = \begin{cases} V'_i(t, u_0^1, u_0^2) & \text{if } t \leq \theta, \\ \tilde{V}_i(t - \theta, V'_1(\theta, u_0^1, u_0^2), V'_2(\theta, u_0^1, u_0^2)) & \text{if } t \geq \theta. \end{cases}$$

Combining (2.20) and (2.29) and setting

$$p_{-1}(d_0) = \frac{1}{2}\pi_{-1}(R_1(d_0)),$$

we obtain, for any  $u_0^1, u_0^2$  such that  $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) \leq R_0$ ,

$$(2.30) \quad \mathbb{P} \left( X_1^a(T, u_0^1, u_0^2) = X_2^a(T, u_0^1, u_0^2), \sum_{i=1}^2 \mathcal{H}(u_i^a(T, u_0^1, u_0^2)) \leq d_0 \right) \geq p_{-1}(d_0),$$

where now

$$V_i^a(\cdot, u_0^1, u_0^2) = (u_i^a(\cdot, u_0^1, u_0^2), W_i^a(\cdot, u_0^1, u_0^2)), \quad X_i^a(\cdot, u_0^1, u_0^2) = P_{N_*} u_i^a(\cdot, u_0^1, u_0^2), \quad i = 1, 2.$$

Clearly, (2.30) implies (2.4).

**Case b:**  $l_0(k) \leq k$ . We now construct a coupling so that (2.3) holds. Since (2.3) depends on the whole history of the coupling and not only on the latest step, (2.3) is proved afterwards when the coupling is constructed on  $[0, \infty)$ .

In this case, we have  $P_{N_*} u_0^1 = P_{N_*} u_0^2$ . We write  $x = P_{N_*} u_0^1 = P_{N_*} u_0^2$ ,  $y_1 = Q_{N_*} u_0^1$  and  $y_2 = Q_{N_*} u_0^2$ .

We apply Proposition 2.8 to

$$\begin{aligned} E &= C((0, T); H_0^1(0, 1)) \times C((0, T); H^{-1}(0, 1)), \\ F &= C((0, T); P_{N_*} H_0^1(0, 1)) \times C((0, T); Q_{N_*} H^{-1}(0, 1)), \\ f_0(u, W) &= (P_{N_*} u, Q_{N_*} W) = (X, \eta), \\ \mu_i &= \mathcal{D}((u(\cdot, u_0^i), W)), \quad \text{on } [0, T]. \end{aligned}$$

We set  $\nu_i = f_0^* \mu_i = \mathcal{D}((X(\cdot, u_0^i), \eta))$  on  $[0, T]$ . We obtain  $(V_i^b(\cdot, u_0^1, u_0^2))_{i=1,2} = (u_i^b(\cdot, u_0^1, u_0^2), W_i^b(\cdot, u_0^1, u_0^2))_{i=1,2}$ , a coupling of  $(\mu_1, \mu_2)$  such that if we set

$$(X_i^b, \eta_i^b) = f_0(V_i^b), \quad i = 1, 2.$$

Then  $(X_i^b, \eta_i^b)(\cdot, u_0^1, u_0^2)_{i=1,2}$  is a maximal coupling of  $(\nu_1, \nu_2)$ . We define  $Y_i^b = Q_{N_*} u_i^b$ ,  $\beta_i^b = P_{N_*} W_i^b$

Let  $\tau$  be a stopping time associated to the process  $(X, \eta)$ .

Let  $\tilde{X}_1^b$  be the unique solution of the truncated equation

$$(2.31) \quad \begin{cases} d\tilde{X}_1^b + G\tilde{X}_1^b dt + 1_{t \leq \tau} f(\tilde{X}_1^b, \Phi(\tilde{X}_1^b, \eta_1^b, (x, y_1))) dt = \sigma_l d\beta_1^b, \\ \tilde{X}_1^b(0) = x. \end{cases}$$

Clearly  $\tilde{X}_1^b = X_1^b$  on  $[0, \tau]$ . We denote by  $\lambda_1$  the distribution of  $(\tilde{X}_1^b, \eta)$  under the probability  $\mathbb{P}$ .

Let  $\tilde{\beta}_1^b(t) = \beta_1^b(t) + \int_0^t d(s) dt$  where

$$d(t) = 1_{t \leq \tau} (\sigma_l)^{-1} \left( f(\tilde{X}_1^b(t), \Phi(\tilde{X}_1^b, \eta_1^b, (x, y_2)))(t) - f(\tilde{X}_1^b(t), \Phi(\tilde{X}_1^b, \eta_1^b, (x, y_1)))(t) \right).$$

We take below a stopping time  $\tau$  such that

$$(2.32) \quad \int_0^T |d(t)|^2 dt \leq M,$$

for a constant  $M$ . Thus Novikov condition holds and Girsanov formula applies. Setting

$$d\tilde{\mathbb{P}} = \exp \left( \int_0^T d(s) dW(s) - \frac{1}{2} \int_0^T |d(s)|^2 dt \right) d\mathbb{P},$$

we know that  $\tilde{\mathbb{P}}$  is a probability under which  $(\tilde{\beta}, \eta)$  is a cylindrical Wiener process.

Furthermore, with such a stopping time  $\tau$ ,  $\tilde{X}_1^b$  is the solution of

$$(2.33) \quad \begin{cases} d\tilde{X}_1^b + G\tilde{X}_1^b dt + 1_{t \leq \tau} f(\tilde{X}_1^b, \tilde{\Phi}(\tilde{X}_1^b, \eta_1^b, (x, y_2))) dt = \sigma_1 d\tilde{\beta}, \\ \tilde{X}_1^b(0) = x. \end{cases}$$

We denote by  $\lambda_2$  the law of  $(\tilde{X}_1^b, \eta)$  under  $\tilde{\mathbb{P}}$ . As in the case a, it is not difficult to see that

$$(2.34) \quad \|\lambda_1 - \lambda_2\|_{var} \leq \frac{1}{2} \sqrt{\mathbb{E} \exp \left( c \int_0^T |d(s)|^2 dt \right) - 1}.$$

This will be helpful to estimate  $\|\nu_1 - \nu_2\|_{var}$ .

**Definition of the coupling on  $[0, \infty)$ .**

We first set

$$u_i(0) = u_0^i, \quad W_i(0) = 0, \quad i = 1, 2.$$

Assuming that we have built  $(u_i, W_i)_{i=1,2}$  on  $[0, kT]$ , then we take  $(V_i^a)_i$  and  $(V_i^b)_i$  as above independent of  $(u_i, W_i)_{i=1,2}$  on  $[0, kT]$  and set for any  $t \in [0, T]$

$$(u_i(kT + t), W_i(kT + t)) =$$

$$(2.35) \quad \begin{cases} V_i^a(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) = \infty \text{ and } \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) \leq R_0, \\ V_i^b(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) \leq k, \\ V_i^0(t, u_1(kT), u_2(kT)) & \text{if } l_0(k) = \infty \text{ and } \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) > R_0, \end{cases}$$

where  $V_i^0(t, u_1(kT), u_2(kT))$  is the trivial coupling. In other words, we take a cylindrical Wiener process  $W$  independent of  $(u_i, W_i)_{i=1,2}$  on  $(0, kT)$  and set

$$V_i^0(t, u_1(kT), u_2(kT)) = ((u(t - kT, u_0^1), W), (u(t - kT, u_0^2), W)).$$

Remark that, when  $l_0(k) = \infty$  and  $\mathcal{H}(u_0^1) + \mathcal{H}(u_0^2) > R_0$ , the choice of the coupling is not very important.

