# A Classification of the Projective Lines over Small Rings II. Non-Commutative Case

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**Abstract**
A list of different types of a projective line over non-commutative rings with unity of order up to thirty-one inclusive is given. Eight different types of such a line are found. With a single exception, the basic characteristics of the lines are identical to those of their commutative counterparts. The exceptional projective line is that defined over the non-commutative ring of order sixteen that features ten zero-divisors and it most pronouncedly differs from its commutative sibling in the number of shared points by the neighbourhoods of three pairwise distant points (three versus zero), that of "Jacobson" points (zero versus five) and in the maximum number of mutually distant points (five versus three).

**Keywords:** Projective Ring Lines – Non-Commutative Rings of Small Orders

This is a short organic supplement to our recent paper [1], where the reader is referred to for the necessary background information on the concept of a projective ring line. Employing the definitions, symbols/notation and strategy of the above-mentioned paper, we have examined in detail the structure of different types of a projective line over small non-commutative rings with unity. As such rings are rather scarce for orders below thirty-two [2,3], we have found only eight line types, whose basic properties and all representative rings are listed in Table 1. Here, the term projective line over $R$ means the *left*-line, i.e. the line whose points are given by the *left* equivalence classes $(\varrho a, \varrho b)$, where $(a, b)$ is admissible over $R$ and $\varrho$ is a unit of $R$ [1].

Comparing Table 1 with Table 3 of [1], one readily notices the exceptional character of the line of 16/10 type, which differs from its commutative counterpart in a number of aspects. This difference stems from the fact that its base ring [3; pp. 433, 531], although having no two-sided ideals, has *three* proper maximal right- (and also left-) ideals to be compared with only *two* proper (and, of course, two-sided) maximal ideals of the corresponding commutative rings ($\cong GF(4) \otimes Z_4$ or $GF(4) \otimes GF(2)[x]/\langle x^2 \rangle$) [1]. A deeper insight into this intriguing difference is acquired when we try to pass to the *right*-line over $R$, i.e., the line whose points are regarded as right equivalence classes $(a\varrho, b\varrho)$. For all rings under consideration except the 16/10 one, this right-line was found to exist and possess the same properties as its left-companion. In the 16/10 case, however, this concept breaks down due to the fact that equivalence classes are not of the same cardinality, which renders it impossible to define consistently the notions of neighbour/distant [4].

Attacking the next order, thirty-two, in the hierarchy seems to represent a truly big computational challenge; this not only because of a large number (ninety-nine in total) of distinct non-

Table 1: The basic types of a projective line over small non-commutative rings with unity. The representative rings are given in the notation of [2] (orders ≤16) and [3] (orders >16). For the line of 16/10 type, the numbers given in brackets correspond to its commutative counterpart.

| Line Type | Cardinalities of Points | | | | | | | Representative Rings |
|---|---|---|---|---|---|---|---|---|
| | Tot | TpI | 1N | ∩2N | ∩3N | Jcb | MD | |
| 27/15 | 48 | 42 | 20 | 6 | 0 | 2 | 4 | 3.20 |
| 24/20 | 72 | 44 | 47 | 28 | 12 | 3 | 3 | 3.22 |
| 16/4 | 20 | 20 | 3 | 0 | 0 | 3 | 5 | 5.105 |
| 16/8 | 24 | 24 | 7 | 0 | 0 | 7 | 3 | 4.83, 4.117, 5.98, 5.101 |
| 16/10 | 35(30) | 26 | 18(13) | 9(4) | 3(0) | 0(5) | 5(3) | 5.96 |
| 16/12 | 36 | 28 | 19 | 8 | 0 | 3 | 3 | 4.68, 4.73, 5.97, 5.100, 5.104, 5.106 |
| 16/14 | 54 | 30 | 37 | 24 | 12 | 1 | 3 | 5.113 |
| 8/6 | 18 | 14 | 9 | 4 | 0 | 1 | 3 | 3.11 |

commutative rings with unity there [5], but also, and mainly, due to the fact that their addition and multiplication tables have not been published/available yet.


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