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Abstract

In this paper we formalize a graph-partitioning problem that arises in the design of SONET networks as a Set-Covering problem. We then improve the general performance ratio of the Greedy Algorithm proved by Chvátal for this particular case.

key-words: set-covering, approximations, telecommunication networks.

Introduction

SONET networks [5] are common and widely used telecommunication networks. For such networks, the first problem to cope with is to partition the edges of the demand graph into sub-graphs called *rings*. Nodes of a same ring are then linked together by two fibers so as to form a cycle and demands are routed automatically around this cycle on one of the fibers, according to possible failures. This is made possible by complex devices called ADM at every node. These devices have a limited capacity of treatment which constraints a ring to a maximum *capacity* C. In the scope of this paper, we consider that this capacity must be at least as big as the sum of the demands within a ring[1].

ADMs have high cost, and in what follows we consider all other costs to be negligible. The cost of a ring is thus its number of nodes, and the global cost of a partition is the sum of the costs of the rings.

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[1]This corresponds to unidirectional rings, that is when a fiber is dedicated to normal work, the other one saved for failure cases; then all demands must go all around the ring, and therefore an ADM may have to deal with the sum of all demands when they are simultaneous.

In our model, a node may be shared by several rings. All demands are supposed to be unitary, and thus cannot be split between several rings, and they all must be satisfied.

With these restrictions, the SONET network design problem is indeed a special case of the well-known Set-Covering problem, as the first section of this paper formalizes it. In the literature, this special case is referred as k-EP (k-Edge Partition) or ADRL (Assignment of Demands to Rings, with Link-costs only). Complexity issues and some linear-time $\mathcal{O}(\sqrt{C})$-approximations are considered in [3, 1], and a tabu method is proposed in [4].

Here, we turn to Chvátal's Greedy Algorithm for set-covering and prove better guarantees for our special case than in the general case, which leads to the best known guarantee for this problem.

1 A Set-Covering problem

First, we formalize the partition problem we deal with. Let $G = (V, E)$ be a simple graph and C a capacity. We aim at finding a minimum-cost partition of the edges of G into rings of capacity at most C. A ring is any subset of E with no more than C edges. Thus, let be:

$$\begin{array}{rcl} \mathcal{A} & = & \{A_i \subset E, |A_i| \leq C\} \\ v_i & = & \text{the number of vertices of } A_i \\ a_{ei} & = & \begin{cases} 1 \text{ if } e \in A_i \\ 0 \text{ otherwise} \end{cases} \end{array}$$

The problem is then:

$$[\textbf{ADRL}] \begin{cases} \min & \sum_{A_i \in \mathcal{A}} v_i x_i & & (\textbf{ADRL}.0) \\ s.t. & \sum_{A_i \in \mathcal{A}} a_{ei} x_i \geq 1 & \forall e \in E & (\textbf{ADRL}.1) \\ & x_i \in \{0, 1\} & \forall A_i \in \mathcal{A} & (\textbf{ADRL}.2) \end{cases}$$

The set $\mathcal{A}$ is the set of all possible rings. Variables x_i are boolean variables, of value 1 if and only if the ring A_i is selected in the solution. The objective (**ADRL**.0) is thus to minimize the sum of the costs of the rings in the solution. Constraint (**ADRL**.1) forces every edge to be covered. This constraint could be an equality without changing the value of an optimal solution.

2 Performance of the Greedy Algorithm

To solve a Set-Covering problem, there exists a very intuitive greedy heuristic: at iteration k, select the ring A_i which minimizes the ratio $\frac{v_i}{|A_i^k|}$, where

$|A_i^k|$ is the number of edges in A_i which are still uncovered at iteration k.

If several such rings exist, we select a ring such that $A_i = A_i^k$, so that an edge is covered exactly once. Such a ring always exists since we allow any subgraph of at most C edges to be a ring. This is done for the sake of simplicity and indeed without loss of generality.

In the following, we call *Greedy* this algorithm.

Chvátal [2] proves that Greedy is a $\mathcal{H}(C)$-approximation algorithm, with $\mathcal{H}(C) = \sum_{k=1}^{C} \frac{1}{k} \leq 1 + \ln(C)$. For our particular case, this guarantee is improved:

Theorem 1 :

The Greedy Algorithm has a worst-case performance-guarantee for ADRL of:

$$\alpha(C) \;=\; \sum_{k=v_C}^{C} \frac{1}{k} + \frac{1}{v_C}\left(\frac{C+1}{C}\left\lfloor\frac{v_C-1}{2}\right\rfloor + 2\left\lceil\frac{v_C-1}{2}\right\rceil\right)$$

where $v_C = \left\lceil \frac{1+\sqrt{1+8C}}{2} \right\rceil$. ◦

In the above theorem, the value v_C corresponds to the minimum number of vertices in a graph with C edges (*cf* appendix B).

