HIGH-FREQUENCY ASYMPTOTICS FOR SUBORDINATED ISOTROPIC FIELDS ON AN ABELIAN COMPACT GROUP[*]

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Abstract

Let $\widetilde{T}(g)$ be a random field indexed by an Abelian compact group G, and suppose that $\widetilde{T}$ has the form $\widetilde{T}(g) = F(T(g))$, where T is Gaussian and isotropic. The aim of this paper is to establish high-frequency central limit theorems for the Fourier coefficients associated to $\widetilde{T}$. The proofs of our main results involve recently established criteria for the weak convergence of multiple Wiener-Itô integrals. Our research is motivated by physical applications, mainly related to the probabilistic modelization of the Cosmic Microwave Background radiation. In this connection, the case of the n-dimensional torus is analyzed in detail.

Key Words – Gaussian fields; Isotropic fields; Central limit theorems; Abelian groups; Multiple Wiener-Itô integrals.

AMS classification – Primary 60B15; Secondary 60F05, 60G60

Running Title - ASYMPTOTICS FOR RANDOM FIELDS ON AN ABELIAN GROUP

1 Introduction

Let G be a connected compact Abelian group. The aim of this paper is to establish central limit theorems (CLTs) for the Fourier coefficients associated to a random field indexed by G, and subordinated to some real-valued isotropic Gaussian field $T = \{T(g) : g \in G\}$. By *isotropic* we mean that, for every $p \geq 1$ and every $h, g_1, ..., g_p \in G$,

$$\{T(hg_1), ..., T(hg_p)\} \overset{law}{=} \{T(g_1), ..., T(g_p)\}, \tag{1}$$

i.e. the finite-dimensional distributions of the "translated" process $g \mapsto T(hg)$ coincide with those of T, for every $h \in G$. As a consequence of the Peter-Weyl theorem (see e.g. [11]), the Gaussian field T always admits the expansion

$$T(g) = \sum_{\pi \in \hat{G}} a_\pi \chi_\pi(g), \quad g \in G, \tag{2}$$

where $\hat{G}$ is the collection of the irreducible unitary representations of G (that is, $\hat{G}$ is the *dual* of G – see e.g. [26]), χ_π is the *character* associated to a given $\pi \in \hat{G}$, and

$$a_\pi \triangleq \int_G T(g) \chi_\pi(g^{-1}) \, dg \tag{3}$$

with dg indicating the Haar measure (a more detailed discussion of the properties of the expansion (3) is deferred to the next section). Now consider a real-valued $F \in L^2(\mathbb{R}, \exp(-x^2/2)\, dx)$, and define the *subordinated field* $F[T]$ as

$$F[T](g) \triangleq F(T(g)), \quad \forall g \in G. \tag{4}$$

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Plainly, for a non-linear transformation F the field $F[T]$ is in general not Gaussian. However, since T is isotropic $F[T]$ is isotropic, and the Peter-Weyl Theorem yields again the spectral expansion

$$F[T](g) = \sum_{\pi \in \hat{G}} \widetilde{a}_{\pi}(F) \chi_{\pi}(g), \quad g \in G, \tag{5}$$

where

$$\widetilde{a}_{\pi}(F) \triangleq \int_G F[T](g) \chi_{\pi}\left(g^{-1}\right) dg. \tag{6}$$

Our aim in this paper is to investigate the asymptotic behavior of the complex-valued variable $\widetilde{a}_{\pi}(F)$, whenever the dual set $\hat{G}$ is infinite. More precisely, we shall establish sufficient (and in many cases, also necessary) conditions for the following CLT to hold:

$$\mathbb{E}\left[\left|\widetilde{a}_{\pi}(F)\right|^2\right]^{-\frac{1}{2}} \widetilde{a}_{\pi}(F) \underset{\{\pi\}}{\overset{law}{\to}} N + iN', \tag{7}$$

where N and N' are two independent centered Gaussian random variables with common variance equal to 1/2. In (7), and for the rest paper, the subscript $\{\pi\}$ means that $\{\pi\} = \{\pi_l : l = 1, 2, ...\}$ is an infinite sequence of elements of $\hat{G}$, and that the limit is taken as $l \to +\infty$. A central limit result such as (7) is called a *high-frequency central limit theorem*, in analogy with the case of G being a the n-dimensional torus $\mathbb{R}^n / (2\pi\mathbb{Z})^n$. Indeed, in this case one has that: (i) $\hat{G}$ can be identified with the class of complex-valued mappings of the type $\vartheta \mapsto \exp(i\mathbf{k}'\vartheta)$, where $\mathbf{k} \in \mathbb{Z}^n$ and $\vartheta \in (0, 2\pi]^n$, (ii) the class $\left\{\widetilde{a}_{\pi}(F) : \pi \in \hat{G}\right\}$ reduces to the collection of the coefficients $\{\widetilde{a}_{\mathbf{k}}(F) : \mathbf{k} \in \mathbb{Z}^n\}$ appearing in the usual Fourier expansion $F[T](\vartheta) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \widetilde{a}_{\mathbf{k}}(F) \exp(i\mathbf{k}'\vartheta)$, and (iii) the subscript $\{\pi\}$ in (7) may be replaced by the condition $\|\mathbf{k}\|_{\mathbb{Z}^n} \to +\infty$, where $\|\cdot\|_{\mathbb{Z}^n}$ stands for the Euclidean norm.

Our work is strongly motivated by physical applications; indeed, nonlinear transformations of Gaussian random fields emerge quite naturally in a variety of physical models. A particularly active area has recently been related to theoretical Cosmology, and more precisely, to so-called inflationary models aimed at the investigation of the dynamics of the gravitational potential around the Big Bang (see for instance [10] and [25]). In this area, the aim is the understanding of the primordial fluctuations which have provided the seeds for the large scale structure of the Universe as it is currently observed, i.e., the formation of structures such as clusters of galaxies, filaments, walls and all those inhomogeneities which have made our own existence possible. The currently favored scenario suggests that the primordial seeds for these inhomogeneities have actually been provided by quantum fluctuations in the gravitational potential, which have then been "freezed" as large scale fluctuations when the Universe experienced a phase of superluminal expansion known as inflation. In these models, the primordial gravitational potential is represented as a Gaussian field undergoing a small nonlinear perturbation, the simplest example being provided by the so-called Bardeen's potential

$$\widetilde{\Phi}(\vartheta) = \Phi(\vartheta) + f_{NL}(\Phi^2(\vartheta) - \mathbb{E}\Phi^2(\vartheta)) \ , \vartheta \in \Theta \ , \tag{8}$$

where $\Phi(\vartheta)$ denotes a zero-mean, isotropic Gaussian random field, with parameter space Θ; the nonlinearity parameter f_{NL} can be usually described explicitly in terms of fundamental physical constants. There is now an enormously vast physical literature on these Gaussian subordinated fields, see for instance [4], [17]; a recent and comprehensive survey is in [3]. The topological structure of Θ can vary across different physical models and it is not unusual to assume that ϑ belongs to the three-dimensional torus $\mathbb{R}^3/(2\pi\mathbb{Z})^3$ (see for instance [7], [8]).

Very recently it has become possible to place tight observational constraints on the predictions of inflationary models, by means of observations on the Cosmic Microwave Background radiation (CMB). CMB can be viewed as a snapshot of the Universe at the time of recombination, i.e. "soon after" the Big Bang (see again [10] for more detailed statements). It is directly related to the primordial gravitational potential, by means of a filtering equation known as the radiation transfer function. In the last few years huge satellite experiments by NASA and ESA have reached the level of resolution where models like (8) can be tested on the observations. A vast literature has focussed on such testing procedures (for instance [6], [16], [18]). An important feature of these procedures is their asymptotic behavior; in this framework, asymptotic is meant in the so-called high resolution sense, i.e. with respect to observations corresponding to frequencies which become higher and higher as the resolution of the experiment improves. On these components much effort for physical investigation is focussing, and it is therefore of fundamental importance to understand what is the high-frequency behavior of Gaussian subordinated fields (see also [1] for other statistical motivations). The present paper is a contribution in this direction; in future work we shall address related issues for random fields defined on homogenous spaces of non-Abelian groups, primarily the rotation group $SO(n)$, see [19].

The proofs of our main results rely on the classic representation of the function $F(.)$ in (4) as an infinite series of Hermite polynomials, and on recently established criteria for the weak convergence of multiple Wiener-Itô integrals – as proved in [20] and [24]. Our methodology, which involves the explicit computation of the norms associated to contraction operators, should be compared with the classic "method of diagrams" (see e.g. [5], [12] and [27]).

The plan of the paper is as follows: in Section 2 we introduce our general setting and we review some background material on random fields on groups. Section 3 is devoted to the statements of our main results, whose proofs are collected in Section 5, which builds upon background material on weak convergence of multiple stochastic integrals which is collected in Section 4. Section 6 addresses some joint convergence issues, whereas Section 7 is devoted to the analysis of general, square integrable transforms. Finally Section 8 specializes our results to the case of the n-dimensional torus, discussing the possible fulfillment of our necessary and sufficient conditions for the CLT by physically motivated models.

2 General setting

Given $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ stand, respectively, for the real and imaginary part of z. Let $(G, \mathbb{G})$ be a topological compact connected Abelian group, where $\mathbb{G}$ is a topology with a countable basis. As in formula (2), we shall denote by $\hat{G}$ the *dual* of G, i.e. $\hat{G}$ is the collection of all the equivalence classes of the unitary irreducible representations of G. The elements of $\hat{G}$ are noted π, σ, ...; the associated characters are written χ_π, χ_σ, and so on. It is well known that, since G is Abelian, every irreducible representation of G has dimension one. Moreover, since G is second countable (and therefore metrizable), $\hat{G}$ is at most countable. Recall also that $\hat{G}$ is itself an Abelian group (which in general fails to be compact), under the commutative group operation

$$(\pi, \sigma) \mapsto \pi\sigma \triangleq \pi \otimes \sigma, \tag{9}$$

where $\otimes$ indicates the tensor product between representations. The identity element of $\hat{G}$ is π_0, i.e. the trivial representation, and $\pi^{-1} = \overline{\pi}$, where $\overline{\pi}$ indicates complex conjugation. By using this notation, $\forall \sigma, \pi \in \hat{G}$ one has the obvious relations

$$\chi_\pi \chi_\sigma = \chi_{\pi\sigma} \quad \text{and} \quad \overline{\chi_\pi} = \chi_{\pi^{-1}} = \chi_{\overline{\pi}}; \tag{10}$$

moreover, by connectedness, χ_π is real-valued if, and only if, $\pi = \pi_0$. Observe that, since every $\pi \in \hat{G}$ has dimension one, the distinction between π and χ_π is immaterial (see e.g. [11, Corollary 4.1.2]). However, part of the results of this paper can be extended to the case of a non-commutative compact group (as the group of rotations $SO(3)$ – see e.g. [19]) and, to facilitate the connection between the two frameworks, we choose to adopt this slightly redundant notation throughout Sections 2 to 7. We note dg the unique Haar measure with mass 1 associated to G, and write $L^2(G) = L^2(G, dg)$ to indicate the space of complex-valued functions on G that are square-integrable with respect to dg. Since G is Abelian, the class $\left\{\chi_\pi : \pi \in \hat{G}\right\}$ is an orthonormal basis of $L^2(G)$. In what follows, G will always indicate a topological compact group such that the cardinality of $\hat{G}$ is infinite. The reader is referred e.g. to [9], [11, Chapter IV] or [14], for every unexplained notion or result concerning group representations.

