DYNAMICAL ANALYSIS OF SCHRODINGER OPERATORS WITH GROWING SPARSE POTENTIALS

Serguei Tcheremchantsev

UMR 6628-MAPMO, Université d'Orléans, B.P. 6759,

F-45067 Orléans Cedex, France

E-mail: Serguei.Tcherem@labomath.univ-orleans.fr

1 Introduction

Consider the discrete Schrödinger operators in $l^2(\mathbf{Z}^+)$:

$$H_\theta\psi(n) = \psi(n-1) + \psi(n+1) + V(n)\psi(n), \tag{1.1}$$

where $V(n)$ is some real function, with boundary condition

$$\psi(0)\cos\theta + \psi(1)\sin\theta = 0, \quad \theta \in (-\pi/2, \pi/2). \tag{1.2}$$

We shall consider the case of sparse potentials. Namely, $V(n) = V_N$, if $n = L_N$ and $V(n) = 0$ else, where L_N is monotone rapidly increasing sequence. Such potentials were studied, in particular, in [G, P, S, SSP, SST, JL, K, KR, Z]. Their interest lies in the fact that the spectrum on $(-2, 2)$ may be singular continuous with nontrivial Hausdorff dimension. In the present paper we will be interested by a particular case of such potentials, considered by Jitomirskaya and Last [JL]. We consider a slightly more general model. Let

$$V(n) = \sum_{N=1}^{\infty} L_N^{\frac{\eta-1}{2\eta}} \delta_{L_N,n} + Q(n) \equiv S(n) + Q(n), \tag{1.3}$$

where L_N is some very fast growing sequence such that

$$L_1 L_2 \cdots L_{N-1} = L_N^{\alpha_N}, \quad \lim_{N\to\infty}\alpha_N = 0,$$

$\eta \in (0, 1)$ is a parameter, and $Q(n)$ is any compactly supported real function (i.e. $Q(n) = 0$ for all $n \geq n_0$). It is well known that the study of operator defined by (1.1)-(refin02) is equivalent to the study of operator with Dirichlet boundary condition $\psi(0) = 0$ and potential

$$V_1(n) = V(n) - \tan\theta\delta_{1,n}.$$

It is clear that $V_1(n) = S(n) + Q_1(n)$, where Q_1 is another compactly supported function. Thus, without loss of generality, we may consider only operators with Dirichlet boundary condition and potentials given by (1.3). We shall denote by H the corresponding operator.

For such model, it is known [SST] that $(-2, 2)$ belongs to the singular continuous spectrum of H, and there may exist some discrete point spectrum outside of $[-2, 2]$. It was shown in [JL] that the Hausdorff dimensionality of the spectrum in $(-2, 2)$ lies between η and $\frac{2\eta}{1+\eta}$ for all boundary conditions (they consider $Q(n) = 0$ in our notations). Moreover, for Lebesgue a.e. θ, the spectrum on $(-2, 2)$ is of exact dimension η. Combes and Mantica [CM] showed that the parking dimension of the spectral measure restricted to $(-2, 2)$ is equal to 1. These spectral results imply dynamical lower bounds in a usual way [L], [GSB]. However, for the considered model this is only partial dynamical information. Some dynamical upper bounds were obtained by Combes and Mantica [CM] (in our proofs we use some ideas of their paper). Krutikov and Remling [KR], [K] studied the behaviour of the Fourier transform of the spectral measure at infinity.

The main motivations of the present paper are the following:

1. To give a rather complete description of the (time-averaged) dynamical behaviour of the considered model related to the singular continuous part of the spectrum (and some strong results in the case of more general initial states). This is the first example of this kind where dynamics is studied in such a detailed manner. Although this model is simple enough, the results suggest what could be done in more complicated cases, namely, for Fibonacci potentials, bounded sparse barriers, random decaying potentials or random polymers.
2. For some initial states ψ we find the exact expression of the intermittency function (see definition below) $\beta_\psi^-(p)$ which is nonconstant in p. To the best of our knowledge, this is the first model where such phenomenon of "quantum intermittency" is rigorously proven.
3. Throughout the paper, we use many different methods to study dynamics and show how their combination gives stronger results. In particular, we further develop the method for proving lower bounds based on Parseval formula [DT], allowing more general initial states ψ than δ_1. We hope that these ideas will be useful in many other cases.
4. For a long time the priority was given to the spectral analysis of operators with s.c. spectrum rather than to the analysis of the corresponding dynamics (and most dynamical bounds were obtained as a consequence of spectral results). In the present paper we show how it is possible to study *direclty* dynamics without virtually any knowledge of the spectral properties. Indeed, the only information we need in our considerations is that $(-2, 2) \in \sigma(H)$. Although we prove that the spectral measure is of exact Hausdorff dimension η on $(-2, 2)$ for *all* boundary conditions (improving the result of Jitomirskaya and Last), this is just a particular simple consequence of our dynamical results.

Let $\psi \in l^2(\mathbf{Z}^+)$ be some initial state (in particular, $\psi = \delta_1$). The time evolution is given by

$$\psi(t) = \exp(-itH)\psi,$$

where $\exp(-itH)$ is the unitary group. We shall be interested by the time-averaged quantities like

$$a_\psi(n,T) = \frac{1}{T}\int_0^\infty dt\,\exp(-t/T)|\psi(t,n)|^2.$$

This definition of time-averaging is virtually equivalent to the Cesaro average, but is more convinient for technical reasons. We consider the time averaging because of the rather irregular behaviour in time of $|\psi(t,n)|^2$ in the case of singular continuous spectrum. For the sparse barriers model one can see it from numerical simulations in [CM]. Moreover, effective analytical methods exist to study time-averaged quantities. Mention that upper bounds for the return probability as $t \to \infty$ without time-averaging are obtained in [K], [KR], and this is difficult technically.

We shall study the inside and outside time-averaged probabilities defined as

$$P_\psi(n \le M, T) = \sum_{n \le M} a_\psi(n,T)$$

and

$$P_\psi(n \ge M, T) = \sum_{n \ge M} a_\psi(n,T)$$

respectively. Here $M > 0$ are some numbers which may depend on T (growing with T). The quantity $P_\psi(n \le M, T)$ can be interpreted as the time-averaged probability to find a system inside an interval $[0, M]$ and similarly for outside probabilities. The obtained results are of the form

$$P_\psi(n \ge M_1(T), T) \ge c > 0, \quad P_\psi(n \le M_2(T), T) \ge c > 0, \tag{1.4}$$

or

$$P_\psi(n \ge M_3(T), T) \ge h(T), \quad P_\psi(n \le M_4(T), T) \ge g(T), \tag{1.5}$$

and similarly for the upper bounds, where $M_i(T) \to +\infty$ are some growing functions, and $h(T), g(T)$ tends to 0 not faster then polynomially. Thus, we control the essential parts of the wavepacket (1.4), as well as polynomially small parts of the wavepacket (1.5) (such bounds for outside probabilities imply lower bounds for the moments of the position operator).

We also consider the more traditional quantities:

$$\langle |X|_\psi^p \rangle(T) = \sum_{n>0} n^p a_\psi(n,T), \quad p > 0,$$

called time-averaged moments of order p of the position operator, as well as their growth exponents $\beta_\psi^\pm(p)$ (both functions non decreasing in p):

$$\beta_\psi^-(p) = \frac{1}{p}\mathrm{limsup}_{T\to\infty}\frac{\log\langle |X|_\psi^p\rangle(T)}{\log T}, \quad p > 0,$$

and similarly for $\beta_\psi^+(p)$. Since

$$\langle |X|_\psi^p \rangle(T) \ge M^p P_\psi(n \ge M, T)$$

for any $M > 0$, it is clear that probabilities and moments are related.

We shall also study the time-averaged return probability:

$$J_\psi(1/T, \mathbf{R}) = \frac{1}{T}\int_0^\infty dt \exp(-t/T)|\langle\psi(t), \psi\rangle|^2.$$

Let us present the main results. Assume first that ψ belongs to the subspace of continuous spectrum of H. Then due to RAGE theorem, the system escape with time (after time-averaging) from any finite interval $[1, M]$ and thus the quantum particle goes to infinity. Since the barriers are very sparse, the picture of motion is rather obvious. If the main part of the wavepacket is far enough from the barriers: $L_{N-1} << n << L_N$ for some N, then the propagation is ballistic (as in the case of the free particle). When the wavepacket reaches a barrier $V(L_N)$ (at time T of order L_N) the motion is slowed down and the process of tunneling through the high barrier begins. The time necessary for the essential part of the wavepacket to go through is about $L_N V^2(L_N) = L_N^{1/\eta}$. During this time the main part of the wavepacket is confined in the interval $[1, L_N]$. For $T >> L_N^{1/\eta}$ most of the wavepacket is on $[L_N + 1, \infty)$ and a new period of ballistic motion begins. It is clear that given a large value of T, it is crucial to locate it with respect to the L_N. Thus, throughout the paper for any T we shall denote by N (depending on T and $N \to \infty$ if $T \to \infty$) the unique value such that $L_N/C \leq T < L_{N+1}/C$ with some $C > 1$. We prefer considering $L_N/C \leq T < L_{N+1}/C$ rather than $L_N \leq T < L_{N+1}$ for the following reason: if $L_N/C \leq T \leq L_N$, the far tail of the wavepacket is already approaching the barrier $V(L_N)$ and the tunneling begins. For simplicity, we take $C = 4$ (of course, one could take any other value).

Let $\psi = f(H)\delta_1 \neq 0$, where f is a complex function from $f \in C_0^\infty([-2+\nu, 2-\nu])$ for some $\nu \in (0, 1)$. The operator $f(H)$ is given by spectral theorem. We shall call these ψ initial states of the first kind.

The followng bounds are proven:
For $T:\ L_N/4 \leq T \leq 2L_N$,

$$C_1 L_N^{1-1/\eta-\alpha_N} \leq P_\psi(n \geq 2L_N, T) \leq C_2 L_N^{1-1/\eta}, \tag{1.6}$$

where $\alpha_N \to 0$ as $N \to \infty$ (i.e. as $T \to \infty$)

These bounds describe the beginning of tunneling.
For $T:\ 2L_N \leq T \leq L_N^{1/\eta}$,

$$C_1 T L_N^{-1/\eta-\alpha_N} \leq P_\psi(n \geq T, T) \leq P_\psi(n \geq 2L_N, T) \leq C_2 T L_N^{-1/\eta},\ \alpha_N \to 0. \tag{1.7}$$

These bounds describe the main part of the tunneling process.

In particular, for $T:\ L_N/4 \leq T \leq L_N^{1/\eta}$

$$P_\psi(n \leq 2L_N, T) \geq ||\psi||^2 - CTL_N^{-1/\eta}. \tag{1.8}$$

Thus, for $T:\ L_N/4 \leq T \leq cL_N^{1/\eta}$ with c small enough, the main part of the wavepacket is located in $[1, 2L_N]$.

Moreover, for $T:\ L_N/4 \le T \le L_N^B$ with some $B > 0$,

$$P_\psi(L_N/4 \le n \le L_N, T) \ge C(B) L_N^{-\alpha_N}. \tag{1.9}$$

For $T:\ L_N/4 \le T \le L_N^{1/\eta}$ the following bounds hold for the time-averaged moments of position operator:

$$C_1 L_N^{-\alpha_N}(L_N^p + T^{p+1} L_N^{-1/\eta}) \le \langle |X|_\psi^p \rangle(T) \le C_2(L_N^p + T^{p+1} L_N^{-1/\eta}). \tag{1.10}$$

The bounds (1.6)-(1.10) are proved in Theorem 3.4 and Theorem 4.3. The upper bound in (1.10) for the moments averaged over the boundary condition θ (1.2) was proved by Combes and Mantica in [CM] for $p \le 2$. Our result holds for all $p > 0$ and all compact potential Q (in particular, for *all* boundary conditions).

The next bounds describe the beginning and the end of ballistic regime. If $T:\ L_N^{1/\eta} \le T \le L_N^{1/\eta+\delta}$ or $T:\ L_{N+1}^{1-\delta} \le T < L_{N+1}/4$ for some $\delta > 0$, then

$$C L_N^{-\alpha_N} \le P_\psi(n \ge T, T) \le P_\psi(n \ge 2L_N, T) \le ||\psi||^2, \tag{1.11}$$

and for the moments

$$C_1 T^p L_N^{-\alpha_N} \le \langle |X|_\psi^p \rangle(T) \le C_2 T^p. \tag{1.12}$$

These bounds are proved in Theorem 4.3.