This results may be detailed so as to give a better idea of the guarantee:

Corollary 1 :

For $C \geq 3$*:*

$$\begin{aligned} \Gamma(C) \;\leq\; \alpha(C) \;&\leq\; \Gamma(C) + \tfrac{3}{2v_C} - \tfrac{1}{2C} - \tfrac{1}{Cv_C} \\ &\leq\; \tfrac{14}{9} + \ln\left(\sqrt{\tfrac{C}{2}}\right) \end{aligned}$$

where $\Gamma(C) = \ln\left(\frac{C}{v_C}\right) - \frac{3}{2v_C} + \frac{1}{C} + \frac{3}{2}$. ◦

This comes from the comparison between $\sum_{k=1}^{x} \frac{1}{k}$ and $\ln(x)$. The properties of composition of equivalents of the logarithm function then straightforwardly lead to this second corollary:

Corollary 2 :

$$\alpha(C) \sim_{+\infty} \ln\left(\sqrt{\frac{C}{2}}\right)$$

◦

Let us now prove those results. The first idea is to use duality, as Chvátal does for its own proof.

Let H be a heuristic for the set-covering problem which returns a feasible solution for which an edge e has a cost[2] θ^e (such a solution has a total cost $\sum_{e\in E}\theta^e$). If there is a α_H such that $\frac{\theta^e}{\alpha_H}$ is a feasible solution of the dual problem, then α_H is a performance ratio for H. This is formalized in the following lemma:

Lemma 1 :
Let the following primal problem P and its dual D be:

$$[P]\left\{\begin{array}{ll}\min & cx\\ s.t. & Ax\geq b\\ & x\geq 0\end{array}\right.\qquad [D]\left\{\begin{array}{ll}\max & yb\\ s.t. & yA\leq c\\ & y\geq 0\end{array}\right.$$

Let $\bar{x}$ be a feasible solution for $[P]$ returned by an algorithm H and let be $\alpha\geq 1$. If there is a θ such that:

1. $y=\frac{\theta}{\alpha}$ *is feasible for* $[D]$ *(*i.e. $\theta A\leq \alpha.c, \theta\geq 0$*) and*
2. $\theta b\geq c\bar{x}$

*then H is a α-approximation for problem $[P]$ (*i.e. *: $\bar{x}\leq \alpha.x$ for all feasible solution x of $[P]$).* ∘

proof :
Let x be a feasible solution of $[P]$. We have:

$$\begin{array}{rcl}\alpha.cx & \overset{(\alpha.c\geq\theta A)}{\geq} & \theta Ax\\ & \overset{(Ax\geq b)}{\geq} & \theta b\\ & \overset{(\theta b\geq c\bar{x})}{\geq} & c\bar{x}\end{array}$$

This is true in particular for an optimal solution, and so $\bar{x}$ is a α-approximation of this optimum. □

Let us re-write these results in our particular case:

$$[Primal]\left\{\begin{array}{lll}\min & \sum_{A_i\in\mathcal{A}} v_i.x_i & \\ s.c. & \sum_{A_i\in\mathcal{A}} a_{ei}.x_i\geq 1 & \forall e\in E\\ & x_i\geq 0 & \end{array}\right.\qquad [Dual]\left\{\begin{array}{lll}\max & \sum_{e\in E} y_e & \\ s.c. & \sum_{e\in A_i} y_e\leq v_i & \forall A_i\in\mathcal{A}\\ & y_e\geq 0 & \end{array}\right.$$

[2]This exact notion of cost of an edge is precised later.

($[Primal]$ is indeed the continuous relaxation of the initial problem. We also drop the constraint $x_i \leq 1$ which is always true for an optimal solution.) Intuitively, y_e is the cost that must not be overpaid to cover the edge e.

Let x be a feasible solution of the primal problem. Define the cost in x of the edge e as: $\theta_x^e = \frac{v_e}{|A^e|}$, where A^e is the unique ring that contains e in the solution x (v^e is the number of vertices — that is the cost — of A^e). With this definition, we evenly divide the cost of a ring among the edges that compose it.

