THERMODYNAMIC FORMALISM AND VARIATIONS OF THE HAUSDORFF DIMENSION OF QUADRATIC JULIA SETS

GUILLAUME HAVARD, MICHEL ZINSMEISTER

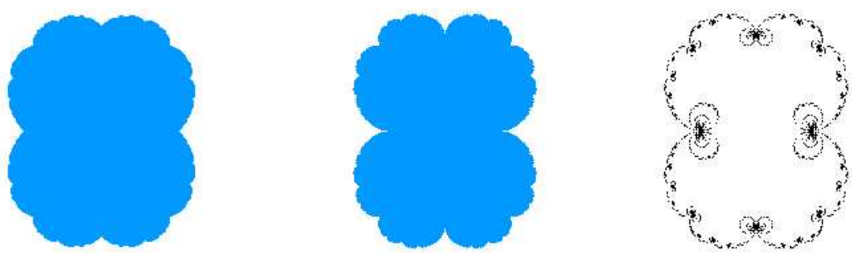
Abstract. Let $d(c)$ denotes the Hausdorff dimension of the Julia set of the polynomial $z \mapsto z^2 + c$. The function d restricted to $[0, +\infty)$ is real analytic in $[0, \frac{1}{4}) \cup (\frac{1}{4}, +\infty)$ ([Ru^2]), is left-continuous at $\frac{1}{4}$ ([Bo,Zi]) but not continuous ([Do,Se,Zi]). We prove that $c \mapsto d'(c)$ tends to $+\infty$ from the left at $\frac{1}{4}$ as $(\frac{1}{4} - c)^{d(\frac{1}{4}) - \frac{3}{2}}$. In particular the graph of d has a vertical tangent on the left at $\frac{1}{4}$, result which comforts the numerical experiments.

1. Introduction and statement of the result

By quadratic family we mean the family of polynomials

$$P_c : z \longmapsto z^2 + c, \qquad c \in \mathbb{C}.$$

As usual K_c and J_c will denote respectively the filled-in Julia set and the Julia set of P_c. We recall that $J_c = \partial K_c$ and that K_c is the set of complex numbers z for which the sequence obtained by induction by $z_0 = z$, $z_{n+1} = P_c(z_n)$ does not converge to ∞.



$c = 0,21$ $\qquad$ $c = 0,25$ $\qquad$ $c = 0,251$

Figure 1. Filled-in Julia sets for different values of the parameter c (*Pictures made with Arnaud Cheritat's program*).

We are interested in the function $c \mapsto d(c)$ where $d(c)$ denotes Hausdorff dimension of J_c. A well known result of Ruelle ([Ru^2]), based on thermodynamic formalism (Bowen Formula) asserts that this function is real-analytic on the complement of the Mandelbrot set $\mathbb{M} = \{c \in \mathbb{C} \mid K_c \text{ is connected}\}$ as well as on every hyperbolic component of $\overset{o}{\mathbb{M}}$.

So d is real-analytic on $]\frac{-3}{4}, \frac{1}{4}[\cup]\frac{1}{4}, \infty[$, since $]\frac{-3}{4}, \frac{1}{4}[$ is the trace on $\mathbb{R}$ of the main cardioïd of $\mathbb{M}$, the hyperbolic component of values of c such that P_c has an attracting

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fixed point, and $]\frac{1}{4},\infty[\subset\mathbb{C}\setminus\mathbb{M}$.
The point $\frac{1}{4}$ corresponds dynamically to a bifurcation; for $0<c<\frac{1}{4}$ P_c has two real fixed points, one attracting and one repelling and they both converge to $\frac{1}{2}$, the parabolic fixed point for $P_{\frac{1}{4}}$, as c goes to $\frac{1}{4}$. When c is greater than $\frac{1}{4}$, P_c possesses two repelling conjugated fixed points.
In [Bo,Zi] it has been proved that $d_{|\mathbb{R}}$ is continuous from the left at $\frac{1}{4}$. On the other hand, parabolic implosion produces a topological discontinuity which induces a jump of the function d; more precisely it is shown in [Do,Se,Zi] that

$$d(\frac{1}{4})<\liminf_{c\to\frac{1}{4}+0}d(c)\leq\limsup_{c\to\frac{1}{4}+0}d(c)<2.$$

Notice that it is not known if d has a limit from the right at $\frac{1}{4}$.
The purpose of the present article is to precise the speed of convergence of $d(c)$ towards $d(\frac{1}{4})$ from the left. From now on we write $\delta=d(\frac{1}{4})$.
Bowen's formula is relatively well adapted to computation and several numerical experiments have been performed [Ga], [Bo,Zi], [McMu]. They all show the same behaviour before $\frac{1}{4}$, namely a vertical tangent at this point. We prove this rigorously modulo the accepted bound $\delta<\frac{3}{2}$. Indeed all quoted numerical values give $\delta\simeq 1,07\pm 0,02$ which is very far from $\frac{3}{2}$. We do not feel like going into tedious computations to prove this, even if it can certainly be done.

