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On symmetric Fraenkel's and small deviations conjectures

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Abstract

We prove the *symmetric* Fraenkel's conjecture, and show that this proof implies the conjecture of Brauner and Crama [2] concerning instances of the maximum deviation just-in-time sequencing with maximum deviation $B^* < \frac{1}{2}$.

1 The Conjectures

Given are n positive integers $d_1 \leq \ldots \leq d_n$, define $D = \sum_{i=1}^{n} d_i$ and $r_i = \frac{d_i}{D}$. Consider the following optimization problem referred to as the maximum deviation just in time sequencing (MDJIT) problem:

$$B^* = \min \max_{i,k} |x_{ik} - kr_i|$$

Subject to

$x_{ik} \leq x_{ik+1}$	for $i = 1, \ldots, n$ and $k = 1, \ldots, D-1$
$\sum_{i=1}^{n} x_{ik} = k$	for $k = 1, \ldots, D$
x_{ik} non-negative integers	for $i = 1, \ldots, n$ and $k = 1, \ldots, D$.

This maximum deviation just-in-time sequencing problem was introduced and studied in the context of just-in-time production systems, see Monden [5]. However, its applications extend to multiprocessor and communication networks where one looks for an optimal routing in queuing networks, see

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Altman, Gaujal, and Hordijk [1]. We also refer the reader to Kubiak [3] and Steiner and Yeomans [6] for efficient algorithms concerning just-in-time sequencing problems.

Brauner and Crama [2] were first to study those instances of the MDJIT problem that result in a maximum deviation being less than $\frac{1}{2}$. More precisely, they searched for those *standard* instances of the MDJIT problem i.e. vectors of positive integers $(d_1, d_2, ..., d_n)$ such that $d_1 \leq \ldots \leq d_n$ and $\gcd(d_1, d_2, ..., d_n) = 1$ that result in the maximum deviation being less than $\frac{1}{2}$. Their investigation led them to the following conjecture.

Conjecture 1 (Small deviations conjecture) *For $n \geq 3$, a standard instance $(d_1, d_2, ..., d_n)$ of the MDJIT problem has optimal value $B^* < \frac{1}{2}$ if and only if $d_i = 2^{i-1}$ for $i = 1, \ldots, n$.*

Actually, two other equivalent versions of Conjecture 1 are as follows. Let $[x]$ be the rounding of x to the closest integer; $x - \frac{1}{2} \leq [x] < x + \frac{1}{2}$, then:

Conjecture 2 *Let $r_1 \leq \ldots \leq r_n$ be $n \geq 3$ rational numbers. Then $\sum_{i=1}^{n} [kr_i] = k$ for any integer k if and only if $r_i = \frac{2^{i-1}}{2^n - 1}$ for $i = 1, \ldots, n$.*

Conjecture 3 (Competition-free instance) *For $n \geq 3$, all D numbers $\lceil \frac{2j-1}{2r_i} \rceil$ where $j = 1, \ldots, d_i$ and $i = 1, \ldots, n$ are different if and only if $r_i = \frac{2^{i-1}}{2^n - 1}$.*

Conjecture 1 (as well as 2 and 3) turns out to be a *symmetric* form of Fraenkel's conjecture which concerns covers of the integers with Beatty sequences. We will establish a link between Fraenkel's conjecture and Conjecture 1. However, we will neither consider Fraenkel's conjecture directly nor use the concept of Beatty sequences. Consequently, we refer the reader to [7] and [9] for an excellent account of the state of the art about Fraenkel's conjecture and Beatty sequences.

A central concept used in our proof are the *balanced words* defined as follows.

Definition 1 (Balanced Word) *A* balanced word *on an alphabet $\{v_1, v_2, \ldots, v_n\}$ is an infinite sequence $\sigma = (s_1, s_2, \ldots)$ such that*

1. *$s_j \in \{v_1, v_2, \ldots, v_n\}$ for all $j \in \mathbb{N}$, and*
2. *if σ_1 and σ_2 are two subsequences consisting of t consecutive elements of σ ($t \in \mathbb{N}$), then the number of occurrences of the letter v_i in σ_1 and σ_2 differs by at most 1, for all $i = 1, 2, \ldots, n$.*

