

BLOW-UP FOR SEMI-LINEAR PARABOLIC PROBLEMS WITH NON CONSTANT COEFFICIENTS

MICHELE GRILLOT

IUFM d'Orléans-Tours et Université d'Orléans MAPMO -

BP 6759 - 45 067 Orléans cedex 02 France

ALICE (CHALJUB)-SIMON

Université d'Orléans MAPMO -

BP 6759 - 45 067 Orléans cedex 02 France

Abstract

In this paper we study the solutions of some semi-linear parabolic problems with non constant coefficients. We prove the existence of solutions which blow up at a finite time, and give the behavior near a point of blow-up.

1 Introduction : notations and main results

In this paper we consider the problem :

$$\begin{cases} u_t + Lu = \lambda a(x)f(u) & \text{for } (x, t) \in \mathbb{R}^N \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N \\ u(x, t) \rightarrow 0 & \text{when } |x| \rightarrow \infty \end{cases} \quad (1)$$

whith $L = -\Delta + c^2$, $c > 0$, $N > 2$ and $\lambda > 0$. The functions a and u_0 are continuous, bounded, strictly positive and tend to zero at infinity. The function f is superlinear. We also assume that f is C^2 with nonnegative values, $f'(x) > 0$ for $x > 0$ and $f''(x) > 0$ for $x > 0$.

The operator L appeared earlier in some elliptic problems related with the equation of Klein-Gordon [8] [9]. The first motivation of this work is the study of the relationship with the elliptic problem. The special form of the right-hand side of (1) is given by sake of simplicity. More general forms can be considered.

In this paper (1) will be written as :

$$\begin{cases} u_t - \Delta u = F(x, u) & \text{for } (x, t) \in \mathbb{R}^N \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N \\ u(x, t) \rightarrow 0 & \text{when } |x| \rightarrow \infty \end{cases} \quad (2)$$

where $F(x, u) = \lambda a(x)f(u) - c^2u$. We consider regular solutions of (2) in the sense of Kaplan [5] : let Ω be an open regular connected, not necessarily bounded set of \mathbb{R}^N and $Q_T = \Omega \times (0, T]$ for $T > 0$. The function u is $C^{2,1}(\overline{Q_T})$ means that u , $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ are defined in Q_T and can be continuously continued up to $\overline{\Omega} \times (0, T]$.

⁰AMS Subject Classifications : 35B40, 35K55.

We use extensively the comparison theorem of Kaplan [5] that we denote in the rest of the paper by Kaplan's theorem.

In section 2, we study the solutions of (2) which blow up at a finite time, and get estimates of the time of blow-up. In fact we cannot use the standard methods to (2), due to the term "a" which tends to zero at infinity. We work in the ball centered at zero of radius $R > 0$ denoted by B_R and we consider the problem :

$$\begin{cases} u_t - \Delta u = F(x, u) & \text{for } (x, t) \in B_R \times (0, +\infty) \\ u(x, 0) = u_0(x) & \text{for } x \in B_R \\ u(x, t) = 0 & \text{when } |x| = R. \end{cases} \quad (3)$$

We prove existence of blow-up for solutions of (3) and then we can conclude with Kaplan's theorem. More precisely, define

$$\alpha_R = \min_{x \in B_R} a(x), \quad (4)$$

(λ_1, ϕ) depending on R such that :

$$\begin{cases} -\Delta \phi = \lambda_1 \phi & \text{in } B_R \\ \phi = 0 & \text{on } \partial B_R \\ \phi > 0 & \text{in } B_R \\ \int_{B_R} \phi \, dx = 1 \end{cases} \quad (5)$$

and s_0 the greatest zero of the function g_R defined on $[0, \infty)$ by :

$$g_R(s) = \lambda \alpha_R f(s) - (\lambda_1 + c^2)s. \quad (6)$$

Troughout this paper we denote by T_b the time of blow-up of a function b and by

$$[0, T(b)] \quad (7)$$

a closed interval on which b is bounded and regular. The main result of this section is the following :

Theorem 1.1 *Fix $R = R_0 > 0$ such that $\int_A^{+\infty} \frac{ds}{f(s)} < \infty$ for $A > 0$ and $\min_{B_R} u_0 > s_0$. Then the solution u of (2) blows up in a finite time $T_u \neq 0$.*

In section 3, we study the blow-up rate and prove that the qualitative properties of the solutions of (2) near a blow-up point is the same as in the constant coefficient case [2] [3]. More precisely we prove the following theorem :