Clearly,  $(u_i, W_i)_{i=1,2}$  is a coupling of  $(u(\cdot, u_0^i))_{i=1,2}$  which is  $l_0$ -Markov. In the following, we write

$$X_i = P_{N_*} u_i, \quad Y_i = Q_{N_*} u_i, \quad \beta_i = P_{N_*} W_i, \quad \eta_i = Q_{N_*} W_i, \quad i = 1, 2.$$

It remains to prove that (2.3) holds.

**Proof of (2.3)**

We are in the situation where the coupling on  $[kT, (k+1)T]$  has been constructed in case b. We use the notation used in the construction of the coupling.

Let us define for  $i = 1, 2$

$$\hat{\tau}_{k,l}^i = \inf \left\{ t \in [0, T] \mid E_{\hat{u}_{i,4}}(kT + t, lT) > \kappa + 1 + d_0^4 + d_0^8 + C'_4(t + (k-l)T) \right\},$$

where  $C'_4$  is given in Proposition 2.1, and

$$\hat{\tau}_{k,l}^3 = \inf \left\{ t \leq T \mid \int_{kT}^{kT+t \wedge \hat{\tau}_{k,l}^1 \wedge \hat{\tau}_{k,l}^2} l(\hat{u}_1(s), \hat{u}_2(s)) \|\hat{r}(s)\|^2 ds > C_*(d_0) e^{a - \frac{\alpha}{2}(k-l)T} \right\},$$

where  $a, d_0, \kappa$  are chosen below,  $C_*(d_0) = C^{**}(C'_4, d_0)$  is given in Lemma 2.3 and

$$\begin{aligned} \hat{u}_i &= u_i \text{ on } [0, kT], \quad \hat{u}_i(kT + \cdot) = \tilde{X}_1^b + \Phi(\tilde{X}_1^b, \eta_1^b, u_i(kT)) \text{ on } [kT, (k+1)T], \\ \hat{r} &= \hat{u}_1 - \hat{u}_2. \end{aligned}$$

We also take  $B = C'_4$  in the definition of  $l_0(k)$ .

Note that, with the notation of case b,  $\hat{u}_1$  (resp.  $\hat{u}_2$ ) is a solution of a truncated NLS equation under the the probability  $\mathbb{P}$  (resp.  $\tilde{\mathbb{P}}$ ). It follows that when  $(\tilde{X}_1^b, \eta_1^b)$  has law  $\lambda_1$  (resp.  $\lambda_2$ ) then  $\hat{u}_1$  (resp.  $\hat{u}_2$ ) is a solution in law of a truncated NLS equation. But if  $(\tilde{X}_1^b, \eta_1^b)$  has law  $\lambda_1$ ,  $\hat{u}_2$  is a solution of a truncated NLS equation with a drift term.

We wish to use the construction described in case b with the stopping time  $\tau = \tau_{k,l}$  given by

$$\tau_{k,l} = \hat{\tau}_{k,l}^1 \wedge \hat{\tau}_{k,l}^2 \wedge \hat{\tau}_{k,l}^3.$$

Then

$$|d(t)| \leq 1_{t \leq \tau_{k,l}} \sigma_0 |f(\tilde{X}_1^b(t), \Phi(\tilde{X}_1^b, \eta_1^b, (x, y_2))(t)) - f(\tilde{X}_1^b(t), \Phi(\tilde{X}_1^b, \eta_1^b, (x, y_1))(t))|$$

and it is not difficult to see that

$$|d(t)|^2 \leq c 1_{t \leq \tau_{k,l}} l (\hat{u}_1(t), \hat{u}_2(t)) \|\hat{r}(t)\|^2.$$

So that, by the definition of  $\tau_{k,l}$ , we get

$$(2.36) \quad \int_0^T |d(t)|^2 dt \leq C^*(d_0) \sigma_0^{-2} \exp\left(a - \frac{\alpha}{2}(k-l)T\right).$$

Hence the Novikov condition is satisfied and (2.10) holds.

Moreover, using the same argument as in the proof of (2.25), we obtain

$$(2.37) \quad \mathbb{P}((X_1^b, \eta_1^b) \neq (X_2^b, \eta_2^b) \text{ or } \tau < T) \leq \|\lambda_1 - \lambda_2\|_{var} + \nu_1(\hat{A}_1^c) + \nu_1(\hat{A}_3^c) + \nu_2(\hat{A}_2^c).$$

where

$$\hat{A}_i = \{(X, \eta) \mid \hat{\tau}_i = T\}, \quad i = 1, 2, 3.$$

It can be seen that for  $i = 1, 2$

$$(2.38) \quad \nu_i(\hat{A}_i^c) = \mathbb{P}\left(\sup_{t \in [0, T]} (E_{u_i, 4}(kT + t, lT) - C'_4(t + (k-l)T)) > \kappa + 1 + d_0^4 + d_0^8 \mid \mathcal{F}_{kT}\right).$$

The third term  $\nu_1(\hat{A}_3^c)$  cannot be written in terms of  $u_1$  and  $u_2$ . Indeed, when  $(\tilde{X}_1^b, \eta_1^b)$  has law  $\nu_1$ ,  $\hat{u}_2$  is a solution of an equation with a drift term.

**Remark 2.12.** *We remark here that Proposition 2.1 is not the Foias-Prodi estimate which is usually used in the coupling method. Here, we have also a drift term  $h$ . This modification is introduced precisely to treat the term  $\nu_1(\hat{A}_3^c)$ . We take  $h(\cdot) = bd(kT + \cdot) = \sigma_1 d(kT + \cdot)$ . This additional term is due to the fact that we introduce a term depending on  $r$  in the truncation. In the preceding papers using this kind of coupling method, this was not necessary and the Foias-Prodi estimate was used to get (2.36). However, this requires a path-wise Foias-Prodi estimate and we do not know if it holds in our situation.*

By (2.38), we have

$$\nu_1(\hat{A}_1^c) + \nu_1(\hat{A}_3^c) + \nu_2(\hat{A}_2^c) \leq 3\mathbb{P}(B_{l,k} \mid \mathcal{F}_{kT})$$

with

$$B_{l,k} = \left\{ \begin{array}{l} \sup_{t \in [0, T]} (E_{u_i,4}(kT + t, lT) - C'_4(t + (k-l)T)) > \kappa + 1 + d_0^4 + d_0^8, \quad i \in \{1, 2\} \\ \text{or } \int_{kT}^{kT + \tau_{k,l}} \left( \sum_{i=1}^2 \mathcal{H}(\hat{u}_i(s))^2 \right) \|\hat{r}(s)\|^2 ds \geq C_*(d_0) e^{2\kappa - \frac{\alpha}{4}(k-l)T} \end{array} \right\}.$$

