

Statistique sur la cyclicité de Modules de Drinfeld de rang 2

Statistics about the cyclicity of a Drinfeld Modules of rank 2

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Résumé

Soit Φ un $\mathbf{F}_q[T]$ -module de Drinfeld de rang 2, sur un corps fini L , une extension de degré n d'un corps fini \mathbf{F}_q . On étudie la cyclicité de la structure de A -module induite par Φ sur L .

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Abstract

Let Φ be a Drinfeld $\mathbf{F}_q[T]$ -module of rank 2, over a finite field $L = \mathbf{F}_{q^n}$. We will study the cyclic property of the structure L^Φ . We will prove that the latter is cyclic only for trivial extensions of \mathbf{F}_q . *To cite this article: Mohamed-Saadbouh Mohamed-Ahmed , C. R. Acad. Sci. Paris, Ser. I ... (...).*

1 Introduction

let K a no empty global field of characteristic p (namely a rational functions field of one indeterminate over a finite field) together with a constant field, the finite field \mathbf{F}_q with p^s elements. We fix one place of K , denoted by ∞ , and call A the ring of regular elements away from the place ∞ . Let L be a commutator

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field of characteristic p , $\gamma : A \rightarrow L$ be a ring A -homomorphism. The kernel of this A -homomorphism is denoted by P . We put $m = [L, A/P]$, the extension degree of L over A/P , and $d = \deg P$. We denote by $L\{\tau\}$ the polynomial ring of τ , namely, the Ore polynomial ring, where τ is the Frobenius of \mathbf{F}_q with the usual addition and where the product is given by the commutation rule : for every $\lambda \in L$, we have $\tau\lambda = \lambda^q\tau$. A Drinfeld A -module $\Phi : A \rightarrow L\{\tau\}$ is a non trivial ring homomorphism and a non trivial embedding of A into $L\{\tau\}$ different from γ . This homomorphism Φ , once defined, define an A -module structure over the A -field L , noted L^Φ , where the name of a Drinfeld A -module for a homomorphism Φ . This structure of A -module depends on Φ and, especially, on his rank. We will make a statistic, analogue to the statistic for elliptic curves by Vladut in[4], about the ordinary Drinfeld A -modules such that the A -modules L^Φ are cyclic, we note by $C(d, m, q)$ the proportion of the number (of isomorphisms of) ordinary Drinfeld A -modules, of rank 2 such that the A -modules structures L^Φ are cyclic, this means : if we note by $\#\{\Phi, \text{isomorphism, ordinary}\}$ the number of classes of L -isomorphisms of an ordinary Drinfeld Modules of rank 2, we have :

$C(d, m, q) = \frac{\#\{\Phi, L^\Phi \text{ cyclic}\}}{\#\{\Phi, \text{isomorphism, ordinary}\}}$ and we note by $C_0(d, m, q)$ the proportion of the number (of isogeny classes of) ordinary Drinfeld A -modules, of rank 2 such that the A -modules L^Φ are cyclic, otherwise, if we note by $\#\{\Phi, \text{isogeny, ordinary}\}$ the number of isogeny classes, of ordinary Drinfeld modules of rank 2, we have : $C_0(d, m, q) = \frac{\#\{\text{isogeny Classes of } \Phi, L^\Phi \text{ cyclic}\}}{\#\{\Phi, \text{isogeny, ordinary}\}}$, $C(d, m, q) = C_0(d, m, q) = 1$ if and only if $m = d = 1$.

This means that, to have a cyclic Drinfeld A -modules we must have a trivial extension L , and we let think, in conjecture form, that for a big q the values of $C(d, m, q)$ and $C_0(d, m, q)$ will tend to 1.