Theorem 1.2 *Let $p > 1$ and $R_0 > 0$. For $f(t) \geq t^p$ for all $t \geq 0$, we have the estimate :*

$$\| u(\cdot, t) \|_{\infty} \geq [(p-1)\lambda \alpha_{R_0}]^{-\frac{1}{p-1}} (T_u - t)^{-\frac{1}{p-1}} \quad (8)$$

for $0 < t < T_u$ with :

$$\| u(\cdot, t) \|_{\infty} = \sup_{x \in \mathbb{R}^N} u(x, t). \quad (9)$$

We also give an upper bound of a solution of (2) in a neighborhood of a blow-up point. Let $R_0 > 0$. We introduce the following assumptions :

$$\Delta u_0 - c^2 u_0 + \lambda a(x) f(u_0) \geq 0 \quad \text{in } B_{R_0} \quad (10)$$

and

$$f'(r)r - f(r) \geq 0 \quad \text{for } r \geq 0. \quad (11)$$

Theorem 1.3 *Assume (10) and (11). Let (X, T_u) be a blow-up point for a solution u of (2) and assume that V is an open neighborhood of X in B_{R_0} such that u is bounded on $\partial V \times [0, T_u)$. Then for every $\eta \in (0, T_u)$, there exists a constant $\delta = \delta(u, \eta)$ such that*

$$u(x, t) \leq \Phi^{-1}[-\delta(T_u - t)] \quad (12)$$

for all $x \in V$ and $t \in (\eta, T_u)$, where Φ^{-1} is the inverse function of a primitive Φ of f .

We precise both last theorems in particular cases :

Theorem 1.4 *Under the assumptions of theorems 1.2 and 1.3, for $f(u) = u^p$ or $f(u) = (1 + u)^p$ with $p > 1$, the function w defined by :*

$$w(x, t) = (T_u - t)^{\frac{1}{p-1}} u(x, t) \quad (13)$$

is bounded on $\mathbb{R}^N \times (0, T_u)$.

In section 4, we give the asymptotic behavior of the solution u of (2) near a blow-up point.

Theorem 1.5 *Assume that $1 < p < \frac{N+2}{N-2}$ and $f(u) = u^p$ or $f(u) = (1 + u)^p$ with $p > 1$. Let (X, T_u) be a blow-up point for u satisfying (2). Then*

$$\lim_{t \rightarrow T_u} (T_u - t)^{\frac{1}{p-1}} u(X + y(T_u - t)^{\frac{1}{2}}, t) = (\lambda (p-1) a(X))^{-\frac{1}{p-1}}. \quad (14)$$

The limit is independent of $y \in \mathbb{R}^N$ and it is uniform on each compact set $|y| \leq C$.

2 Existence of blow-up for solutions of (2)

2.1 Upper bound for u_R and u

We assume there exists a regular solution u_R of (3). We have

$$a(x) \leq \|a\|_{\infty}.$$

Consider the differential problem :

$$\begin{cases} \frac{dz}{dt} = \lambda \|a\|_\infty f(z) - c^2 z \\ z(0) \geq \sup_{x \in \mathbb{R}^N} u_0(x) \\ z(t) \geq 0 \end{cases} \quad (15)$$

By the change $z(t) = e^{-c^2 t} v(t)$

$$\begin{cases} \frac{dv}{dt} = e^{c^2 t} \lambda \|a\|_\infty f(e^{-c^2 t} v(t)) \geq 0 \\ v(0) \geq \sup_{x \in \mathbb{R}^N} u_0(x) \end{cases}$$

and then $v(t) \geq 0$ and $z(t) \geq 0$ on their interval of definition. We use the notation introduced in (7). Choosing $z(0) \geq \sup_{x \in \mathbb{R}^N} u_0(x)$, we get a solution z of (15) regular and bounded on $[0, T(z)]$. By Kaplan's theorem, we obtain :

$$u_R(x, t) \leq z(t) \quad \text{for all } x \in \overline{B_R} \quad \text{and } t \in (0, \min(T(u_R), T(z))). \quad (16)$$

As z does not depend on R , the inequality (16) is true for every R . Now we look at u . Assume there exists a regular solution of (2). In B_R , u satisfies

$$\begin{cases} u_t - \Delta u = F(x, u) \\ u(x, 0) = u_0(x) \\ u(x, t) \geq 0 \quad \text{for } |x| = R. \end{cases}$$

By Kaplan's theorem in B_R we have :

$$u \geq u_R \quad \text{in } B_R \times (0, \min(T(u), T(u_R))). \quad (17)$$

As $u_R(x, t) = 0$ outside of B_R , the inequality (17) is true in \mathbb{R}^N . On the other side, Kaplan's theorem gives :

$$u(x, t) \leq z(t) \quad \text{in } \mathbb{R}^N \times [0, \min(T(u), T(z))].$$

Finally we get :

$$\text{For all } R > 0 \quad u_R \leq u \leq z \quad \text{in } \mathbb{R}^N \times (0, \min(T(u), T(u_R), T(z))). \quad (18)$$

2.2 Existence of blowing up solutions

2.2.1 Blow-up for u_R

Following the original idea of Kaplan [5], we show that $\sup_{x \in \overline{B_R}} u_R(x, t)$ is bounded from below. This estimate allows us to prove the existence of blow-up for u_R . We assume that u_R exists as a regular solution of (3). Define

$$\hat{u}_R(t) = \int_{B_R} u_R(x, t) \phi(x) dx \quad (19)$$

where ϕ is defined in (5). Then

$$\hat{u}_R(0) \geq \inf_{\overline{B}_R} u_0(x). \quad (20)$$

Multiplying the equation (3) of u_R by ϕ and integrating over B_R , we find :

$$\frac{d\hat{u}_R}{dt} = \lambda \int_{B_R} a f(u_R) \phi \, dx - (\lambda_1 + c^2) \hat{u}_R.$$