Let us write

$$\begin{aligned} & \mathbb{P}(B_{l,k} \mid l_0(l) = l) \\ & \leq \sum_{i=1,2} \mathbb{P} \left( \sup_{t \in [0, T]} (E_{u_i,4}(kT + t, lT) - C'_4(t + (k-l)T)) > \kappa + 1 + d_0^4 + d_0^8 \mid l_0(l) = l \right) \\ & + \mathbb{P} \left( \int_{kT}^{kT + \tau_{k,l}} \left( \sum_{i=1}^2 \mathcal{H}(\hat{u}_i(s))^2 \right) \|\hat{r}(s)\|^2 ds \geq C_*(d_0) e^{2\kappa - \frac{\alpha}{4}(k-l)T} \mid l_0(l) = l \right). \end{aligned}$$

Using Proposition 2.4 with  $\kappa = \rho$  and solutions starting at  $lT$  and replacing  $T$  by  $kT$  we get, since  $l_0(l) = l$  implies  $\mathcal{H}(u_i(lT)) \leq d_0$ ,

$$\begin{aligned} & \mathbb{P} \left( \sup_{t \in [0, T]} (E_{u_i,4}(kT + t, lT) - C'_4(t + (k-l)T)) > \kappa + 1 + d_0^4 + d_0^8 \mid \mathcal{F}_{lT} \right) \\ & \leq \mathbb{P} \left( \sup_{t \in [0, T]} (E_{u_i,4}(kT + t, lT) - C'_4(t + (k-l)T)) > \kappa + 1 + \mathcal{H}(u_i(lT))^4 + \mathcal{H}(u_i(lT))^8 \mid \mathcal{F}_{lT} \right) \\ & \leq K_{4,q+1} (\kappa + (k-l)T)^{-q-1}. \end{aligned}$$

It follows

$$\begin{aligned} & \mathbb{P} \left( \sup_{t \in [0, T]} (E_{u_i,4}(kT + t, lT) - C'_4(t + (k-l)T)) > \kappa + 1 + d_0^4 + d_0^8 \mid l_0(l) = l \right) \\ & \leq K_{4,q+1} (\kappa + (k-l)T)^{-q-1}. \end{aligned}$$

Similarly, by Lemma 2.3, with  $h(t) = \sigma_l d(kT + t) 1_{t \leq \tau}$  which clearly satisfies (2.6) and  $\rho = a = \kappa$ , we have

$$\begin{aligned} & \mathbb{P} \left( \int_{kT}^{kT + \tau_{k,l}} \left( \sum_{i=1}^2 \mathcal{H}(\hat{u}_i(s))^2 \right) \|\hat{r}(s)\|^2 ds \geq C_*(d_0) e^{2\kappa - \frac{\alpha}{4}(k-l)T} \mid l_0(l) = l \right) \\ & \leq e^{-\kappa - \frac{\alpha}{2}(k-l)T} \\ & \leq c (\kappa + (k-l)T)^{-q-1}. \end{aligned}$$

Gathering these estimates, we obtain

$$\mathbb{P}(B_{l,k} \mid l_0(l) = l) \leq c (\kappa + (k-l)T)^{-q-1}.$$

By (2.37), (2.10), and (2.36), we obtain for  $k \geq l$  and on  $l_0(k) = l$

$$\begin{aligned} & \mathbb{P}((X_1, \eta_1) \neq (X_2, \eta_2) \text{ on } [kT, (k+1)T] \text{ or } B_{k,l} \mid \mathcal{F}_{kT}) \\ & \leq \|\lambda_1 - \lambda_2\|_{var} + 3\mathbb{P}(B_{l,k} \mid \mathcal{F}_{kT}) \\ & \leq \sqrt{\mathbb{E} \exp \left( c \int_0^T |d(s)|^2 dt \right)} - 1 + 3\mathbb{P}(B_{l,k} \mid \mathcal{F}_{kT}) \\ & \leq C^*(d_0) \sigma_0^{-1} e^{\kappa - \frac{\alpha}{4}(k-l)T} + 3\mathbb{P}(B_{l,k} \mid \mathcal{F}_{kT}). \end{aligned}$$

We have

$$\{l_0(k) = l\} \cap \{(X_1, \eta_1) = (X_2, \eta_2) \text{ on } [kT, (k+1)T]\} \cap B_{l,k}^c \subset \{l_0(k+1) = l\}.$$

Therefore, integrating over  $l_0(k) = l$  gives for  $T \geq T_1(d_0)$  and for  $k > l$

$$\mathbb{P}(l_0(k+1) \neq l, l_0(k) = l | l_0(l) = l) \leq C^*(d_0)\sigma_0^{-1}e^{\kappa - \frac{\kappa}{4}(k-l)T} + 3\mathbb{P}(B_{l,k} | l_0(l) = l).$$

Which implies that there exists  $\kappa > 0$  sufficiently large and  $d_0 > 0$  sufficiently small such that for any  $T > 0$

$$(2.39) \quad \mathbb{P}(l_0(k+1) \neq l, l_0(k) = l | l_0(l) = l) \leq \frac{1}{4}(1 + (k-l)T)^{-q}.$$

Remark that

$$\mathbb{P}(l_0(k) \neq l | l_0(l) = l) \leq \sum_{n=l}^{k-1} \mathbb{P}(l_0(n+1) \neq l, l_0(n) = l | l_0(l) = l),$$

so that, applying (2.39), we obtain

$$\mathbb{P}(l_0(k) \neq l | l_0(l) = l) \leq \frac{1}{4} + \frac{1}{T^{q+1/2}} \sum_{n=1}^{\infty} \frac{1}{k^q} \leq \frac{1}{4} + C_q \frac{1}{T^q},$$

which implies that there exists  $T_q > 0$  such that for  $T \geq T_q$

$$(2.40) \quad \mathbb{P}(l_0(k) = l | l_0(l) = l) \geq \frac{1}{2},$$

Combining (2.39) and (2.40), we establish (2.3).

**2.6. Conclusion.** We have just shown that the coupling constructed in section 2.5 satisfies (2.3) and (2.4) which are precisely (2.13) and (2.14). The constants used in (2.3) have been chosen in the preceding subsection. The random variables  $l_0(k)$  clearly satisfy (2.11) and, as already mentioned, (2.15) is implied by Lemma 2.5. Finally, (2.12) is a consequence of Proposition 2.1 with  $h = 0$  and Tchebychev inequality.

We deduce that Theorem 2.9 can be applied. Moreover (1.5) is a consequence of Lemma 2.5. This ends the proof of Theorem 1.1.