2 Cyclicity Statistics for the A -module L^Φ

We define $C(d, m, q)$ as been the ration of the number of (isomorphism classes of) Drinfeld modules of rank 2 with cyclic structure L^Φ to the number of L -isomorphisms classes of ordinary Drinfeld modules of rank 2, noted by $\#\{\Phi, \text{isomorphism, ordinary}\}$: $C(d, m, q) = \frac{\#\{\Phi, L^\Phi \text{ cyclic}\}}{\#\{\Phi, \text{isomorphism, ordinary}\}}$, as same, we define $N(d, m, q)$ as been the ration of the number of (isogeny classes of) Drinfeld modules of rank 2 with not cyclic structure L^Φ to the number of L -isomorphisms isogeny of ordinary Drinfeld modules of rank 2, noted by $\#\{\Phi, \text{isogeny, ordinary}\}$: $N(d, m, q) = \frac{\#\{\Phi, L^\Phi \text{ noncyclic}\}}{\#\{\Phi, \text{isogeny, ordinary}\}}$. We remark that : $0 \leq C(d, m, q), N(d, m, q) \leq 1$. J. Yu in [3], was proved that the charecteristic polynomial P_Φ of a Drinfeld module Φ can be given by : $P_\Phi(X) = X^2 - cX + \mu P^m$, such that $\mu \in \mathbf{F}_q^*$, and $c \in A$, where $\deg c \leq \frac{m \cdot d}{2}$ by the Hasse-Weil analogue in this case.

Since the no cyclicity of the structure L^Φ needs the fact that $i^2 \mid P_\Phi(1)$ and $i_2 \mid (c-2)$, it is natural to introduce i (so i_2) in the calculus of $C(d, m, q)$ and $N(d, m, q)$. We fix the characteristic polynomial P_Φ , this means that we fix the isogeny classes of Φ , and we define :

Definition 2.1 We note by $n(P_\Phi, i_2) = \#\{\Phi : L^\Phi = \frac{A}{(i_1)} \oplus \frac{A}{(i_2)}\}$.

Remark 1 The number $n(P_\Phi, i_2)$ is equal to the number of isomorphism classes of Φ whole the A -module $L^\Phi \simeq \frac{A}{(i_1)} \oplus \frac{A}{(i_2)}$, in one isogeny classes, from where is coming the correspondence between Φ and i_2 .

For $n(P_\Phi, i_2)$ we have :

Lemma 2.2 Let $P_\Phi(X) = X^2 - cX + \mu P^m$ be the characteristic polynomial of an ordinary Drinfeld A -module Φ of rank 2, and let i_2 be an unitary polynomial of A . Then if $i_2 \mid c-2$ we have : $n(P_\Phi, i_2) \geq 1$, else $n(P_\Phi, i_2) = 0$.

We can deduct :

Corollary 2.3 With the above notations :

$$\#\{\Phi, L^\Phi \text{ noncyclic}\} = \sum_{P_\Phi} \sum_{i_2, i_2^2 \mid P_\Phi(1)} n(P_\Phi, i_2) \cdot \#\{i_2, i_2^2 \mid P_\Phi(1) \text{ and } i_2 \mid (c-2)\},$$

$$\#\{\Phi, L^\Phi \text{ cyclic}\} = \sum_{P_\Phi} \sum_{i_2, i_2^2 \mid P_\Phi(1)} n(P_\Phi, i_2) \cdot \#\{i_2, i_2^2 \nmid P_\Phi(1) \text{ and } i_2 \mid (c-2)\},$$

and if we note by $n_0(P_\Phi, i_2) = \#\{\text{isogeny classes of } \Phi : L^\Phi = \frac{A}{(i_1)} \oplus \frac{A}{(i_2)}\}$, we have : $n_0(P_\Phi, i_2) = 1$.

We note now by $\#\{\Phi, \text{isogeny, ordinary}\}$ the number of isogeny classes, for an ordinary module Φ , then we define : $N_0(d, m, q) = \frac{\#\{\text{isogeny classes of } \Phi, L^\Phi \text{ notcyclic}\}}{\#\{\Phi, \text{isogeny, ordinary}\}}$, the same for $C_0(d, m, q) = \frac{\#\{\text{isogeny classes of } \Phi, L^\Phi \text{ cyclic}\}}{\#\{\Phi, \text{isogeny, ordinary}\}}$,

We can so announce the following lemma :