For a finite word S on alphabet $\{v_1, v_2, \ldots, v_n\}$, let us denote by $|S|$ the length of S and by $|S|_i$ the number of occurrences of the letter v_i in S. Also let us define the rate r_i of the letter v_i as the fraction $r_i = \frac{|S|_i}{|S|}$. We assume $r_1 \leq \ldots \leq r_n$. For a finite word S, let $S^* = (S, S, \ldots)$ be the infinite repetition of S. An infinite word W is called *periodic* if $W = S^*$ for some finite word S. An infinite word W is called *periodic* and *balanced* if W is balanced and $W = S^*$ for some finite word S. A finite word S is called *symmetric* if $S = S^R$, where S^R is a mirror reflection of S. An infinite word W will be called *periodic*, *symmetric* and *balanced* if W is balanced and $W = S^*$ for some finite symmetric word S. Tijdeman [9] and Altman et al. [1] show that Fraenkel's conjecture is equivalent to the following:

Conjecture 4 (Fraenkel's conjecture for periodic, balanced words) *There exists a periodic, balanced word on $n \geq 3$ letters with rates $r_1 < r_2 < \ldots < r_n$ if and only if $r_i = \frac{2^{i-1}}{2^n - 1}$.*

The challenging problem in Conjecture 4 is to prove the necessity since it is easy to construct a periodic, balanced word with the rates $r_i = \frac{2^{i-1}}{2^n-1}$. In this paper, we shall prove that Fraenkel's conjecture restricted to periodic, symmetric and balanced words holds true, and that consequently Conjecture 1 holds true.

Theorem 1 (Fraenkel's symmetric case) *There exists a periodic, symmetric and balanced word on $n \geq 3$ letters with rates $r_1 < r_2 < \ldots < r_n$, if and only if the rates verify $r_i = \frac{2^{i-1}}{2^n-1}$.*

The key observation is that in a symmetric and balanced word, there is a letter a with rate $r_a > \frac{1}{2}$. Before giving details of the proof in the next section we now show the aforementioned link between the symmetric Fraenkel's conjecture and the small deviations conjecture. We begin with some necessary definitions. Let x_{ik} for $i = 1, \ldots, n$ and $k = 1, \ldots, D$ be a solution of the MDJIT problem with maximum deviation $B^* < \frac{1}{2}$. This solution generates a finite word $S = s_1 \ldots s_D$ on $\{v_1, v_2, \ldots, v_n\}$ as follows: $s_k = v_j$ if and only if $x_{jk} - x_{jk-1} = 1$. The following proposition establishes the link between Conjecture 1 and Theorem 1.

Proposition 1 *A sequence S of the MDJIT problem with $B^* < \frac{1}{2}$ generates a word $S^* = (S, S, \ldots)$ which is periodic, symmetric and balanced.*

Proof: First, S^* is periodic by definition.

Second, S^* is balanced by *Proposition 8* in [2].

Finally, [2] shows that, if $B^* < \frac{1}{2}$, then $x_{ik} = [kr_i]$ and hence

$$\begin{aligned} x_{iD-k} - x_{iD-k-1} &= [(D-k)r_i] - [(D-k-1)r_i] \\ &= d_i - [kr_i] - d_i + [(k+1)r_i] \\ &= [(k+1)r_i] - [kr_i] \\ &= x_{ik+1} - x_{ik}. \end{aligned}$$

This proves the symmetry of S for $B^* < \frac{1}{2}$. □

2 Proof of the small deviations conjecture

This section will present a proof of the Fraenkel's symmetric case and consequently the small deviations conjecture. The proof is based on the concept of balanced words which has emerged as the most elegant and useful tool in the quest for the proof of Fraenkel's conjecture, see Tijdeman [9] and Altman et al. [1], up to now.

The fact that the numbers $d_i = 2^{i-1}$ for $i = 1, 2 \ldots n$, $n > 2$, result in periodic, symmetric and balanced sequences was proved by Brauner and Crama [2] and will not be repeated in this paper. Thus, it will only remain to show here that periodic, symmetric and balanced sequences build with at least three different letters are necessarily periodic, infinite repetitions of a symmetric, finite word where letter i occurs exactly $d_i = 2^{i-1}$ times. The proof of this fact is motivated to some extend by a geometric proof of the conjecture given by Kubiak [4].