### 3. PROOF OF THEOREM 2.9

**3.1. Reformulation of the problem.** We already noticed that it is sufficient to establish (2.19).

Let us denote by  $k$  the unique integer such that  $t \in (2(k-1)T, 2kT]$ . Then

$$\begin{aligned} \mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_0(1+t-(k-1)T)^{-q}\right) &\leq \mathbb{P}(l_0(2k) \geq k) + \\ &\mathbb{P}\left(d_E(u_1(t), u_2(t)) > c_0(1+t-(k-1)T)^{-q} \text{ and } l_0(2k) < k\right). \end{aligned}$$

Thus applying (2.12), using  $2(t-(k-1)T) > t$ , it follows

$$(3.1) \quad \mathbb{P}\left(d_E(u_1(t), u_2(t)) > 2^q c_0(1+t)^{-q}\right) \leq \mathbb{P}(l_0(2k) \geq k) + 2^q c_0(1+t)^{-q}.$$

In order to estimate  $\mathbb{P}(l_0(2k) \geq k)$ , we introduce the following notation

$$l_0(\infty) = \limsup l_0.$$

Taking into account (2.11), we obtain that for  $l < \infty$

$$\{l_0(\infty) = l\} = \{l_0(k) = l, \text{ for any } k \geq l\}.$$

We deduce

$$(3.2) \quad \mathbb{P}(l_0(2k) \geq k) \leq \mathbb{P}(l_0(\infty) \geq k).$$

Taking into account (3.1), (3.2) and using a Chebyshev inequality, it is sufficient to obtain that there exist  $c_5 > 0$  such that

$$(3.3) \quad \mathbb{E}(l_0(\infty)^q) \leq c_5 (1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)).$$

**3.2. Definition of a sequence of stopping times.** Let

$$\tau = \min \{t \in T\mathbb{N} \mid \mathcal{H}(u_1(t)) + \mathcal{H}(u_2(t)) \leq R_0\}.$$

Then, the Lyapunov structure (2.15) implies that there exist  $\delta_0 > 0$  and  $c_6 > 0$  such that

$$(3.4) \quad \mathbb{E}(\exp(\delta_0 \tau)) \leq c_6 (1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)).$$

For a proof, see the proof of (1.56) at the end of the subsection 1.4 of [26].

We set

$$\hat{\sigma} = \min \{k \in \mathbb{N} \setminus \{0\} \mid l_0(k) > 1\}, \quad \sigma = \hat{\sigma}T.$$

Clearly  $\hat{\sigma} = 1$  if the two solutions do not get coupled at time 0 or  $T$ . Otherwise, they get coupled at 0 or  $T$  and remain coupled until  $\sigma$ .

From now, we fix  $q_1 \in (q, q_0 - 1)$ . Let us assume for the moment that there exists  $p_\infty$  such that if  $\mathcal{H}_0 \leq R_0$ , then

$$(3.5) \quad \begin{cases} \mathbb{E}(\sigma^{q_1} 1_{\sigma < \infty}) \leq K, \\ \mathbb{P}(\sigma = \infty) \geq p_\infty > 0. \end{cases}$$

The proof is given at the end of this section.

Now we build a sequence of stopping times

$$\tau_0 = \tau,$$

$$\hat{\sigma}_{k+1} = \min \{l \in \mathbb{N} \setminus \{0\} \mid lT > \tau_k \text{ and } l_0(l)T > \tau_k + T\}, \quad \sigma_{k+1} = \hat{\sigma}_{k+1}T$$

$$\tau_{k+1} = \sigma_{k+1} + \tau_0 \theta_{\sigma_{k+1}},$$

where  $(\theta_t)_t$  is the shift operator. The idea is the following. We wait the time  $\tau_k$  to enter the ball of radius  $R_0$ . Then, if we do not start coupling at time  $\tau_k$ , we try to couple at time  $\tau_k + T$ . If we fail to start coupling at time  $\tau_k$  or  $\tau_k + T$  we set  $\sigma_k = \tau_k + T$  else we set  $\sigma_k$  the time the coupling fails ( $\sigma_k = \infty$  if the coupling never fails). Then if  $\sigma_k < \infty$ , we retry to couple after entering in the ball of radius  $R_0$ . The fact that  $R_0 \geq d_0$  implies that  $l_0(\tau_k) \in \{\tau_k, \infty\}$ .

Note that we clearly have  $l_0(\tau_k) \in \{\tau_k, \infty\}$  and  $l_0(\sigma_k) \in \{\sigma_k, \infty\}$ , and the  $l_0$ -Markov property implies the strong Markov property when conditioning with respect to  $\mathcal{F}_{\tau_k}$  or  $\mathcal{F}_{\sigma_k}$ .

We infer from the  $l_0$ -Markov property of  $V$  that

$$\sigma_{k+1} = \tau_k + \sigma_0 \theta_{\tau_k},$$

which implies

$$\tau_{k+1} = \tau_k + \rho_0 \theta_{\tau_k}, \quad \text{where } \rho = \sigma + \tau_0 \theta_\sigma.$$

**3.3. Polynomial estimate on  $\rho$ .** We first establish that there exist  $K_0$  such that for any  $V_0$  such that  $\mathcal{H}_0 \leq R_0$

$$(3.6) \quad \mathbb{E}_{V_0} (\rho^{q_1} 1_{\rho < \infty}) \leq K_0.$$

Notice that for any  $V_0$  such that  $\mathcal{H}_0 \leq R_0$ ,

$$(3.7) \quad \mathbb{E}_{V_0} (\rho^{q_1} 1_{\rho < \infty}) \leq c (\mathbb{E}_{V_0} (\sigma^{q_1} 1_{\sigma < \infty}) + \mathbb{E} ((\tau \circ \theta_\sigma)^{q_1} 1_{\tau \circ \theta_\sigma < \infty} 1_{\sigma < \infty})).$$

Applying the  $l_0$ -Markovian property and (3.4), we obtain

$$\mathbb{E} ((\tau \circ \theta_\sigma)^{q_1} 1_{\tau \circ \theta_\sigma < \infty} 1_{\sigma < \infty} | \mathcal{F}_\sigma) \leq c_6 (1 + \mathcal{H}(u_1(\sigma)) + \mathcal{H}(u_2(\sigma))) 1_{\sigma < \infty},$$

which implies by applying the Lyapunov structure (2.15)

$$(3.8) \quad \mathbb{E} ((\tau \circ \theta_\sigma)^{q_1} 1_{\tau \circ \theta_\sigma < \infty}) \leq c_6 (1 + 2K'(R_0 + \mathbb{E}(\sigma 1_{\sigma < \infty}))).$$

Applying (3.5) and (3.8) to (3.7), we obtain (3.6).