We now consider a centered real-valued Gaussian random field $T = \{T(g) : g \in G\}$ which is isotropic in the sense of relation (1), and we shall assume for simplicity that $\mathbb{E}\left[T(g)^2\right] = 1$. As discussed in the introduction, the Peter-Weyl theorem implies that the spectral expansion (2) holds, where the convergence takes place in $L^2(\Omega \times G, \mathbb{P} \times dg)$. Note also that, for every fixed $g \in G$, the RHS of (2) converges in $L^2(\mathbb{P})$ (see e.g. [22] for general results concerning decompositions of isotropic fields).

Due to the isotropic and Gaussian assumptions, the class of random variables $\left\{a_\pi : \pi \in \hat{G}\right\}$ appearing in (3) has a special structure (compare [2]). This point is summarized in the following Lemma.

Lemma 1 *The family* $\left\{a_\pi : \pi \in \hat{G}\right\}$ *is composed of complex-valued Gaussian random variables such that*

1. $a_\pi = \overline{a}_{\pi^{-1}}$ *for every* $\pi \in \hat{G}$ *(in particular,* a_{π_0} *is real);*
2. *For any* $\pi, \sigma \in \hat{G}$ *such that* $\pi \notin \left\{\sigma, \sigma^{-1}\right\}$*, the coefficients* a_π *and* a_σ *are independent;*
3. *For every* $\pi \neq \pi_0$*, the random variables* $\Re(a_\pi)$ *and* $\Im(a_\pi)$ *are Gaussian, independent, centered and identically distributed (in particular,* $\mathbb{E}\Re(a_\pi)^2 = \mathbb{E}\Im(a_\pi)^2$*);*
4. *By noting*
$$C_\pi \triangleq \mathbb{E}|a_\pi|^2 = 2\mathbb{E}\left(\Re(a_\pi)^2\right) = 2\mathbb{E}\left(\Im(a_\pi)^2\right), \quad \pi \in \hat{G}, \tag{11}$$
one has $C_\pi = C_{\pi^{-1}}$ *and* $\sum_{\pi \in \hat{G}} C_\pi < +\infty$.

Proof. Point 1 is a consequence of (3). The isotropic assumption implies that $\forall \pi, \sigma \in \hat{G}$ such that $\pi \neq \sigma$, $\mathbb{E}\left[a_\pi \overline{a_\sigma}\right] = 0$. It follows that, if $\pi \notin \left\{\sigma, \sigma^{-1}\right\}$, $0 = \mathbb{E}\left[a_\pi \overline{a_\sigma}\right] = \mathbb{E}\left[a_\pi \overline{a_{\sigma^{-1}}}\right] = \mathbb{E}\left[a_\pi a_\sigma\right]$, thus giving Point 2. Now fix $\pi \neq \pi_0$. Point 2, implies that

$$\begin{aligned} 0 &= \mathbb{E}\left[a_\pi \overline{a_{\pi^{-1}}}\right] = \mathbb{E}\left[a_\pi a_\pi\right] \\ &= \mathbb{E}\left(\Re(a_\pi)^2\right) - \mathbb{E}\left(\Im(a_\pi)^2\right) + 2i\mathbb{E}\left(\Re(a_\pi)\Im(a_\pi)\right), \end{aligned}$$

giving immediately Point 3. Point 4 follows by combining Point 1 and Point 3. ■

Remarks – (a) The law of a collection of random variables $\left\{a_\pi : \pi \in \hat{G}\right\} \in \mathbb{C}^{\hat{G}}$ satisfying Points 1-3 of Lemma 1 is completely determined by the coefficients C_π defined in (11).

(b) Given a collection $\left\{a_\pi : \pi \in \hat{G}\right\} \in \mathbb{C}^{\hat{G}}$, satisfying Points 1-3 of Lemma 1 and such that $\sum_{\pi \in \hat{G}} C_\pi < +\infty$, we may always define a real-valued Gaussian isotropic random field $\overline{T}$ by setting $\overline{T}(g) = \sum_\pi a_\pi \chi_\pi(g)$.

Throughout the paper, we will systematically work under the following assumption.

Assumption I – Let $\left\{a_\pi : \pi \in \hat{G}\right\}$ be the Fourier coefficients defined in formula (3), and let $\{C_\pi : \pi \in \hat{G}\}$ be given by (11). Then, $C_\pi > 0$ for every $\pi \in \hat{G}$ (or, equivalently, $a_\pi \neq 0$, a.s.-$\mathbb{P}$, for every $\pi \in \hat{G}$).

Assumption I is a mild regularity condition on the behavior of the spectral density of T. Basically, it ensures that every field of the type $g \mapsto F(T(g))$, where F is a polynomial, admits an expansion of the type (5) such that $\widetilde{a}_\pi(F) \neq 0$ for every $\pi \in \hat{G}$, and therefore that the asymptotic behavior of the $\widetilde{a}_\pi(F)$'s is not trivial at the limit. Observe that the results of this paper extend easily to the case of a Gaussian field T, such that $a_\pi \neq 0$ for infinitely many π's (at the cost of some heavier notation).

We now note $L_0^2\left(\mathbb{R}, \exp\left(-x^2/2\right)dx\right)$ the class of real-valued functions on $\mathbb{R}$, such that $\int_{\mathbb{R}} F(x)\, e^{-x^2/2}dx = 0$. For a fixed $F \in L_0^2\left(\mathbb{R}, \exp\left(-x^2/2\right)dx\right)$, we define the (centered) subordinated field $F[T]$ as in (4).

As indicated in the introduction, $F[T]$ is isotropic and admits the spectral representation (5), where the convergence of the series takes place in $L^2(\Omega \times G, \mathbb{P} \times dg)$, and, for every fixed $g \in G$, in $L^2(\mathbb{P})$. It is evident that the coefficients $\widetilde{a}_\pi(F)$, $\pi \in \hat{G}$, defined in (6) are complex-valued, centered and square integrable random variables for every π, and also that $\Im(\widetilde{a}_{\pi_0}(F)) = 0$. Moreover, by arguments similar to those used in the proof of Lemma 1

$$\mathbb{E}\left[\Re(\widetilde{a}_\pi(F))\Im(\widetilde{a}_\pi(F))\right] = 0 \tag{12}$$

for every $\pi \in \hat{G}$, and also

$$\widetilde{a}_\pi(F) = \overline{\widetilde{a}_{\pi^{-1}}(F)} \quad \text{and} \quad \Re(\widetilde{a}_\pi(F)) \overset{law}{=} \Im(\widetilde{a}_\pi(F)). \tag{13}$$

In general, $\Re(\widetilde{a}_\pi(F))$ and $\Im(\widetilde{a}_\pi(F))$ are not independent.

Remark – The results of this paper extend immediately to (not necessarily centered) functions $F \in L^2\left(\mathbb{R}, \exp\left(-x^2/2\right)dx\right)$, by considering the function $F' = F - \int F(x)(2\pi)^{-\frac{1}{2}} e^{-x^2/2} dx$.

We are interested in studying the asymptotic behavior of the coefficients $\widetilde{a}_\pi$ along some infinite sequence $\{\pi_l : l \geq 1\} \subset \hat{G}$. In particular, we shall determine conditions on the coefficients $\{C_\pi\}$ in (11) ensuring that, for a fixed F, the central limit theorem (7) holds, where $N, N' \sim \mathcal{N}(0, 1/2)$ are independent. The first series of results involves Hermite polynomials.

3 Necessary and sufficient conditions for Hermite transformations (statements)

We start by giving a exhaustive characterization of the CLT (7), when F is an *Hermite polynomial* of arbitrary order $m \geq 2$. Recall (see e.g. [15, p. 20]) that the sequence $\{H_m : m \geq 0\}$ of Hermite polynomials is defined through the relation

$$H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m}\left(e^{-\frac{x^2}{2}}\right), \quad x \in \mathbb{R}, \ m \geq 0; \tag{14}$$

it is well known that the sequence $\left\{(m!)^{-1/2} H_m : m \geq 0\right\}$ constitutes an orthonormal basis of the space $L^2\left(\mathbb{R}, (2\pi)^{-1/2} e^{-\frac{x^2}{2}} dx\right)$.

To state our main results, we need to introduce some further notation. For $\pi \in \hat{G}$ and $m \geq 1$, define the coefficient $\widehat{C}_{\pi,m}$ as

$$\begin{aligned}
\widehat{C}_{\pi,m} &\triangleq \sum_{\sigma_1 \in \hat{G}} \cdots \sum_{\sigma_m \in \hat{G}} \left\{C_{\sigma_1} C_{\sigma_2} \cdots C_{\sigma_m}\right\} \mathbf{1}_{\sigma_1 \cdots \sigma_m = \pi} && (15)\\
&= \sum_{\substack{\sigma_1, \ldots, \sigma_m \in \hat{G} \\ \sigma_1 \cdots \sigma_m = \pi}} C_{\sigma_1} C_{\sigma_2} \cdots C_{\sigma_m} && (16)\\
&= \sum_{\sigma_1, \ldots, \sigma_{m-1} \in \hat{G}} C_{\sigma_1} C_{\sigma_2} \cdots C_{(\sigma_1 \cdots \sigma_{m-1})^{-1}\pi}. && (17)
\end{aligned}$$

Note that, in (15)-(17), $\hat{G}$ is regarded as an Abelian group, with group operation given by (9), and that $\widehat{C}_{\pi,q} = \widehat{C}_{\pi^{-1},q}$. Moreover, $\widehat{C}_{\pi,1} = C_\pi$ and, for every $m \geq 2$ and $q = 1, \ldots, m-1$,

$$\widehat{C}_{\pi,m} = \sum_{\mu \in \hat{G}} \widehat{C}_{\mu,q} \widehat{C}_{\pi\mu^{-1},m-q}. \tag{18}$$

In the statements of the subsequent results, we systematically adopt the same notation and conventions pinpointed in the introduction (see formula (7)), that is: when no further specification is given, $\{\pi\} = \{\pi_l : l = 1, 2, ...\}$ stands for a fixed sequence of elements of $\hat{G}$, and all limits are taken as $l \to +\infty$.

Theorem 2 *Fix $m \geq 2$, and define the random variable $\widetilde{a}_\pi (H_m)$ according to (6) and (14), i.e.*

$$\widetilde{a}_\pi (H_m) = \int_G H_m (T (g)) \chi_\pi \left(g^{-1}\right) dg. \tag{19}$$

Then,

$$\mathbb{E}\left[|\widetilde{a}_\pi (H_m)|^2\right] = m!\widehat{C}_{\pi,m}, \tag{20}$$

where $\widehat{C}_{\pi,m}$ is defined as in (15). Moreover, the following four asymptotic conditions are equivalent:

1.
$$\frac{\widetilde{a}_\pi (H_m)}{\sqrt{m!\widehat{C}_{\pi,m}}} \underset{\{\pi\}}{\overset{law}{\to}} N + iN', \tag{21}$$
where $N, N' \sim \mathcal{N}(0, 1/2)$ are independent;

2.
$$\left[m!\widehat{C}_{\pi,m}\right]^{-2} \mathbb{E}\left[\Re (\widetilde{a}_\pi (H_m))^4\right] \underset{\{\pi\}}{\to} \frac{3}{4}, \quad \text{and} \quad \left[m!\widehat{C}_{\pi,m}\right]^{-2} \mathbb{E}\left[\Im (\widetilde{a}_\pi (H_m))^4\right] \underset{\{\pi\}}{\to} \frac{3}{4}; \tag{22}$$

3.
$$\widehat{C}_{\pi,m}^{-2} \sum_{\lambda \in \hat{G}} \widehat{C}^2_{\lambda,q} \widehat{C}^2_{\pi\lambda^{-1},m-q} \underset{\{\pi\}}{\to} 0, \quad \forall q = 1, ..., m-1; \tag{23}$$

4.
$$\max_{q=1,...,m-1} \frac{\sup_{\lambda \in \hat{G}} \widehat{C}_{\lambda,q} \widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu \in \hat{G}} \widehat{C}_{\mu,q} \widehat{C}_{\pi\mu^{-1},m-q}} \underset{\{\pi\}}{\to} 0. \tag{24}$$

The proof of Theorem 2 is the object of the subsequent sections.