As

$$\begin{aligned} \int_{B_R} a f(u_R) \phi \, dx &\geq \alpha_R \int_{B_R} f(u_R) \phi \, dx \\ &\geq \alpha_R f \left(\int_{B_R} u_R \phi \, dx \right) \end{aligned}$$

with α_R defined in (4), and by use of Jensen's inequality. Finally we find a differential inequality for \hat{u}_R :

$$\frac{d\hat{u}_R}{dt} \geq \lambda \alpha_R f(\hat{u}_R) - (\lambda_1 + c^2) \hat{u}_R \quad (21)$$

with (20). Consider the differential problem :

$$\begin{cases} \frac{d\zeta_R}{dt} &= \lambda \alpha_R f(\zeta_R) - (\lambda_1 + c^2) \zeta_R \\ \zeta_R(0) &= \inf_{x \in \overline{B}_R} u_0(x). \end{cases} \quad (22)$$

Kaplan's theorem gives $\hat{u}_R(t) \geq \zeta_R(t)$ for $t \in [0, \min(T(\hat{u}_R), T(\zeta_R))]$. As $\hat{u}_R(t) \leq \sup_{x \in \overline{B}_R} u_R(x, t)$, we get :

$$\sup_{x \in \overline{B}_R} u_R(x, t) \geq \zeta_R(t). \quad (23)$$

Inequality (23) is true for bounded functions; now if ζ_R blows up for $T_{\zeta_R} > T(\zeta_R)$, by continuity we get blow-up results for u_R . Recall that s_0 and g_R are defined in (6). Then

Proposition 2.1 *If $\int_A^{+\infty} \frac{ds}{f(s)} < \infty$ and if $\inf_{\overline{B}_R} u_0 > s_0$, then ζ_R blows up at a finite time*

$$T_{\zeta_R} = \int_{\zeta_R(0)}^{+\infty} \frac{ds}{\lambda \alpha_R f(s) - (\lambda_1 + c^2)s}.$$

Note that the integral is convergent as f is superlinear. The second condition says that u_0 must be "big enough". Now the inequality (23) shows that u_R must blow up for a finite time T_{u_R} , if ζ_R does. Using the function z introduced in (15), we know that $u_R(x, t) \leq z(t)$ for $t \in (0, \min(T(u_R), T(z)))$. Let $h(s) = \lambda \|a\|_{\infty} f(s) - c^2 s$, and denote by s_1 the greatest zero of h . We get :

Proposition 2.2 *If $z(0) > s_1$, then z blows up in a finite time*

$$T_z = \int_{z(0)}^{+\infty} \frac{ds}{\lambda \|a\|_{\infty} f(s) - c^2 s}.$$

We can choose $z(0) > s_1$, and then get a function z which blows up at $t = T_z$. Finally we get :

Proposition 2.3 *Under the conditions of propositions 2.1 and 2.2, we get blow-up for u_R in a finite time T_{u_R} satisfying*

$$\int_{z(0)}^{+\infty} \frac{ds}{\lambda \|a\|_{\infty} f(s) - c^2 s} \leq T_{u_R} \leq \int_{\inf_{\overline{B_R}} u_0}^{+\infty} \frac{ds}{\lambda \alpha_R f(s) - (\lambda_1 + c^2) s}.$$

2.2.2 Blow-up for u

We can deduce from the preceding subsection conditions for the explosion of u . We assume that the conditions of proposition 2.3 are satisfied for $R = R_0$. We have :

$$u_{R_0}(x, t) \leq u(x, t) \leq z(t) \quad \text{for } (x, t) \in \overline{B_{R_0}} \times (0, \min(T(u_{R_0}), T(u), T(z)))$$

and

$$\sup_{x \in \overline{B_{R_0}}} u(x, t) \geq \sup_{x \in \overline{B_{R_0}}} u_{R_0}(x, t).$$

Then theorem 1.1 results from the inequality $T_z \leq T_u \leq T_{\zeta_{R_0}}$.

2.2.3 Special cases

First we consider $f(u) = u^p$ with $p > 1$. Then (22) becomes

$$\begin{cases} \frac{d\zeta}{dt} = \lambda \alpha_R \zeta^p - (\lambda_1 + c^2) \zeta \\ \zeta(0) = \inf_{x \in B_R} u_0(x) \end{cases}$$

By the change $\zeta(t) = e^{-\mu t} g(t)$ with $\mu = \lambda_1 + c^2$, (22) becomes :

$$\begin{cases} \frac{dg}{dt} = \lambda \alpha_R e^{-(p-1)\mu t} g^p \\ g(0) = \inf_{x \in B_R} u_0(x). \end{cases} \quad (24)$$