In what follows, we only consider solutions returned by Greedy, and thus we omit the index x and write θ^e for the cost of the edge e. Notice that, as an edge belongs to a unique ring in a solution, we have: $\theta^e = \sum_{A_i \in \mathcal{A}} a_{ei} \frac{v_i}{|A_i|} x_i$.

Now, let us prove that $\theta = (\theta^e)_{e \in E}$ verifies the hypotheses of lemma 1. We have[3]:

$$\begin{aligned}\sum_{e \in E} \theta^e &= \sum_{e \in E} \sum_{A_i \in \mathcal{A}} a_{ei} \frac{v_i}{|A_i|} x_i \\ &= \sum_{A_i \in \mathcal{A}} \left(\frac{v_i}{|A_i|} x_i \sum_{e \in E} a_{ei} \right) \\ &= \sum_{A_i \in \mathcal{A}} \frac{v_i}{|A_i|} x_i |A_i| = \sum_{A_i \in \mathcal{A}} v_i x_i\end{aligned}$$

and hence θ verifies hypothesis 2 of lemma 1. If besides we suppose that, for an integer α:

$$\forall A_i \in \mathcal{A} \quad : \quad \sum_{e \in A_i} \theta^e \leq \alpha . v_i \qquad (1)$$

then θ also verifies the hypothesis 1. So, it is sufficient to prove that the value of $\alpha(C)$ of theorem 1 is valid for equation (1) to conclude that Greedy is a $\alpha(C)$-approximation for $[Primal]$, and thus for ADRL.

Here are now two lemmas which allow to bound the cost of an edge according to the iteration it is selected by Greedy.

Lemma 2 :

*Let A_i be an element of $\mathcal{A}$. Let $e_1, e_2, \ldots e_{|A_i|}$ be the edges in A_i numbered from 1 to $|A_i|$, with respect to the order of selection by Greedy. (*i.e. *there are at most $k-1$ edges of A_i selected before e_k). We have:*

$$\theta^{e_k} \leq \frac{v_i}{|A_i| - k + 1}$$

∘

[3] One should be careful not to confuse notations. A^e is the unique ring that contains e in the solution, whereas A_i is the i^{th} ring of $\mathcal{A}$. If $e \in A_i$, then $A_i = A^e$.

proof :

When the edge e_k is selected, at most $k-1$ edges had been taken from A_i by Greedy. So, there remain at least $|A_i| - k + 1$ edges for at most v_i vertices in A_i. These edges form a ring A_i^k of cost $\frac{v_i}{|A_i|-k+1}$ by edge. If e_k has a higher cost, this implies that it has been selected with a more expensive ring, which contradicts the selection rule of Greedy. □

Lemma 3 :

Let A_i be an element of $\mathcal{A}$. Let S be a sub-set of A_i with $v_i - 1$ edges. We have:

$$\sum_{e \in S} \theta^e \leq \frac{C+1}{C} \left\lfloor \frac{v_i - 1}{2} \right\rfloor + 2 \left\lceil \frac{v_i - 1}{2} \right\rceil$$

for every solution returned by Greedy. ∘

proof :

First, remove from the graph G every edge that is selected by Greedy for a cost lower than (or equal to) $\frac{C+1}{C}$. Let a_0 be the number of such edges in S. We note S' the set S with these edges removed.

Cycles of C (or less) edges would be removed for a cost $1 \leq \frac{C+1}{C}$ per edge. A connected component of C (or more) edges would have at least C edges removed for a cost lower than $\frac{C+1}{C}$ per edge. So, remains in G only a forest F of which trees have strictly less than C edges. Let t be the number of trees of $F \cap S'$ (a tree of F may induce several trees in $F \cap S'$).