Remarks – (a) Since $H_1 (x) = x$,

$$\begin{aligned} \frac{\widetilde{a}_\pi (H_1)}{\sqrt{\widehat{C}_{\pi,1}}} &= \frac{\int_G T (g) \chi_\pi \left(g^{-1}\right) dg}{\sqrt{C_\pi}} = \frac{a_\pi}{\sqrt{C_\pi}} \overset{law}{=} N + iN', \\ N, N' &\sim \mathcal{N}(0, 1/2) \text{ independent,} \end{aligned}$$

where we have used Lemma 1, (3) and the fact that $\widehat{C}_{\pi,1} = C_\pi$.

(b) (*An interpretation of condition* (24) *in terms of random walks on groups*) Note $C_* \triangleq \sum_\pi C_\pi$, and consider a sequence of independent and identically distributed $\hat{G}$-valued random variables $\{X_j : j \geq 1\}$, such that

$$\mathbb{P}[X_1 = \pi] = \frac{C_\pi}{C_*}, \quad \forall \pi \in \hat{G}.$$

We associate to the sequence $\{X_j\}$ the *$\hat{G}$-valued random walk* $Z = \{Z_m : m \geq 0\}$, defined as $Z_0 = \pi_0$, and $Z_m = X_1 X_2 \cdots X_m$ $(m \geq 1)$. Then, it is easily seen that, $\forall m \geq 2$, $\forall q = 1, ..., m-1$ and $\forall \pi \in \hat{G}$, the ratio appearing in (24) can be rewritten as

$$\frac{\sup_{\lambda \in \hat{G}} \widehat{C}_{\lambda,q} \widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu \in \hat{G}} \widehat{C}_{\mu,q} \widehat{C}_{\pi\mu^{-1},m-q}} = \frac{\sup_{\lambda \in \hat{G}} \mathbb{P}[Z_q = \lambda, Z_m = \pi]}{\mathbb{P}[Z_m = \pi]} = \sup_{\lambda \in \hat{G}} \mathbb{P}[Z_q = \lambda \mid Z_m = \pi],$$

so that the CLT (21) holds if, and only if,

$$\sup_{\lambda\in\hat{G}}\mathbb{P}\left[Z_q=\lambda\mid Z_m=\pi\right]\underset{\{\pi\}}{\rightarrow}0, \tag{25}$$

for every $q=1,...,m-1$. Condition (25) can be interpreted as follows. For every $\pi\in\hat{G}$, define a "bridge" of length m, from π_0 to π, by conditioning Z to equal π at time m. Then, (25) is verified if, and only if, the probability that the bridge hits λ at time q converges to zero, uniformly on λ, as π moves along the sequence $\{\pi\}$. Plainly, when (25) is verified for every $q=1,...,m-1$, one also has that

$$\sup_{\lambda_1,...,\lambda_{m-1}\in\hat{G}}\mathbb{P}\left[Z_1=\lambda_1,...,Z_{m-1}=\lambda_{m-1}\mid Z_m=\pi\right]\underset{\{\pi\}}{\rightarrow}0,$$

meaning that, asymptotically, there is no "privileged path" of length m linking π_0 to π.

Now recall that $H_2(x)=x^2-1$: by using the fact that, for $\pi\neq\pi_0$, $\int_G\chi_\pi(g)\,dg=0$, we deduce from Theorem 2 the following criterion for squared isotropic Gaussian fields on commutative groups.

Corollary 3 *Let, for* $\hat{G}\ni\pi\neq\pi_0$,

$$\widetilde{a}_\pi(H_2)=\int_G\left(T^2(g)-1\right)\chi_\pi\left(g^{-1}\right)dg=\int_G T^2(g)\,\chi_\pi\left(g^{-1}\right)dg.$$

Then,

$$\mathbb{E}\left[\left|\widetilde{a}_\pi(H_2)\right|^2\right]=2\widehat{C}_{\pi,2}=\sum_{\lambda\in\hat{G}}C_\lambda C_{\lambda^{-1}\pi},$$

and the following conditions are equivalent

1.
$$\left(2\widehat{C}_{\pi,2}\right)^{-\frac{1}{2}}\widetilde{a}_\pi(H_2)\overset{law}{\underset{\{\pi\}}{\rightarrow}}N+iN',$$

   with $N,N'\sim\mathcal{N}(0,1/2)$ *independent;*

2.
$$\frac{\sup_{\lambda\in\hat{G}}C_\lambda C_{\pi\lambda^{-1}}}{\sum_{\mu\in\hat{G}}C_\mu C_{\pi\mu^{-1}}}\underset{\{\pi\}}{\rightarrow}0\quad. \tag{26}$$

Analogously, from the relation $H_3(x)=x^3-3x$ we obtain

Corollary 4 *For* $\pi\in\hat{G}$, *let*

$$\widetilde{a}_\pi(H_3)=\int_G\left(T^3(g)-3T(g)\right)\chi_\pi\left(g^{-1}\right)dg.$$

The following conditions are equivalent

1.
$$\left(6\widehat{C}_{\pi,3}\right)^{-\frac{1}{2}}\widetilde{a}_\pi(H_3)\overset{law}{\underset{\{\pi\}}{\rightarrow}}N+iN',$$

   with $N,N'\sim\mathcal{N}(0,1/2)$ *independent;*

2.
$$\lim_{\{\pi\}}\frac{\sup_{\lambda\in\hat{G}}\widehat{C}_{\lambda,2}C_{\pi\lambda^{-1}}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,2}C_{\pi\mu^{-1}}}=\lim_{\{\pi\}}\frac{\sup_{\lambda\in\hat{G}}C_\lambda\widehat{C}_{\pi\lambda^{-1},2}}{\sum_{\mu\in\hat{G}}C_\mu\widehat{C}_{\pi\mu^{-1},2}}=0.$$

Our strategy to prove Theorem 2 is to represent each $\widetilde{a}_\pi(H_m)$ as a complex-valued functional of a centered Gaussian measure, having the special form of a multiple Wiener-Itô integral. To do this, we need to recall several crucial facts concerning multiple stochastic integrals of real-valued kernels, and then to establish some useful extensions to the case of complex-valued random variables.

4 Ancillary results about multiple Wiener-Itô integrals

In this section, we summarize some basic properties of multiple Wiener-Itô integrals. The reader is referred e.g. to [15, Chapter VII] for any explained definition or result.

Real kernels – Let $(A, \mathcal{A}, \mu)$ be a finite measure space, with μ positive, finite and non-atomic. For $d \geq 1$, we define $L^2_{\mathbb{R}}\left(\mu^d\right)$ and $L^2_{s,\mathbb{R}}\left(\mu^d\right)$ to be the Hilbert spaces, respectively of square-integrable, and square-integrable and symmetric real-valued functions on A^d, with respect to the product measure μ^d. As usual, $L^2_{\mathbb{R}}\left(\mu^1\right) = L^2_{s,\mathbb{R}}\left(\mu^1\right) = L^2_{\mathbb{R}}(\mu) = L^2_{\mathbb{R}}(A, \mathcal{A}, \mu)$.

We note $\mathbf{W} = \left\{\mathbf{W}(h) : h \in L^2(\mu)\right\}$ a centered *isonormal Gaussian process* over $L^2(\mu)$. This means that $\mathbf{W}$ is a centered Gaussian family indexed by $L^2(\mu)$ and such that

$$\mathbb{E}\left[\mathbf{W}(h')\,\mathbf{W}(h)\right] = \int_A h'(a)\,h(a)\,\mu(da) \triangleq (h', h)_{L^2(\mu)},$$

for every $h, h' \in L^2(\mu)$. For every $f \in L^2_{s,\mathbb{R}}\left(\mu^d\right)$, we define $I_d(f)$ to be the the multiple Wiener-Itô integral (MWII) of f with respect to $\mathbf{W}$, i.e.

$$I_d(f) = I_d^{\mathbf{W}}(f) = \int_A \cdots \int_A f(a_1, ..., a_d)\,\mathbf{W}(da) \cdots \mathbf{W}(da), \tag{27}$$

where the multiple integration in (27) implicitly excludes diagonals. Recall that

$$\mathbb{E}\left[I_d(f)\,I_{d'}(g)\right] = d!\,\delta_{d,d'}\,(f, g)_{L^2_{\mathbb{R}}(\mu^d)}, \tag{28}$$

where δ is the Kronecker symbol, and therefore the application $f \mapsto I_d(f)$ defines an isomorphism between the dth Wiener chaos associated to $\mathbf{W}$, and the space $L^2_{\mathbb{R}}\left(\mu^d\right)$, endowed with the modified norm $\sqrt{d!}\,\|\cdot\|_{L^2_{\mathbb{R}}(\mu^d)}$. A fundamental relation between objects such as (27) and the Hermite polynomials introduced in (14) is the following: for every $h \in L^2_{\mathbb{R}}(\mu)$ such that $\|h\|_{L^2_{\mathbb{R}}(\mu)} = 1$, and every $m \geq 1$,

$$H_m(I_1(h)) = I_m(h \otimes \cdots \otimes h), \tag{29}$$

where the tensor product inside the second integral is defined as

$$h \otimes \cdots \otimes h\,(a_1, ..., a_m) = h(a_1) \cdots h(a_m) \in L^2_{s,\mathbb{R}}(\mu^m),$$

$\forall a_1, ..., a_m \in A^m$.

For every $d \geq 2$, every $f \in L^2_{s,\mathbb{R}}\left(\mu^d\right)$ and every $q = 1, ..., d-1$, we define the (not necessarily symmetric) *contraction kernel* $f \otimes_q f \in L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)$ as

$$\begin{aligned} & f \otimes_q f\left(x_1, ..., x_{2(d-q)}\right) \\ \triangleq\ & \int_{A^q} f(a_1, ..., a_q, x_1, ..., x_{d-q})\, f\left(a_1, ..., a_q, x_{d-q+1}, ..., x_{2(d-q)}\right) \mu(da_1) \cdots \mu(da_q). \end{aligned} \tag{30}$$

The following CLT, which has been proved in [20] (for the Part A) and [24] (for the Part B), concerns sequences (of vectors of) MWIIs such as (27). It is the crucial element in the proof of Theorem 2.

Theorem 5 (Nualart and Peccati, 2005; Peccati and Tudor, 2004) *(A) Fix $d \geq 2$, and let $f_k \in L^2_{s,\mathbb{R}}\left(\mu^d\right)$, $k \geq 1$. If the variance of $I_d(f_k)$ converges to 1 ($k \to +\infty$) the following three conditions are equivalent: (i) $I_d(f_k)$ converges in law to a standard Gaussian random variable $N(0,1)$, (ii)*

$E\left[I_d(f_k)^4\right] \to 3$, *(iii) for every* $q = 1, ..., d-1$, *the contraction kernel* $f_k \otimes_q f_k$ *converges to* 0 *in* $L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)$.