Finally, if $T:\ L_N^{1/\eta+\delta} \le T \le L_{N+1}^{1-\delta}$, the motion is exactly ballistic. Namely, for any $\theta > 0$ there exists $\tau > 0$ small enough (independent of T) such that

$$||\psi||^2 - \theta \le P_\psi(n \ge \tau T, T) \le ||\psi||^2, \tag{1.13}$$

for T large enough, and

$$C_1 T^p \le \langle |X|_\psi^p \rangle(T) \le C_2 T^p. \tag{1.14}$$

Moreover, for $T:\ L_N^{1/\eta} \le T < L_{N+1}/4$,

$$P_\psi(n \le 2L_N, T) \le C L_N^{1/\eta+\alpha_N} T^{-1}. \tag{1.15}$$

The bounds (1.13)-(1.14) are proved in Theorem 4.4, and (1.15) follows from (3.7) of Lemma 3.3.

For the time-averaged return probability for any $T:\ L_N/4 \le T < L_{N+1}/4$ the bounds hold (Theorem 4.2):

$$\frac{C}{L_N(1 + T L_N^{-1/\eta})} \le J_\psi(1/T, \mathbf{R}) \le \frac{C L_N^{\alpha_N}}{L_N(1 + T L_N^{-1/\eta})}. \tag{1.16}$$

A related result (Lemma 3.3) states that

$$a_\psi(n, T) = \langle |\psi|^2(t, n) \rangle(T) \le \frac{C L_N^{\alpha_N}}{L_N(1 + T L_N^{-1/\eta})} \tag{1.17}$$

for any n.

As a particular corollary of our bounds for the time-averaged moments, we obtain the exact expression for the functions $\beta_\psi^\pm(p)$ (the result for $\beta_\psi^+(p)$ follows also from $\dim_P(\mu_\psi) = 1$ proved in [CM]):

$$\beta_\psi^-(p) = \frac{p+1}{p+1/\eta},\ \beta_\psi^+(p) = 1,\ p > 0. \tag{1.18}$$

Thus, the upper bound for $\beta_\psi^-(p)$, obtained in [CM] for $p \leq 2$ and for a.e. boundary conditions, gives in fact the exact expression of $\beta_\psi^-(p)$ for all $p > 0$ and all boundary conditions, as it was conjectured in [CM]. The result (1.18) is important from two points of view:

1. This is the first example where nontrivial (i.e. nonconstant) function $\beta_\psi^-(p)$ is rigorously calculated.

2. It implies (Corollary 4.5) that the restriction of the spectral measure on $(-2, 2)$ is of exact Hausdorff dimension η. This result holds for all compact potentials Q and thus, in particular, for all boundary conditions θ in (1.2). This improves the result of [JL], where it was proven only for Lebesgue-a.e. θ.

Consider now more general initial states ψ, for example, $\psi = \delta_1$. The problem is that we have no control of the discrete spectrum outside $(-2, 2)$. Thus, it is possible that some part of the wavepacket remains well localized at any time. On the other hand, it is also possible that the part of the wavepacket related to the discrete spectrum moves quasiballistically (the well known example is the one of [DRJLS]).

As a consequence, we cannot prove nontrivial upper bounds for the outside probabilities and for the moments, and we cannnot prove that all the wave function escapes from $[1, L_N]$ as $T >> L_N^{1/\eta}$.

However, the part of the wavepacket corresponding to the continuous spectrum (if nonzero) behaves in the same manner. It escapes from any interval $[1, M]$, moves ballistically between the barriers, tunnels through the barrier etc. Therefore, we are able to prove nontrivial lower bounds for outside probabilities and for the moments.

Consider $\psi = f(H)\delta_1 \neq 0$, where f is some bounded Borel complex function such that for some interval $S = [E_0 - \nu, E_0 + \nu] \subset [-2+\nu, 2-\nu]$, f is C^∞ on S and $|f(x)| \geq c > 0$ on S. We call these ψ initial states of the second kind. In particular, previously considered ψ and $\psi = \delta_1$ verify this condition.

For ψ described above, the following bounds hold (proved essentially in Theorem 4.3 and Theorem 4.4):

The first bound in (1.6), and the first and the second bounds in (1.7) remain true. Instead of (1.8) we prove that for some $\delta > 0$ small enough and $T:\ L_N/4 \leq T \leq \delta L_N^{1/\eta}$,

$$P_\psi(n \leq 2L_N, T) \geq c_1 > 0.$$

The bound (1.9) remains true as well as the first bound in (1.10). The bound (1.11) and the first bound in (1.12) hold (we do not have a priori ballistic upper bound for the considered ψ, except the case where f is smooth, in particular, $\psi = \delta_1$). Instead of (1.13), one has the bound

$$P_\psi(n \geq \tau T, T) \geq c_1 > 0.$$

The first bound in (1.14) follows. For the time-averaged return probability the lower bound in (1.16) holds (Theorem 4.2). For the functions $\beta_\psi^\pm(p)$, one has lower bounds

$$\beta_\psi^-(p) \geq \frac{p+1}{p+1/\eta}, \ \ \beta_\psi^+(p) \geq 1.$$

One can ask whether the smoothness condition on f is relevant. As for the upper bounds for moments and outside probabilities, it seems essential. Some results, namely, Lemma 2.1, Corollary 2.6, Lemma 3.3 and Theorem 4.2, hold for nonsmooth f. Probably, lower bounds for outside probabilities and for the moments (for both kinds of ψ) can be proved without smoothness of f.

The paper is organized as following. In Section 2 we first prove upper bounds for the transfer matrices with complex energies $T(n, 0; z)$ associated to the equation $Hu = zu$. With this result we obtain some lower bounds for probabilities and for the moments (Theorem 2.4) using Parseval formula. The combination of this method with the traditional approach going back to Guarneri, allows us to obtain some control of the essential part of the wavepacket (Corollary 2.6) as well as better lower bound for the time-averaged moments (Corollary 2.7 and Theorem 2.8). The approach of Section 2 can be applied to a more general class of models, where the transfer matrix has nontrivial upper bound like

$$||T(n, 0; E + i\varepsilon)|| \leq g_\Delta(n), \quad E \in \Delta, \ \varepsilon \in (0, 1).$$

Here Δ is any compact interval in $(-2, 2)$, and the function $g_\Delta(n)$, growing not too fast, does not depend on $E \in \Delta, \ \varepsilon \in (0, 1)$. In particular, $g_\Delta = C(\Delta)n^\alpha$ with some $\alpha > 0$ is possible (Theorem 2.9).

The bounds of Theorem 2.4 show the importance of the integrals

$$I(\Delta, \varepsilon) = \varepsilon \int_\Delta dE \mathrm{Im}^2 F(E + i\varepsilon), \ \Delta \subset (-2, 2),$$

where F denotes the Borel transform of spectral measure. Good lower bounds for $I(\Delta, \varepsilon)$ imply better lower bounds for probabilities and thus for the moments. These integrals are closely related to the time-averaged return probabilities and to the correlation dimensions of the spectral measure restricted to $(-2, 2)$.

In Section 3, which is specific to the considered model with growing sparse potentials, we obtain upper bounds for inside (Lemma 3.3) and outside probabilities and moments (Theorem 3.4). These results are proved for $\psi = f(H)\delta_1$ with f compactly supported on $(-2, 2)$ (and moreover $f \in C_0^\infty$ in Theorem 3.4). When considering the inside probabilities, we obtain some upper bound for $\mathrm{Im}F(x + i\varepsilon)$, $x \in (-2, 2)$. It implies a very simple proof of the fact that for any $\delta > 0, \nu > 0$ the spectral measure is uniformly $\eta - \delta$-Hölder continuous on $[-2 + \nu, 2 - \nu]$ (the result which follows also from the proofs of [JL]).

In Section 4 we first use the obtained upper bounds for outside probabilities to obtain lower bound for the integrals $I(\Delta, \varepsilon)$ which is virtually optimal (Corollary 4.1). Together

with Theorem 2.4, it implies better lower bounds for probabilities and for the moments (which are optimal for ψ of the first kind up to the factors like $L_N^{\alpha_N}$, where $\alpha_N \to 0$). It implies also bounds for the time-averaged return probabilities (Theorem 4.2). The upper bounds of Section 3 are also used (Theorem 4.4) to control the essential part of the wavepacket on $[1, 2L_N]$ and on $[\tau T, +\infty)$ with some $\tau > 0$. Finally, we show that the obtained upper bounds for the moments imply that the restriction of spectral measure on $(-2, 2)$ is of exact Hausdorff dimension η.
Acknowledgments. I would like to thank F. Germinet for useful discussions.

2 Direct lower bounds for probabilities and moments

Define the time-averaged quantities (which we call probabilities) of the form

$$P_\psi(n \geq M, T) = \sum_{n \geq M} \langle |\psi(t,n)|^2 \rangle (T) \equiv \sum_{n \geq M} \frac{1}{T} \int_0^{+\infty} dt e^{-t/T} |\exp(-itH)\psi(n)|^2$$

and similarly for $P_\psi(n \leq M, T)$, $P_\psi(L \leq n \leq M, T)$, where M, L may depend on T. We shall call $P_\psi(n \geq M, T)$ outside and $P_\psi(n \leq M, T)$ inside probabilities respectively.

Throughout the paper we shall consider two kinds of initial states ψ:
1. $\psi = f(H)\delta_1$, where $f \in C_0^\infty([-2+\nu, 2-\nu])$ for some $\nu > 0$ and $f(x_0) \neq 0$ for some x_0. We shall call these ψ initial states of the first kind.
2. $\psi = f(H)\delta_1$ where $f : \mathbf{R} \to \mathbf{C}$ is a bounded Borel function such that for some $[E_0 - \nu, E_0 + \nu] \subset [-2+\nu, 2-\nu]$, with $\nu > 0$,

$$f \in C^\infty([E_0 - \nu, E_0 + \nu]) \text{ and } |f(x)| \geq c > 0, \; x \in [E_0 - \nu, E_0 + \nu]. \tag{2.1}$$

In particular, one can take $\psi = \delta_1$. We shall call these ψ initial states of the second kind. One can observe that any ψ of the first kind belongs to the second kind.

In the case of any ψ we shall denote by μ_ψ the corresponding spectral measure, and by $\mu \equiv \mu_{\delta_1}$ the measure of the state δ_1. Observe that

$$d\mu_\psi(x) = |f(x)|^2 d\mu(x).$$

Let ψ be any vector and μ_ψ its spectral measure. For any Borel set Δ and $\varepsilon > 0$ define the following integrals:

$$J_\psi(\varepsilon, \Delta) = \int_\Delta d\mu_\psi(x) \int_{\mathbf{R}} d\mu_\psi(y) R((x-y)/\varepsilon),$$

where $R(w) = 1/(1+w^2)$. These quantities will play an important role in the sequel. What one can observe is the following identity (which can be easily proved using spectral theorem):

$$\frac{1}{T}\int_0^\infty dt \exp(-t/T)|\langle\psi(t),\psi\rangle|^2 = J_\psi(\varepsilon,\mathbf{R}),\quad \varepsilon = 1/T. \tag{2.2}$$

Thus, $J_\psi(\varepsilon,\mathbf{R})$ coincides with the time-averaged return probability.

The first statement is of a rather general nature, and holds in fact for any self-adjoint operator H.

Lemma 2.1 *Let H be some self-adjoint operator in $l^2(\mathbf{N})$ and ψ any vector such that $c_1 = \mu_\psi(\Delta) > 0$, where Δ is some Borel set. Let $M(T) = c_1^2/(16J_\psi(T^{-1},\Delta))$. Then*

$$P_\psi(n \geq M(T),T) \geq c_1/2 > 0.$$

Proof. The result follows rather directly from [T] and is obtained using the traditional approach developed by Guarneri-Combes-Last. For the sake of completeness we shall give the main lines of the proof. Define

$$\rho = X_\Delta\psi,\ \chi = \psi - \rho,$$

where X_S is the spectral projector of the operator H on the set S. One has $\rho \neq 0$ since $\|\rho\|^2 = \mu_\psi(\Delta) = c_1 > 0$. One shows [T] that for any $M > 0$,

$$P_\psi(n \geq M,T) \geq ||\rho||^2 - 2|D(M,T)|, \tag{2.3}$$

where

$$D(M,T) = \frac{1}{T}\int_0^{+\infty} dt \exp(-t/T)\sum_{n<M}\psi(t,n)\overline{\rho(t,n)}.$$

Lemma 2.1 of [T] (with $h(u) = \exp(-u),\ u > 0$) implies

$$|D(M,T)| \leq \int_\Delta d\mu_\psi(x)\sqrt{b(x,T)S_M(x)}, \tag{2.4}$$

where

$$b(x,T) = \int_{\mathbf{R}} d\mu_\psi(u)R((T(x-u)) = \varepsilon \mathrm{Im} F_{\mu_\psi}(x+i\varepsilon),\ \varepsilon = \frac{1}{T}, \tag{2.5}$$

F_{μ_ψ} is the Borel transform of spectral measure, $S_M(x) = \sum_{n<M}|u_\psi(n,x)|^2$, and $u_\psi(n,x)$ are generalised eigenfunctions associated to the state ψ. Since

$$\int_{\mathbf{R}} d\mu_\psi(x)|u_\psi(n,x)|^2 \leq 1$$

for any n, the bound (2.4) and Cauchy-Schwarz inequality yield

$$|D(M,T)|^2 \leq MJ_\psi(\varepsilon,\Delta),\ \varepsilon = 1/T. \tag{2.6}$$

Let us take $M = ||\rho||^4(16J_\psi(\varepsilon,\Delta))^{-1}$. It follows from (2.3), (2.6) that

$$P_\psi(n \geq M,T) \geq ||\rho||^2/2 = c_1/2 > 0.$$

The proof is completed.