Let a_1 be the number of edges of S' which are isolated in F, and a_2 the number of edges which are not; we have: $v_i - 1 = a_0 + a_1 + a_2$, or:

$$\left(a_1 + \frac{a_2}{2}\right) + \left(a_0 + \frac{a_2}{2}\right) = v_i - 1 \qquad (\clubsuit)$$

There are t disjoint trees on $v_i - 1 - a_0$ edges: they require at least $v_i - 1 - a_0 + t$ vertices among those of S', and S' has at most v_i vertices since it is included in A_i. As a consequence: $v_i - 1 - a_0 + t \leq v_i$, and hence: $t \leq a_0 + 1$. Because $t \geq a_1$ (an isolated edge is a tree), we obtain: $a_1 \leq a_0 + 1$, which can be detailed since $t = a_1$ if and only if $a_2 = 0$. All in all we get:

$$\begin{array}{ll} & a_2 > 0 \text{ and } a_1 < a_0 + 1 \ (\textit{i.e.} : a_1 \leq a_0) \\ \text{or} & \\ & a_2 = 0 \text{ and } a_1 \leq a_0 + 1 \end{array}$$

Besides, according to equation ($\clubsuit$), we have: $(a_1 + \frac{a_2}{2}) + (a_0 + \frac{a_2}{2}) = v_i - 1$. From which:

- if $a_2 > 0$ then $a_1 \leq a_0$ and $(a_1 + \frac{a_2}{2}) \leq \frac{v_i - 1}{2} \leq \left\lceil \frac{v_i - 1}{2} \right\rceil$;
- if $a_2 = 0$ then $a_1 \leq a_0 + 1$ and $a_1 = (a_1 + \frac{a_2}{2}) \leq \frac{v_i}{2}$. Since a_1 and v_i are integers: $(a_1 + \frac{a_2}{2}) \leq \left\lfloor \frac{v_i}{2} \right\rfloor = \left\lceil \frac{v_i - 1}{2} \right\rceil$.

In all cases:

$$(a_1 + \frac{a_2}{2}) \leq \left\lceil \frac{v_i - 1}{2} \right\rceil \qquad (\spadesuit)$$

Notice now that every edge e isolated in F is selected with a cost of $\theta^e = 2$. Any other edge of F may be selected with the whole tree to which it belongs for a cost $\theta^e \leq \frac{1}{2}(2 + \frac{C+1}{C})$ (this tree has $2 \leq p \leq C$ edges and thus costs $\frac{p+1}{p} \leq \frac{3}{2} \leq \frac{1}{2}(2 + \frac{C+1}{C})$). This leads to a bound on the total cost for the edges of S:

$$\sum_{e \in S} \theta^e \leq \tfrac{C+1}{C} a_0 + 2a_1 + \tfrac{1}{2}(2 + \tfrac{C+1}{C})a_2 = \tfrac{C+1}{C}(a_0 + \tfrac{a_2}{2}) + 2(a_1 + \tfrac{a_2}{2})$$

together with equation ($\clubsuit$):

$$\sum_{e \in S} \theta^e \leq \frac{C+1}{C}(v_i - 1) + \left(2 - \frac{C+1}{C}\right)\left(a_1 + \frac{a_2}{2}\right)$$

and with the inequality ($\spadesuit$), as $\frac{C+1}{C} \leq 2$, we obtain:

$$\begin{aligned} \sum_{e \in S} \theta^e &\leq \tfrac{C+1}{C}(v_i - 1) + \left(2 - \tfrac{C+1}{C}\right)\left\lceil \tfrac{v_i - 1}{2} \right\rceil \\ &\leq \tfrac{C+1}{C}\left(v_i - 1 - \left\lceil \tfrac{v_i-1}{2} \right\rceil\right) + 2\left\lceil \tfrac{v_i-1}{2} \right\rceil \end{aligned}$$

that is:

$$\sum_{e \in S} \theta^e \leq \frac{C+1}{C}\left\lfloor \frac{v_i - 1}{2} \right\rfloor + 2\left\lceil \frac{v_i - 1}{2} \right\rceil$$

□

Let us prove now, with an induction on C, that inequality (1) holds for $\alpha = \alpha(C)$ as defined in theorem 1. What we have to prove is thus:

$$\forall A_i \in \mathcal{A} : \frac{1}{v_i} \sum_{e \in A_i} \theta^e \leq \sum_{k=v_C}^{C} \frac{1}{k} + \frac{1}{v_C}\left(\frac{C+1}{C}\left\lfloor \frac{v_C - 1}{2} \right\rfloor + 2\left\lceil \frac{v_C - 1}{2} \right\rceil\right) \quad (1')$$

(we assume an empty sum — when $v_C > C$ — to be zero.)