(B) *Fix integers* $p \geq 2$ *and* $1 \leq d_1 \leq \cdots \leq d_p$. *Consider a sequence of vectors*

$$\left(f_k^{(1)}, f_k^{(2)}, ..., f_k^{(p)}\right), \quad k \geq 1,$$

such that, for each k, $f_k^{(j)} \in L^2_{s,\mathbb{R}}\left(\mu^{d_j}\right)$, $j = 1, ..., p$, *and*

$$\lim_n \mathbb{E}\left[I_{d_j}\left(f_k^{(j)}\right) I_{d_i}\left(f_k^{(i)}\right)\right] = \delta_{i,j},$$

where δ *is the Kronecker symbol. Then, if* $\forall\ j = 1, ..., p$ *the sequence* $\left\{f_k^{(j)} : k \geq 1\right\}$ *satisfies either one of conditions (i)-(iii) of Part A (with* d_j *substituting* d*), as* $k \to +\infty$,

$$\left(I_{d_1}\left(f_k^{(1)}\right), ..., I_{d_p}\left(f_k^{(p)}\right)\right) \overset{law}{\to} \mathbf{N}_p,$$

where $\mathbf{N}_p = (N_1, ..., N_p) \sim \mathcal{N}_p(0, \mathbb{I}_p)$ *is a* p*-dimensional vector of independent, centered standard Gaussian random variables.*

Complex kernels – For $n \geq 1$ and $d \geq 1$, $L^2_{\mathbb{C}}\left(\mu^d\right)$ and $L^2_{s,\mathbb{C}}\left(\mu^d\right)$ are the Hilbert spaces, respectively of square integrable and square integrable and symmetric complex-valued functions with respect to the product Lebesgue measure. For every $g \in L^2_{s,\mathbb{C}}\left(\mu^d\right)$ with the form $g = a + ib$, where $a, b \in L^2_{s,\mathbb{R}}\left(\mu^d\right)$, we set $I_d(g) = I_d(a) + iI_d(b)$. Note that, by (28),

$$\mathbb{E}\left[I_d(g)\overline{I_{d'}(f)}\right] = d!\delta_{d,d'}(g, f)_{L^2_{\mathbb{C}}(\mu^d)}. \tag{31}$$

Also, a random variable such as $I_d(g)$ is real valued if, and only if, g is real valued. For every pair and $g_k = a_k + ib_k \in L^2_{s,\mathbb{C}}\left(\mu^d\right)$, $k = 1, 2$, and every $q = 1, ..., d-1$, we set

$$\begin{aligned} & g_1 \otimes_q g_2\left(x_1, ..., x_{2(d-q)}\right) \\ = & \int_{A^q} g_1(a_1, ..., a_q, x_1, ..., x_{d-q})\, g_2\left(a_1, ..., a_q, x_{d-q+1}, ..., x_{2(d-q)}\right) \mu(da_1)\cdots\mu(da_q) \\ = & a_1 \otimes_q a_2 - b_1 \otimes_q b_2 + i(a_1 \otimes_q b_2 + b_1 \otimes_q a_2). \end{aligned} \tag{32}$$

The following result is an extension of Theorem 5.

Proposition 6 *Suppose that the sequence* $g_l = a_l + ib_l \in L^2_{s,\mathbb{C}}\left(\mu^d\right)$, $l \geq 1$, *is such that*

$$\lim_{l\to+\infty} d!\, \|a_l\|^2_{L^2_{\mathbb{R}}(\mu^d)} = \lim_{l\to+\infty} d!\, \|b_l\|^2_{L^2_{\mathbb{R}}(\mu^d)} \to \frac{1}{2} \quad \text{and} \quad (a_l, b_l)_{L^2_{\mathbb{R}}(\mu^d)} = 0. \tag{33}$$

Then, the following conditions are equivalent: as $l \to +\infty$,

1. $I_d(g_l) \overset{law}{\to} N + iN'$, *where* $N, N' \sim \mathcal{N}(0, 1/2)$ *are independent;*
2. $g_l \otimes_q \overline{g_l} \to 0$ *and* $g_l \otimes_q g_l \to 0$ *in* $L^2_{\mathbb{C}}\left(\mu^{2(d-q)}\right)$ *for every* $q = 1, ..., d-1$;
3. $g_l \otimes_q \overline{g_l} \to 0$ *in* $L^2_{\mathbb{C}}\left(\mu^{2(d-q)}\right)$ *for every* $q = 1, ..., d-1$;
4. $a_l \otimes_q a_l \to 0$, $b_l \otimes_q b_l \to 0$ *and* $a_l \otimes_q b_l \to 0$ *in* $L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)$ *for every* $q = 1, ..., d-1$;
5. $a_l \otimes_q a_l \to 0$, $b_l \otimes_q b_l \to 0$ *in* $L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)$ *for every* $q = 1, ..., d-1$;

6. *$\mathbb{E}\left[I_d(a_l)^4\right] \to 3/4$, $\mathbb{E}\left[I_d(b_l)^4\right] \to 3/4$ and $\mathbb{E}\left[I_d(a_l)^2 I_d(b_l)^2\right] \to 1/4$;*

7. *$\mathbb{E}\left[I_d(a_l)^4\right] \to 3/4$, $\mathbb{E}\left[I_d(b_l)^4\right] \to 3/4$.*

Proof. Note first that, due to (28) and the second part of (33),

$$\mathbb{E}\left[I_d(b_l)\, I_d(a_l)\right] = (a_l, b_l)_{L^2_{s,\mathbb{R}}(\mu^d)} = 0, \quad l \geq 1.$$

Now, ($7 \to 1$) holds because of (33) and Part B of Theorem 5. ($5 \leftrightarrow 1 \to 6$) is again a consequence of (33) and Part B of Theorem 5 (note that (33) implies that all moments of the real and imaginary parts of $I_d(a_l)$ and $I_d(b_l)$ are uniformly bounded). ($2 \longleftrightarrow 4$) derives from (32). ($2 \to 3$), ($4 \to 5$) and ($6 \to 7$) and are obvious. ($5 \to 4$) is a consequence of

$$\begin{aligned}
\|a_l \otimes_q b_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} &= \int_{A^{d-q}}\int_{A^{d-q}}\int_{A^q}\int_{A^q} a_l(\mathbf{s}_q, \mathbf{a}_{d-q})\, b_l(\mathbf{s}_q, \mathbf{b}_{d-q})\, a_l(\mathbf{t}_q, \mathbf{a}_{d-q}) \\
&\quad b_l(\mathbf{t}_q, \mathbf{b}_{d-q})\, \mu^{d-q}(d\mathbf{a}_{d-q})\, \mu^{d-q}(d\mathbf{b}_{d-q})\, \mu^q(d\mathbf{s}_q)\, \mu^q(d\mathbf{t}_q) \\
&= \left((a_l \otimes_{d-q} a_l), (b_l \otimes_{d-q} b_l)\right)_{L^2_{\mathbb{R}}(\mu^{2q})},
\end{aligned}$$

where $\mathbf{s}_q$ stands for a vector of the type

$$(s_1, ..., s_q), \quad \text{with } s_j \in A, \quad j = 1, ..., q,$$

and $\mu^q(d\mathbf{s}_q) = \mu(ds_1)\cdots\mu(ds_q)$ (similar conventions apply to $\mathbf{a}_{d-q}$, $\mathbf{b}_{d-q}$ and $\mathbf{t}_q$). We are left with the implication ($3 \to 2$), which is a consequence of the relation

$$\|g_l \otimes_q \overline{g_l}\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} \geq \|g_l \otimes_q g_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)}, \quad \forall l \geq 1. \tag{34}$$

To prove (34), just write

$$\begin{aligned}
\|g_l \otimes_q \overline{g_l}\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} &= \|a_l \otimes_q a_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} + \|b_l \otimes_q b_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} \\
&\quad + 2\left(a_l \otimes_q a_l, b_l \otimes_q b_l\right)_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} + 2\|a_l \otimes_q b_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} \\
&\quad - 2\left(a_l \otimes_q b_l, b_l \otimes_q a_l\right)_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)}
\end{aligned}$$

and

$$\begin{aligned}
\|g_l \otimes_q g_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} &= \|a_l \otimes_q a_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} + \|b_l \otimes_q b_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} \\
&\quad - 2\left(a_l \otimes_q a_l, b_l \otimes_q b_l\right)_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} + 2\|a_l \otimes_q b_l\|^2_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} \\
&\quad + 2\left(a_l \otimes_q b_l, b_l \otimes_q a_l\right)_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)},
\end{aligned}$$

and finally

$$\begin{aligned}
&2\left(a_l \otimes_q a_l, b_l \otimes_q b_l\right)_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} - 2\left(a_l \otimes_q b_l, b_l \otimes_q a_l\right)_{L^2_{\mathbb{R}}\left(\mu^{2(d-q)}\right)} \\
= \;& 2\left(a_l \otimes_{d-q} b_l, a_l \otimes_{d-q} b_l\right)_{L^2_{\mathbb{R}}(\mu^{2q})} - 2\left(a_l \otimes_{d-q} b_l, b_l \otimes_{d-q} a_l\right)_{L^2_{\mathbb{R}}(\mu^{2q})} \\
= \;& \|a_l \otimes_{d-q} b_l\|^2_{L^2_{\mathbb{R}}(\mu^{2q})} + \|b_l \otimes_{d-q} a_l\|^2_{L^2_{\mathbb{R}}(\mu^{2q})} - 2\left(a_l \otimes_{d-q} b_l, b_l \otimes_{d-q} a_l\right)_{L^2_{\mathbb{R}}(\mu^{2q})} \\
= \;& \|a_l \otimes_{d-q} b_l - b_l \otimes_{d-q} a_l\|^2_{L^2_{\mathbb{R}}(\mu^{2q})} \geq 0.
\end{aligned}$$

■

5 Proof of Theorem 2

Let $\left\{C_\pi : \pi \in \hat{G}\right\}$ be defined as in (11). We start by considering a collection of complex-valued and square integrable functions $\left\{f_\pi : \pi \in \hat{G}\right\} \subset L^2_{\mathbb{C}}(\mu)$, with the following properties: (i) $\Im(f_{\pi_0}) = 0$, (ii) $f_\pi = \overline{f}_{\pi^{-1}}$, (iii) $\int_A f_\pi(a) f_\pi(a) \mu(da) = 0$, $\forall \pi \neq \pi_0$, (iv) both $\Re(f_\pi)$ and $\Im(f_\pi)$ are orthogonal (in $L^2_{\mathbb{R}}(\mu)$) to $\Re(f_\sigma)$ and $\Im(f_\sigma)$ for every $\sigma \notin \left\{\pi, \pi^{-1}\right\}$, (v) $\int_A |f_\pi(a)|^2 \mu(da) = C_\pi$. Note that

$$\int_A f_\pi(a) f_\pi(a) \mu(da) = \int_A \left(\Re(f_\pi(a))^2 - \Im(f_\pi(a))^2\right) \mu(da) + 2i \int_A \Re(f_\pi(a)) \Im(f_\pi(a)) \mu(da),$$

and therefore (iii) holds if, and only if,

$$C_\pi = \int_A |f_\pi(a)|^2 \mu(da) = 2\int_A \Re(f_\pi(a))^2 \mu(da) = 2\int_A \Im(f_\pi(a))^2 \mu(da), \quad \forall \pi \neq \pi_0,$$

and $\int_A \Re(f_\pi(a)) \Im(f_\pi(a)) \mu(da) = 0$ for every π.