In the sequel we shall also need the following integrals:

$$I_\psi(\varepsilon,\Delta)=\varepsilon\int_\Delta dE\mathrm{Im}^2F_\psi(E+i\varepsilon)=\varepsilon^3\int_\Delta dE\left(\int_{\mathbf{R}}\frac{d\mu_\psi(u)}{\varepsilon^2+(E-u)^2}\right)^2,$$

where ψ is some state and F_ψ denotes the Borel transform of its spectral measure. In fact, the integrals $I_\psi(\varepsilon,\Delta)$ and $J_\psi(\varepsilon,\Delta)$ are closely related.

Lemma 2.2 *Let $0<\varepsilon<1$, $\Delta=[a,b]$ some bounded interval. The uniform estimate holds:*

$$J_\psi(\varepsilon,\Delta)\le C(\Delta)I_\psi(\varepsilon,\Delta). \tag{2.7}$$

Proof. For simplicity we shall omit the dependence on ψ in the proof. The definition of I implies

$$I(\varepsilon,\Delta)=\varepsilon^3\int_{\mathbf{R}}d\mu(x)\int_{\mathbf{R}}d\mu(u)\int_\Delta\frac{dE}{((u-E)^2+\varepsilon^2)((x-E)^2+\varepsilon^2)}.$$

Thus

$$I(\varepsilon,\Delta)\ge\int_\Delta d\mu(x)\int_{\mathbf{R}}d\mu(u)f(x,u,\varepsilon), \tag{2.8}$$

where

$$f(x,u,\varepsilon)=\varepsilon^3\int_a^b\frac{dE}{((u-E)^2+\varepsilon^2)((x-E)^2+\varepsilon^2)},\ \ \Delta=[a,b].$$

One changes the variable $t=(E-x)/\varepsilon$ in the integral over E:

$$f(x,u,\varepsilon)=\int_A^B\frac{dt}{(t^2+1)((t+s)^2+1)},$$

where $A=(a-x)/\varepsilon, B=(b-x)/\varepsilon,\ \ s=(x-u)/\varepsilon$. Since one integrates in (2.8) over $x\in[a,b]$, and $0<\varepsilon<1$, one can easily see that

$$f(x,u,\varepsilon)\ge c/(s^2+1)$$

with uniform positive constant. The bound (2.8) yields

$$I(\varepsilon,\Delta)\ge c\int_\Delta d\mu(x)\int_{\mathbf{R}}d\mu(u)R((x-y)/\varepsilon)=cJ(\varepsilon,\Delta). \tag{2.9}$$

As a basis of our proofs we shall use the following

Lemma 2.3 *Let $x\in[-2+\nu,2-\nu]$ with some $\nu>0$, $\varepsilon\in[0,1)$. The following uniform bounds hold under condition $n\varepsilon\le K$ for some $K>0$.*
a) If $n<L_N$, then

$$||T(n,0;x+i\varepsilon)||\le C(K,\nu)L_N^{\alpha_N},\ \ \alpha_N\to 0. \tag{2.10}$$

If $n:\ L_N\le n<L_{N+1}$, then

$$||T(n,0;x+i\varepsilon)||\le C(K,\nu)L_N^{\frac{1-\eta}{2\eta}+\alpha_N},\ \ \alpha_N\to 0. \tag{2.11}$$

Proof. Assume first that $Q(n) \equiv 0$. Then one can easily see that for any $n: \ L_m \leq n < L_{m+1}$ with some $m \geq 1$,

$$T(n,0;z) = T_0(n-L_m,z)A(L_m,z)T_0(L_m-L_{m-1}+1,z)A(L_{m-1},z)\cdots A(L_1,z)T_0(L_1-1,z). \tag{2.12}$$

Here $T_0(k,z) = A_0(z)^k$ is the free transfer matrix with

$$A_0(z) = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \tag{2.13}$$

and

$$A(L_k,z) = \begin{pmatrix} z - L_k^{\frac{1-\eta}{2\eta}} & -1 \\ 1 & 0 \end{pmatrix}.$$

For real $x \in [-2+\nu, 2-\nu]$ one can show by direct calculations that $||T_0(k,x)|| \leq C$ uniformly in x, k. For complex $z = x + i\varepsilon$ one represents $A_0(z) = A_0(x) + i\varepsilon D$ with

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Developing $A_0(z)^k$, one sees that one has still

$$||T_0(k,z)|| = ||A_0(z)^k|| \leq C \tag{2.14}$$

while $k\varepsilon \leq K$. As to $A(L_k,z)$, it can be bounded by

$$||A(L_k,z)|| \leq C L_k^{\frac{1-\eta}{2\eta}}, \tag{2.15}$$

since $x \in [-2+\nu, 2-\nu]$, $\varepsilon \in [0,1)$. The statement of the Lemma follows directly from the bounds (2.12)-(2.15) and the sparseness condition:

$$L_1 L_2 \cdots L_m \equiv L_{m+1}^{\nu_{m+1}}, \quad \nu_m \to 0.$$

For more details see the similar proof in [JL]. If one add the finite range perturbation $Q(n)$, it is clear that the norms $||T(n,0;z)||$ remain bounded by the same expressions (2.10), (2.11) with different constants. The proof is completed.

In the next statement we shall use notation $I(\varepsilon,\Delta) = I_{\delta_1}(\varepsilon,\Delta)$. In all statements of the paper α_N denotes sequences such that $\lim \alpha_N = 0$ (not necessarily the same).

Theorem 2.4 *Assume that ψ is of the second kind (in particular, $\psi = \delta_1$). Let $\Delta = [E_0 - \nu/2, E_0 + \nu/2]$, where ν comes from (2.1).*
1. Let $L_N \leq T < L_{N+1}/4$ for some N. The uniform in T bound holds:

$$P_\psi(n \geq T, T) \geq cTL_N^{\frac{\eta-1}{\eta}-\alpha_N}(I(1/T,\Delta) + \frac{1}{T}) \geq cL_N^{\frac{\eta-1}{\eta}-\alpha_N}. \tag{2.16}$$

2. Let $L_N/4 \le T \le 4L_N$. Then

$$P_\psi(n > L_N, T) \ge cL_N^{2-\frac{1}{\eta}-\alpha_N}(I(1/T,\Delta)+\frac{1}{T}) \ge cL_N^{\frac{\eta-1}{\eta}-\alpha_N}. \qquad (2.17)$$

3. Let $L_N/4 \le T \le L_N^B$ with some $B > 1$. Then the uniform bound holds:

$$P_\psi(L_N/4 \le n \le L_N, T) \ge c_B L_N^{1-\alpha_N}(I(1/T,\Delta)+\frac{1}{T}) \ge c_B L_N^{1-\alpha_N}T^{-1}. \qquad (2.18)$$

In all bounds (2.16)-(2.18), $c > 0$ and $\lim_{N\to\infty}\alpha_N = 0$.

Proof. We shall follow the ideas of [DT]. The starting point is the Parseval formula:

$$\langle|\psi(t,n)|^2\rangle(T) = \frac{\varepsilon}{\pi}\int_{\mathbf{R}} dE|(R(E+i\varepsilon)\psi)(n)|^2,\ \varepsilon = (2T)^{-1}, \qquad (2.19)$$

where $R(z) = (H - zI)^{-1}$.
a) We begin with $\psi = \delta_1$. Let $u(n,z) = (R(z)\delta_1)(n)$. It is well known [KKL] that

$$(u(n+1,z), u(n,z))^T = T(n,0,z)(F(z),-1)^T,\ n \ge 0, \qquad (2.20)$$

where $T(n,0,z)$ is the transfer matrix associated to the equation $Hu = zu$ and F is the Borel transform of the spectral measure $F(z) = \int_{\mathbf{R}} \frac{d\mu_{\delta_1}(x)}{x-z}$. Let $E \in [-2+\delta, 2-\delta]$ with some $\delta \in (0,1)$, $z = E + i\varepsilon$, $\varepsilon = (2T)^{-1}$. Assume first that $L_N \le T \le L_{N+1}/4$. The bound (2.11) of Lemma 2.3 and (2.20) imply (since $||T|| = ||T^{-1}||$) for any $L_N \le n \le 2T$

$$|u(n+1,z)|^2 + |u(n,z)|^2$$

$$\ge ||T(n,0,z)||^{-2}(|F(z)|^2+1) \ge a(\delta)L_N^{\frac{\eta-1}{\eta}-2\alpha_N}((\mathrm{Im}^2 F(z)+1), \qquad (2.21)$$

where $\alpha_N \to 0$. Summation in (2.21) over $n : T \le n \le 2T$ and integration over $E \in [-2+\delta, 2-\delta]$ in (2.19) yields (2.16) with $\Delta = [-2+\delta, 2-\delta]$. We have used a simple bound $I(u/2,\Delta) \ge 1/8I(u,\Delta)$, which directly follows from the definition of integrals I. If $L_N/4 \le T \le 4L_N$, one considers $n : 2L_N \le n \le 3L_N$ to get (2.17). The bound (2.18) is proved in a similar manner using the bound (2.10) of Lemma 2.3 and summating over $n : L_N/4 \le n < L_N - 1$.
b) Assume now that ψ is such that $\psi = g(H)\delta_1$, $g(x) \in C_0^\infty(S)$, where $S = [E_0 - \nu, E_0 + \nu] \subset [-2+\nu, 2-\nu]$ for some $\nu \in (0,1)$. Assume also that $g(x) \equiv 1$, $x \in [E_0 - 3\nu/4, E_0 + 3\nu/4]$. Consider the decomposition

$$\delta_1 = \psi + \chi,\ \psi = g(H)\delta_1,\ \chi = (1-g(H))\delta_1.$$

Let $L_N \le T \le L_{N+1}/4$. Since

$$|R(z)\psi(n)|^2 \ge 1/2|R(z)\delta_1(n)|^2 - |R(z)\chi(n)|^2,$$

integration over $\Delta = [E_0 - \nu/2, E_0 + \nu/2]$ and summation over $n : T \le n \le 2T$ yields (using the proof of a)):

$$P_\psi(n \ge T, T) \ge cTL_N^{\frac{\eta-1}{\eta}-\alpha_N}(I(1/(2T),\Delta)+1/T) - c/T \int_\Delta dE \sum_{n\ge T} |R(E+i\varepsilon)\chi(n)|^2 \quad (2.22)$$

To bound from above $|R(E+i\varepsilon)\chi(n)|, \ \ E \in \Delta$, we shall use now the following result from [GK]:

$$|(u(H)\delta_m)(n)| \le C_k |||u|||_{k+2}(1+|n-m|^2)^{-k/2}, \quad (2.23)$$

for any integer $k > 0$, where u is some smooth complex function,

$$|||u|||_k = \sum_{r=0}^{k} \int_{\mathbf{R}} dx |u^{(r)}(x)|(1+|x|^2)^{(r-1)/2},$$

and the constants in (2.23) are independent of u and H. Although the result of [GK] is stated in the continuous case, one can easily see that the result holds in the discrete case for any self-adjoint operator H.