**3.4. Conclusion.** Applying a convexity inequality, we obtain

$$\mathbb{E} (\tau_k^{q_1} 1_{\tau_k < \infty}) \leq (k+1)^{(q_1-1)^+} \left( \mathbb{E} \tau^{q_1} + \sum_{n=0}^{k-1} \mathbb{E} (\rho \circ \theta_{\tau_n})^{q_1} 1_{\rho \circ \theta_{\tau_n} < \infty} \right).$$

Combining the  $l_0$ -Markov property, (3.4) and (3.6) gives

$$(3.9) \quad \mathbb{E} (\tau_k^{q_1} 1_{\tau_k < \infty}) \leq C(k+1)^{1 \vee q_1} (1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)).$$

Now, we are able to estimate  $\mathbb{E} (l_0(\infty)^q)$

$$\mathbb{E} (l_0(\infty)^q) \leq c \left( 1 + \sum_{n=0}^{\infty} \mathbb{E} (\tau_n^q 1_{\tau_n < \infty} 1_{k_0=n}) \right),$$

where

$$k_0 = \inf \{k \in \mathbb{N} \mid \sigma_{k+1} = \infty\}.$$

Then, applying an Holder inequality, we obtain

$$\mathbb{E} (l_0(\infty))^q \leq c \left( 1 + \sum_{n=0}^{\infty} \mathbb{E} (\tau_n^{pq} 1_{\tau_n < \infty})^{\frac{1}{p}} (\mathbb{P}(k_0 = n))^{\frac{1}{p'}} \right).$$

Using the second inequality of (3.5) and  $\tau < \infty$ , we obtain from the  $l_0$ -Markov property that

$$(3.10) \quad \mathbb{P}(k_0 > n) \leq (1 - p_\infty)^n.$$

It follows that  $k_0 < \infty$  almost surely and that

$$l_0(\infty) \in \{\tau_{k_0}, \tau_{k_0} + 1\}.$$

Therefore  $l_0(\infty) < \infty$  almost surely and applying (3.9), we obtain that if  $pq = q_1$

$$\mathbb{E} (l_0(\infty))^q \leq C \left( \sum_{n=0}^{\infty} (n+1)^{\frac{1}{p} \vee q} (1 - p_\infty)^{\frac{n}{p'}} \right) (1 + \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2)).$$

Thus (3.3) is established and we can conclude.

**3.5. Proof of (3.5).** Now we establish (3.5). There are two cases. The first case is  $l_0(0) = 0$ . Then, applying (2.13), we obtain that

$$\mathbb{P}(\sigma = \infty) \geq \prod_{k=0}^{\infty} \mathbb{P}(l_0(k+1) = 0 | l_0(k) = 0) \geq \prod_{k=0}^{\infty} p_k.$$

The second case is  $l_0(0) = \infty$ . Then

$$\mathbb{P}(\sigma = \infty) \geq \mathbb{P}(l_0(1) = 1) \prod_{k=1}^{\infty} \mathbb{P}(l_0(k+1) = 1 | l_0(k) = 1).$$

Since  $\mathcal{H}_0 \leq R_0$ , then applying (2.13) and (2.14)

$$\mathbb{P}(\sigma = \infty) \geq \prod_{k=-1}^{\infty} p_k.$$

Since  $p_k > 0$  and  $1 - p_k$  decreases to 0 faster than  $k^{-q_0}$  with  $q_0 > 1$ , then the product converges and in the two cases

$$(3.11) \quad \mathbb{P}(\sigma = \infty) \geq p_{\infty} = \prod_{k=-1}^{\infty} p_k > 0.$$

Notice that (2.13) implies

$$\begin{aligned} \mathbb{P}(\sigma = n) &\leq \mathbb{P}(l_0(n+1) \neq n | l_0(n) = 0) + \mathbb{P}(l_0(n+1) \neq n | l_0(n) = 1), \\ &\leq 2c_1 (1 + (n-1)T)^{-q_0}, \end{aligned}$$

which gives the first inequality of (3.5) and allows to conclude because  $q_1 < q_0 - 1$ .

#### 4. PROOF OF THE A PRIORI ESTIMATES

As already mentioned, the various computations made in this section are not rigorous. Indeed, the solutions are not smooth enough to apply Ito formula. A suitable approximation could be used to justify the result rigorously.

##### **Ito Formula for $|u|^6$**

Applying Ito Formula to  $|u|^6$ , we obtain

$$d|u|^6 + 6\alpha |u|^6 dt = 6|u|^4 (u, bdW) + 12|u|^2 |b^*u|^2 dt + 3B_0 |u|^4 dt.$$

Since  $b^*$  is a bounded operator on  $L^2(0, 1)$ ,

$$12|u|^2 |b^*u|^2 \leq 12B_0 |u|^4.$$

We deduce

$$12|u|^2 |b^*u|^2 + 3B_0 |u|^4 \leq \alpha |u|^6 + C,$$

and

$$(4.1) \quad d|u|^6 + 5\alpha |u|^6 dt \leq 6|u|^4 (u, bdW) + C dt.$$

##### **Ito Formula for $\mathcal{H}$**

Applying Ito Formula to  $\mathcal{H}_*$ , we obtain

$$(4.2) \quad d\mathcal{H}_*(u) + \alpha \left( \|u\|^2 - |u|_4^4 \right) dt = dM_* + B_1 dt + I_* dt,$$

where

$$\begin{aligned} dM_* &= \left( Au - |u|^2 u, bdW \right), \\ I_* &= - \sum_{n=1}^{\infty} b_n^2 \int_{[0,1]} \left( 2\Re(u(t, x) \overline{e_n(x)})^2 + |e_n(x)|^2 |u(t, x)|^2 \right) dx. \end{aligned}$$

Note that, since  $b^*A$  is a bounded operator from  $L^2(0, 1)$  to  $H_0^1(0, 1)$ ,  $M_*$  is well defined. Recalling that  $|e_n|_{\infty} = 1$ , we obtain

$$I_* \leq 3B_0 |u|^2 \leq \alpha c_0 |u|^6 + C.$$

Recalling that  $|\cdot|_4^4 \leq \frac{1}{4} \|\cdot\|^2 + c_0 |\cdot|^6$ , we infer from (4.1), (4.2) and the last inequality that

$$(4.3) \quad d\mathcal{H}(u) + \frac{3}{2}\alpha\mathcal{H}(u)dt \leq dM_1 + C_1dt,$$

where

$$dM_1 = dM_* + 6c_0 |u|^4 (u, bdW).$$

**Ito Formula for  $\mathcal{H}^k$**

Applying Ito Formula to  $\mathcal{H}^k$  for  $k \in \mathbb{N} \setminus \{0\}$ , we obtain similarly as above

$$(4.4) \quad d\mathcal{H}(u)^k + \frac{3}{2}\alpha k\mathcal{H}(u)^k dt \leq dM_k + k\mathcal{H}(u)^{k-1}C_1dt + \frac{k(k-1)}{2}\mathcal{H}(u)^{k-2}d\langle M_1 \rangle,$$

where

$$dM_k = k\mathcal{H}(u)^{k-1}dM_1.$$

Note that, since  $b^*A$  is a bounded operator from  $L^2(0,1)$  to  $H_0^1(0,1)$  and  $b^*$  is bounded from  $L^1(0,1)$  to  $L^2(0,1)$ ,

$$\left| b^* \left( Au - |u|^2 u \right) \right|^2 \leq 4B_1 \|u\|^2 + cB_1 |u|_3^6,$$

it follows from a Gagliardo-Nirenberg inequality

$$\left| b^* \left( Au - |u|^2 u \right) \right|^2 \leq cB_1 \left( \|u\|^2 + |u|^{10} \right).$$