The class $\left\{f_\pi : \pi \in \hat{G}\right\}$ can be constructed as follows. Let $\{..., \pi_{-1}, \pi_0, \pi_1, \pi_2, ...\}$ be any two-sided enumeration of $\hat{G}$, such that π_0 is the trivial representation as before, and $\pi_j = \pi_{-j}^{-1}$ for every $j = 1, 2, ...$ Then, consider an orthonormal basis $\{e_k : k = ..., -1, 0, 1, 2, ...\}$ of $L^2_{\mathbb{R}}(\mu)$, and set $f_{\pi_0} = e_0$ and, for $j \geq 1$,

$$f_{\pi_j} = \sqrt{\frac{C_{\pi_j}}{2}} \times (e_j + ie_{-j}) \quad \text{and} \quad f_{\pi_{-j}} = \sqrt{\frac{C_{\pi_j}}{2}} \times (e_j - ie_{-j})$$

(with this notation, one has plainly that $C_{\pi_j} = C_{\pi_{-j}}$).

The next Lemma is easily verified.

Lemma 7 *The following identity in law holds*

$$\left\{I_1(f_\pi) : \pi \in \hat{G}\right\} \overset{law}{=} \left\{a_\pi : \pi \in \hat{G}\right\}, \tag{35}$$

where the coefficients a_π are given by (3), and therefore

$$T(g) \overset{law}{=} \sum_{\pi \in \hat{G}} I_1(f_\pi) \chi_\pi(g), \quad g \in G, \tag{36}$$

where the identity in law is in the sense of stochastic processes. As a consequence, for every $F \in L^2\left(\mathbb{R}, \exp\left(-x^2/2\right) dx\right)$ and $g \in G$

$$\widetilde{a}_\pi(F) \overset{law}{=} \int_G F\left[\sum_{\pi \in \hat{G}} I_1(f_\pi) \chi_\pi(g)\right] \chi_\pi\left(g^{-1}\right) dg, \tag{37}$$

where $\widetilde{a}_\pi(F)$ is defined as in (6).

Since, for any $\pi \in G$,

$$\sum_{\pi \in \hat{G}} C_\pi = \sum_{\pi \in \hat{G}} \|f_\pi\|^2_{L^2_{\mathbb{C}}(\mu)} = \sum_{\pi \in \hat{G}} \|f_\pi \chi_\pi(g)\|^2_{L^2_{\mathbb{C}}(\mu)} < +\infty,$$

for every *fixed* $g \in G$ and any sequence of finite subsets $\hat{G}_N \subset \hat{G}$ such that $\hat{G}_N \uparrow \hat{G}$, the sequence

$$\sum_{g \in \hat{G}_N} f_\pi(\cdot)\,\chi_\pi(g) \in L^2_{\mathbb{C}}(\mu), \quad N \geq 1,$$

converges (as $N \to +\infty$) in $L^2_{\mathbb{C}}(\mu)$ to a certain function

$$h^g(\cdot) \triangleq \sum_{\pi \in \hat{G}} f_\pi(\cdot)\,\chi_\pi(g) \in L^2_{\mathbb{C}}(\mu) \tag{38}$$

(we stress that in (38) g is a fixed parameter). Note that the properties of the f_π's imply that h^g is real-valued, and also that the mapping $(x, g) \mapsto h^g(x)$ is jointly measurable. By using the linearity of MWIIs, we deduce from (36) that, as stochastic processes,

$$T(g) \overset{law}{=} I_1(h^g), \quad g \in G, \tag{39}$$

and therefore (37) implies that for every π,

$$\widetilde{a}_\pi(F) \overset{law}{=} \int_G F[I_1(h^g)]\,\chi_\pi(g)\,dg. \tag{40}$$

Now fix $m \geq 2$, and consider the mth Hermite polynomial H_m. Since

$$1 = \mathbb{E}\left[T(g)^2\right] = \mathbb{E}\left[I_1(h^g)^2\right] = \|h^g\|^2_{L^2_{\mathbb{R}}(\mu)},$$

we deduce from (29) that, for every $g \in G$, $H_m[I_1(h^g)] = I_m(h^g \otimes \cdots \otimes h^g)$. Thus, by using (40) in the case $F = H_m$ and by interchanging deterministic and stochastic integration,

$$\begin{aligned}
\widetilde{a}_\pi(H_m) &\overset{law}{=} \int_G H_m[I_1(h^g)]\,\chi_\pi\left(g^{-1}\right)dg && (41)\\
&= \int_G I_m(h^g \otimes \cdots \otimes h^g)\,\chi_\pi\left(g^{-1}\right)dg \\
&= I_m\left(\int_G \{h^g \otimes \cdots \otimes h^g\}\,\chi_\pi\left(g^{-1}\right)dg\right) && (42)\\
&= I_m\left(\widetilde{h}_{m,\pi}\right),
\end{aligned}$$

where

$$\widetilde{h}_{m,\pi} \triangleq \int_G \{h^g \otimes \cdots \otimes h^g\}\,\chi_\pi\left(g^{-1}\right)dg \in L^2_{s,\mathbb{C}}(\mu^m). \tag{43}$$

Remark – Since the Haar measure dg has finite mass, the "stochastic Fubini theorem" applied in (42) can be justified by standard arguments. See for instance [21, Lemma 13].

The function $\widetilde{h}_{m,\pi}$ can be made explicit by means of (38), i.e.

$$\begin{aligned}
&\widetilde{h}_{m,\pi}(x_1, ..., x_m) \\
&= \int_G \left\{\sum_{\sigma_1 \in \hat{G}} f_{\sigma_1}(x_1)\,\chi_{\sigma_1}(g) \times \cdots \times \sum_{\sigma_m \in \hat{G}} f_{\sigma_m}(x_m)\,\chi_{\pi_m}(g)\right\}\chi_\pi\left(g^{-1}\right)dg && (44)\\
&= \sum_{\sigma_1 \in \hat{G}} \sum_{\sigma_2 \in \hat{G}} \cdots \sum_{\sigma_{m-1} \in \hat{G}} f_{\sigma_1}(x_1)\,f_{\sigma_2}(x_2) \times \cdots \times f_{\pi(\sigma_1 \cdots \sigma_{m-1})^{-1}}(x_m), \quad (x_1, ..., x_m) \in A^m,
\end{aligned}$$

where we used (10) and the orthogonality between characters of non-equivalent representations. By using (41) and (31),

$$\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(H_{m}\right)\right|^{2}\right]=m!\left\|\widetilde{h}_{m,\pi}\right\|_{L_{\mathbb{C}}^{2}(\mu^{m})}^{2}=m!\sum_{\substack{\sigma_{1},...,\sigma_{m}\in\hat{G}\\ \sigma_{1}\cdots\sigma_{m}=\pi}}C_{\sigma_{1}}C_{\sigma_{2}}\cdots C_{\sigma_{m}}=m!\widehat{C}_{\pi,m},$$

thus proving (20). Now define

$$\widetilde{\widetilde{a}}_{\pi}\left(H_{m}\right)\triangleq\frac{\widetilde{a}_{\pi}\left(H_{m}\right)}{\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(H_{m}\right)\right|^{2}\right]^{\frac{1}{2}}}\overset{law}{=}I_{m}\left(\widetilde{\widetilde{h}}_{m,\pi}\right),\tag{45}$$

where

$$\widetilde{\widetilde{h}}_{m,\pi}\triangleq\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(H_{m}\right)\right|^{2}\right]^{-\frac{1}{2}}\widetilde{h}_{m,\pi}=\left(m!\widehat{C}_{\pi,m}\right)^{-1/2}\widetilde{h}_{m,\pi}.\tag{46}$$

Since (12) and (13) hold (with $F=H_m$), it is clear that, for $\pi\in\hat{G}$,

$$m!\left(\Re\left(\widetilde{\widetilde{h}}_{m,\pi}\right),\Im\left(\widetilde{\widetilde{h}}_{m,\pi}\right)\right)_{L_{\mathbb{R}}^{2}(\mu^{m})}=\mathbb{E}\left[\Re\left(\widetilde{\widetilde{a}}_{\pi}\left(H_{m}\right)\right)\Im\left(\widetilde{\widetilde{a}}_{\pi}\left(H_{m}\right)\right)\right]=0$$

and also $m!\left\|\widetilde{\widetilde{h}}_{m,\pi}\right\|_{L_{\mathbb{C}}^{2}(\mu^{m})}^{2}=\mathbb{E}\left[\left|\left(\widetilde{\widetilde{a}}_{\pi}\left(H_{m}\right)\right)\right|^{2}\right]=1$, so that

$$\begin{aligned}\mathbb{E}\left[\Re\left(\widetilde{\widetilde{a}}_{\pi}\left(H_{m}\right)\right)^{2}\right] &= \mathbb{E}\left[\Im\left(\widetilde{\widetilde{a}}_{\pi}\left(H_{m}\right)\right)^{2}\right]=m!\left\|\Re\left(\widetilde{\widetilde{h}}_{m,\pi}\right)\right\|_{L_{\mathbb{C}}^{2}(\mu^{m})}^{2}\\ &= m!\left\|\Im\left(\widetilde{\widetilde{h}}_{m,\pi}\right)\right\|_{L_{\mathbb{C}}^{2}(\mu^{m})}^{2}=\frac{1}{2}.\end{aligned}$$

It follows that all the assumptions of Proposition 6 are satisfied, with $d=m$, $g_l=\widetilde{\widetilde{h}}_{m,\pi_l}$, and therefore $a_l=\Re\left(\widetilde{\widetilde{h}}_{m,\pi_l}\right)$ and $b_{n,l}=\Im\left(\widetilde{\widetilde{h}}_{m,\pi_l}\right)$ (recall that, in the statement of Theorem 2, $\{\pi\}$ stands for a sequence of the form $\{\pi_l : l\geq 1\}$). As a consequence, in view of (45), we deduce from the implications $(1\leftrightarrow 7)$ in Proposition 6 that the convergence in law (21) holds if, and only if, (22) is verified. We have therefore proved that Conditions 1 and 2 in Theorem 2 are equivalent.

To conclude the proof, we start by observing that, thanks e.g. to the implications $(7\longleftrightarrow 1\longleftrightarrow 3)$ in Proposition 6, either one of conditions (21) and (22) is equivalent to the following:

$$\widetilde{\widetilde{h}}_{m,\pi}\otimes_{q}\overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)}\underset{\{\pi\}}{\rightarrow}0,\text{ in }L_{\mathbb{C}}^{2}\left(\mu^{2(m-q)}\right),\quad\forall q\in\{1,...,m-1\}.\tag{47}$$

It follows that the equivalence of Conditions 1, 2 and 3 in Theorem 2 is established, once it is shown that (47) is true if, and only if, condition (23) is verified for every $q=1,...,m-1$. Start with $q=m-1$.