We shall take $u_{E+i\varepsilon}(x) = \frac{\chi(x)}{x-E-i\varepsilon}$, where $\chi(x) = 1 - g(x)$, and $z = E + i\varepsilon$ is considered as a parameter. Thus, $R(E+i\varepsilon)\chi(n) = (u_{E+i\varepsilon}(H)\delta_1)(n)$. The definition of f implies that $\chi(x) = 0$ for any $x \in [E_0 - 3\nu/4, E_0 + 3\nu/4]$. One can easily show that $|||u_{E+i\varepsilon}|||_k \le C(k,\nu)$ for any k and any $E \in \Delta, \ \varepsilon > 0$ with uniform constants. Thus, (2.23) implies $|R(E+i\varepsilon)\chi(n)| \le C(k)n^{-k}$ and

$$\sum_{n\ge T} |R(E+i\varepsilon)\chi(n)|^2 \le C(k)T^{-k}. \quad (2.24)$$

Taking k large enough, we see that (2.22), (2.24) imply the same bound (2.16), since $T \ge L_N$ and thus the integral in (2.22) is small with respect to the first term. The bounds (2.17) and (2.18) can be proved in the same manner.

c) Let now ψ be any vector of the second kind. Let g be some function verifying conditions of the part b), that is, $g \in C_0^\infty(S)$, $S \equiv [E_0-\nu, E_0+\nu]$, $g(x) = 1, \ x \in [E_0-3\nu/4, E_0+3\nu/4]$. One can write

$$g(x) = l(x)f(x),$$

where $l(x) = 0$ if $|x - E_0| > \nu$ and $l(x) = g(x)/f(x), \ \ |x - E_0| \le \nu$. The facts that $f \in C^\infty(S), \ g \in C_0^\infty(S)$ and $|f(x)| \ge c > 0$ on S imply that $l \in C_0^\infty(S)$. Again, due to (2.23), the kernel of $l(H)$ is fast decaying in $|n - m|$, so that for any $k > 0$,

$$|R(E+i\varepsilon)g(H)\delta_1(n)|^2 \le \sum_m \frac{C_k}{1+|n-m|^k} |R(E+i\varepsilon)f(H)\delta_1(m)|^2.$$

Therefore, for any $L > 0$,

$$A(2L,T) \equiv 1/T \sum_{n\ge 2L} \int_\Delta dE |R(E+i\varepsilon)g(H)\delta_1(n)|^2 \le$$

$$1/T \int_{\Delta} dE \sum_{m} h_k(m,T)|R(E+i\varepsilon)f(H)\delta_1(m)|^2,$$

where

$$h_k(m,T) = \sum_{n\geq 2L} \frac{C_k}{1+|n-m|^k}.$$

Let us split the sum over m into two with $m < L$ and $m \geq L$. One observes that $h_k(m,T) \leq C_k L^{1-k}$ in the first case and one uses trivial bound $h_k(m,T) \leq C_k$ in the second case. Thus, we get

$$A(2L,T) \leq C_k L^{1-k} + C_k/T \int_{\Delta} dE \sum_{m\geq L} |R(E+i\varepsilon)f(H)\delta_1(m)|^2, \tag{2.25}$$

where we used the fact that

$$\varepsilon \sum_{m} \int_{\mathbf{R}} dE |R(E+i\varepsilon)\psi(m)|^2 = \pi ||\psi||^2.$$

Let us assume first that $L_N \leq T \leq L_{N+1}/4$. One can easily see from the proofs of the part b) that the quantity $A(2T,T)$ is bounded from below by the r.h.s. of (2.16) (only the constant changes). Taking $k > 1/\eta$, using (2.25) and Parseval equality, we get (2.16) for $\psi = f(H)\delta_1$. For (2.17) the proof is similar with $L = 2L_N$. To prove (2.18), one considers

$$A(T) \equiv 1/T \sum_{L_N/2\leq n\leq 3L_N/4} \int_{\Delta} dE |R(E+i\varepsilon)g(H)\delta_1(n)|^2 \leq$$

$$1/T \int_{\Delta} dE \sum_{m} h_k(m)|R(E+i\varepsilon)f(H)\delta_1(m)|^2 \tag{2.26}$$

with $h_k(m) = \sum_{L_N/2\leq n\leq 3L_N/4} C_k(1+|n-m|^k)^{-1}$. Splitting the sum over m in (2.26) into three with $m < L_N/4$, $m > L_N$ and $L_N/4 \leq m \leq L_N$, one shows that the first two are bounded from above by $C_k L_N^{1-k}$ and the third by

$$C/T \int_{\Delta} \sum_{L_N/4\leq m\leq L_N} |R(E+i\varepsilon)f(H)\delta_1(m)|^2.$$

On the other hand, $A(T)$ is bounded from below by the r.h.s. of (2.18) (the proof is identical with the one of the part b), only the constants change). Since $T \leq L_N^B$, taking k large enough we get the bound (2.18) for $\psi = f(H)\delta_1$. The proof is completed.

Corollary 2.5 *Let $\Delta = [-2+\nu, 2-\nu]$ with some $\nu > 0$. Let $\varepsilon > 0$ and N be such that $L_N/4 \leq T \equiv 1/\varepsilon < L_{N+1}/4$. The following estimate holds:*

$$J(\varepsilon,\Delta) \leq CI(\varepsilon,\Delta) \leq \frac{CL_N^{\alpha_N}}{L_N + TL_N^{\frac{\eta-1}{\eta}}} \tag{2.27}$$

with uniform in T constants and $\lim \alpha_N = 0$. Here the integrals J, I correspond to $\psi = \delta_1$.

Proof. We shall use the bounds of Theorem 2.4 for $\psi = \delta_1$. In this case, as it follows from the part a) of the proof, (2.16), (2.17), (2.18) hold with $\Delta = [-2+\nu, 2-\nu]$. Moreover, (2.18) holds for all $L_N/4 \le T \le L_{N+1}/4$ without restriction. On the other hand, all the quantities $P_\psi(n \ge T, T), P_\psi(n \ge 2L_N, T)$, $P_\psi(L_N/4 \le n \le L_N, T)$ are bounded from above by 1. We thus obtain the last inequality in (2.27). The first inequality is that of Lemma 2.2. The proof is completed.

Corollary 2.6 *Let ψ be any vector of the second kind (but f is not necessarily smooth on $\Delta = [E_0 - \nu, E_0 + \nu]$). Then*

$$P_\psi(n \ge M(T), T) \ge c > 0, \text{ for } M(T) = CL_N^{-\alpha_N}\left(L_N + TL_N^{\frac{\eta-1}{\eta}}\right), \tag{2.28}$$

where again $L_N/4 \le T < L_{N+1}/4$ and $\alpha_N \to 0$.

Proof. Since $|f(x)| \ge c > 0,\ x \in \Delta$ and $\Delta \subset (-2,2) \subset \sigma(H)$, it is clear that $\mu_\psi(\Delta) \ge c^2\mu(\Delta) > 0$. On the other hand, since f is bounded, by (2.27),

$$J_\psi(\varepsilon, \Delta) \le CJ(\varepsilon, \Delta) \le C\frac{L_N^{\alpha_N}}{L_N + TL_N^{\frac{\eta-1}{\eta}}}. \tag{2.29}$$

The result now follows from (2.29) and Lemma 2.1. The proof is completed.

Generally speaking, to obtain better lower bound for $M(T)$, one should better estimate from above the integrals $J(\varepsilon, \Delta)$. Similarly, to get better lower bounds for probabilities (Theorem 2.4), one should bound from below the integrals $I(\varepsilon, \Delta)$. These quantities are both closely related to the correlation dimensions $D^{\pm}(2)$ [T] of the spectral measure restricted to Δ. To get good bounds for I, J, one should have a rather good knowledge of the fine structure of the spectral measure. In the Section 4 we shall use the obtained upper bound for the outside probabilities to obtain optimal lower bounds for $I(\varepsilon, \Delta)$. The idea is the following: upper bound on outside probabilities => upper bound on $M(T)$ such that $P_\psi(n \ge M(T), T) \ge c > 0$ => lower bound on J => lower bound on I. This method, however, is specific to the considered model with unbounded sparse potentials.

Consider now applications of the obtained results for probabilities to the time-averaged moments of the position operator:

$$\langle |X|_\psi^p \rangle(T) \equiv \sum_n |n|^p \langle |\psi(t,n)|^2 \rangle(T), \quad p > 0.$$

An immediate consequence of Lemma 2.1 and Theorem 2.4 is the following.

Corollary 2.7 *Let ψ be of the second kind, $p > 0, T : L_N/4 \le T < L_{N+1}/4$ for some N. The bounds hold:*

$$\langle |X|_\psi^p \rangle(T) \ge CJ(\varepsilon, \Delta)^{-p} + C\left(L_N^{p+1-\alpha_N} + T^{p+1}L_N^{\frac{\eta-1}{\eta}-\alpha_N}\right)(I(\varepsilon, \Delta) + 1/T) \tag{2.30}$$

$$\ge C(L_N + TL_N^{\frac{\eta-1}{\eta}})^p L_N^{-p\alpha_N} + CT^p L_N^{\frac{\eta-1}{\eta}-\alpha_N}, \tag{2.31}$$

where $\varepsilon = 1/T$ and $\alpha_N \to 0$.

Proof. One observes that $\langle |X|_\psi^p\rangle(T)\ge M^p P_\psi(n\ge M,T)$ for any M,T. The bound (2.30) now follows directly from Lemma 2.1, Theorem 2.4 and Lemma 2.2. The bound (2.31) follows directly from (2.30) and (2.29) (since $T\ge L_N$, the term with L_N^{p+1}/T is smaller than L_N^p, so we don't keep it in (2.31)). The proof is completed.

What is interesting is the following observation: even if one has no additional information about integrals I,J, one can obtain the bound better than (2.31), optimising (2.30) as a sum of two related via Lemma 2.2 terms.

Theorem 2.8 *Let ψ be of the second kind. Let $p>0$, $T: L_N/4\le T<L_{N+1}/4$. The uniform in T estimate holds:*

$$\langle |X|_\psi^p\rangle(T)\ge CL_N^{-\alpha_N}\left(L_N^p+T^pL_N^{\frac{p}{p+1}\frac{\eta-1}{\eta}}\right), \tag{2.32}$$

where $\alpha_N\to 0$. In particular,

$$\beta_\psi^-(p)\ge\frac{(p+1)}{p+1/\eta},\quad \beta_\psi^+(p)\ge 1. \tag{2.33}$$

Proof. The bound (2.30) of Corollary 2.7 and Lemma 2.2 imply

$$\langle |X|_\psi^p\rangle(T)\ge C\left(z^{-p}+L_N^{-\alpha_N}(L_N^{p+1}+T^{p+1}L_N^{\frac{\eta-1}{\eta}})z\right),$$

where $z=I(\varepsilon,\Delta)$. The function $f(z)=z^{-p}+Kz,\ z>0$, is bounded from below by $c(p)K^{\frac{p}{p+1}}$. The bound (2.32) follows. To prove the second statement, define $s=\frac{p(1-\eta)}{(p+1)\eta}$. Considering $T: L_N/4\le T\le L_N^{\frac{p+s}{p}}$ and $T: L_{N+1}/4>T\ge L_N^{\frac{p+s}{p}}$, one can easily see from (2.32) that in both cases

$$\langle |X|_\psi^p\rangle(T)\ge cL_N^{-\alpha_N}T^{\frac{p^2}{p+s}}\ge cT^{-\alpha_N+\frac{p^2}{p+s}}.$$

The first bound of (2.33) follows. To see that $\beta^+(p)\ge 1$ for any $p>0$, it is sufficient to take the sequence $T_N=L_N$ in (2.32). The proof is completed.

Remark 1. A priori we don't have upper bounds for the moments. However, if ψ is such that ballistic upper bound holds, then $\beta_\psi^+(p)=1$ for any p.

Remark 2. In somewhat paradoxal manner, one can obtain better *lower* bounds for the moments, if one has good *upper* bounds. This can be done in the following way. Assume that

$$\langle |X|_\psi^r\rangle(T)\le h_r(T),\quad r>0,$$

with some nontrivial $h_r(T)$ (that is, better than ballistic). Then the bound (2.30) implies some nontrivial lower bound $J\ge A(r,\varepsilon)$ and upper bound $I\le B(r,\varepsilon)$. The result of Lemma 2.2 yields $I\ge CA(r,\varepsilon)$ and $J\le CB(r,\varepsilon)$. These two bounds (with any values $r=r_1$ and $r=r_2$ respectively) can be inserted into (2.30). Finally, one can optimise the obtained bound (for a given $p>0$), choosing appropriate values of r_1,r_2. Most probably, one should take r_1 small and r_2 large.

The methods developed in this section, as it was mentioned in Introduction, can be applied to more general models. For example, one can prove the following statement. It can be applied, in particular, to the operators f with bounded sparse potentials considered in [Z], [GKT] and gives a better result.