For $C = 1$ or $C = 2$, the only rings have one edge less than their number of vertices; that is: $\forall A_i \in \mathcal{A} : |A_i| = v_i - 1$. Lemma 3 then applies with $S = A_i$. For rings such that $v_i = v_c$, the inequality (1') then holds. The other rings are indeed only single edges for $C = 2$ and then we have:

$$\frac{1}{v_i} \sum_{e \in A_i} \theta^e = 1 \leq \frac{1}{v_C}\left(\frac{C+1}{C}\left\lfloor \frac{v_C - 1}{2} \right\rfloor + 2\left\lceil \frac{v_C - 1}{2} \right\rceil\right) = \frac{7}{6}$$

and inequality (1') also holds. Therefore the induction hypothesis is true for $C = 1$, $C = 2$.

Let $C \geq 3$ and assume that inequality (1') holds for every $C', 1 \leq C' < C$. Let A_i be a ring. Several cases may happen:

Case 1: $|A_i| = C' \leq C - 1$

The induction hypothesis is that inequality (1') holds for every C'. Since α is an increasing function of C, and v_i is positive, one has:

$$\sum_{e \in A_i} \theta^e \leq \alpha(C').v_i \leq \alpha(C).v_i$$

Case 2: $|A_i| = C$

Ring A_i has $v_i = v_C + x$ vertices. Remember that v_C is the minimum number of vertices for a graph with C edges; therefore $x \geq 0$.

Lemma 2 proves that the $C - v_i + 1$ first edges of A_i are selected for a total cost of at most $\sum_{k=1}^{C-v_i+1} \frac{v_i}{C-k+1}$; lemma 3 proves that the $v_i - 1$ remaining edges have a total cost of at most $\frac{C+1}{C} \left\lfloor \frac{v_i-1}{2} \right\rfloor + 2 \left\lceil \frac{v_i-1}{2} \right\rceil$. It implies that:

$$\begin{aligned} \frac{1}{v_i} \sum_{e \in A_i} \theta^e &\leq \sum_{k=v_i}^{C} \frac{1}{k} + \frac{1}{v_i} \left(\frac{C+1}{C} \left\lfloor \frac{v_i - 1}{2} \right\rfloor + 2 \left\lceil \frac{v_i - 1}{2} \right\rceil \right) \\ &= \alpha(C) - \beta(C, x) \end{aligned}$$

where:

$$\begin{aligned} \beta(C, x) = & \sum_{k=v_C}^{v_i-1} \frac{1}{k} \\ & - \frac{1}{v_C + x} \left(\frac{C+1}{C} \left\lfloor \frac{v_C + x - 1}{2} \right\rfloor + 2 \left\lceil \frac{v_C + x - 1}{2} \right\rceil \right) \\ & + \frac{1}{v_C} \left(\frac{C+1}{C} \left\lfloor \frac{v_C - 1}{2} \right\rfloor + 2 \left\lceil \frac{v_C - 1}{2} \right\rceil \right) \end{aligned}$$

It is thus suffisant to prove that $\beta(C, x)$ is always positive to get that equation (1) holds.

If $x = 0$ then $\beta(C, 0) = 0$.

If $x > 1$ then (note that $\frac{1}{C}(C+1+C-1)=2$) :

$$\begin{aligned}
\beta(C,x) \geq\ & \frac{1}{v_C} + \frac{1}{v_C+1} \\
& + \frac{C+1}{C}\left[\frac{1}{v_C}\Bigg(\underbrace{\left\lfloor\frac{v_C-1}{2}\right\rfloor + \left\lceil\frac{v_C-1}{2}\right\rceil}_{=v_C-1}\Bigg)\right. \\
& \qquad \left. - \frac{1}{v_C+x}\Bigg(\underbrace{\left\lfloor\frac{v_C+x-1}{2}\right\rfloor + \left\lceil\frac{v_C+x-1}{2}\right\rceil}_{=v_C+x-1}\Bigg)\right] \\
& + \frac{C-1}{C}\Bigg(\frac{1}{v_C}\underbrace{\left\lceil\frac{v_C-1}{2}\right\rceil}_{\geq\frac{v_C-1}{2}} - \frac{1}{v_C+x}\underbrace{\left\lceil\frac{v_C+x-1}{2}\right\rceil}_{\leq\frac{v_C+x}{2}}\Bigg) \\
\geq\ & \frac{1}{v_C} + \frac{1}{v_C+1} + \frac{C+1}{C}\left[1 - \frac{1}{v_C} - 1 + \frac{1}{v_C+x}\right] + \frac{C-1}{C}\left(\frac{1}{2} - \frac{1}{2v_C} - \frac{1}{2}\right) \\
\geq\ & \frac{1}{v_C+1} - \frac{1}{2v_C}\underbrace{\left(2\frac{C+1}{C} + \frac{C-1}{C} - 2\right)}_{=\frac{C+1}{C}\leq\frac{4}{3}\text{ for }C\geq 3} + \underbrace{\frac{C+1}{C(v_C+x)}}_{\geq 0} \\
\geq\ & \frac{1}{v_C+1} - \frac{2}{3v_C} \\
\geq\ & \frac{v_C-2}{3v_C(v_C+1)} \\
\geq\ & 0
\end{aligned}$$