Indeed,

$$\begin{aligned}
&\widetilde{\widetilde{h}}_{m,\pi} \otimes_{m-1} \overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)} (x_1, x_2) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \int_A \cdots \int_A \widetilde{h}_{m,\pi} (a_1, ..., a_{m-1}, x_1) \overline{\left(\widetilde{h}_{m,\pi}\right)} (a_1, ..., a_{m-1}, x_2) \, \mu (da_1) ... \mu (da_m) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \sum_{\pi_1 \in \hat{G}} \sum_{\pi_2 \in \hat{G}} \cdots \sum_{\pi_{m-1} \in \hat{G}} C_{\pi_1} C_{\pi_2} \cdots C_{\pi_{m-1}} \times f_{\pi(\pi_1 \cdots \pi_{m-1})^{-1}} (x_1) \, f_{\pi^{-1}(\pi_1 \cdots \pi_{m-1})} (x_2) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \sum_{\lambda \in \hat{G}} \sum_{\substack{\pi_1, ..., \pi_{m-1} \in \hat{G} \\ \pi_1 \cdots \pi_{m-1} = \lambda}} C_{\pi_1} C_{\pi_2} \cdots C_{\pi_{m-1}} \times f_{\pi\lambda^{-1}} (x_1) \, f_{\pi^{-1}\lambda} (x_2) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \sum_{\lambda \in \hat{G}} \widehat{C}_{\lambda,m-1} f_{\pi\lambda^{-1}} (x_1) \, f_{\pi^{-1}\lambda} (x_2) = \left(m!\widehat{C}_{\pi,m}\right)^{-1} \sum_{\lambda \in \hat{G}} \widehat{C}_{\lambda,m-1} f_{\pi\lambda^{-1}} (x_1) \, \overline{f_{\pi\lambda^{-1}} (x_2)},
\end{aligned}$$

yielding

$$\left\| \widetilde{\widetilde{h}}_{m,\pi} \otimes_{m-1} \overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)} \right\|^2_{L^2_{\mathbb{C}}(\mu^2)} = \left(m!\widehat{C}_{\pi,m}\right)^{-2} \sum_{\lambda \in \hat{G}} \widehat{C}^2_{\lambda,m-1} C_{\pi\lambda^{-1}} C_{\pi^{-1}\lambda} = \left(m!\widehat{C}_{\pi,m}\right)^{-2} \sum_{\lambda \in \hat{G}} \widehat{C}^2_{\pi,m-1} C^2_{\pi\lambda^{-1}},$$

thus proving that (23) holds for $q = m - 1$ if, and only if, $\widetilde{\widetilde{h}}_{m,\pi} \otimes_{m-1} \overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)} \underset{\{\pi\}}{\to} 0$. Now suppose $m \geq 3$, and fix $q = 1, ..., m - 2$. In this case,

$$\begin{aligned}
&\widetilde{\widetilde{h}}_{m,\pi} \otimes_q \overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)} \left(x_1, ..., x_{2(m-q)}\right) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \int_A \cdots \int_A \widetilde{h}_{m,\pi} (a_1, ..., a_q, x_1, ..., x_{m-q}) \times \\
& \qquad \times \overline{\left(\widetilde{h}_{m,\pi}\right)} \left(a_1, ..., a_q, x_{m-q+1}, ..., x_{2(m-q)}\right) \mu (da_1) \cdots \mu (da_q) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \sum_{\pi_1, ..., \pi_q \in \hat{G}} C_{\pi_1} \cdots C_{\pi_q} \times \\
& \qquad \times \sum_{\rho_1, ..., \rho_{m-1-q}} \sum_{\sigma_1, ..., \sigma_{m-1-q}} \prod_{r=1}^{m-q-1} f_{\rho_r} (x_r) \, f_{\sigma_r^{-1}} (x_{m-q+r}) \\
& \qquad\qquad \times f_{\pi(\pi_1 \cdots \pi_q)^{-1} (\rho_1 \cdots \rho_{m-1-q})^{-1}} (x_{m-q}) \, f_{\pi^{-1}(\pi_1 \cdots \pi_q)(\sigma_1 \cdots \sigma_{m-1-q})} \left(x_{2(m-q)}\right) \\
= \;& \left(m!\widehat{C}_{\pi,m}\right)^{-1} \sum_{\rho_1, ..., \rho_{m-1-q}} \sum_{\sigma_1, ..., \sigma_{m-1-q}} \prod_{r=1}^{m-q-1} f_{\rho_r} (x_r) \, f_{\sigma_r^{-1}} (x_{m-q+r}) \\
& \qquad \times \sum_{\lambda \in \hat{G}} \widehat{C}_{\lambda,q} f_{\pi\lambda^{-1} (\rho_1 \cdots \rho_{m-1-q})^{-1}} (x_{m-q}) \, f_{\pi^{-1}\lambda(\sigma_1 \cdots \sigma_{m-1-q})} \left(x_{2(m-q)}\right),
\end{aligned}$$

and some calculations yield

$$\left\| \widetilde{\widetilde{h}}_{m,\pi} \otimes_q \overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)} \right\|^2_{L^2_{\mathbb{C}}\left(\mu^{2(m-q)}\right)} = \left(m!\widehat{C}_{\pi,m}\right)^{-2} \sum_{\lambda \in \hat{G}} \widehat{C}^2_{\lambda,q} \widehat{C}^2_{\pi\lambda^{-1},m-q}. \tag{48}$$

Relation (48) shows in particular that, for $q = 1, ..., m - 2$, $\widetilde{\widetilde{h}}_{m,\pi} \otimes_q \overline{\left(\widetilde{\widetilde{h}}_{m,\pi}\right)} \underset{\{\pi\}}{\to} 0$ if, and only if, (23) is verified. To see that Conditions 3 and 4 in the statement of Theorem 2 are equivalent, use (18) to write

$$\begin{aligned}
\widehat{C}_{\pi,m}^{-2}\sum_{\lambda\in\hat{G}}\widehat{C}_{\lambda,q}^{2}\widehat{C}_{\pi\lambda^{-1},m-q}^{2} &= \sum_{\lambda\in\hat{G}}\left\{\frac{\widehat{C}_{\lambda,q}\widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,q}\widehat{C}_{\pi\mu^{-1},m-q}}\right\}^{2} \\
&\leq \sup_{\lambda\in\hat{G}}\left\{\frac{\widehat{C}_{\lambda,q}\widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,q}\widehat{C}_{\pi\mu^{-1},m-q}}\right\}\sum_{\lambda\in\hat{G}}\left\{\frac{\widehat{C}_{\lambda,q}\widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,q}\widehat{C}_{\pi\mu^{-1},m-q}}\right\} \\
&= \sup_{\lambda\in\hat{G}}\left\{\frac{\widehat{C}_{\lambda,q}\widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,q}\widehat{C}_{\pi\mu^{-1},m-q}}\right\},
\end{aligned}$$

and also

$$\begin{aligned}
\max_{q=1,\ldots,m-1}\widehat{C}_{\pi,m}^{-2}\sum_{\lambda\in\hat{G}}\widehat{C}_{\lambda,q}^{2}\widehat{C}_{\pi\lambda^{-1},m-q}^{2} &= \max_{q=1,\ldots,m-1}\sum_{\lambda\in\hat{G}}\left\{\frac{\widehat{C}_{\lambda,q}\widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,q}\widehat{C}_{\pi\mu^{-1},m-q}}\right\}^{2} \\
&\geq \max_{q=1,\ldots,m-1}\sup_{\lambda\in\hat{G}}\left\{\frac{\widehat{C}_{\lambda,q}\widehat{C}_{\pi\lambda^{-1},m-q}}{\sum_{\mu\in\hat{G}}\widehat{C}_{\mu,q}\widehat{C}_{\pi\mu^{-1},m-q}}\right\}^{2}.
\end{aligned}$$

This concludes the proof of Theorem 2. ■

In Section 7, we will establish a CLT of the type (7) for functions $F\in L_0^2\left(\mathbb{R},e^{-x^2/2}dx\right)$ that are not necessarily Hermite polynomials. As a first step, in the next section we prove a result concerning the joint convergence of vectors of coefficients of the type $\widetilde{a}_\pi\left(H_m\right)$.

6 Joint convergence of the $\widetilde{\widetilde{a}}_\pi\left(H_m\right)$

Fix integers $p\geq 2$ and $2\leq m_1<\cdots<m_p$, and define, for $\pi\in\hat{G}$, the vectors

$$\left(\widetilde{a}_\pi\left(H_{m_1}\right),...,\widetilde{a}_\pi\left(H_{m_p}\right)\right)\quad\text{and}\quad\left(\widetilde{\widetilde{a}}_\pi\left(H_{m_1}\right),...,\widetilde{\widetilde{a}}_\pi\left(H_{m_p}\right)\right),$$

according respectively to (6) and (45).

Theorem 8 *Suppose that, for any $j=1,...,p$, the coefficients $\left\{C_\pi:\pi\in\hat{G}\right\}$ (as defined in (11)) verify either one of conditions (21)-(24) (with m_j substituting m). Then,*

$$\left\{T\quad;\quad\left(\widetilde{\widetilde{a}}_\pi\left(H_{m_1}\right),...,\widetilde{\widetilde{a}}_\pi\left(H_{m_p}\right)\right)\right\}\underset{\{\pi\}}{\overset{law}{\to}}\left\{T\quad;\quad\left(N_1+iN_1',...,N_p+iN_p'\right)\right\}\tag{49}$$

where $\mathbf{N}_p=\left(N_1,...,N_p\right)$ and $\mathbf{N}_p'=\left(N_1',...,N_p'\right)$ are two independent vectors of $\mathcal{N}\left(0,1/2\right)$ i.i.d. random variables, such that $\mathbf{N}_p$ and $\mathbf{N}_p'$ are independent of T. On the other hand, if the asymptotic relation (49) holds, then conditions (21)-(24) are necessarily satisfied.

Remark – The convergence relation (49) is meant in the sense of finite dimensional distributions, that is: (49) is true if, and only if, for any $k\geq 1$ and every $\left(g_1,...,g_k\right)\in G^k$,

$$\left(T\left(g_1\right),...,T\left(g_k\right),\widetilde{\widetilde{a}}_\pi\left(H_{m_1}\right),...,\widetilde{\widetilde{a}}_\pi\left(H_{m_p}\right)\right)\underset{\{\pi\}}{\overset{law}{\to}}\left(T\left(g_1\right),...,T\left(g_k\right),N_1+iN_1',...,N_p+iN_p'\right)\tag{50}$$

Proof. For some $k \geq 1$, consider vectors $(g_1, ..., g_k) \in G^k$ and $(\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$. Then, arguments analogous to the ones adopted in the proof of Theorem 2 show that

$$\left(\sum_{i=1}^{k} \lambda_i T(g_i), \widetilde{\widetilde{a}}_\pi (H_{m_1}), ..., \widetilde{\widetilde{a}}_\pi (H_{m_p}) \right) \overset{law}{=} \left(I_1 \left(\Sigma_{i=1}^k \lambda_i h^{g_i} \right), I_{m_1} \left(\widetilde{\widetilde{h}}_{m_1,\pi} \right), ..., I_{m_p} \left(\widetilde{\widetilde{h}}_{m_p,\pi} \right) \right), \tag{51}$$

where the h^{g_i}'s are given by (38), and the kernels $\widetilde{\widetilde{h}}_{m_j,\pi}$, $j = 1, ..., p$, are defined in (46). Note that the kernel $\Sigma_{i=1}^k \lambda_i h^{g_i}$ (which does not depend on π) is real-valued, and therefore $I_1 \left(\Sigma_{i=1}^k \lambda_i h^{g_i} \right)$ is a real-valued Gaussian random variable. Also, by construction the following relations hold: (i) $\forall j = 1, ..., p$, $\Re \left(I_{m_j} \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) = I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right)$ and $\Im \left(I_{m_j} \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) = I_{m_i} \left(\Im \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right)$, and

$$\begin{aligned} \mathbb{E} \left[I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) I_{m_j} \left(\Im \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) \right] &= 0 \\ \mathbb{E} \left[I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right)^2 \right] &= \mathbb{E} \left[I_{m_j} \left(\Im \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right)^2 \right] = \frac{1}{2}; \end{aligned} \tag{52}$$