Theorem 2.9 *Let H be any operator in $l^2(\mathbf{Z}^+)$ such that the corresponding transfer matrix verifies the condition:*

$$||T(n,0;E+i\varepsilon)|| \le C(\delta)n^{\alpha}, \quad \alpha > 0,$$

for any $E \in [-2+\delta, 2-\delta]$, $\varepsilon \in (0,1)$ and n such that $n\varepsilon \le K$, $K > 0$. Let ψ be of the second kind (in particular, $\psi = \delta_1$). For any T the bounds hold:

$$P_\psi(n \ge T, T) \ge T^{1-2\alpha}(I(1/T, \Delta) + 1/T) \ge CT^{-2\alpha}, \tag{2.34}$$

$$\langle |X|_\psi^p \rangle(T) \ge CI^{-p}(1/T,\Delta) + T^{p+1-2\alpha} I(1/T,\Delta) \ge C(p)T^{p-2p\alpha/(p+1)}. \tag{2.35}$$

Thus,

$$\beta_\psi^-(p) \ge 1 - \frac{2\alpha}{p+1}$$

(this bound is nontrivial only beginning from $p > 2\alpha - 1$).

Proof. The bound (2.34) is obtained following the proof of Theorem 2.4. The first inequality in (2.35) follows from the proof of Corollary 2.7, and the second from the proof of Theorem 2.8.

3 Dynamical upper bounds

In this section we shall establish some upper bounds for the inside and outside probabilities and the moments. It is clear that one cannot consider the same class of initial states ψ as in the previous section. The problem is that we do not have dynamical control of the possible pure point spectrum outside $(-2,2)$. Thus, we shall consider only $\psi = f(H)\delta_1$ such that $\mathrm{supp} f \subset (-2,2)$. Moreover, to control the decay at infinity (when considering outside probabilities), we shall assume that the function f is infinitely smooth (recall that we call these ψ initial states of the first kind).

We begin with the inside probabilities. Let $\psi = f(H)\delta_1$, where f is a bounded Borel function such that $\mathrm{supp} f \subset \Delta = [-2+\nu, 2-\nu]$ for some $\nu > 0$. Following the proof of Lemma 2.1, one can show that for any $K, M > 0$,

$$\sum_{n=K}^{n=K+M} \langle |\psi(t,n)|^2 \rangle(T) \le C\sqrt{MJ(\varepsilon,\Delta)} \le C\sqrt{\frac{ML_N^{\alpha_N}}{L_N + TL_N^{\frac{\eta-1}{\eta}}}}. \tag{3.1}$$

In fact, a slightly better result can be obtained using the upper bound for the imaginary part of the Borel transform of spectral measure. Such a bound represent an independent interest since it provides an upper bound for the measure of intervals and thus a lower bound for Hausdorff and packing dimensions of the spectral measure.

Lemma 3.1 *Let μ be the spectral measure of the state $\psi = \delta_1$ and $F(z)$ its Borel transform. For any $\nu \in (0,1)$ there exists constant $C(\nu)$ such that for all $x \in [-2+\nu, 2-\nu]$ and $\varepsilon : \frac{4}{L_{N+1}} < \varepsilon \leq \frac{4}{L_N}$ the bound holds:*

$$\frac{1}{2\varepsilon}\mu([x-\varepsilon, x+\varepsilon]) \leq \mathrm{Im}F(x+i\varepsilon) \leq C(\nu)L_N^{\alpha_N}\left(\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}\right)^{-1}, \tag{3.2}$$

where $\alpha_N \to 0$.

Proof. It is well known that

$$\mathrm{Im}F(z) = \mathrm{Im}z||R(z)\delta_1||^2 = \mathrm{Im}z\sum_{n=1}^{\infty}|u(n,z)|^2,$$

where $F(z)$ is the Borel transform of μ. The first inequality in (2.21) implies

$$\mathrm{Im}F(z) \geq c\mathrm{Im}z(\mathrm{Im}^2F(z)+1)\sum_{n=1}^{\infty}||T(n,0,z)||^{-2}. \tag{3.3}$$

Let $x \in [-2+\nu, 2-\nu]$, $\varepsilon \in (4/L_{N+1}, 4/L_N]$, $z = x+i\varepsilon$. We can summate over $n : 1 \leq n < L_N$ and over $n : L_N \leq n \leq K/\varepsilon$ with suitable K ($K = 8$ for $1/(2L_N) \leq \varepsilon \leq 4/L_N$ and $K = 1$ for $\varepsilon < 1/(2L_N)$, for example) using the upper bounds for $||T||$ of Lemma 2.3. Thus, we obtain from (3.3):

$$\mathrm{Im}F(z) \geq C(\nu)\varepsilon\mathrm{Im}^2F(z)L_N^{-2\alpha_N}(L_N + \varepsilon^{-1}L_N^{\frac{\eta-1}{\eta}}).$$

Since $\mathrm{Im}F(x+i\varepsilon) \geq 1/(2\varepsilon)\mu([x-\varepsilon, x+\varepsilon])$, the result follows.

Remark. The proof is rather simple because we have from the very beginning the upper bound for $||T(n,0,z)||$ for complex z. In most applications, however, one has such bounds only for real z, and one should proceed in a more complicated way using the Jitomirskaya-Last method [JL].

As a first direct consequence of this result, one can obtain the already known upper bounds (2.27) on $I(\varepsilon,\Delta), J(\varepsilon,\Delta)$. Indeed, for $\Delta = [-2+\nu, 2-\nu]$,

$$J(\varepsilon,\Delta) = \int_\Delta d\mu(x)b(x,T),$$

where

$$b(x,T) = \varepsilon\mathrm{Im}F(x+i\varepsilon) \leq C(\nu)\varepsilon L_N^{\alpha_N}\left(\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}\right)^{-1} \tag{3.4}$$

due to (3.2). The bound for $J(\varepsilon,\Delta)$ follows. Next,

$$I(\varepsilon,\Delta) = \varepsilon\int_\Delta dE\mathrm{Im}^2F(E+i\varepsilon) = \int_\Delta dEb(E,T)\mathrm{Im}F(x+i\varepsilon)$$

The bound (3.4) and

$$\int_{\mathbf{R}} dE \mathrm{Im} F(x+i\varepsilon) = \mu(\mathbf{R}) = 1$$

imply the bound for $I(\varepsilon, \Delta)$.

Before stating the next corollary, let us recall the definition of the lower and upper Hausdorff dimension of Borel measure:

$$\dim_*(\mu) = \inf\{\dim(S) \mid \mu(S) > 0\},$$

$$\dim^*(\mu) = \inf\{\dim(S) \mid \mu(S) = \mu(\mathbf{R})\},$$

where $\dim(S)$ denotes Hausdorff dimension of the set S. Thus, the measure gives zero weight to any set S with $\dim(S) < \dim_*(\mu)$ and for any $\varepsilon > 0$ is supported by some set S with $\dim(S) < \dim^*(\mu) + \varepsilon$. The measure is of exact Hausdorff dimension if $\dim_*(\mu) = \dim^*(\mu)$. It is known (see [T] for the referencies) that

$$\dim_*(\mu) = \mu - \mathrm{essinf} \gamma^-(x) = \sup\{\alpha \mid \gamma^-(x) \geq \alpha \ \mu - a.s.\}, \tag{3.5}$$

$$\dim^*(\mu) = \mu - \mathrm{esssup} \gamma^-(x) = \inf\{\alpha \mid \gamma^-(x) \leq \alpha \ \mu - a.s.\}. \tag{3.6}$$

Here $\gamma^-(x)$ is the lower local exponent of μ:

$$\gamma^-(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu([x-\varepsilon, x+\varepsilon])}{\log \varepsilon}.$$

For the packing dimension similar formulae hold (see [T] for details).

Corollary 3.2 *1. For any $\delta \in (0,1)$, $\nu > 0$ the spectral measure μ of the state $\psi = \delta_1$ is uniformly $\eta - \delta$-Hölder continuous on $[-2+\nu, 2-\nu]$. In particular, for μ', the restriction of μ on $(-2,2)$, $\dim_*(\mu') \geq \eta$.*
2. The packing dimension of μ is 1.

Proof. Let $\varepsilon \in (4/L_{N+1}, 4/L_N]$ for some N. One can easily see that

$$\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}} \geq \varepsilon^{1-\eta}.$$

Therefore, Lemma 3.1 implies

$$\mu([x-\varepsilon, x+\varepsilon]) \leq C(\delta)\varepsilon^{\eta} L_N^{\alpha_N} \leq C_1(\delta)\varepsilon^{\eta-\alpha_N}$$

for any $x \in [-2+\nu, 2-\nu]$. Since $\lim \alpha_N = 0$, the uniform $\eta - \delta$-continuity of μ restricted to $[-2+\nu, 2-\nu]$ follows. As a particular consequence, $\gamma^-(x) \geq \eta$ for all $x \in (-2,2)$. The equality (3.5) implies $\dim_*(\mu') \geq \eta$.

Taking $\varepsilon_N = 1/L_N$, we obtain from Lemma 3.1 that

$$\mu([x-\varepsilon_N, x+\varepsilon_N]) \leq C(\delta)\varepsilon_N^{1-\alpha_N}.$$

Therefore, for the upper local exponents of the measure we have

$$\gamma^+(x) \equiv \text{limsup}_{\varepsilon\to 0}\frac{\log\mu([x-\varepsilon, x+\varepsilon])}{\log\varepsilon} \geq 1.$$

The fact that $\dim_P(\mu) = 1$ follows [T]. The proof is completed.
Remark. These results are not new. The fact that $\dim_*(\mu') \geq \eta$ is proved in [JL] and $\dim_P(\mu) = 1$ in [CM]. Our proof, however, is more simple. Moreover, the upper bound (3.2) contains more information.

Lemma 3.3 *Let $\psi = f(H)\delta_1$, where f is bounded Borel function such that* supp$f \subset \Delta = [-2+\nu, 2-\nu]$ *for some $\nu > 0$. Let $T:\ L_N/4 \leq T < L_{N+1}/4$ for some N.*
1. For any n the bound holds:

$$\langle|\psi(t,n)|^2\rangle(T) \leq C\frac{L_N^{\alpha_N}}{L_N + TL_N^{\frac{\eta-1}{\eta}}}. \tag{3.7}$$

2. Define $M(T) = L_N^{-\delta}(L_N + TL_N^{\frac{\eta-1}{\eta}})$ with some $\delta > 0$. Then

$$P_\psi(n \leq M(T), T) \leq CL_N^{-\delta/2}$$

for T large enough and thus

$$P_\psi(n \geq M(T), T) \geq ||\psi||^2 - CL_N^{-\delta/2}.$$

3. For the time-averaged return probability the bound holds:

$$\frac{1}{T}\int_0^\infty dt\exp(-t/T)|\langle\psi(t),\psi\rangle|^2 \leq C\frac{L_N^{\alpha_N}}{L_N + TL_N^{\frac{\eta-1}{\eta}}}. \tag{3.8}$$

Proof. Using spectral Theorem in a standard way (see [T], for example), one first shows that

$$\begin{aligned} 2\langle|\psi(t,n)|^2\rangle(T) = \int_{\mathbf{R}}\int_{\mathbf{R}} d\mu_\psi(x)d\mu_\psi(y)u_\psi(n,x)\overline{u_\psi(n,y)}R(T(x-y)) \leq \\ 2\int_{\mathbf{R}} d\mu_\psi(x)|u_\psi(n,x)|^2 b_\psi(x,T), \end{aligned} \tag{3.9}$$

where

$$b_\psi(x,T) = \int_{\mathbf{R}} d\mu_\psi(u)R(T(x-u)) = \varepsilon \text{Im} F_{\mu_\psi}(x+i\varepsilon),\ \varepsilon = 1/T.$$

Since f is bounded and supp$f \subset \Delta$, we get

$$\int_{\mathbf{R}} d\mu_\psi(x)|u_\psi(n,x)|^2 b_\psi(x,T) \leq C\int_\Delta d\mu_\psi(x)b(x,T)|u_\psi(n,x)|^2.$$

The bound (3.2) and

$$\int_{\mathbf{R}} d\mu_\psi(x)|u_\psi(n,x)|^2 \leq 1$$

yield (3.7). The second statement of Lemma directly follows. For the return probabilities the result follows from the bound

$$J_\psi(\varepsilon, \mathbf{R}) \leq CJ(\varepsilon, \Delta)$$

and the established upper bound for $J(\varepsilon, \Delta)$ (Corollary 2.5). The proof is completed.