Integrating the differential equation (24) between 0 and t , and taking $t = T_g$, we get :

$$1 - e^{-(p-1)\mu T_g} = \frac{\mu}{\lambda \alpha_R g^{p-1}(0)}$$

which gives the condition for existence of blow-up for g :

$$\inf_{x \in B_R} u_0(x) = g(0) > \left[\frac{\lambda \alpha_R}{\lambda_1 + c^2} \right]^{\frac{1}{p-1}}$$

and the time of blow-up for g and ζ_R :

$$T_g = T_{\zeta_R} = -\frac{1}{\mu(p-1)} \ln \left[1 - \frac{\mu}{\lambda \alpha_R g^{p-1}(0)} \right].$$

We proceed in a similar way for z . In (15) we make the change : $z(t) = e^{-c^2 t} v(t)$ and get:

$$\begin{cases} \frac{dv}{dt} = \lambda \|a\|_\infty e^{-(p-1)c^2 t} v^p \\ v(0) \geq \sup_{\mathbb{R}^N} u_0. \end{cases}$$

Integrating this differential equation between 0 and t and taking $t = T_v$, we get :

$$v(0) > \left[\frac{\lambda \|a\|_\infty}{c^2} \right]^{\frac{1}{p-1}}$$

and

$$T_z = T_v = -\frac{1}{(p-1)c^2} \ln \left(1 - \frac{c^2}{\lambda \|a\|_\infty v^{p-1}(0)} \right).$$

Then

$$-\frac{1}{(p-1)c^2} \ln \left(1 - \frac{c^2}{\lambda \|a\|_\infty z^{p-1}(0)} \right) \leq T_{u_R} \leq -\frac{1}{\mu(p-1)} \ln \left(1 - \frac{\mu}{\lambda \alpha_R g^{p-1}(0)} \right).$$

Next, if $f(u) = (1+u)^p$ or $f(u) = 1+u^p$ with $p > 1$, we have $f(u) > u^p$ for $u > 0$ and so we get the same bound from below of u_R and the same bound from above of T_{u_R} . On the other side, if $f(t) = (1+t)^p$ we can integrate the associated differential equation to get a bound from above of u_R and a bound from below of T_{u_R} . Precisely, we have :

$$\begin{cases} \frac{dv}{dt} = \lambda \|a\|_\infty e^{c^2 t} (1 + e^{-c^2 t} v)^p \leq \lambda \|a\|_\infty e^{c^2 t} (1+v)^p \\ v(0) \geq \sup_{\mathbb{R}^N} u_0. \end{cases}$$

Considering

$$\begin{cases} \frac{d\nu}{dt} = \lambda \|a\|_\infty e^{c^2 t} (1+\nu)^p \\ \nu(0) = v(0) \end{cases}$$

we get :

$$e^{c^2 T_\nu} = e^{c^2 T_v} = 1 + \frac{c^2}{(p-1)\lambda \|a\|_\infty} \times \frac{1}{[1+\nu(0)]^{p-1}}.$$

Let us notice that no necessary condition for the blow-up of ν appears in this proof. Finally we have :

$$\frac{1}{c^2} \ln \left[1 + \frac{c^2}{(p-1)\lambda \|a\|_\infty} \times \frac{1}{[1+\nu(0)]^{p-1}} \right] \leq T_{u_R} \leq -\frac{1}{(p-1)\mu} \ln \left[1 - \frac{\mu}{\lambda \alpha_R g^{p-1}(0)} \right].$$

3 Estimate of blow-up rate

In this section we give an estimate of a solution u of (2) with respect to $(T_u - t)$.

3.1 Lower bound

Proof of theorem 1.2 : First we give a minoration of u_{R_0} . Recall that \hat{u}_{R_0} defined in (19) satisfies (20)-(21).

First we study the case : $f(u) = u^p$. We consider the differential problem (22) with the change : $\zeta(t) = e^{-\mu t} g(t)$, $\mu = \lambda_1 + c^2$, we get :

$$\begin{cases} \frac{dg}{dt} = \lambda \alpha_{R_0} e^{-(p-1)\mu t} g^p \\ g(0) = \inf_{B_{R_0}} u_0 \end{cases}$$

By integration :

$$g(t) = \left(\frac{\mu}{\lambda \alpha_{R_0}} \right) \left[e^{-(p-1)\mu t} - e^{-(p-1)\mu T_g} \right]^{-\frac{1}{p-1}}$$

for $0 < t < T_g$ and by the mean-value theorem :

$$g(t) = [(p-1)\lambda \alpha_{R_0}]^{-\frac{1}{p-1}} (T_g - t)^{-\frac{1}{p-1}} e^{\mu \theta} \quad (t < \theta < T_g)$$

for $0 < t < T_g$. Now

$$\hat{u}_{R_0}(t) \geq [(p-1)\lambda \alpha_{R_0}]^{-\frac{1}{p-1}} (T_g - t)^{-\frac{1}{p-1}}.$$