(ii) $\forall 1 \leq k \neq j \leq p$,

$$\begin{aligned} \mathbb{E} \left[I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) I_{m_k} \left(\Im \left(\widetilde{\widetilde{h}}_{m_k,\pi} \right) \right) \right] &= \mathbb{E} \left[I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) I_{m_k} \left(\Re \left(\widetilde{\widetilde{h}}_{m_k,\pi} \right) \right) \right] \\ &= \mathbb{E} \left[I_{m_j} \left(\Im \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) I_{m_k} \left(\Im \left(\widetilde{\widetilde{h}}_{m_k,\pi} \right) \right) \right] = 0; \end{aligned} \tag{53}$$

(iii) $\forall j = 1, ..., p$,

$$\mathbb{E} \left[I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) I_1 \left(\Sigma_{i=1}^k \lambda_i h^{g_i} \right) \right] = \mathbb{E} \left[I_{m_j} \left(\Im \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right) I_1 \left(\Sigma_{i=1}^k \lambda_i h^{g_i} \right) \right] = 0. \tag{54}$$

Now suppose that either one of conditions (21)-(24) hold $\forall m_j$ $(j = 1, ..., p)$. Then, Theorem 2 implies that $\forall j = 1, ..., p$,

$$\lim_{\{\pi\}} \mathbb{E} \left[I_{m_j} \left(\Re \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right)^4 \right] = \lim_{\{\pi\}} \mathbb{E} \left[I_{m_j} \left(\Im \left(\widetilde{\widetilde{h}}_{m_j,\pi} \right) \right)^4 \right] = \frac{3}{4}, \tag{55}$$

so that Part B of Theorem 5, together with (52)-(54), yield that,

$$\begin{aligned} &\left(I_1 \left(\Sigma_{i=1}^k \lambda_i h^{g_i} \right), \Re \left(I_{m_1} \left(\widetilde{\widetilde{h}}_{m_1,\pi} \right) \right), \Im \left(I_{m_1} \left(\widetilde{\widetilde{h}}_{m_1,\pi} \right) \right), ... \right. \\ &\qquad \left. ..., \Re \left(I_{m_p} \left(\widetilde{\widetilde{h}}_{m_p,\pi} \right) \right), \Im \left(I_{m_p} \left(\widetilde{\widetilde{h}}_{m_p,\pi} \right) \right) \right) \\ &\qquad \underset{\{\pi\}}{\overset{law}{\to}} \left(I_1 \left(\Sigma_{i=1}^k \lambda_i h^{g_i} \right), N_1, N_1', ..., N_p, N_p' \right), \end{aligned} \tag{56}$$

where the vectors $\mathbf{N}_p = (N_1, ..., N_p)$ and $\mathbf{N}_p' = \left(N_1', ..., N_p' \right)$ are defined in the statement of Theorem 8. Now note that, due to (51), the asymptotic relation (56) holds $\forall (\lambda_1, ..., \lambda_k)$ if, and only if, (49) is verified. The proof of the first part of Theorem 8 is therefore concluded. To prove the last part of the statement, use the equivalence between (49) and (56) to show that (49) implies that (55) holds for every $j = 1, ..., p$. But, due to (45) and (51), (55) is equivalent to the condition: for every $j = 1, ..., p$,

$$\left[m_j! \widehat{C}_{\pi,m_j} \right]^{-2} \mathbb{E} \left[\Re \left(\widetilde{a}_\pi (H_{m_j}) \right)^4 \right] \underset{\{\pi\}}{\to} \frac{3}{4}, \quad \text{and} \quad \left[m_j! \widehat{C}_{\pi,m_j} \right]^{-2} \mathbb{E} \left[\Im \left(\widetilde{a}_\pi (H_{m_j}) \right)^4 \right] \underset{\{\pi\}}{\to} \frac{3}{4},$$

so that the proof is concluded by using once again Theorem 2. ■

Now define $\mathfrak{A} \triangleq \left\{ a_\pi : \pi \in \hat{G} \right\}$, where the a_π's are defined according to (3). An immediate consequence of Theorem 8 is the following result.

Corollary 9 *Fix a vector of integers* $2 \leq m_1 < \cdots < m_p$, *and suppose that* $\forall j = 1, ..., p$,

$$\widetilde{\widetilde{a}}_\pi \left(H_{m_j} \right) \underset{\{\pi\}}{\overset{law}{\to}} N + iN', \tag{57}$$

where $N, N' \sim \mathcal{N}(0, 1/2)$ *are independent. Then,*

$$\left(\mathfrak{A} \ ; \ \widetilde{\widetilde{a}}_\pi (H_{m_1}), ..., \widetilde{\widetilde{a}}_\pi \left(H_{m_p} \right) \right) \underset{\{\pi\}}{\overset{law}{\to}} \left\{ \mathfrak{A} \ ; \ \left(N_1 + iN_1', ..., N_p + iN_p' \right) \right\},$$

where $\mathbf{N}_p = (N_1, ..., N_p)$ *and* $\mathbf{N}_p' = \left(N_1', ..., N_p' \right)$ *are two independent vectors of* $\mathcal{N}(0, 1/2)$ *i.i.d. random variables, such that* $\mathbf{N}_p$ *and* $\mathbf{N}_p'$ *are independent of* $\mathfrak{A}$.

Proof. Due to Theorem 2, (57) holds for every $j = 1, ..., p$, if, and only if, either one of conditions (22)-(24) are verified for every $j = 1, ..., p$, with m_j replacing m. The conclusion is achieved by using Theorem 8, as well as the fact that, by (2) and (3), $\sigma(\mathfrak{A}) = \sigma(T)$. ■

7 A CLT for general $F \in L_0^2 \left(\mathbb{R}, e^{-x^2/2} dx \right)$

We now establish a CLT such as (7) for a general real-valued function $F \in L_0^2 \left(\mathbb{R}, e^{-x^2/2} dx \right)$. Since the sequence of normalized Hermite polynomials $\left\{ (m!)^{-1/2} H_m : m \geq 0 \right\}$ defined by (14) is an orthonormal basis for $L^2_{\mathbb{R}}(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2} dx)$, the function F admits a unique representation of the form

$$F(x) = \sum_{m=1}^{\infty} \frac{c_m(F)}{m!} H_m(x), \quad x \in \mathbb{R}, \tag{58}$$

where the coefficients $c_m(F)$, $m = 1, 2...$, are such that

$$c_m(F) = \int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} H_m(x) F(x) \, dx, \quad \text{and} \quad \Sigma_{m \geq 1} \frac{c_m(F)^2}{m!} < +\infty \tag{59}$$

(note that the sum in (58) starts from $m = 1$ since F is centered, i.e. $F \in L_0^2 \left(\mathbb{R}, e^{-x^2/2} dx \right)$). As a consequence, the coefficients $\widetilde{a}_\pi(F)$, $\pi \in \hat{G}$, defined in (6) can be written as

$$\widetilde{a}_\pi(F) = \sum_{m=1}^{\infty} \frac{c_m(F)}{m!} \int_G H_m(T(g)) \chi_\pi \left(g^{-1} \right) dg = \sum_{m=0}^{\infty} \frac{c_m(F)}{m!} \widetilde{a}_\pi(H_m) \tag{60}$$

where the series converges in $L^2_{\mathbb{C}}(\mathbb{P})$, and the $\widetilde{a}_\pi(H_m)$'s are given by (19). By combining Theorem 2 and Theorem 8, from (60) we deduce the following result.

Theorem 10 *For every* $\pi \neq \pi_0$,

$$\mathbb{E}\left[|\widetilde{a}_\pi(F)|^2 \right] = \sum_{m=1}^{\infty} \left(\frac{c_m(F)}{m!} \right)^2 \mathbb{E}\left[|\widetilde{a}_\pi(H_m)|^2 \right] = \sum_{m=1}^{\infty} \frac{c_m(F)^2}{m!} \widehat{C}_{\pi,m}. \tag{61}$$

Suppose moreover that the following relations hold

1. *For every* $m \geq 1$,
$$\lim_{\{\pi\}} \frac{m!\widehat{C}_{\pi,m}}{\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(F\right)\right|^{2}\right]} \to \sigma_{m}^{2} \in (0,+\infty)\,;$$
2. $\sum_{m\geq 1}\left\{c_{m}\left(F\right)/m!\right\}^{2}\sigma_{m}^{2} \triangleq \sigma^{2}\left(F\right) < +\infty$;
3. *For every* $m \geq 2$, *the coefficients* $\left\{C_{\pi} : \pi \in \hat{G}\right\}$ *given by (11) verify either one of conditions (23) and (24);*
4. $\lim_{p\to+\infty}\overline{\lim}_{\{\pi\}}\sum_{m=p+1}^{\infty}\left\{c_{m}\left(F\right)^{2}/m!\right\}\widehat{C}_{\pi,m} = 0.$

Then,
$$\widetilde{\widetilde{a}}_{\pi}\left(F\right) \triangleq \frac{\widetilde{a}_{\pi}\left(F\right)}{\sqrt{\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(F\right)\right|^{2}\right]}} \overset{law}{\underset{\{\pi\}}{\longrightarrow}} \left(\sigma^{2}\left(F\right)\right)^{\frac{1}{2}} \times \left\{N + iN'\right\},$$
where $N, N' \sim \mathcal{N}\left(0,1/2\right)$ *are independent Gaussian random variables.*

Proof. Fix $p \geq 1$. Assumptions 1 and 3 in the statement imply, thanks to Theorem 8, that
$$\frac{1}{\sqrt{\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(F\right)\right|^{2}\right]}}\left(\widetilde{a}_{\pi}\left(H_{1}\right),...,\widetilde{a}_{\pi}\left(H_{p}\right)\right) \overset{law}{\underset{\{\pi\}}{\longrightarrow}} \left(\sqrt{\sigma_{1}^{2}} \times \left(N_{1}+iN_{1}'\right),...,\sqrt{\sigma_{p}^{2}} \times \left(N_{p}+iN_{p}'\right)\right),$$
where $\mathbf{N}_{p} = \left(N_{1},...,N_{p}\right)$ and $\mathbf{N}_{p}' = \left(N_{1}',...,N_{p}'\right)$ are two independent vectors of $\mathcal{N}\left(0,1/2\right)$ i.i.d. random variables. In particular, it follows that
$$\begin{aligned}
\Phi^{(p)}\left(\pi\right) &\triangleq \frac{\sum_{m=1}^{p}\left\{c_{m}\left(F\right)/m!\right\}\widetilde{a}_{\pi}\left(H_{m}\right)}{\sqrt{\mathbb{E}\left[\left|\widetilde{a}_{\pi}\left(F\right)\right|^{2}\right]}} \overset{law}{\underset{\{\pi\}}{\longrightarrow}} \sum_{m=1}^{p}\left\{c_{m}\left(F\right)/m!\right\} \times \left\{\sqrt{\sigma_{m}^{2}} \times \left(N_{m}+iN_{m}'\right)\right\} \\
&\overset{law}{=} \left[\sum_{m=1}^{p}\left\{c_{m}\left(F\right)/m!\right\}^{2}\sigma_{m}^{2}\right]^{\frac{1}{2}} \times \left\{N_{1}+iN_{1}'\right\}.
\end{aligned}$$
Now take a uniformly bounded Lipschitz function $g : \mathbb{C} \mapsto \mathbb{R}$, with Lipschitz coefficient equal to one. Then
$$\begin{aligned}
&\left|\mathbb{E}\left[g\left(\widetilde{\widetilde{a}}_{\pi}\left(F\right)\right)\right] - \mathbb{E}\left[g\left(\left(\sigma^{2}\left(F\right)\right)^{\frac{1}{2}} \times \left\{N+iN'\right\}\right)\right]\right| \\
\leq\; &\left|\mathbb{E}\left[g\left(\widetilde{\widetilde{a}}_{\pi}\left(F\right)\right)\right] - \mathbb{E}\left[g\left(\Phi^{(p)}\left(\pi\right)\right)\right]\right| \\
&+\left|\mathbb{E}\left[g\left(\left(\sum_{m=1}^{p}\left\{c_{m}\left(F\right)/m!\right\}^{2}\sigma_{m}^{2}\right)^{\frac{1}{2}} \times \left\{N_{1}+iN_{1}'\right\}\right)\right] - \mathbb{E}\left[g\left(\Phi^{(p)}\left(\pi\right)\right)\right]\right| \\
&+\left|\mathbb{E}\left[g\left(\left(\sum_{m=1}^{p}\left\{c_{m}\left(F\right)/m!\right\}^{2}\sigma_{m}^{2}\right)^{\frac{1}{2}} \times \left\{N_{1}+iN_{1}'\right\}\right)\right] - \mathbb{E}\left[g\left(\left(\sigma^{2}\left(F\right)\right)^{\frac{1}{2}} \times \left\{N+iN'\right\}\right)\right]\right|.
\end{aligned} \tag{62}$$
Now recall that $\{\pi\}$ stands for a sequence of the type $\{\pi_{l} : l \geq 1\}$, and replace π with π_{l} in (62). Then, by first taking the limit as $l \to +\infty$, and then the limit as $p \to +\infty$ in the RHS of (62), we deduce from Assumptions 2 and 4 in the statement that the LHS (62) converges to zero as $l \to +\infty$. This concludes the proof. ■