The situation is more difficult with the upper bounds for outside probabilities. We shall consider the initial state ψ of the form $\psi = f(H)\delta_1$, where $f \in C_0^\infty([-2+\nu, 2-\nu])$ with some $\nu \in (0, 1/2)$. For smooth f it is well known that the function $\psi(n)$ decays at infinity faster than any inverse power and moreover, for the moments of the time-averaged position operator, the ballistic upper bound holds:

$$\langle |X|_\psi^p \rangle(T) \leq C(p)T^p, \quad p > 0. \tag{3.10}$$

The following statement holds (where we use some ideas of [CM] in the proof).

Theorem 3.4 *Consider ψ of the first kind. Let T be such that $L_N/4 \leq T \leq L_N^{1/\eta}$ for some N.*
1. For any $p \geq 0$ the following bound holds

$$\sum_{n \geq 2L_N} n^p \langle |\psi(t,n)|^2 \rangle(T) \leq C(p)T^{p+1}L_N^{-1/\eta} \tag{3.11}$$

In particular,

$$P_\psi(n \geq 2L_N, T) \leq CTL_N^{-1/\eta} \tag{3.12}$$

and

$$\langle |X|_\psi^p \rangle(T) \leq CL_N^p + CT^{p+1}L_N^{-1/\eta}. \tag{3.13}$$

2. Let $T : L_N/4 \leq L_{N+1}^{1-\delta}$ with some $\delta > 0$. For $M > 2L_N$ and any $A > 0$ the uniform bound holds:

$$P_\psi(2L_N \leq n \leq M, T) \leq C\frac{M}{T + L_N^{1/\eta}} + \frac{C_A}{L_N^A}. \tag{3.14}$$

Proof. First of all, observe that the ballistic upper bound (3.10) implies

$$\langle |\psi(t,n)|^2 \rangle(T) \leq C(r)T^r n^{-r}$$

for any $r > 0$. Therefore, taking r large enough, we obtain

$$\sum_{n \geq T^2} n^p \langle |\psi(t,n)|^2 \rangle(T) \leq C(r,p)T^{2p+2-r} \leq C(p,A)T^{-A}$$

for any $A > 0$. Thus, to prove (3.11), it is sufficient to consider the sum over $n :\ 2L_N \leq n \leq T^2$. We use again the Parseval formula:

$$\langle |\psi(t,n)|^2 \rangle(T) = \frac{\varepsilon}{\pi} \int_{\mathbf{R}} dE |(R(E + i\varepsilon)f(H)\delta_1)(n)|^2, \quad \varepsilon = \frac{1}{2T}. \tag{3.15}$$

Define $\Delta = [-2+\nu/2, 2-\nu/2]$, where $f \in C_0^\infty([-2+\nu, 2-\nu])$. We shall denote by $a_1(n,T)$ the integral over $\mathbf{R}\setminus\Delta$ in (3.15), and by $a_2(n,T)$ the integral over Δ. Since $f(x) = 0$, $|x| \geq 2-\nu$, one can show, as in the proof of the part b) of Theorem 2.4 (bounds (2.22)-(2.24)), that

$$|R(E+i\varepsilon)f(H)\delta_1(n)| \leq \frac{C(k,\nu)}{E(1+|n|^2)^{k/2}}$$

for any integer $k>0$ and all $E \in \mathbf{R}\setminus\Delta$ with uniform in n, E, ε constants. Therefore,

$$a_1(n,T) \leq \frac{C(k,\nu)}{T}(1+|n|^2)^{-k} \tag{3.16}$$

for any $k>0$. In particular, taking k large enough, we obtain

$$\sum_{n\geq 2L_N} n^p a_1(n,T) \leq C(p,A)L_N^{-A} \tag{3.17}$$

for any $A>0$.

Consider now the term $a_2(n,T)$. Since $R(z)f(H) = f(H)R(z)$, one can write it as follows:

$$a_2(n,T) = \frac{\varepsilon}{\pi}\int_\Delta dE|(f(H)R(E+i\varepsilon)\delta_1)(n)|^2. \tag{3.18}$$

Since $f \in C_0^\infty([-2,2])$, it follows again from results of [GK] that for any $\chi \in l^2(\mathbf{N})$

$$|(f(H)\chi)(n)|^2 \leq C(k)\sum_m (1+|n-m|^2)^{-k}|\chi(m)|^2$$

Inserting this bound in (3.18) yields after integration:

$$a_2(n,T) \leq C(k)\varepsilon\sum_m (1+|n-m|^2)^{-k}\int_\Delta dE|(R(E+i\varepsilon)\delta_1)(m)|^2 \tag{3.19}$$

for any $k>0$. Denote by $a_{21}(n,T)$ the sum in (3.19) over $m: m \leq L_N$, by $a_{22}(n,T)$ the sum over $m:\ L_N < m \leq T^2+L_N$ and by $a_{23}(n,T)$ the sum over $m:\ m > T^2+L_N$. It is clear that for any $A>0$,

$$\sum_{2L_N\leq n\leq T^2} n^p(a_{21}(n,T)+a_{23}(n,T)) \leq C(p,A)L_N^{-A}\varepsilon\sum_m\int_\Delta |(R(E+i\varepsilon)\delta_1)(m)|^2. \tag{3.20}$$

The fact that

$$\frac{\varepsilon}{\pi}\sum_m\int_{\mathbf{R}} dE|(R(E+i\varepsilon)\delta_1)(m)|^2 = ||\delta_1||^2 = 1$$

and (3.20) yield

$$\sum_{2L_N\leq n\leq T^2} n^p(a_{21}(n,T)+a_{23}(n,T)) \leq C(p,A)L_N^{-A}. \tag{3.21}$$

The summation over n in the expression of $a_{22}(n,T)$ yields

$$\sum_{2L_N \le n \le T^2} n^p a_{22}(n,T) \le C\varepsilon \sum_{L_N < m \le T^2 + L_N} m^p \int_\Delta dE |(R(E+i\varepsilon)\delta_1)(m)|^2. \tag{3.22}$$

To bound from above the r.h.s. of (3.22), we shall introduce in $l^2(\mathbf{N})$ operator

$$H_N = H_0 + V_N, \quad V_N(n) = F(n \le L_N)V(n)$$

with compactly supported potential and thus absolutely continuous spectrum on $(-2,2)$. We denote by $R(z)$ and $R_N(z)$ the resolvents of H and H_N respectively. One can see that for N large enough (so that $Q(n)$ disappear),

$$(H - H_N)\phi(n) = \sum_{k=N+1}^{\infty} \delta_{L_k}(n) V(L_k)\phi(L_k). \tag{3.23}$$

The resolvent equation implies that for any complex $z = E + i\varepsilon$,

$$||R(z)\delta_1 - R_N(z)\delta_1|| \le \frac{1}{\varepsilon}||(H - H_N)R_N(z)\delta_1||. \tag{3.24}$$

To bound from above the r.h.s. of (3.22) and the r.h.s. of (3.24), we need to control $g(n) = R_N(z)\delta_1(n)$ for $n > L_N$. In fact, a rather explicit expression can be obtained. Since $(H_N - z)g = \delta_1$ and $V_N(n) = 0$ for $n > L_N$,

$$g(n-1) + g(n+1) - zg(n) = 0, \quad n > L_N.$$

Thus,

$$(g(n+1), g(n))^T = T_0(n - L_N, z)(g(L_N+1), g(L_N))^T, \quad n \ge L_N, \tag{3.25}$$

where $T_0(m,z) = A_0(z)^m$ is the free transfer matrix with $A_0(z)$ given by (2.13). Since $E \in \Delta = [-2+\nu/2, 2-\nu/2]$, the matrix $A_0(z)$ has two complex eigenvalues

$$\lambda_{1,2} = \frac{1}{2}(z \pm \sqrt{z^2 - 4})$$

with corresponding eigenvectors $e_i = (\lambda_i, 1)^T$, $i = 1, 2$. It follows from (3.25) that

$$(g(n+1), g(n))^T = C_1 \lambda_1^{n-L_N} e_1 + C_2 \lambda_2^{n-L_N} e_2, \quad n \ge L_N,$$

with some complex C_1, C_2. Since $\mathrm{Im} z = \varepsilon > 0$, one of the two eigenvalues, say, λ_1, is such that $|\lambda_1| < 1$ and then $|\lambda_2| > 1$. On the other hand, since $g = R_N(z)\delta_1$, it should be square integrable in n. Therefore, $C_2 = 0$ and

$$(g(n+1), g(n))^T = C\lambda_1^{n-L_N}(\lambda_1, 1)^T.$$

Finally, we obtain that

$$g(n) \equiv (R_N(z)\delta_1)(n) = \lambda_1^{n-L_N} g(L_N) \tag{3.26}$$

for any $n \geq L_N$. One can see from the expression of λ_1 that

$$\exp(-c_1\varepsilon) \leq |\lambda_1| \leq \exp(-c\varepsilon) \tag{3.27}$$

with uniform $c_1, c > 0$ for all $E \in \Delta,\ \varepsilon \in (0,1)$.

Let us return to the resolvent $R(z)$. Using the trivial bound $|g(L_N)| \leq 1/\varepsilon$, one gets from (3.23)-(3.24) and (3.26)-(3.27):

$$||R(z)\delta_1 - R_N(z)\delta_1||^2 \leq \varepsilon^{-4} \sum_{k=N+1}^{\infty} V^2(L_k) \exp(-2c\varepsilon(L_k - L_N). \tag{3.28}$$

Since $T = \frac{1}{2\varepsilon} \leq L_{N+1}^{1-\delta}$ in all three statements of the Theorem, $V(L_k) = L_k^{(1-\eta)/2\eta}$, and L_k is a very fast growing sequence, it is easy to check that

$$||R(z)\delta_1 - R_N(z)\delta_1||^2 \leq C\exp(-1/\varepsilon^\alpha) \tag{3.29}$$

with some $\alpha > 0$ for all $E \in \Delta,\ \varepsilon \in [L_{N+1}^{\delta-1}, 4L_N^{-1}]$. Thus, the bounds (3.22) and (3.29) imply

$$\sum_{2L_N \leq n \leq T^2} n^p a_{22}(n,T) \leq C/\varepsilon^{2p}\exp(-1/\varepsilon^\alpha) + C\varepsilon \sum_{L_N \leq m \leq T^2 + L_N} m^p \int_\Delta dE|R_N(E+i\varepsilon)\delta_1(m)|^2. \tag{3.30}$$

It follows from (3.26)-(3.27) and $\varepsilon \leq 2/L_N$ that

$$\sum_{L_N \leq m \leq T^2 + L_N} m^p |R_N(E+i\varepsilon)\delta_1(m)|^2 \leq C\varepsilon^{-p-1}|R_N(E+i\varepsilon)\delta_1(L_N)|^2. \tag{3.31}$$

To bound $R_N(E+i\varepsilon)\delta_1(L_N)$, one can use the result of Lemma 4 in [CM]. For the sake of completeness we shall give here a simple and slightly different proof. Namely, we shall show that

$$|(R_N(E+i\varepsilon)\delta_1)(L_N)|^2 \leq C(\Delta)\frac{1}{1+\varepsilon L_N^{1/\eta}}\mathrm{Im}F_N(E+i\varepsilon), \tag{3.32}$$

where $E \in \Delta$ and F_N denotes the Borel transform of the spectral measure associated to the state δ_1 and operator H_N. First, it follows from (3.26)-(3.27) that

$$\frac{1}{\varepsilon}\mathrm{Im}F_N(E+i\varepsilon) = ||R_N(E+i\varepsilon)\delta_1||^2 \geq \sum_{m > L_N} |g(m)|^2 \geq \frac{C}{\varepsilon}|g(L_N)|^2.$$

Therefore,

$$|g(L_N)|^2 \leq C\mathrm{Im}F_N(E+i\varepsilon). \tag{3.33}$$

Let $L_{N-1} < n < L_{N+1}$. The definition of $g = R_N(z)\delta_1$ implies

$$g(n+1) + g(n-1) - zg(n) = 0, \quad n \neq L_N, \tag{3.34}$$

$$g(L_N+1) + g(L_N-1) + (V(L_N) - z)g(L_N) = 0. \tag{3.35}$$

It is clear that for $n > L_N$,

$$(g(n+1), g(n))^T = T_0(n - L_N, z)(g(L_N+1), g(L_N))^T, \tag{3.36}$$

and for $n < L_N - 1$

$$(g(n+1), g(n))^T = T_0(n - L_N + 1, z)(g(L_N), g(L_N - 1))^T, \tag{3.37}$$

where $T_0(m, z)$ is the free transfer matrix. Since $z = E + i\varepsilon, E \in \Delta$, its norm is uniformly bounded for $|m| \leq K/\varepsilon$. Using the fact that $||T^{-1}|| = ||T||$ and $\varepsilon \leq 2/L_N$, we thus get that for $2L_N > n > L_N$,

$$|g(n+1)|^2 + |g(n)|^2 \geq c(|g(L_N+1)|^2 + |g(L_N)|^2).$$

with uniform $c > 0$. Summating this bound, one obtains

$$cL_N(|g(L_N+1)|^2 + |g(L_N)|^2) \leq 2||g||^2 = 2/\varepsilon \mathrm{Im} F_N(E + i\varepsilon). \tag{3.38}$$

Similarly, summation over $L_N/2 < n < L_N$ yields (since $L_N > 2L_{N-1}$):

$$c/2L_N(|g(L_N)|^2 + |g(L_N - 1)|^2) \leq 2/\varepsilon \mathrm{Im} F_N(E + i\varepsilon). \tag{3.39}$$

Thus, (3.38)-(3.39) yield

$$|g(L_N - 1)|^2 + |g(L_N + 1)|^2 \leq \frac{C}{\varepsilon L_N} \mathrm{Im} F_N(E + i\varepsilon). \tag{3.40}$$

It follows from (3.35) that

$$|V(L_N) - z|^2 |g(L_N)|^2 \leq \frac{C}{\varepsilon L_N} \mathrm{Im} F_N(E + i\varepsilon).$$

Since $|z| \leq 3$ and $V(L_N) = L_N^{(1-\eta)/2\eta}$, we obtain

$$|g(L_N)|^2 \leq C(\Delta)\varepsilon^{-1} L_N^{-1/\eta} \mathrm{Im} F_N(E + i\varepsilon). \tag{3.41}$$

The bound (3.32) follows from (3.33) and (3.41).