If $x = 1$ then the first three inequalities above hold if removing the term $\frac{1}{v_C+1}$. We get:

$$\begin{aligned}
\beta(C,1) \geq\ & -\frac{1}{2v_C}\left(2\frac{C+1}{C} + \frac{C-1}{C} - 2\right) + \frac{C+1}{C(v_C+x)} \\
\geq\ & \frac{C+1}{C}\left(\frac{1}{v_C+1} - \frac{1}{2v_C}\right) \\
\geq\ & \frac{C+1}{C}\frac{v_C-1}{2v_C(v_C+1)} \\
\geq\ & 0
\end{aligned}$$

From the two cases above, it follows that equation (1) holds for $\alpha(C)$. As a consequence, by the induction principle, this ends the proof of theorem 1.

With such performances, the *Greedy* algorithm has the best known guarantees for the ADRL problem. Moreover those guarantees are pessimistic, for the proved ratio does not seem to be tight.

We now prove the two corollaries. The first one is a consequence of the following lemma:

Lemma 4 :

$$\forall n \in \mathbb{N}, \forall p \in \mathbb{N}, p \geq n : \ln\left(\frac{p}{n}\right) + \frac{1}{p} \leq \sum_{k=n}^{p} \frac{1}{k} \leq \ln\left(\frac{p}{n}\right) + \frac{1}{n}$$

∘

proof :

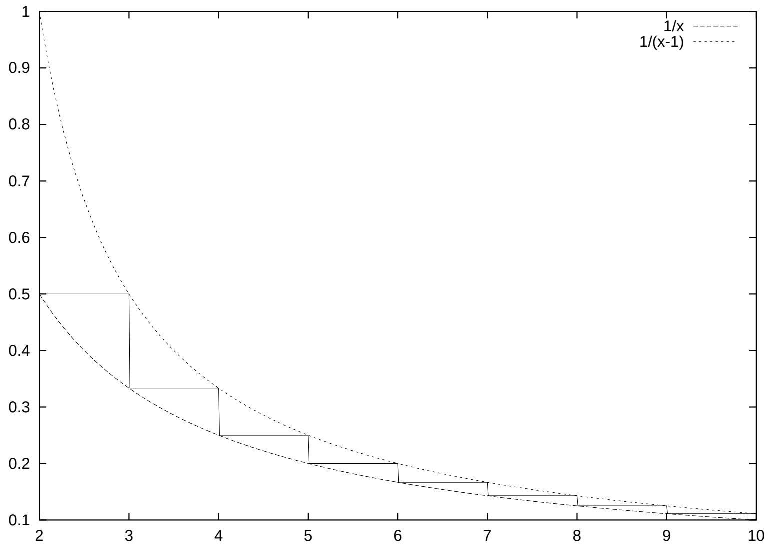

Figure 1: Comparison of the areas of $\int_n^p \frac{1}{x}$, $\sum_n^p \frac{1}{x}$ and $\int_{n+1}^{p+1} \frac{1}{x-1}$

This interval is a mere comparison of areas. As a matter of fact, for $p \geq n$, we have (*cf* Figure 1) :

$$\int_n^p \frac{1}{t} dt + \frac{1}{p} \leq \sum_{k=n}^{p} \frac{1}{k} \leq \int_{n+1}^{p+1} \frac{1}{t-1} dt + \frac{1}{n} = \int_n^p \frac{1}{t} dt + \frac{1}{n}$$

i.e. :

$$\ln\left(\frac{p}{n}\right) + \frac{1}{p} \leq \sum_{k=n}^{p} \frac{1}{k} \leq \ln\left(\frac{p}{n}\right) + \frac{1}{n}$$