8 The n-dimensional torus

In this last section, we focus on the case of G being the n-dimensional torus $\mathbb{R}^n/(2\pi\mathbb{Z})^n$, which we parameterize as $(0,2\pi]^n$ with addition $mod(2\pi)$ as the group operation. In this case, the dual space $\hat{G}$ is the class of all applications of the type $\vartheta \mapsto \exp(i\mathbf{k}'\vartheta)$ where $\vartheta = (\vartheta_1,...,\vartheta_n) \in (0,2\pi]^n$ and $\mathbf{k} = (k_1,...,k_n) \in \mathbb{Z}^n$ (here, we identify $\hat{G}$ with the class of its associated characters). By using the notation introduced in (15)-(18), we have also that, for every $\mathbf{k} \in \mathbb{Z}^n$,

$$\widehat{C}_{\mathbf{k},m} \triangleq \sum_{\mathbf{j}_1 \in \mathbb{Z}^n} \cdots \sum_{\mathbf{j}_m \in \mathbb{Z}^n} \left\{ C_{\mathbf{j}_1} \cdots C_{\mathbf{j}_m} \right\} \mathbf{1}_{\mathbf{j}_1 + \cdots + \mathbf{j}_m = \mathbf{k}}.$$

Moreover, for any fixed $\mathbf{l}^* \in \overline{\mathbb{Z}^n}$, condition (24) in the statement of Theorem (2) can be rewritten as: when $\mathbf{l} \to \mathbf{l}^*$,

$$\frac{\sup_{\mathbf{j}\in\mathbb{Z}^n} \widehat{C}_{\mathbf{j},m-q} \widehat{C}_{\mathbf{j}-\mathbf{l},q}}{\sum_{\mathbf{a}\in\mathbb{Z}^n} \widehat{C}_{\mathbf{a},m-q} \widehat{C}_{\mathbf{a}-\mathbf{l},q}} \to 0, \quad \forall q = 1,...,m-1. \tag{63}$$

Remark – Condition (63) bears a clear resemblance with Lindeberg-type assumptions for the Central Limit Theorem in a martingale difference setting, see for instance [13]. Indeed, in some very simple cases (i.e. quadratic transformations of Gaussian random fields on the 1-dimensional torus) it seems possible to derive sufficient conditions for the CLT by means of martingale approximations and the extension to complex-valued variables of convergence results for the real-valued martingale difference sequences. However, this approach is clearly unfeasible for general nonlinear transforms of Gaussian random fields on higher-dimensional tori or on abstract Abelian groups.

As discussed in the introduction, of particular interest for physical applications is the case where $\|\mathbf{l}\| \to \infty$, that is, when we analyze the behavior of high-frequency components. We discuss two examples to illustrate the application of our results; in both cases we assume that $C_0 = 0$ to simplify the discussion.

Example 1 (*Algebraic decay on the circle*) – With this example we show that the CLT fails for general Hermite transformations, when the angular power spectrum decays algebraically. We take $n = 1$ (merely for notational simplicity) and for all $l \in \mathbb{Z}\setminus\{0\}$, we assume there exist positive constants $c_2 > c_1$ and $\alpha > 1$ such that

$$c_1|l|^{-\alpha} \leq C_l \leq c_2|l|^{-\alpha} \ ;$$

of course we thus cover any model of the form $C_l = 1/h(|l|)$, where $h(l) = h_0 + h_1 l + ... + h_p l^{p-1} > 0$ for all $l > 0$ and $1/h(|l|)$ is summable.

We have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} C_k C_{l-k} &\leq c_2 \left\{ \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \frac{1}{(k+l)^\alpha} + \sum_{k=1}^{l-1} \frac{1}{k^\alpha} \frac{1}{(l-k)^\alpha} + \sum_{k=l+1}^{\infty} \frac{1}{k^\alpha} \frac{1}{(k-l)^\alpha} \right\} . \\
&\leq c_2 \left\{ \frac{2}{l^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} + \sum_{k=1}^{l-1} \frac{1}{k^\alpha} \frac{1}{(l-k)^\alpha} \right\} \\
&\leq c_2 \left\{ \frac{2}{l^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} + 2 \sum_{k=1}^{[l/2]+1} \frac{1}{k^\alpha} \frac{1}{(l-k)^\alpha} \right\} \\
&\leq c_2 \left\{ \frac{2}{l^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^\alpha} + \frac{2}{(l/2)^\alpha} \sum_{k=1}^{[l/2]+1} \frac{1}{k^\alpha} \right\} \leq \frac{c_{22}}{l^\alpha} .
\end{aligned}$$

On the other hand it is immediate to see that

$$\sum_{k=-\infty}^{\infty} C_k C_{l-k} \geq \sup_{k\in\mathbb{Z}} C_k C_{l-k} \geq C_1 C_{l-1} \geq \frac{c_1^2}{|l-1|^\alpha} \geq \frac{c_{12}}{|l|^\alpha} \text{ , some } c_{12} > 0 \ .$$

Arguing by induction, we have thus shown that there exist positive sequences $c_{2q} > c_{1q}$, $q = 2, 3, ...$ such that

$$c_{1q}|l|^{-\alpha} \leq \widehat{C}_{l,q} \leq c_{2q}|l|^{-\alpha} \ ,$$

and

$$\frac{\max_k \widehat{C}_{k,1}\widehat{C}_{l-k,m-1}}{\widehat{C}_{l,m}} \geq \frac{c_1 c_{1,m-1}}{c_{2,m}} > 0 \text{ for all } l \in \mathbb{Z}\backslash\left\{0\right\} \ ,$$

whence the necessary conditions for the Central Limit Theorem (63) fail for each $m \geq 2$.

Remark – Analogous examples where the CLT fails could be easily provided for $n > 1$, by considering for instance the spectral function

$$C_{l_1 \ldots l_n} = \frac{1}{h(|l_1|, ..., |l_p|)} \ ,$$

for $h(., ..., .)$ a multivariate polynomial which takes nonnegative values on the positive integers. A polynomial decay of the power spectrum is common in physical models for the large scale structure of the Universe, for instance in the highly popular Harrison-Zeldovich model (see [25]).

Example 2 (*Exponential decay on the circle*) – With this example we show that the CLT holds for arbitrary Hermite transformations when the angular power spectrum decays exponentially, up to multiplicative algebraic factors. Assume we have

$$c_1 h(|l|)\exp(-\vartheta|l|) \leq C_l \leq c_2 h(|l|)\exp(-\vartheta|l|) \ , \ l \in \mathbb{Z}\backslash\left\{0\right\} \ , \tag{64}$$

for strictly positive constants ϑ and $c_2 > c_1$, and where $h(l) = h_0 + h_1 l + ... + h_p l^{p-1} > 0$.

Then

$$\widehat{C}_{l,2} \geq \sum_{k=1}^{l-1} C_k C_{l-k} \geq c_1 \sum_{k=1}^{l-1} h(k)h(l-k)\exp(-\vartheta|l|) \geq c_{12}|l|^{2p+1}\exp(-\vartheta|l|) \ ,$$

for some constant $c_{12} > 0$. Iterating this argument, we obtain by induction

$$\widehat{C}_{l,q} = \sum_{k_1=-\infty}^{\infty} C_{k_1}\widehat{C}_{l-k_1,q-1} \geq \sum_{k_1=1}^{l-1} C_{k_1}\widehat{C}_{l-k_1,q-1} \geq c_{1q}|l|^{qp+q-1}\exp(-\vartheta|l|) \ , \ c_{1p} > 0 \ .$$

On the other hand, we have also

$$\widehat{C}_{\lambda,q} \leq c_{2q}|l|^{qp+q-1}\exp(-\vartheta|l|)\exp(-\vartheta|l|) \ , \text{ some } c_{2p} > 0 \ ,$$

because

$$\begin{aligned} \widehat{C}_{l,2} &= \sum_{|k|\leq 2l}\left\{C_k C_{l-k}\right\} + \sum_{|k|>2l}\left\{C_k C_{l-k}\right\} \leq c_2'|l|^{2p+1}\exp(-\vartheta|l|) + c_2'' C_l \sum_{|k|>2l} C_k \\ &\leq c_{22}|l|^{2p+1}\exp(-\vartheta|l|) \ , \end{aligned}$$

and then the argument is completed by induction. Hence we have

$$\begin{aligned} \sup_{\lambda\in\mathbb{Z}} \widehat{C}_{\lambda,m-q}\widehat{C}_{l-\lambda,q} &\leq c_{2q}|l|^{qp+q-1}|l|^{(m-q)p+m-q-1}\exp(-\vartheta|l|) = c_{2q}|l|^{mp+m-2}\exp(-\vartheta|l|) \\ \sum_{\mu\in\mathbb{Z}} \widehat{C}_{\mu,m-q}\widehat{C}_{l-\mu,q} &= \widehat{C}_{l,m} \geq c_{1q}|l|^{mp+m-1}\exp(-\vartheta|l|) \ , \end{aligned}$$

whence it is immediate to see that (63) follows.

Remark – Analogous examples where a CLT of the type (21) holds for every $m \geq 2$ could be easily provided for $n > 1$, considering for instance the spectral function

$$C_{l_1 \ldots l_n} = h(|l_1|, ..., |l_p|)\exp(-\vartheta_1|l_1| ... - \vartheta_n|l_n|) \ ,$$

for $h(.,...,.)$ a multivariate polynomial which takes nonnegative values on the positive integers. An exponential decay of the angular power spectrum at very high frequencies is expected in physical models for the CMB random field, due to the so-called Silk damping (or diffusion damping) effect (see [10]).

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