Lemma 2.4 With the notations above, we have :

$$N_0(d, m, q) = \frac{\#\{i_2, i_2^2 \mid P_\Phi(1) \text{ and } i_2 \mid (c-2)\}}{\#\{\Phi, \text{isogeny, ordinary}\}},$$

$$N(d, m, q) = \frac{\sum_{P_\Phi} \sum_{i_2, i_2^2 \mid P_\Phi(1)} n(P_\Phi, i_2) \cdot \#\{i_2, i_2^2 \mid P_\Phi(1) \text{ and } i_2 \mid (c-2)\}}{\#\{\Phi, \text{isomorphism, ordinary}\}},$$

$$C_0(d, m, q) = \frac{\#\{i_2, i_2^2 \mid P_\Phi(1) \text{ et } i_2 \mid (c-2)\}}{\#\{\Phi, \text{isogeny, ordinary}\}}, \quad C(d, m, q) = \frac{\sum_{P_\Phi} \sum_{i_2, i_2^2 \mid P_\Phi(1)} n(\Phi, i_2) \cdot \#\{i_2, i_2^2 \mid P_\Phi(1) \text{ and } i_2 \mid (c-2)\}}{\#\{\Phi, \text{isomorphism, ordinary}\}},$$

and $N(d, m, q) + C(d, m, q) = 1$, $N_0(d, m, q) + C_0(d, m, q) = 1$.

The calculus of $\#\{\Phi, \text{isogeny, ordinary}\}$, for an ordinary A -module Φ , has been calculated in [1], as been :

Proposition 2.5 *Let $L = F_{q^n}$ and P the A -characteristic of L . We put $m = [L : A/P]$ and $d = \deg P$:*

- (1) *m is odd and d is odd : $\#\{\Phi, \text{isogeny, ordinary}\} = (q-1)(q^{\lfloor \frac{m}{2}d \rfloor + 1} - q^{\lfloor \frac{m-2}{2}d \rfloor + 1} + 1)$.*
- (2) *$m \cdot d$ even : $\#\{\Phi, \text{isogeny, ordinary}\} = (q-1)(\frac{q-1}{2}q^{\frac{m}{2}d} - q^{\frac{m-2}{2}d} + 1)$.*

As for the number L -isomorphisms classes, we will need the following result, for the proof and more details see [3] :

Proposition 2.6 *Let L be a finite extension of degree n over \mathbf{F}_q , then the number of L -isomorphisms classes of a Drinfeld A -module of rank 2 over L is $(q-1)q^n$ if n is odd and $q^{n+1} - q^n + q^2 - q$ else.*

And to calculate the number of L -isomorphisms classes for an ordinary Drinfeld modules, we will need to calculate the number of L -isomorphisms classes for a supersingular Drinfeld modules and subtract it from the global number of L -isomorphisms classes, for this, we have by [3] :

Proposition 2.7 *Let L be a finite extension of n degrees over \mathbf{F}_q , then the number of L -isomorphisms classes of a supersingular Drinfeld A -module of rank 2, over L is $(q^{n_2} - 1)$, where $n_2 = \text{pgcd}(2, n)$.*

The calculus of $C(d, m, q)$ will be calculated in function of the values of d and m which are two major values to determinate c because $\deg c \leq \frac{m \cdot d}{2}$. And to calculate the number of L -isomorphisms classes existing in each isogeny classes, we need the following Definition for more information, see [4] :

Definition 2.8 *Let L be a finite extension of degree n over \mathbf{F}_q , we define $W(F)$ as been :*

$$W(F) = \sum_{\Phi, F = \text{Frobenius}(\Phi, L)} \text{Weigh}(\Phi) \quad \text{where : } \text{Weigh}(\Phi) = \frac{q-1}{\#\text{Aut}_L \Phi}.$$

$W(F)$ is the sum of weights(noted $\text{Weigh}(\Phi)$) of number of L -isomorphisms classes existing in each isogeny classes of the module Φ which the Frobenius is F . And to calculate $\#\text{Aut}_L \Phi$ we have the following lemma :

Lemma 2.9 *Let Φ be an ordinary Drinfeld A -module of rank 2, over a finite field $L = F_{q^n}$, then : $\#Aut_L \Phi = q - 1$.*

By the previous lemma, we can see that $\text{Weight}(\Phi) = \frac{q-1}{\#Aut_L \Phi} = 1$, that means :

Corollary 2.10 *In the case of ordinary Drinfeld modules of rank 2, $W(F)$ is the number of L -isomorphisms classes existing in each isogeny classes.*