As $T_{\hat{u}_{R_0}} \geq T_g$, we obtain :

$$\hat{u}_{R_0}(t) \geq [(p-1)\lambda \alpha_{R_0}]^{-\frac{1}{p-1}} (T_{\hat{u}_{R_0}} - t)^{-\frac{1}{p-1}}$$

for $0 < t < T_{\hat{u}_{R_0}}$. As $\sup_{x \in B_{R_0}} u_{R_0}(x, t) \geq \hat{u}_{R_0}(t)$ with $T_{u_{R_0}} \leq T_{\hat{u}_{R_0}}$, we get :

$$\| u_{R_0}(\cdot, t) \|_{\infty} \geq [(p-1)\lambda \alpha_{R_0}]^{-\frac{1}{p-1}} (T_{u_{R_0}} - t)^{-\frac{1}{p-1}} \quad (25)$$

for $0 < t < T_{u_{R_0}}$. As $u(x, t) \geq u_{R_0}(x, t)$ for $x \in \mathbb{R}^N$, we finally obtain :

$$\| u(\cdot, t) \|_{\infty} \geq [(p-1)\lambda \alpha_{R_0}]^{-\frac{1}{p-1}} (T_u - t)^{-\frac{1}{p-1}} \quad (26)$$

for $0 < t < T_u$.

Now if $f(u) > u^p$ for $u \geq 0$, we get again :

$$\begin{cases} \frac{d\hat{u}_{R_0}}{dt} \geq \lambda \alpha_R (\hat{u}_{R_0})^p - (\lambda_1 + c^2) \hat{u}_{R_0} \\ \hat{u}_{R_0}(0) \geq \inf_{x \in B_{R_0}} u_0(x) \end{cases} \quad (27)$$

and the proof is still valid.

3.2 Lower bound of a solution of (2) in a neighborhood of a blow-up point

In this section we prove theorems 1.3 and 1.4. Let us consider a solution u of (2), we restrict ourselves to B_{R_0} as the blow-up occurs in B_{R_0} . We assume (10). For instance if $u_0 = U_0$, where U_0 is a positive constant, and if $f(t) = t^p$, we find that (10) is verified if

$$U_0 \geq \left[\frac{c^2}{\lambda \inf_{x \in B_{R_0}} a(x)} \right]^{\frac{1}{p-1}}$$

and furthermore U_0 must satisfy the condition of proposition 2.1. It is possible to choose U_0 in such a way. In the case $f(t) = (1+t)^p$, it is easy to see that we can choose $u_0 = U_0$ constant satisfying (10), but the value of U_0 is not explicit.

Now we give here a proof widely inspired by Friedmann and MacLeod [2]. Let δ be a positive real and consider the function J defined for $(x, t) \in B_{R_0} \times (0, T_u)$ by :

$$J(x, t) = u_t(x, t) - \delta f(u(x, t)).$$

Because of (10), we have : $u_t(x, t) > 0$ in $B_{R_0} \times (0, T_u)$ (see [7]).

Lemma 3.1 *Under the condition (11), J satisfies in $B_{R_0} \times (0, T_u)$ the differential inequality*

$$J_t - \Delta J + c^2 J - \lambda a(x) f'(u) J \geq 0. \quad (28)$$

Proof : A direct computation gives :

$$J_t - \Delta J + c^2 J - \lambda a(x) f'(u) J = \delta f''(u) |\nabla u|^2 + \delta c^2 [f'(u)u - f(u)]$$

and the result holds.

The condition (11) says that f is greater than a linear function. In fact, the solutions of the equation $f'(u)u - f(u) = 0$ are the linear functions (in particular $f(0) = 0$). We can say also that $xf'(x) - f(x) \geq 0$ is equivalent to the following : the function $x \mapsto \frac{f(x)}{x}$, for $x > 0$, is an increasing function.

We see also that $f(x) = x^p$ satisfies the inequality (11) for $x \geq 0$, and that $f(x) = (1+x)^p$ satisfies (11) for $x \geq 1$.

Lemma 3.2 *Let (X, T_u) be a blow-up point for u solution of (2) and assume that V is an open neighborhood of X in B_{R_0} such that u is bounded on $\partial V \times [0, T_u)$. Then for every $\eta \in (0, T_u)$, there exists $\delta = \delta(\eta, u) > 0$ such that $J \geq 0$ in $V \times (\eta, T_u)$.*

Proof : As $u_t > 0$ in $\bar{V} \times (0, T_u)$, there exists a constant $C_u > 0$ such that $u_t \geq C_u > 0$ in $\bar{V} \times (0, T_u)$. As u is bounded on $\partial V \times [0, T_u)$, so is $f(u)$, and there exists δ_1 such that :

$$u_t - \delta_1 f(u) \geq C_u - \delta_1 f(u) \geq 0$$

on $\partial V \times [0, T_u)$. On the other hand, let x be in V and η in $(0, T_u)$. The function $x \mapsto u(x, \eta)$ is bounded on \bar{V} by definition of T_u . Then there exists a constant $\delta_2(u, \eta) > 0$ such that :

$$J(x, \eta) = u_t(x, \eta) - \delta_2 f(u(x, \eta)) \geq 0$$

for $x \in \bar{V}$. By Kaplan's theorem, comparing J and 0 on $V \times [\eta, T(u)]$, $T(u) < T_u$, we conclude that $J \geq 0$ in $V \times [\eta, T(u)]$ and the result holds by continuity.