□

In our case, for $C \geq 3$ (and thus $v_c \leq C$), we have:

$$\alpha(C) = \sum_{k=v_C}^{C} \frac{1}{k} + \underbrace{\frac{1}{v_C}\left(\frac{C+1}{C}\left\lfloor\frac{v_C-1}{2}\right\rfloor + 2\left\lceil\frac{v_C-1}{2}\right\rceil\right)}_{\gamma(C)}$$

and the previous lemma proves:

$$\ln\left(\frac{C}{v_C}\right) + \frac{1}{C} + \gamma(C) \leq \alpha(C) \leq \ln\left(\frac{C}{v_C}\right) + \frac{1}{v_C} + \gamma(C) \qquad (2)$$

now:

$$\begin{aligned}
\gamma(C) &= \frac{1}{v_C}\left(\left(2 - \frac{C-1}{C}\right)\left\lfloor\frac{v_C-1}{2}\right\rfloor + 2\left\lceil\frac{v_C-1}{2}\right\rceil\right) \\
&= \frac{1}{v_C}\left(2\left(\left\lfloor\frac{v_C-1}{2}\right\rfloor + \left\lceil\frac{v_C-1}{2}\right\rceil\right) - \frac{C-1}{C}\left\lfloor\frac{v_C-1}{2}\right\rfloor\right) \\
&= \frac{1}{v_C}\left(2v_C - 2 - \underbrace{\frac{C-1}{C}}_{\leq 1}\underbrace{\left\lfloor\frac{v_C-1}{2}\right\rfloor}_{\frac{v_C-2}{2}\leq\cdot\leq\frac{v_C-1}{2}}\right)
\end{aligned}$$

from that:

$$\gamma(C) \geq \frac{1}{v_C}\left(2v_C - 2 - \frac{v_C-1}{2}\right) = \frac{3}{2} - \frac{3}{2v_C}$$

using (2), we obtain:

$$\alpha(C) \geq \ln\left(\frac{C}{v_C}\right) + \frac{1}{C} + \frac{3}{2} - \frac{3}{2v_C} = \Gamma(C)$$

which is the announced lower bound. We also have:

$$\begin{aligned}
\gamma(C) &\leq \frac{1}{v_C}\left(2v_C - 2 - \frac{C-1}{C}\frac{v_C-2}{2}\right) \\
&\leq \frac{1}{v_C}\left(2v_C - 2 - \frac{v_C}{2} + 1 + \frac{v_C}{2C} - \frac{1}{C}\right) \\
&\leq \frac{3}{2} - \frac{1}{v_C} + \frac{1}{2C} - \frac{1}{Cv_C} \\
&\leq \frac{3}{2} + \frac{1}{2C} - \frac{C+1}{Cv_C}
\end{aligned}$$

using equation (2), we obtain:

$$\alpha(C) \leq \Gamma(C) + \frac{3}{2v_C} - \frac{1}{2C} - \frac{1}{Cv_C}$$

which is the announced upper bound.

Remarking that $\frac{C}{v_C} = \frac{C}{\left\lceil\frac{1+\sqrt{1+8C}}{2}\right\rceil} \leq \frac{C}{\frac{\sqrt{8C}}{2}} = \sqrt{\frac{C}{2}}$, we get:

$$\alpha(C) \leq \ln\left(\sqrt{\frac{C}{2}}\right) + \frac{3}{2} + \frac{1}{2C} - \frac{1}{Cv_C} \leq \ln\left(\sqrt{\frac{C}{2}}\right) + \frac{14}{9}$$

($\frac{1}{2C} - \frac{1}{Cv_C} = \frac{v_C-2}{2Cv_C}$ is maximum for $C = 3$)

For the equivalent in corollary 2, notice that in the inequalities of corollary 1 the term $\ln\left(\frac{C}{v_C}\right)$ is the only one which tends to infinity. So, $\alpha(C) \sim_{+\infty} \ln\left(\frac{C}{v_C}\right)$.

Moreover, if two functions f_1 and f_2 are strictly positive, equivalent at infinity, and tend to infinity, then $\ln(f_1) \sim_{+\infty} \ln(f_2)$. Here, these hypothesis hold for $\frac{C}{v_C}$ and $\sqrt{\frac{C}{2}}$ and this ends the proof by transitivity of the equivalence relation.

3 Conclusion

We prove that Chvátal's greedy algorithm for the set-covering problem, when used in the special case of ADRL, has an improved performance guarantee, which is the best known guarantee for this particular case. However the sub problem of finding the best ring in the demand graph is $\mathcal{NP}$-hard in the general case (reduction to CLIQUE) and thus the algorithm is unlikely to be polynomial.