Definition 2.11 *Let D be an imaginary discriminant and let l a polynomial for which the square is a divisor of D and let $h(\frac{D}{l^2})$ the number of classes of the order for which the discriminant is $\frac{D}{l^2}$. We define the number of classes of Hurwitz for an imaginary discriminant D , noted $H(D)$ by : $H(D) = \sum_l \sum_{l^2|D} h(\frac{D}{l^2})$.*

Lemma 2.12 *If α is an integral element over A , for which $O = A[\alpha]$ is an A -order, then $\text{disc}(A[\alpha])$ is equal to the discriminant of the minimal polynomial of α .*

What is interesting for us is the calculus of the $\text{disc}(A[F])$ and since $\text{disc}(A[F]) = \text{disc}(P_\Phi)$. To calculate the number of classes $W(F)$, we have the following result, for proof see [4].

Proposition 2.13 *Let L be a finite extension of degree n of a field F_q and F the Frobenius of L , then :*

$$W(F) = H(\text{disc}(A[F])).$$

It remains for us to calculate $n(\Phi, i_2)$:

Lemma 2.14 *Let P_Φ be the characteristic polynomial of an ordinary Drinfeld A -module of rank 2, over a finite field L such that $L^\Phi = \frac{A}{(i_1)} \oplus \frac{A}{(i_2)}$, and let Δ the discriminant of the characteristic polynomial of the Frobenius F , then : $n(P_\Phi, i_2) = H(O(\Delta/i_2^2))$.*

3 Application

1) $d = m = 1$, in this case $L = A/P = \mathbf{F}_q$, the A -module $L^\Phi = A/P$ is cyclic, so $C(1, 1, q) = 1$. Conversely :

Theorem 3.1 *Let $L = F_{q^n}$ and P the A -characteristic of L , $m = [L, A/P]$ and $d = \text{deg}P$. Then : $C_0(d, m, q) = C(d, m, q) = 1 \Leftrightarrow m = d = 1$.*

2) $m = 1$ et $d = 2$, In this case $n = m.d = 2$, and $n_2 = 2 \Rightarrow \#\{\Phi, \text{isomorphism, ordinary}\} = q^3 - q - (q^2 - 1) = q^3 - q^2 - q + 1$. $C_0(2, 1, q) = \frac{q(q-1)-5}{q(q-1)-2}$,

$$C(2, 1, q) = \frac{q^3 - q^2 - q + 1 - \left[\frac{q-1}{2} \sum_{P_\Phi} \sum_{i_2, i_2^2 | A - 4\mu P} H(O(\frac{A-4\mu P}{i_2^2})) + (q-1) \sum_{P_\Phi} \sum_{i_2, i_2^2 | c^2 - 4\mu P} H(O(\frac{c^2 - 4\mu P}{i_2^2})) \right]}{q^3 - q^2 - q + 1}.$$

3) $m = 2$ and $d = 1$, in this case $n = m.d = 2$, and $n_2 = 2 \Rightarrow \#\{\Phi, \text{isomorphism, ordinary}\} = q^3 - q - (q^2 - 1) = q^3 - q^2 - q + 1$. $C_0(1, 2, q) = \frac{(q-1)q-4}{(q-1)q-2}$,

$$C(1, 2, q) = \frac{q^3 - q^2 - q + 1 - \sum_{P_\Phi} \sum_{i_2, i_2^2 | c^2 - 4\mu P} H(O(\frac{c^2 - 4\mu P}{i_2^2}))}{q^3 - q^2 - q + 1}.$$

By the calculus of $C_0(d, m, q)$ and $C(d, m, q)$ for $m.d \leq 2$, we have : $\lim_{q \rightarrow \infty} C_0(1, 1, q) = \lim_{q \rightarrow \infty} C_0(1, 2, q) = \lim_{q \rightarrow \infty} C_0(2, 1, q) = 1$, $\lim_{q \rightarrow \infty} C(1, 1, q) = \lim_{q \rightarrow \infty} C(1, 2, q) = \lim_{q \rightarrow \infty} C(2, 1, q) = 1$. By the results above, we can give the following conjecture :

Conjecture 3.2 *Let $L = F_{q^n}$ and P the A -characteristic of L , $m = [L, A/P]$ and $d = \text{deg}P$. Then :*

$$\lim_{q \rightarrow \infty} C(d, m, q) = \lim_{q \rightarrow \infty} C_0(d, m, q) = 1.$$

References

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