Proof of Theorem 1.3 : By lemma 3.2, we get : $u_t \geq \delta f(u)$; assuming $f(u) \neq 0$, we have

$$\frac{u_t}{f(u)} \geq \delta.$$

By integration in the interval $(t, t') \subset (0, T_u)$, and taking $t' = T_u$, we get :

$$\Phi(u(t)) \leq -\delta(T_u - t).$$

Φ is a monotone function, which has an inverse function Φ^{-1} , and the result holds.

Proof of Theorem 1.4 : If $f(r) = r^p$, then $\Phi(r) = -1/[(p-1)r^{p-1}]$ and $\Phi^{-1}(r) = [-(p-1)r]^{-1/(p-1)}$, $r < 0$.

If $f(r) = (1+r)^p$, then $\Phi(r) = -1/[(p-1)(1+r)^{p-1}]$ and $\Phi^{-1}(r) = -1 + [-(p-1)r]^{-1/(p-1)}$, $r < 0$.

4 Asymptotic behavior

To prove theorem 1.5 we follow the idea of Giga and Kohn [3]. To study u near a point (X, T_u) , we introduce the rescaled function w of theorem 1.4 :

$$w(y, s) = (T_u - t)^{\frac{1}{p-1}} u(x, t) \tag{29}$$

with

$$\begin{cases} x - X = (T_u - t)^{\frac{1}{2}} y \\ T_u - t = e^{-s}. \end{cases} \tag{30}$$

For $f(u) = u^p$, $p > 1$, the function w solves :

$$w_s - \frac{1}{\rho} \nabla \cdot (\rho \nabla w) + \frac{1}{p-1} w + c^2 e^{-s} w = \lambda a(X + e^{-\frac{s}{2}} y) w^p \tag{31}$$

in $\mathbb{R}^N \times (\sigma_0, +\infty)$ where $\sigma_0 = -\ln T_u$ and $\rho(y) = \exp(-|y|^2/4)$. And for $f(u) = (1+u)^p$, $p > 1$, equation (31) is replaced by :

$$w_s - \frac{1}{\rho} \nabla \cdot (\rho \nabla w) + \frac{1}{p-1} w + c^2 e^{-s} w = \lambda a(X + e^{-\frac{s}{2}} y) (e^{-\frac{s}{p-1}} + w)^p. \tag{32}$$

Before proving theorem 1.5, we establish two lemmas which concern L^2 -estimates of w_s and ∇w . Note that in lemma 4.1 the condition $p < (N+2)/(N-2)$ is not needed. We denote by M a bound from above of w , which exists by preceding theorem.

Lemma 4.1 *There exists a real number $L > 0$ which only depends on p, c, λ, a, T_u, M and $\int_{\mathbb{R}^N} |\nabla w|^2(y, \sigma_0) \rho(y) dy$, such that :*

$$\int_{\sigma_0}^{+\infty} \int_{\mathbb{R}^N} w_s^2 \rho \, dy \, ds \leq L . \quad (33)$$

Proof : *Case 1* : $f(u) = u^p$, $p > 1$. Multiplying equation (31) by $w_s \rho$ and integrating on any ball B_R , we obtain for $s > \sigma_0$:

$$\begin{aligned} & \int_{B_R} w_s^2 \rho \, dy + \int_{B_R} \left[-w_s \nabla \cdot (\rho \nabla w) + \left(\frac{1}{p-1} + c^2 e^{-s} \right) w w_s \rho \right] dy \\ & = \lambda \int_{B_R} a(X + e^{-\frac{s}{2}} y) w^p w_s \rho \, dy . \end{aligned} \quad (34)$$

Since

$$\int_{B_R} -w_s \nabla \cdot (\rho \nabla w) dy = \int_{B_R} -w \nabla \cdot (\rho \nabla w_s) + \int_{\partial B_R} \nabla w_s \cdot \nu w \rho d\sigma - \int_{\partial B_R} \nabla w \cdot \nu w_s \rho d\sigma ,$$

this implies when R tends to infinity :

$$\int_{\mathbb{R}^N} -w_s \nabla \cdot (\rho \nabla w) dy = \int_{\mathbb{R}^N} -w \nabla \cdot (\rho \nabla w_s) dy .$$

Then

$$\frac{1}{2} \frac{d}{ds} \left(\int_{\mathbb{R}^N} |\nabla w|^2 \rho dy \right) = - \int_{\mathbb{R}^N} w_s \nabla \cdot (\rho \nabla w) dy .$$

This and relation (34) lead us to :

$$\begin{aligned} & \int_{\mathbb{R}^N} w_s^2 \rho \, dy + \frac{1}{2} \frac{d}{ds} \left(\int_{\mathbb{R}^N} |\nabla w|^2 \rho dy \right) + \int_{\mathbb{R}^N} \left(\frac{1}{p-1} + c^2 e^{-s} \right) w w_s \rho \, dy \\ & = \lambda \int_{\mathbb{R}^N} a(X + e^{-\frac{s}{2}} y) w^p w_s \rho \, dy . \end{aligned}$$