Now, we write

$$d\langle M_1 \rangle \leq 2 \left| b^* \left( Au - |u|^2 u \right) \right|^2 + 72c_0^2 |u|^8 |b^*u|^2,$$

and deduce that

$$d\langle M_1 \rangle \leq cB_1 \left( \|u\|^2 + |u|^{10} \right),$$

and

$$(4.5) \quad d\langle M_1 \rangle \leq cB_1 \left( 1 + \mathcal{H}(u)^{\frac{5}{3}} \right).$$

Gathering (4.4) and (4.5) and using once more an arithmetico-geometric inequality, we obtain

$$(4.6) \quad d\mathcal{H}(u)^k + \alpha k\mathcal{H}(u)^k dt \leq dM_k + C_k'' dt.$$

**Proof of Lemma 2.5**

Multiplying (4.6) by  $e^{\alpha kt}$  yields

$$d \left( e^{\alpha kt} \mathcal{H}(u)^k \right) \leq e^{\alpha kt} dM_k + e^{\alpha kt} C_k'' dt.$$

By integration we obtain

$$e^{\alpha kt} \mathcal{H}(u(t))^k \leq \mathcal{H}(u_0)^k + \int_0^t e^{\alpha ks} dM_k(s) + \frac{C_k''}{\alpha k} e^{\alpha kt}$$

and

$$\mathcal{H}(u(t))^k \leq \mathcal{H}(u_0)^k e^{-\alpha kt} + \int_0^t e^{-\alpha k(t-s)} dM_k(s) + \frac{C_k''}{\alpha k},$$

which yields, by taking the expectation, the first inequality of Lemma 2.5.

Let  $M > 0$  and  $\tau \leq M$  be a bounded stopping time. Then, integrating (4.6) between 0 and  $\tau$  and taking the expectation yields

$$\mathbb{E} \left( \mathcal{H}(u(\tau))^k \right) \leq \mathcal{H}(u_0)^k + C_k'' \mathbb{E}(\tau),$$

which is the second inequality of Lemma 2.5 for bounded stopping times. Assume now that  $\tau$  is a general stopping time. We consider the second inequality of Lemma 2.5 for the stopping time  $\tau \wedge M$  and we take the limit when  $M \rightarrow \infty$ . The second inequality of Lemma 2.5 for  $\tau$  follows from Fatou Lemma.

**Proof of Proposition 2.4**

We first note that

$$d \langle M_k \rangle = k^2 \mathcal{H}(u)^{2(k-1)} d \langle M_1 \rangle,$$

so that, taking into account (4.5),

$$(4.7) \quad d \langle M_k \rangle \leq c_k (1 + \mathcal{H}(u)^{2k}) ds$$

Taking the expectation of (4.6), we obtain for any  $k \geq 1$

$$(4.8) \quad \mathbb{E} \int_0^t \mathcal{H}(u(s))^k dt \leq C_k (\mathcal{H}(u_0)^k + t).$$

Hence, for any  $p \geq 1$ ,

$$(4.9) \quad \mathbb{E} \langle M_k \rangle^p (t) \leq C_{k,p} (\mathcal{H}(u_0)^{2kp} + t^p).$$

Applying the maximal martingale inequality and taking into account (4.6), we infer from (4.9) the first inequality of Proposition 2.4.

Applying the maximal martingale inequality on  $[n, n + 1]$ ,  $n \geq 0$ , we have

$$\mathbb{P} \left( \sup_{[n, n+1]} M_k > a + \mathcal{H}(u_0)^{2k} + n + 1 \right) \leq c_p \frac{\mathbb{E} \langle M_k \rangle^{p+1} (n + 1)}{(a + \mathcal{H}(u_0)^{2k} + n + 1)^{2p+2}}.$$

It follows from (4.9) that

$$(4.10) \quad \mathbb{P} \left( \sup_{[n, n+1]} M_k > a + \mathcal{H}(u_0)^{2k} + n + 1 \right) \leq \frac{c_p C'_{k,p+1}}{(a + \mathcal{H}(u_0)^{2k} + n + 1)^{p+1}}.$$

Now, summing (4.10) over  $n \geq T$ , for  $T$  integer, we obtain that for any  $(p, k) \in (\mathbb{N} \setminus \{0\})^2$  there exists  $K_{k,p}$  such that

$$(4.11) \quad \mathbb{P} \left( \sup_{t \in [T, \infty)} M_k(t) > 1 + a + \mathcal{H}(u_0)^{2k} + t \right) \leq K_{k,p} (a + T)^{-p}, \quad T > 0.$$

Taking into account (4.6), this implies the second inequality of Proposition 2.4.

**Proof of Proposition 2.6**

Combining Lemma 2.5 applied to  $\tau = t$  and Chebyshev's inequality, we obtain

**Lemma 4.1.** *Let  $(u_i, W_i)_{i=1,2}$  be a couple of solutions of (1.1), (1.2) such that  $W_1$  and  $W_2$  are two cylindrical Wiener process on  $L^2([0, 1])$ . If  $R_0 \geq \left( \sum_{i=1}^2 \mathcal{H}(u_0^i) \right) \vee C_1$ , then*

$$\mathbb{P} (\mathcal{H}(u_1(t)) + \mathcal{H}(u_2(t)) \geq 4C_1) \leq \frac{1}{2},$$

provided  $t \geq \theta_1(R_0) = \frac{1}{\alpha} \ln \frac{R_0}{C_1}$ .

It follows from Lemma 4.1 that it is sufficient to establish Proposition 2.6 for  $R_0 = 4C_1$  and  $t = T_{-1}(R_0, R_1)$  (instead of  $t \geq T_{-1}(R_0, R_1)$ ). From now on, we only consider the case  $R_0 = 4C_1$ .

Let  $T, \delta > 0$ . Applying Chebyshev inequality, we obtain  $N_{-2} = N_{-2}(T, \delta) \in \mathbb{N}$  such that

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|bQ_{N_{-2}}W(t)\|_3 > \frac{\delta}{2} \right) \leq \frac{2}{\delta} \sum_{n > N_{-2}} \mu_n^3 b_n^2 \leq \frac{1}{2}.$$

Moreover  $P_{N_{-2}}W$  is a finite dimensional brownian motion and it is classical that

$$\pi_{-3}(T, \delta, N_{-2}) = \mathbb{P} \left( \sup_{t \in [0, T]} |P_{N_{-2}}W(t)| \leq \frac{\delta}{2} \|b\|_{\mathcal{L}_2(L_2(D), H^3(D))}^{-1} \right) > 0.$$

Writing

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|bW(t)\|_3 \leq \delta \right) \geq \mathbb{P} \left( \sup_{t \in [0, T]} \|bQ_{N_{-2}}W(t)\|_3 \leq \frac{\delta}{2} \right) \pi_{-3}(T, \delta, N_{-2}),$$

it follows

$$(4.12) \quad \pi_{-2}(T, \delta) = \mathbb{P} \left( \sup_{t \in [0, T]} \|bW(t)\|_3 \leq \delta \right) > 0.$$

It thus suffices to prove that there exists  $T_{-1}(R_1), \delta_{-1}(R_1) > 0$  such that

$$(4.13) \quad \left\{ \sup_{t \in [0, T_{-1}]} \|bW(t)\|_3 \leq \delta_{-1} \right\} \subset \left\{ \mathcal{H}(u(T_{-1}, u_0)) \leq \frac{1}{2}R_1 \right\},$$

provided  $\mathcal{H}(u_0) \leq R_0$ .