We can finish now the proof of the first part of the Theorem.

It follows from (3.30),(3.31) and (3.32) that

$$\sum_{2L_N \leq n \leq T^2} n^p a_{22}(n, T) \leq C/\varepsilon^{2p} \exp(-1/\varepsilon^\alpha) +$$

$$C\varepsilon^{-p-1} L_N^{-1/\eta} \int_\Delta dE \mathrm{Im} F_N(E + i\varepsilon) \leq C\varepsilon^{-p-1} L_N^{-1/\eta}, \tag{3.42}$$

since $\varepsilon \leq 2L_N^{-1}$ and

$$\int_\Delta dE \mathrm{Im} F_N(E + i\varepsilon) \leq \int_{\mathbf{R}} dE \mathrm{Im} F_N(E + i\varepsilon) = \pi \mu_N(\mathbf{R}) = \pi.$$

The bound (3.11) of Theorem follows from Parseval equality, (3.17), (3.21) (one takes $A = 1/\eta$) and (3.42). Taking $p = 0$, we obtain the bound for outside probabilities. Since

$$\langle |X|_\psi^p \langle (T) \leq (2L_N)^p ||\psi||^2 + \sum_{n \geq 2L_N} n^p \langle |\psi(t,n)|^2 \rangle (T),$$

the upper bound for the moments follows.

The proof of the second statement is similar. One defines $a_1(n,T)$ and $a_2(n,T)$ in the same manner. The bound (3.17) yields

$$\sum_{2L_N \leq n \leq M} a_1(n,T) \leq C_A L_N^{-A}.$$

Next, one denotes as $a_{21}(n,T), a_{22}(n,T)$ and $a_{23}(n,T)$ the sums in (3.19) over $m \leq L_N$, $m : L_N < m < 2M$ and $m : m \geq 2M$ respectively. The bound (3.21) yields

$$\sum_{2L_N \leq n \leq M} a_{21}(n,T) \leq C_A L_N^{-A}.$$

Similarly, one shows that

$$\sum_{2L_N \leq n \leq M} a_{23}(n,T) \leq C_A L_N^{-A}.$$

Thus, it is sufficient to bound from above the r.h.s. of

$$\sum_{2L_N \leq n \leq M} a_{22}(n,T) \leq C\varepsilon \sum_{L_N < m < 2M} \int_\Delta dE |(R(E+i\varepsilon)\delta_1)(m)|^2.$$

The same consideration as in the proof of the part 1 yields

$$\sum_{2L_N \leq n \leq M} a_{22}(n,T) \leq C\exp(-1/\varepsilon^\alpha) + C\varepsilon \sum_{L_N < m < 2M} \int_\Delta dE |(R_N(E+i\varepsilon)\delta_1)(m)|^2, \quad (3.43)$$

where

$$(R_N(E+i\varepsilon)\delta_1)(m) = g(L_N)\lambda_1^{m-L_N}, \ m \geq L_N. \quad (3.44)$$

The bounds (3.32) and (3.44) imply

$$\varepsilon \sum_{L_N < m < 2M} |(R_N(E+i\varepsilon)\delta_1)(m)|^2 \leq C\frac{M\varepsilon}{1+\varepsilon L_N^{1/\eta}} \mathrm{Im} F_N(E+i\varepsilon).$$

Inserting this bound in (3.43), we obtain

$$\sum_{2L_N \leq n \leq M} a_{22}(n,T) \leq C\frac{M\varepsilon}{1+\varepsilon L_N^{1/\eta}}.$$

The bound (3.14) follows.

4 Improved lower bounds

The result of Theorem 3.4 allows us to have total control of the integrals I, J (up to factor like $L_N^{\alpha_N}$).

Corollary 4.1 *Let Δ be nonempty interval such that $\Delta \subset [-2+\nu, 2-\nu]$ for some $\nu > 0$. There exist positive constants uniform in ε such that for all $\varepsilon : 4L_{N+1}^{-1} < \varepsilon \leq 4L_N^{-1}$,*

$$\frac{C\varepsilon}{\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}} \leq J(\varepsilon, \Delta) \leq CI(\varepsilon, \Delta) \leq \frac{CL_N^{\alpha_N}\varepsilon}{\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}}, \tag{4.1}$$

where $\alpha_N \to 0$.

Proof. Pick a slightly smaller interval $\Delta' \subset \Delta$. Let f be a function from $C_0^\infty(\Delta)$ such that $0 \leq f(x) \leq 1$ and $f(x) = 1$ on Δ'. Define $\psi = f(H)\delta_1$. The result of Lemma 2.1 applied to interval Δ' yields

$$P_\psi(n \geq M_1(T), T) \geq D_1 c_1 > 0, \tag{4.2}$$

where $c_1 = \mu_\psi(\Delta') = \mu(\Delta') > 0$, and

$$M_1(T) = c_1^2/(16 J_\psi(\varepsilon, \Delta')), \quad \varepsilon = 1/T. \tag{4.3}$$

On the other hand, Theorem 3.4 implies

$$P_\psi(n \geq 2L_N, T) \leq D_2 T L_N^{-1/\eta}, \quad T \leq L_N^{1/\eta}. \tag{4.4}$$

If $T \leq \frac{D_1 c_1}{2D_2} L_N^{1/\eta} \equiv \gamma L_N^{1/\eta}$, then (4.2), (4.4) imply $P_\psi(n \geq 2L_N, T) < P_\psi(n \geq M_1(T), T)$ and thus $M_1(T) < 2L_N$. It follows from (4.3) that

$$\frac{C}{L_N} \leq J_\psi(\varepsilon, \Delta') \leq J_\psi(\varepsilon, \mathbf{R}) \tag{4.5}$$

for $\varepsilon \in [\varepsilon_0, 4L_N^{-1}]$, where $\varepsilon_0 = (\gamma)^{-1} L_N^{-1/\eta}$. Recall the equality

$$J_\psi(\varepsilon, \mathbf{R}) = \varepsilon \int_0^\infty dt \exp(-\varepsilon t) |\langle \psi(t), \psi \rangle|^2. \tag{4.6}$$

The crucial observation is that $J_\psi(\varepsilon, \mathbf{R})/\varepsilon$ is decreasing in ε. Therefore, it follows from (4.5)-(4.6) that

$$J_\psi(\varepsilon, \mathbf{R}) \geq \varepsilon/\varepsilon_0 J_\psi(\varepsilon_0, \mathbf{R}) \geq C\varepsilon L_N^{\frac{1-\eta}{\eta}} \tag{4.7}$$

for all $\varepsilon \leq \varepsilon_0$. The bounds (4.5), (4.7) imply that

$$J_\psi(\varepsilon, \mathbf{R}) \geq \frac{C\varepsilon}{\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}}$$

for all $\varepsilon \in (4L_{N+1}^{-1}, 4L_N^{-1}]$ with suitable constant. Since $0 \le f(x) \le 1$ and $f(x) = 0$ for x outside of Δ, the definition of J_ψ, J implies

$$J_\psi(\varepsilon, \mathbf{R}) \le \int_\Delta d\mu(x) \int_\Delta d\mu(y) R((x-y)/\varepsilon) \le J(\varepsilon, \Delta).$$

Thus, we get

$$J(\varepsilon, \Delta) \ge \int_\Delta d\mu(x) \int_\Delta d\mu(y) R((x-y)/\varepsilon) \ge \frac{C\varepsilon}{\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}}. \tag{4.8}$$

The first inequality in (4.1) follows. The second and the third inequalities follow from Lemma 2.2 and Corollary 2.5.

As a direct consequence of this result, one gets lower bound for the time-averaged return probabilities $J_\psi(1/T, \mathbf{R})$. In fact, if the measure μ_ψ has a nontrivial point part: $\mu_\psi(\{E_0\}) = \gamma > 0$ for some E_0, then clearly $J_\psi(\varepsilon, \mathbf{R}) \ge \gamma^2 > 0$ for any ε. The situation is more interesting if the measure is continuous, in our case if $\operatorname{supp}\mu_\psi \subset (-2, 2)$.

Theorem 4.2 *Assume that $\psi = f(H)\delta_1$, where f is a bounded Borel function*
a) supported on $[-2+\nu, 2-\nu]$ for some $\nu > 0$
b) such that $|f(x)| \ge c > 0$ on some interval $\Delta \subset [-2+\nu, 2-\nu]$. Then

$$\frac{C\varepsilon}{\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}} \le J_\psi(1/T, \mathbf{R}) \le \frac{C\varepsilon L_N^{\alpha_N}}{\varepsilon L_N + L_N^{\frac{\eta-1}{\eta}}}, \ \varepsilon = 1/T,$$

for $T: \ L_N/4 \le T < L_{N+1}/4$. If only the condition b) is fulfilled, then only the lower bound for $J_\psi(\varepsilon, \mathbf{R})$ holds.

Proof. The upper bound is proved in Lemma 3.3. Since

$$J_\psi(\varepsilon, \mathbf{R}) \ge c^4 \int_\Delta d\mu(x) \int_\Delta d\mu(y) R((x-y)/\varepsilon),$$

the second inequality in (4.8) yields lower bound. The proof is completed.

One observes that the integral

$$T J_\psi(1/T, \mathbf{R}) = \int_0^\infty dt \exp(-t/T) |\langle \psi(t), \psi \rangle|^2$$

grows linearly for $T: \ L_N \le T \le L_N^{1/\eta}$ and remains stable for $T: \ L_N^{1/\eta} < T < L_{N+1}$ (up to factors like $CL_N^{\alpha_N}$). Since the main contribution to the integral comes from interval $[0, T]$, one can conjecture that the return probability $R_\psi(t) = |\langle \psi(t), \psi \rangle|^2 = |\widehat{\mu}_\psi(t)|^2$ is essentially constant of order L_N^{-1} if $t \in [L_N, L_N^{1/\eta}]$, and is small (decaying at least as $1/t$) if $t \in [L_N^{1/\eta+\delta}, L_{N+1}]$.

The obtained lower bounds for $I(\varepsilon, \Delta)$ imply also improved lower bounds for probabilities and moments.

Theorem 4.3 *Let ψ be of the second kind. Then*
1. For $T:\ L_N \le T \le L_N^{1/\eta}$:

$$P_\psi(n \ge T, T) \ge C T L_N^{-1/\eta - \alpha_N}$$

and for $T:\ L_N^{1/\eta} \le T < L_{N+1}/4$,

$$P_\psi(n \ge T, T) \ge C L_N^{-\alpha_N}.$$

2. For $T:\ L_N/4 \le T \le L_N^{1/\eta}$,

$$P_\psi(L_N/4 \le n \le L_N, T) \ge C L_N^{-\alpha_N}.$$

As a consequence, for $T:\ L_N/4 \le T \le L_N^{1/\eta}$,

$$\langle |X|_\psi^p \rangle(T) \ge C(L_N^p + T^{p+1} L_N^{-1/\eta}) L_N^{\alpha_N},$$

and for $T:\ L_N^{1/\eta} \le T < L_{N+1}/4$,

$$\langle |X|_\psi^p \rangle(T) \ge C T^p L_N^{-\alpha_N}.$$

The results for probabilities follow directly from Theorem 2.4 and Corollary 4.1. The bound $\langle |X|_\psi^p \rangle(T) \ge M^p P_\psi(n \ge M, T)$ for any M yields the result for the moments. The proof is completed.