Now consider $\tau > \sigma_0$ and integrate this relation on $[\sigma_0, \tau]$:

$$\begin{aligned} & \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} w_s^2 \rho \, dy + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2(y, \tau) \rho dy = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2(y, \sigma_0) \rho dy \\ & - \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} \left(\frac{1}{p-1} + c^2 e^{-s} \right) w w_s \rho \, dy \, ds + \lambda \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} a(X + e^{-\frac{s}{2}} y) w^p w_s \rho \, dy \, ds . \end{aligned} \quad (35)$$

To obtain (33), we have to bound the second and third terms of the right hand side of (35). Using an integration by parts, we have :

$$\begin{aligned}
& - \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} \left(\frac{1}{p-1} + c^2 e^{-s} \right) w w_s \rho dy ds \\
& = - \int_{\mathbb{R}^N} \left(\left[\left(\frac{1}{p-1} + c^2 e^{-s} \right) \frac{w^2}{2} \right]_{\sigma_0}^{\tau} - c^2 \int_{\sigma_0}^{\tau} e^{-s} \frac{w^2}{2} ds \right) \rho dy \\
& \leq \int_{\mathbb{R}^N} \left(\frac{1}{p-1} + c^2 T_u \right) \frac{w^2}{2}(y, \sigma_0) \rho dy + c^2 \frac{M^2}{2} T_u \int_{\mathbb{R}^N} \rho dy \\
& \leq \left(\frac{1}{2(p-1)} + c^2 T_u \right) M^2 \int_{\mathbb{R}^N} \rho dy .
\end{aligned} \tag{36}$$

Finally, we also have :

$$\begin{aligned}
& \lambda \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} a(X + e^{-\frac{s}{2}} y) w^p w_s \rho dy ds \\
& = \lambda \int_{\mathbb{R}^N} \left[a(X + e^{-\frac{s}{2}} y) \frac{w^{p+1}}{p+1} \right]_{\sigma_0}^{\tau} \rho dy + \frac{\lambda}{2} \int_{\mathbb{R}^N} \left(\int_{\sigma_0}^{\tau} e^{-\frac{s}{2}} \nabla a(X + e^{-\frac{s}{2}} y) \cdot y \frac{w^{p+1}}{p+1} ds \right) \rho dy \\
& \leq \lambda \|a\|_{\infty} \frac{M^{p+1}}{p+1} \int_{\mathbb{R}^N} \rho dy + \frac{\lambda}{2} \|\nabla a\|_{\infty} \frac{e^{-\frac{\sigma_0}{2}} M^{p+1}}{2} \int_{\mathbb{R}^N} |y| \rho dy .
\end{aligned} \tag{37}$$

Combining (35)-(36) and (37) we derive (33).

Case 2 : $f(u) = (1+u)^p$, $p > 1$. The only difference is that (37) is replaced by :

$$\begin{aligned}
& \lambda \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} a(X + e^{-\frac{s}{2}} y) (e^{-\frac{s}{p-1}} + w)^p w_s \rho dy ds \\
& = \lambda \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} a(X + e^{-\frac{s}{2}} y) (e^{-\frac{s}{p-1}} + w)^p \left(-\frac{1}{p-1} e^{-\frac{s}{p-1}} + w_s \right) \rho dy ds \\
& + \frac{\lambda}{p-1} \int_{\sigma_0}^{\tau} \int_{\mathbb{R}^N} a(X + e^{-\frac{s}{2}} y) (e^{-\frac{s}{p-1}} + w)^p e^{-\frac{s}{p-1}} \rho dy ds .
\end{aligned}$$

The first term can be treated integrating by parts as in (37) and the second term is bounded.

Now we give an estimate of the gradient of w . We only treat the case $f(u) = u^p$, $p > 1$. The other case can be treated similarly, as in the previous lemma. We introduce the energy function E for w as follows : for $s \geq \sigma_0$:

$$\begin{aligned}
E[w](s) & = \int_{\mathbb{R}^N} \left(\frac{1}{2} |\nabla w|^2 + \left(\frac{1}{p-1} + c^2 e^{-s} \right) \frac{w^2}{2} - \frac{\lambda}{p+1} a(X + e^{-\frac{s}{2}} y) w^{p+1} \right) \rho |y|^2 dy \\
& - \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{1}{2} |y|^2 - N \right) w^2 \rho dy .
\end{aligned}$$

Lemma 4.2 *Assume that $1 < p < (N+2)/(N-2)$. Then there exists a real number $\tilde{L} > 0$ which only depends on p, c, λ, a, T_u, M and $E[w](\sigma_0)$, such that :*

$$\int_{\sigma_0}^{+\infty} \int_{\mathbb{R}^N} |\nabla w|^2 (1 + |y|^2) \rho dy ds \leq \tilde{L} . \tag{38}$$