**Proof of (4.13)**

Let us set

$$v = u(\cdot, u_0) - bW,$$

then

$$(4.14) \quad \frac{dv}{dt} + \alpha v + iAv - i|bW + v|^2(bW + v) = -(\alpha + iA)bW.$$

Taking the scalar product between (4.14) and  $2v$ , we obtain

$$\frac{d|v|^2}{dt} + 2\alpha|v|^2 = 2 \left( v, i|bW + v|^2(bW + v) - (\alpha + iA)bW \right).$$

Since

$$\left( v, i|bW + v|^2 v \right) = 0,$$

applying Hölder inequalities and Sobolev Embedding  $H^1(D) \subset L^\infty(0, 1)$ , we deduce

$$\frac{d|v|^2}{dt} + 2\alpha|v|^2 \leq c \|bW\|_3 \left( 1 + \|bW\|_3^2 \right) \left( 1 + \|v\|^3 \right).$$

Applying Ito Formula to  $|v|^6$ , we deduce

$$(4.15) \quad \frac{d|v|^6}{dt} + 6\alpha|v|^6 \leq c \|bW\|_3 \left( 1 + \|bW\|_3^2 \right) \left( 1 + \|v\|^9 \right).$$

Taking the scalar product between (4.14) and  $Av - |v|^2 v$ , we obtain

$$\frac{d\mathcal{H}_*(v)}{dt} + \alpha \|v\|^2 = - \left( Av - |v|^2 v, (\alpha + iA)bW \right) + \alpha \left( |v + bW|^2 (v + bW), v \right).$$

Since

$$I_1 = \alpha \left( \left( |v + bW|^2 (v + bW), v \right) - |v|_4^4 \right) = \alpha \left( |v + bW|^2 (v + bW) - |v|^2 v, v \right),$$

we obtain

$$(4.16) \quad \frac{d\mathcal{H}_*(v)}{dt} + \alpha \left( \|v\|^2 - |v|_4^4 \right) = I_1 + I_2,$$

where

$$I_2 = - \left( Av - |v|^2 v, (\alpha + iA) bW \right).$$

Recalling that for any  $z, h \in \mathbb{C}^2$

$$\left| |z+h|^2 (z+h) - |z|^2 z \right| \leq |h| \left( |z|^2 + |h|^2 \right),$$

and applying Hölder inequality and the Sobolev Embedding  $H^1(D) \subset L^\infty(0,1)$ , we obtain

$$I = I_1 + I_2 \leq c \|bW\|_3 \left( 1 + \|v\|^3 \right) \left( 1 + \|bW\|_3^2 \right).$$

It follows from (4.15), (4.16) and the last inequality that

$$(4.17) \quad \frac{d\mathcal{H}(v)}{dt} + 2\alpha\mathcal{H}(v) \leq c \|bW\|_3 \left( 1 + \|bW\|_3^2 \right) \left( 1 + \mathcal{H}(v)^5 \right).$$

Let  $T, \delta, M > 0$  and assume that

$$\sup_{t \in [0, T]} \|bW(t)\|_3 \leq \delta.$$

We set

$$\tau = \inf \{ t \in [0, T] \mid \mathcal{H}(v) \leq 3R_0 \}.$$

Integrating (4.17), we obtain

$$(4.18) \quad \mathcal{H}(v(t)) \leq e^{-2\alpha t} R_0 + \frac{c}{2\alpha} \delta (1 + \delta^2) (1 + R_0^5),$$

provided  $t \leq \tau$ .

Now we choose  $\delta \leq \delta_{-2}(R'_1) > 0$  such that

$$\frac{c}{2\alpha} \delta (1 + \delta^2) (1 + R_0^5) \leq R'_1 \wedge R_0.$$

It follows from (4.18) that

$$\tau = T,$$

and that

$$\mathcal{H}(v(T)) \leq 2R'_1,$$

provided

$$T \geq \frac{1}{2\alpha} \ln \left( \frac{R'_1}{R_0} \right).$$

In order to conclude, we remark that

$$\mathcal{H}(u(T)) \leq c (\mathcal{H}(bW(T)) + \mathcal{H}(v(T))) \leq c(\delta^2(1 + \delta^4) + R'_1).$$

Then, choosing  $\delta$  and  $R'_1$  sufficiently small, we obtain (4.13).

5. PROOF OF THE FOIAS-PRODI ESTIMATES

The aim of this section is to establish Proposition 2.1.

$L^2$  estimates

Taking into account (2.5), we deduce that the difference of the two solutions  $r = u_1 - u_2$  satisfies the equation

$$(5.1) \quad \frac{dr}{dt} + \alpha r + iAr = iQ_N \left( |u_1|^2 u_1 - |u_2|^2 u_2 \right).$$

Applying Ito Formula to  $|r|^2$ , we obtain

$$\frac{d|r|^2}{dt} + 2\alpha |r|^2 = 2 \left( ir, |u_2|^2 u_2 - |u_1|^2 u_1 \right).$$

Since

$$\left| |u_2|^2 u_2 - |u_1|^2 u_1 \right| \leq c \left( \sum_{i=1}^2 |u_i|^2 \right) |r|,$$

it follows

$$\frac{d|r|^2}{dt} + 2\alpha |r|^2 \leq c \int_{[0,1]} \left( \sum_{i=1}^2 |u_i|^2 \right) |r|^2 dx.$$

Using the Sobolev Embedding  $H^1(0, 1) \subset L^\infty(0, 1)$ , we obtain

$$(5.2) \quad \frac{d|r|^2}{dt} + 2\alpha |r|^2 \leq c \left( \sum_{i=1}^2 \mathcal{H}(u_i) \right) |r|^2.$$

We deduce as in the proof of (4.3)

$$d\mathcal{H}(u_i) + \frac{3}{2}\alpha\mathcal{H}(u_i)dt \leq dM_1^i + C_1 dt + 1_{i=1}(G, h)dt,$$

where

$$\begin{cases} dM_1^i &= \left( Au_i - |u_i|^2 u_i, bdW_i \right) + 6c_0 |u_i|^4 (u_i, bdW_i), \\ G &= Au_1 - |u_1|^2 u_1 + 6c_0 |u_1|_{L^2}^4 u_1. \end{cases}$$