Yet these results give insight of what can be expected from a set-covering approach of this problem. For instance column generation methods, for which finding a good column correspond to finding a good ring, are promising and their performances should be compared with the methods already tried, like tabu search.

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A Some numerical values

Figures 2 et 3 show the evolution of the value of α, respectively for small and big values of C.

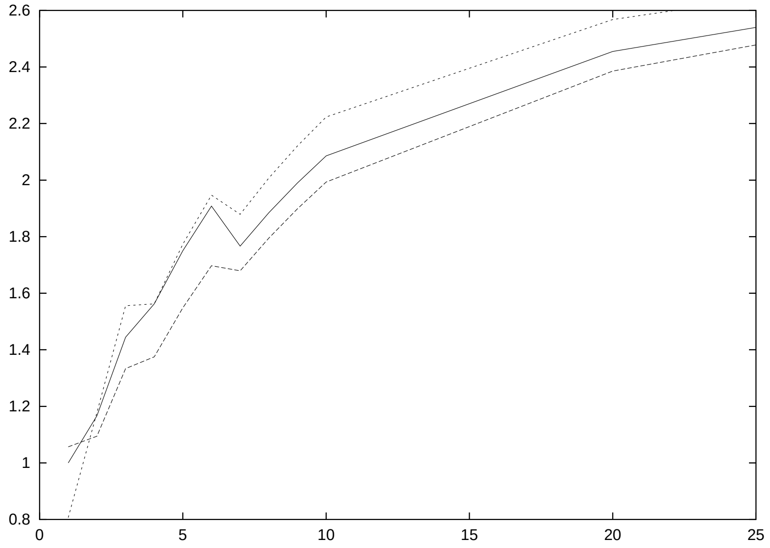

Figure 2: Evolution of the value of $\alpha(C)$ depending on C. Plain line: $\alpha(C)$, dotted line: lower and upper bounds from corollary 1.

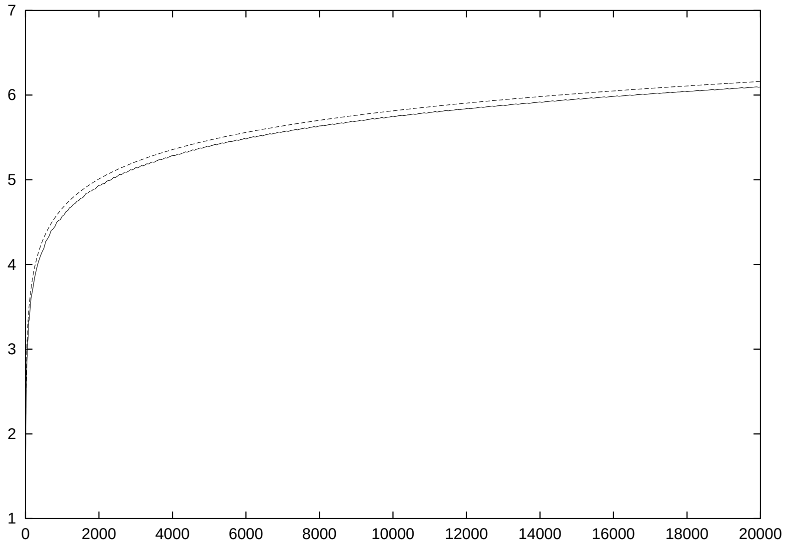

Figure 3: Evolution of the value of $\alpha(C)$ depending on C. Plain line: $\alpha(C)$, dotted line: $\frac{14}{9} + \ln\left(\sqrt{\frac{C}{2}}\right)$.

B Minimum number of vertices in a graph with C edges

Property 1 :

The minimum number of vertices v_C of a graph with C edges is:

$$v_c = \left\lceil \frac{1 + \sqrt{1 + 8C}}{2} \right\rceil$$

∘

proof :

A graph with v_C vertices is contained in the complete graph of v_C vertices. Therefore, the minimum v_C is reached for the smallest complete graph with at least C edges.

A complete graph with v vertices has $\frac{v(v-1)}{2}$ edges. And thus v_C is the solution of:

$$\min \left\{ v : v \geq 0, \frac{v(v-1)}{2} \geq C \right\}$$

This leads to a second-degree equation which straightforwardly gives:

$$v_c = \left\lceil \frac{1 + \sqrt{1 + 8C}}{2} \right\rceil$$

□

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