The result of the Theorem tells, in particular, that for $T \ge L_N^{1/\eta}$, some (not too small) part of the wavepacket has gone through the barrier and moves ballistically:

$$P_\psi(n \ge T, T) \ge C L_N^{-\alpha_N} \ge C T^{-\alpha_N}, \quad \alpha_N \to 0. \tag{4.9}$$

On the other hand, Corollary 1.6 implies that for $T \ge L_N^{1/\eta + \delta}$ with some $\delta > 0$,

$$P_\psi(n > L_N, T) \ge P_\psi(n \ge T L_N^{\frac{\eta-1}{\eta} - \alpha_N}, T) \ge c > 0.$$

Thus, some essential (and not small) part of the wavepacket is on the right of L_N. This part will continue to move ballistically up to the next barrier located at $n = L_{N+1}$. Therefore, one can expect the bound like

$$P_\psi(n \ge T, T) \ge c_1 > 0$$

for $T > L_N^{1/\eta + \delta}$, which is better than just (4.9). The following statement confirm this conjecture. Slightly modifying the proof, we show also that

$$P_\psi(n \le 2L_N, T) \ge c_2 > 0$$

for $T \le \tau L_N^{1/\eta}$ with $\tau > 0$ small enough. This is better than $P_\psi(n \le 2L_N, T) \ge C L_N^{-\alpha_N}$, which follows from Theorem 4.3.

Theorem 4.4 *The following statements hold:*
1. Assume that ψ is of the second kind. For any $\delta > 0$ there exist $\tau > 0, c_1 > 0$ such that for $T: \ L_N^{1/\eta+\delta} \leq T < L_{N+1}^{1-\delta}$ with N large enough,

$$P_\psi(n \geq \tau T, T) \geq c_1 > 0.$$

If ψ is of the first kind, for any $\theta > 0$ one can choose τ so that

$$P_\psi(n \geq \tau T, T) \geq ||\psi||^2 - \theta.$$

In both cases, for such T,

$$\langle |X|_\psi^p \rangle (T) \geq C(p) T^p, \quad p > 0.$$

2. Let ψ be of the second kind. There exists $\tau > 0$ small enough such that

$$P_\psi(n \leq 2L_N, T) \geq c_2 > 0$$

for all $T: \ L_N/4 \leq T \leq \tau L_N^{1/\eta}$. If ψ is of the first kind, then better bound holds:

$$P_\psi(n \leq 2L_N, T) \geq ||\psi||^2 - CTL_N^{-1/\eta}. \tag{4.10}$$

Proof. Recall that $\psi = f(H)\delta_1$, where f is a bounded Borel function, $f \in C^\infty(S)$, $S = [E_0 - \nu, E_0 + \nu] \subset [-2 + \nu, 2 - \nu]$ and $|f(x)| \geq c > 0$ on S. Let h be some function such that $0 \leq h(x) \leq 1$, $h \in C_0^\infty([E_0 - \gamma, E_0 + \gamma])$ and $h(x) = 1, \ x \in [E_0 - \theta, E_0 + \theta]$, where $0 < \theta < \gamma < \nu$. Define $g(x) = f(x)h(x)$. It is clear that $g \in C_0^\infty([E_0 - \gamma, E_0 + \gamma])$. Let

$$\rho = g(H)\delta_1, \ \chi = \psi - \rho = (f(H) - g(H))\delta_1.$$

As $|f(x)| \geq c > 0$ on S,

$$\alpha \equiv ||\rho||^2 \geq c^2 \mu([E_0 - \theta, E_0 + \theta]) > 0.$$

Since

$$\langle \rho, \chi \rangle = \int d\mu(x) |f(x)|^2 h(x)(1 - h(x))$$

and f bounded, choosing the parameter γ in the definition of h close enough to θ, one can ensure that

$$|\langle \rho, \chi \rangle \leq ||\rho||^2/4 = \alpha/4. \tag{4.11}$$

Let $\rho(t) = \exp(-itH)\rho, \ \chi(t) = \exp(-itH)\chi$ and $\psi(t) = \exp(-itH)\psi$. For any $n \in \mathbf{N}$,

$$|\psi(t,n)|^2 = |\rho(t,n)|^2 + |\chi(t,n)|^2 + 2\mathrm{Re}(\rho(t,n)\overline{\chi(t,n)}).$$

Let $M > 0$. Summation over $n \leq M$ and time-averaging yield for any $T > 0$:

$$\sum_{n \leq M} \langle |\psi(t,n)|^2 \rangle (T) \leq$$

$$||\chi||^2 + \sum_{n \le M} \langle |\rho(t,n)|^2 \rangle(T) + 2||\chi|| \left(\sum_{n \le M} \langle |\rho(t,n)|^2 \rangle(T) \right)^{1/2} . \tag{4.12}$$

We have used the fact that $||\chi(t)|| = ||\chi||$ and the Cauchy-Schwarz inequality. The condition (4.11) implies that $||\chi||^2 \le ||\psi||^2 - \alpha/2$. Therefore, (4.12) yields

$$P_\psi(n \le M, T) \le ||\psi||^2 - \alpha/2 + P_\rho(n \le M, T) + C(P_\rho(n \le M, T))^{1/2} \tag{4.13}$$

Thus, if $P_\rho(n \le M, T) \le \eta$, where η is small enough (depending on α), then $P_\psi(n \ge M, T) \ge \alpha/4 > 0$.

Let $M > 2L_N$. To bound from above $P_\rho(n \le M, T)$, we shall write

$$P_\rho(n \le M, T) = P_\rho(n \le 2L_N, T) + P_\rho(2L_N < n \le M, T)$$

Recall that $\rho = g(H)\psi$, where $g \in C_0^\infty([E_0 - \gamma, E_0 + \gamma])$ and

$$[E_0 - \gamma, E_0 + \gamma] \subset [E_0 - \nu, E_0 + \nu] \subset [-2 + \nu, 2 - \nu].$$

Therefore, all upper bounds of the previous section hold for ρ. Since $T \ge L_N^{1/\eta + \delta}$, the bound (3.7) of Lemma 3.3 yields

$$P_\rho(n \le 2L_N) \le CL_N^{\alpha_N - \delta} \le CL_N^{-\delta/2} \tag{4.14}$$

for N large enough. On the other hand, the bound (3.14) of Theorem 3.4 implies

$$P_\rho(2L_N \le n \le M, T) \le C\frac{M}{T} + C_A L_N^{-A} \tag{4.15}$$

for any $A > 0$. The bounds (4.14)-(4.15) yield

$$P_\rho(n \le M, T) \le C\frac{M}{T} + \beta_N, \quad \beta_N \to 0.$$

It is clear that taking $M = \tau T$ with $\tau > 0$ small enough, for N large enough we get $P_\rho(n \le M, T) \le \eta$ and thus $P_\psi(n \ge M, T) \le \alpha/4 > 0$.

In the case of ψ of the first kind the proof is more simple. One can directly estimate $P_\psi(n \le 2L_N, T)$ and $P_\psi(2L_N \le n \le M, T)$ as in (4.14), (4.15). Taking τ small enough, one obtains for T large enough that $P_\psi(n \le \tau T, T) \le \theta$. For the moments the bound directly follows.

To prove the second part of the Theorem, one shows the bound similar to (4.13):

$$P_\psi(n \ge M, T) \le ||\psi||^2 - \alpha/2 + P_\rho(n \ge M, T) + C(P_\rho(n \ge M, T))^{1/2}.$$

Taking $M = 2L_N$ and using the bound (3.12) of Theorem 3.4 for the state ρ, we get

$$P_\rho(n \ge 2L_N, T) \le CTL_N^{-1/\eta}.$$

One sees that for $L_N/4 \le T \le \tau L_N^{1/\eta}$ with τ small enough,

$$P_\rho(n \ge 2L_N, T) + C(P_\rho(n \ge 2L_N, T))^{1/2} \le \alpha/4.$$

Thus, $P_\psi(n \le 2L_N, T) \ge \alpha/4 > 0$. In the case of ψ of the first kind, the bound (4.10) follows directly from the bound (3.12) of Theorem 3.4. The proof is completed.

Corollary 4.5 *Let ψ be of the first kind.*
1. The equalities hold:

$$\beta_\psi^-(p) = \frac{p+1}{p+1/\eta}, \quad \beta_\psi^+(p) = 1.$$

2. The measure μ_ψ and the restriction of μ_{δ_1} to $(-2,2)$, have exact Hausdorff dimension η.

Proof. The first statement is proved using the bounds for the moments of Theorem 2.8 and Theorem 3.4 and considering $L_N \leq T \leq L_N^\alpha$ and $L_N^\alpha \leq T < L_{N+1}$ with suitable α as in the proof of Theorem 2.8. The bound $\dim_*(\mu_\psi) \geq \eta$ was proved by Jitomirskaya and Last (see also Corollary 3.2 for a more simple proof). On the other hand, one has the well known inequality $\beta_\psi^-(p) \geq \dim^*(\mu_\psi)$ for all $p > 0$ (which follows from the results of [L]). Since $\beta_\psi^-(p) = \frac{p+1}{p+1/\eta}$, letting $p \to 0$ we obtain the upper bound $\dim^*(\mu_\psi) \leq \eta$. Thus, μ_ψ has exact Hausdorff dimension η. Since it is true for any f of the first kind, it is true for the restriction of μ_{δ_1} to $(-2,2)$. The proof is completed.

References

[CM] J.M. Combes, G. Mantica, Fractal dimensions and quantum evolution associated with sparse potential Jacobi matrices, in *Long time behaviour of classical and quantum systems*, Ed. S. Graffi, A. Martinez. Series on concrete and appl. math. World Scientific, vol. 1, 107-123, 2001.

[DT] D. Damanik, S. Tcheremchantsev , Power-law bounds on transfer matrices and quantum dynamics in one dimension, Comm. Math. Phys. **236** (2003) , 513-534.

[DRJLS] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank-one perturbations, and localization, J. d'Analyse Math. **69** (1996), 153-200.

[G] A. Ya. Gordon, Deterministic potential with a pure point spectrum, Math. Notes **48** (1990), 1197-1203.

[GKT] F. Germinet, A. Kiselev, S. Tcheremchantsev, Transfer matrices and transport for 1D Schrödinger operators with singular spectrum, preprint.

[GK] F. Germinet, A. Klein, Decay of operator-valued kernels of functions of Schrodinger and other operators. Proc. Amer. Math. Soc. To appear.

[GSB] I. Guarneri, H. Schulz-Baldes, Lower bounds on wave-packet propagation by packing dimensions of spectral measure. Math. Phys. Elec. J. **5** pap. 1 (1999).

[JL] S. Jitomirskaya, Y. Last, Power-law subordinacy and singular spectra. I. Half-line operators. *Acta Math.* **183**, 171-189 (1999).

[K] D. Krutikov, Asymptotics of the Fourier transform of the spectral measure for Schrödinger operators with bounded and unbounded sparse potentials, J. of Physics **A 35** (2002), 6393-6417.

[KKL] R. Killip, A. Kiselev, Y. Last, Dynamical upper bounds on wavepacket spreading, Am. J. Math. **125 (5)** (2003), 1165-1198.

[KR] D. Krutikov, C. Remling, Schrödinger operators with sparse potentials: asymptotics of the Fourier transform of the spectral measure, Comm. Math. Phys. **223** (2001), 509-532.

[L] Y. Last, Quantum dynamics and decompositions of singular continuous spectrum, J. Funct. Anal. **142** (1996), 405-445.

[P] D.B. Pearson, Singular continuous measures in scattering theory, Comm. Math. Phys. **60** (1978), 13-36.

[S] B. Simon, Operators with singular continuous spectrum, VII. Examples with borderline time decay, Comm. Math. Phys. **176** (1996), 713-722.

[SSP] B. Simon, T. Spencer, Trace class perturbations and the absence of absolutely continuous spectrum, Comm. Math. Phys. **125** (1989), 113-126.

[SST] B. Simon, G. Stolz, Operators with singular continuous spectrum, V. Sparse potentials, Proc. Amer. Math. Soc. **124** (1996), 2073-2080.

[T] S. Tcheremchantsev, Mixed lower bounds for quantum transport, J. Funct. Anal. **197**, (2003), 247-282.

[Z] A. Zlatos, Sparse potentials with fractional Hausdorff dimension, preprint.