The proof follows the idea of propositions 4.1, 4.2 and 4.3 of [3] and we only give the derivative of E :

$$\begin{aligned} \frac{d}{ds}E[w](s) &= - \int_{\mathbb{R}^N} w_s^2 |y|^2 \rho dy - \frac{c^2}{2} e^{-s} \int_{\mathbb{R}^N} w^2 |y|^2 \rho dy \\ &- (p+3) \int_{\mathbb{R}^N} w_s (\nabla w \cdot y) \rho dy + \frac{\lambda e^{-\frac{s}{2}}}{2(p+1)} \int_{\mathbb{R}^N} \nabla a(X + e^{-\frac{s}{2}} y) \cdot y w^{p+1} |y|^2 \rho dy \\ &- \int_{\mathbb{R}^N} \left[\frac{p-1}{4} |y|^2 + \frac{1}{2} (N+2 - p(N-2)) \right] |\nabla w|^2 \rho dy \\ &- \frac{p-1}{2} c^2 e^{-s} \int_{\mathbb{R}^N} \left(\frac{1}{2} |y|^2 - N \right) w^2 \rho dy - \lambda e^{-\frac{s}{2}} \int_{\mathbb{R}^N} \nabla a(X + e^{-\frac{s}{2}} y) \cdot y w^{p+1} \rho dy . \end{aligned}$$

Proof of theorem 1.5 : Let (s_j) be any sequence tending to infinity. Consider the function w_j defined on $\mathbb{R}^N \times (\sigma_0 - s_j, +\infty)$ by : $w_j(y, s) = w(y, s + s_j)$. The function w_j is bounded by M and it is a respective solution of

$$w_{js} - \frac{1}{\rho} \nabla \cdot (\rho \nabla w_j) + \frac{1}{p-1} w_j + c^2 e^{-s-s_j} w_j = \lambda a(X + e^{-\frac{s+s_j}{2}} y) w_j^p$$

for $f(u) = u^p$ and

$$w_{js} - \frac{1}{\rho} \nabla \cdot (\rho \nabla w_j) + \frac{1}{p-1} w_j + c^2 e^{-s-s_j} w_j = \lambda a(X + e^{-\frac{s+s_j}{2}} y) \left(e^{-\frac{s+s_j}{p-1}} + w_j \right)^p$$

for $f(u) = (1+u)^p$. Using the L^q -regularity theory for parabolic equations (see [6]), we deduce that ∇w_j , $D^2 w_j$ and w_{js} are bounded in $L^q(B_R \times (-R, +\infty))$ for each $q \in (1, +\infty)$ and $R > 0$ (when s_j is large enough), the bound being independent of j . By Sobolev's inequality and Schauder's estimates (see [1]) we obtain that $(D^2 w_j)$ and (w_{js}) are Hölder continuous on each $B_R \times (-R, +\infty)$, uniformly with respect to j .

By the Arzela-Ascoli theorem and a diagonal argument, there exists a subsequence, still denoted by w_j , converging uniformly to a function l on each $B_R \times (-R, +\infty)$. This function l is in $C^{2,1}(\mathbb{R}^{N+1})$ and it is solution of

$$l_s - \frac{1}{\rho} \nabla \cdot (\rho \nabla l) + \frac{1}{p-1} l = \lambda a(X) l^p .$$

Because of lemmas 4.1 and 4.2, we have :

$$\begin{aligned} \int_{-R}^{+\infty} \int_{B_R} |\nabla w_j|^2 \rho dy ds &= \int_{-R+s_j}^{+\infty} \int_{B_R} |\nabla w|^2 \rho dy ds \rightarrow 0 \\ \int_{-R}^{+\infty} \int_{B_R} w_{js}^2 \rho dy ds &= \int_{-R+s_j}^{+\infty} \int_{B_R} w_{js}^2 \rho dy ds \rightarrow 0 \end{aligned}$$

as $j \rightarrow +\infty$, for all $R > 0$. Thus, l is independent of both y and s and satisfies

$$\frac{1}{p-1} l = \lambda a(X) l^p .$$

Finally because of theorem 2.1 of [4], the limit (14) holds.

References

- [1] A. Friedmann, *Partial differential equations of parabolic type*, Prentice Hall, Englewood Cliffs, NJ (1964).
- [2] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, *Indiana Univ. Math. J.* 34 (1985), 425-447.
- [3] Y. Giga and R. V. Kohn, *Characterizing blow-up using similarity variables*, *Indiana Univ. Math. J.* 36 (1987), 1-40.
- [4] Y. Giga and R. V. Kohn, *Nondegeneracy of blow-up for semilinear heat equations*, *Comm. Pure Appl. Math.* 42 (1989), 845-884.
- [5] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, *Comm. Pure Appl. Math.* 16 (1963), 305-333.
- [6] O. A. Ladyzenskaia, V. A. Solonnikov and N. N. Uralceva, *Linear and quasilinear equations of parabolic type*, AMS, Providence, RI (1968).
- [7] D. Sattinger, *Topics in stability and bifurcation theory*, *Lect. notes math.* 309, Springer, New York (1973).
- [8] A. Simon (A. Chaljub-Simon) and P. Volkmann, *Existence of ground states with exponential decay for semi-linear elliptic equations in \mathbb{R}^N* , *J. Diff. Eq.* 76 (1988), 374-390.
- [9] A. Simon (A. Chaljub-Simon) and P. Volkmann, *Existence de deux solutions positives pour un problème elliptique à paramètre dans \mathbb{R}^N* , *Topological Methods in Nonlin. Anal.* 3 (1994), 295-306.