Lemma 1 *In a symmetric, balanced word S^* every pair of consecutive letters contains at least one v_n (or equivalently $r_n \geq \frac{1}{2}$).*

Proof: For a symmetric S, the first and last letters of S are the same, say x. Therefore, the last letter of the first period S of S^* and the first letter of the second period S of S^* make up the word xx, and thus, since S^* is balanced, every consecutive pair of letters in S^* must contain an x. Consequently, $r_x \geq \frac{1}{2}$ and $x = v_n$. □

Lemma 2 *In a balanced word W, let T and U be any two subwords starting and ending with a non-x and both containing the same number k of non-x's. Then the difference between the number of x's in T and U is at most one.*

Proof: By contradiction. Let m be the number of x's in T and $m - t$ their number in U, where $t \geq 2$. Then, $|T| = k + m$ and $|U| = k + m - t$. Let us delete the first and last letters of T, and extend U by adding to it $t-2$ letters that immediately follow U in W. As a result, we get two words of length $k+m-2$, where the first still contains m letters x, and the second at most $m-t+t-2=m-2$ letters x. This leads to a contradiction since W is balanced. Thus, $t \leq 1$ which proves the lemma. $\square$

Lemma 3 ([1], Proposition 2.29) *Consider a balanced word $W = S^*$ in which letter v_n has rate $r_n \geq 1/2$. Then, the word $W' = S'^*$ obtained by deletion of all letters v_n from W is also balanced.*

Proof: Let T and U be subwords of W beginning and ending with non-v_n such that the deletion of all v_n's from these subwords generates subwords T' and U' of W' of the same length. By Lemma 2, the difference between the number of v_n in T and U is at most 1. If it is 1, then extend the shorter word by adding the letter that immediately follows this word in W. By Lemma 1, this letter must be v_n. We thus get two subwords of W, say U'' and T'', of the same length. Now, for any letter $x \neq v_n$, we observe that x occurs the same number of times in T' (resp. U') as it does in T'' (resp. U''). Since W is balanced, the number of occurrences of x in U'' and T'' differs by at most 1, which proves that W' is balanced. $\square$

Tijdeman [9] recently proved the previous result replacing "$r_n \geq \frac{1}{2}$" by "$r_n \geq \frac{1}{3}$".

Lemma 4 *Let S^* be a symmetric, balanced word on $n \geq 3$ letters. If $v_{n-1}v_n^k v_{n-1}$ is a subword of S^*, then $k = 2$.*

Proof: Consider, a sequence of $k \geq 0$ consecutive v_n's surrounded by two v_{n-1}:

$$S^* = \ldots v_{n-1} \underbrace{v_n \ldots v_n}_{k} v_{n-1} \ldots \tag{1}$$

We have $k > 0$ since any sequence of two consecutive letters contains a v_n. Since $n \geq 3$, there is a letter $x \neq v_n, v_{n-1}$ in S^*. Therefore, if $k = 1$, the sequence S^* contains a subword $v_{n-1}v_n v_{n-1}$ with two letters v_{n-1} and a subword $v_n x v_n$ without letter v_{n-1} which shows that S^* is not balanced and leads to a contradiction. Hence, we have $k \geq 2$.

Since S^* is balanced, then in every subword of k letters, there are at least $(k-1)$ v_n's. Therefore, we observe that any letter $x \neq v_n, v_{n-1}$ must be prefixed and suffixed by $k-1$ v_n's, i.e.,

$$S^* = \ldots \underbrace{v_n \ldots v_n}_{k-1} x \underbrace{v_n \ldots v_n}_{k-1} \ldots \tag{2}$$

On the one hand, (1) implies that every sequence of $k+2$ letters contains at least one v_{n-1}. On the other hand, (2) implies that there is a sequence of $(k-1)+1+(k-1)$ letters without v_{n-1}. Therefore,

$$(k-1)+1+(k-1) < k+2$$

and consequently $k \leq 2$, which ends the proof. □

Now we are ready to prove Theorem 1 by induction on n.

Proof of the theorem: Let $IH(n)$ be the following property.
$IH(n)$: Every symmetric and balanced word with $n \geq 3$ letters has rates $r_i = \frac{2^{i-1}}{2^n-1}$.