It follows from Sobolev Embeddings and Hölder inequalities that

$$\|G\|_{-1} \leq c(1 + \mathcal{H}(u_1))^{\frac{5}{6}}.$$

Hence we deduce from (2.6) that

$$(G, h) \leq c(1 + \mathcal{H}(u_1) + \mathcal{H}(u_2))^4.$$

Taking into account (5.2), it follows

$$(5.3) \quad dZ_1 + 2\alpha Z_1 dt \leq c \left( 1 + \sum_{i=1}^2 \mathcal{H}(u_i)^4 \right) |r|^2 dt + |r|^2 dM_\#,$$

where

$$Z_1 = \left( \sum_{i=1}^2 \mathcal{H}(u_i) \right) |r|^2$$

and

$$dM_\# = dM_1^1 + dM_1^2.$$

**Ito Formula for  $J$**

Now we rewrite (5.1) in the form

$$(5.4) \quad \frac{dr}{dt} + \alpha r + iAr = -i\frac{1}{2}Q_N \left( (|u_1|^2 + |u_2|^2)r + \Re((u_1 + u_2)\bar{r})(u_1 + u_2) \right).$$

Applying Ito Formula to  $J_*(u_1, u_2, r)$ , we obtain

$$(5.5) \quad \begin{aligned} dJ_* + 2\alpha J_* dt &= g(u_1, u_2, r)dt + g(u_2, u_1, r)dt + \psi(u_1, u_2, r)(bdW_1) \\ &+ \psi(u_1, u_2, r)(h(t))dt + \psi(u_2, u_1, r)(bdW_2) + I_1(r)dt + dI_2(r, dt), \end{aligned}$$

where

$$\begin{aligned} g(u_1, u_2, r) &= \begin{cases} 2 \int_{[0,1]} \left( \Re(\bar{u}_1(\alpha u_1 + iAu_1 - i|u_1|^2 u_1)) |r|^2 \right) dx \\ + 2 \int_{[0,1]} \Re(\bar{r}(u_1 + u_2)) \Re(\bar{r}(\alpha u_1 - iAu_1 + i|u_1|^2 u_1)) dx \end{cases}, \\ \psi(u_1, u_2, r)(h) &= 2 \int_{[0,1]} \left( \Re(\bar{u}_1 h) |r|^2 \right) dx + 2 \int_{[0,1]} \Re(\bar{r}(u_1 + u_2)) \Re(\bar{r}h) dx, \\ I_1(r) &= -\sum_{n=1}^{\infty} b_n^2 \int_{[0,1]} \left( |e_n|^2 |r|^2 + \Re(e_n \bar{r})^2 \right) dx, \\ dI_2(r, t) &= -\sum_{p,q=1}^{\infty} b_p b_q \left( \left( \int_{[0,1]} \Re(e_p \bar{r}) \Re(e_q \bar{r}) dx \right) d\langle (W_1, e_p), (W_2, e_q) \rangle \right). \end{aligned}$$

Applying an integration by part to  $Au_1$ , Hölder inequality and the Sobolev Embedding  $H^{\frac{3}{4}}(0, 1) \subset L^\infty(0, 1)$ , we obtain

$$(5.6) \quad g(u_1, u_2, r) \leq \left( 1 + \sum_{i=1}^2 \|u_i\|^6 \right) \|r\| \|r\|_{\frac{3}{4}}.$$

We deduce from Hölder inequality that

$$\psi(u_1, u_2, r)(h(t)) \leq \left( \sum_{i=1}^2 |u_i|_\infty \right) |h(t)| |r|_4^2.$$

Taking into account (2.6) and applying the Sobolev Embeddings  $H^1(0, 1) \subset L^\infty(0, 1)$  and  $H^{\frac{1}{2}}(0, 1) \subset L^4(0, 1)$ , we obtain

$$(5.7) \quad \psi(u_1, u_2, r)(h(t)) \leq c\kappa_0 \left( 1 + \sum_{i=1}^2 \mathcal{H}(u_i)^{\frac{7}{2}} \right) \|r\|_{\frac{1}{2}}^2.$$

Recalling that  $|e_n|_\infty = 1$ , we obtain

$$(5.8) \quad I_1(r) \leq 3B_0 |r|^2.$$

Note that we have no information on the law of the couple  $(W_1, W_2)$ . Hence, we cannot compute  $d\langle (W_1, e_p), (W_2, e_q) \rangle$ . However we know that

$$d|\langle (W_1, e_p), (W_2, e_q) \rangle| \leq dt.$$

Hence

$$d|I_2(r, t)| = \left( \int_{[0,1]} \Re \left( \sum_{n=1}^{\infty} (b_n e_n) \bar{r} \right)^2 dx \right) dt.$$

Applying the following Schwartz inequality

$$\left( \sum_{n=1}^{\infty} b_n \right)^2 \leq \left( \sum_{n=1}^{\infty} \mu_n b_n^2 \right) \left( \sum_{n=1}^{\infty} \frac{1}{\mu_n} \right) \leq cB_1,$$

we deduce from  $|e_n|_\infty = 1$  that

$$(5.9) \quad d|I_2(r, t)| \leq cB_1 |r|^2 dt.$$

Combining (5.5), (5.6), (5.7), (5.8), and (5.9), we obtain

$$(5.10) \quad dJ_* + 2\alpha J_* dt \leq c \left( 1 + \sum_{i=1}^2 \mathcal{H}(u_i)^4 \right) \|r\| \|r\|_{\frac{3}{4}} dt + dM_{\#\#},$$

where

$$dM_{\#\#} = (\psi(u_1, u_2, r)(bdW_1) + \psi(u_2, u_1, r)(bdW_2)).$$

Summing (5.3) and (5.10), we obtain

$$(5.11) \quad dJ + 2\alpha J dt \leq c \left( 1 + \sum_{i=1}^2 \mathcal{H}(u_i)^4 \right) \|r\| \|r\|_{\frac{3}{4}} dt + dM,$$

where

$$dM = dM_{\#\#} + c_1 |r|^2 dM_{\#}.$$

**Conclusion**

Since  $\|r\|_{\frac{3}{4}} \leq \mu_{N+1}^{-\frac{1}{8}} \|r\|$  then there exists  $\Lambda > 0$  such that

$$(5.12) \quad dJ + \left( 2\alpha - \frac{\Lambda}{\mu_{N+1}^{\frac{1}{8}}} l(u_1, u_2) \right) J dt \leq dM.$$

Multiplying (5.12) by  $\exp\left(2\alpha s - \Lambda \mu_{N+1}^{-\frac{1}{8}} \int_0^s l(u_1(s'), u_2(s')) ds'\right)$ , we obtain that

$$(5.13) \quad J_{FP}^N(t \wedge \tau) \leq \int_0^{t \wedge \tau} \exp\left(\frac{3}{2}\alpha s - \Lambda \mu_{N+1}^{-\frac{1}{8}} \int_0^s l(u_1(s'), u_2(s')) ds'\right) dM(s).$$

Fatou Lemma allows to conclude.

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