- $IH(3)$ holds as proved in [8].
- Assume that $IH(n-1)$ holds, and consider a symmetric and balanced word S^* on n letters. By Lemma 1, $r_n \geq \frac{1}{2}$. By Lemma 3, S'^* obtained from S^* by deletion of all occurrences of v_n is balanced. Obviously, S'^* is also periodic and symmetric. To complete the induction we now show that there are exactly two v_n between any two consecutive v_{n-1} in S^*. By Lemma 4, every subword of S^* of four letters must contain a v_{n-1}. Thus between any two consecutive v_{n-1} in S^* there are at most three letters. By Lemma 1, $v_{n-1}v_{n-1}$ is forbidden in S^*, and moreover if $v_{n-1}xv_{n-1}$ is a subword of S^* for some letter x, then $x = v_n$. However, by Lemma 4, $v_{n-1}v_nv_{n-1}$ is forbidden in S^*. Consequently, between any two consecutive v_{n-1} letters in S^* there are either two or three letters. Since, again by Lemma 1, any subword of S^* of two letters must contain a v_n, then between any two consecutive v_{n-1}'s in S^*, we have either v_nv_n or v_nxv_n for some letter x, i.e.

  $$v_{n-1}v_nv_nv_{n-1} \qquad \text{or} \qquad v_{n-1}v_nxv_nv_{n-1}.$$

  As Lemma 4 implies that $x \neq v_n, v_{n-1}$, there are exactly two v_n's between any two consecutive v_{n-1}'s in S^* and hence $|S| = |S'| + 2|S'|_{n-1}$ and $r_n = 2r_{n-1}$. Finally, using $IH(n-1)$, we have

  $$\begin{aligned} r_i &= r'_i\frac{|S'|}{|S|} = r'_i\left(\frac{|S'|}{|S'|+2|S'|_{n-1}}\right) \\ &= r'_i\frac{1}{1+2r'_{n-1}} = \frac{2^{i-1}}{2^{n-1}-1}\frac{1}{1+2\frac{2^{n-2}}{2^{n-1}-1}} = \frac{2^{i-1}}{2^n-1} \end{aligned}$$

  for $i = 1, 2 \ldots n-1$ and $r_n = 2r_{n-1} = \frac{2^{n-1}}{2^n-1}$ which concludes the induction. □

3 Conclusions

We still seem to be far away from proving Fraenkel's conjecture with the approach presented in this paper since it remains challenging to prove that, in a balanced word with distinct rates, there is a letter a with rate $r_a \geq \frac{1}{2}$. It is worth observing that our proof of Theorem 1 does not require the assumption that all ratios are different, therefore Fraenkel's symmetric case can in fact be formulated without this assumption.

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References

[1] E. Altman and B. Gaujal and A. Hordijk (2000), Balanced sequences and optimal routing, *Journal of the ACM* **47** 752-775.

[2] N. Brauner and Y. Crama, Facts and questions about the maximum deviation just-in-time scheduling problem, 2001, Les cahiers du Laboratoire Leibniz-IMAG, no 25, France.

[3] W. Kubiak and S.P. Sethi (1991) A Note on "Level schedules for mixed-model assembly lines in just-in-time production systems," *Management Science* **6** 137-154.

[4] W. Kubiak On small deviations conjecture. Faculty of Business Administration, Memorial University of Newfoundland, Research Report March 2002. Also presented at the *Scheduling in Computer and Manufacturing Systems* Seminar, Dagstuhl, Germany, 2002.

[5] Y. Monden (1983) *Toyota Production Systems* Industrial Engineering and Management Press, Norcross, GA.

[6] G. Steiner and S. Yeomans (1993) Level schedules for mixed-model, just-in-time production processes, *Management Science* **39** 401-418.

[7] R. TIJDEMAN, Exact covers of balanced sequences and Fraenkel's conjecture, Algebraic Number Theory and Diophantine Analysis, F. Halter-Koch and R. F. Tichy, Walter de Gruyter, Berlin, New York, 2000,467-483.

[8] R. TIJDEMAN (1996), On complementary triples of Sturmian bisequences, Indagationes Math.,7,419-424.

[9] R. TIJDEMAN (2000), Fraenkel's conjecture for six sequences, Discrete Mathematics, 222, 223-234.

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