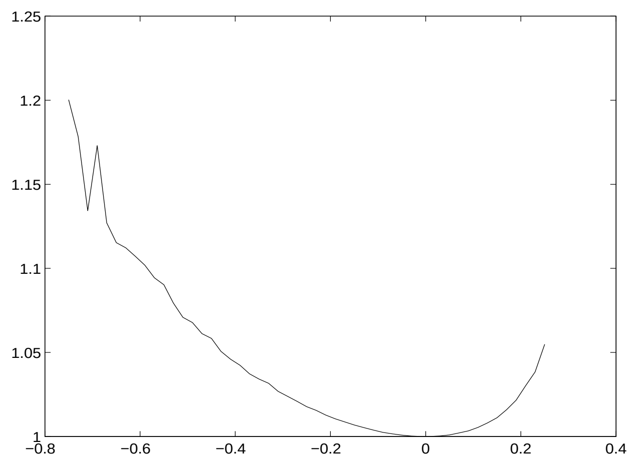

FIGURE 2. The graph of d obtained in [Bo,Zi].

Theorem 1.1. *There exist $c_0<\frac{1}{4}$ and $K>1$ such that*

$$\forall c\in[c_0,\frac{1}{4}[\qquad \frac{1}{K}\leq(\frac{1}{4}-c)^{(\frac{3}{2}-\delta)}d'(c)\leq K.$$

Remark 1.2. Notice that it implies

$$\delta-d(c)\sim(\frac{1}{4}-c)^{\delta-\frac{1}{2}}.$$

2. Bowen's formula and thermodynamic formalism.

The Julia set J_0 admits a natural Markov partition corresponding to dyadic development of the argument measured in "numbers of turns"; namely the partition into two half circles $[0, \frac{1}{2}[$ and $[\frac{1}{2}, 1[$ which will be denoted by **0** and **1** respectively. On the other hand the polynomials P_c, $0 \leq c \leq \frac{1}{4}$ admit a Böttcher coordinate, i.e. there exists $\phi_c : \mathbb{C} \setminus \overline{\mathbb{D}} = \mathbb{C} \setminus K_0 \longrightarrow \mathbb{C} \setminus K_c$ holomorphic bijective satisfying $\phi_c(z) = z + \cdots$ at infinity and conjugating P_0 into P_c. The sets J_c being for $0 \leq c \leq \frac{1}{4}$ Jordan curves, the ϕ_c's have homeomorphic extensions from J_0 to J_c, which allow to define a Markov partition for all J_c's. This Markov partition has of course refining at all orders n; more specifically J_c is for every n a union of 2^n cylinders γ two of these cylinders having closure intersecting at at most one point. We denote by $\Gamma_n(c)$ the set of all cylinders of order n for J_c.
If $c < \frac{1}{4}$ then P_c is hyperbolic and Kœbe distortion theorem applies to all cylinders : there exists a constant $c \geq 1$ independent of x or n (but not on c!) such that

$$\frac{1}{K}\text{diam } \gamma_n(z) \leq |(P_c^{\circ n})'(z)|^{-1} \leq K \text{diam } \gamma_n(z) \tag{2.1}$$

where $\gamma_n(z)$ is the cylinder of order n containing z. Writing, for a cylinder γ of order n, if f is a continuous function on $\overline{\gamma}$

$$f(\gamma) = \sup_{z \in \overline{\gamma}} f(z),$$

we define

$$\Pi_n(t) = \sum_{\gamma \in \Gamma_n(c)} |(P_c^n)'|^{-t}(\gamma) \tag{2.2}$$

Then we have (see [Bow], [Wa1], [Ru1]) that

$$\Pi(t) = \lim_{n \to +\infty} \frac{\log \Pi_n(t)}{n} \tag{2.3}$$

exists and define a convex function on $\mathbb{R}$ strictly decreasing from $+\infty$ to $-\infty$. Bowen's formula [Bow] asserts that $d(c)$ is the unique real t such that $\Pi(t) = 0$. Using Böttcher coordinate, (2.2) becomes

$$\Pi_n(t) = \sum_{\gamma \in \Gamma_n(0)} e^{S_n(-t \log |2\phi_c|)(\gamma)} \tag{2.4}$$

where

$$S_n(\varphi)(z) = \sum_{k=0}^{n-1} \varphi(z^{2^k}).$$

The function Π then appears to be the pressure of the continuous function $-t \log |2\phi_c|$. For a continuous function φ on J_0 the pressure $P(\varphi)$ is defined (see [Wa1]) by

$$P(\varphi) = \lim_{n \to +\infty} \frac{1}{n} \log \left(\sum_{\gamma \in \Gamma_n(0)} e^{S_n(\varphi)(\gamma)} \right). \tag{2.5}$$

Notice that ϕ_c (and thus $\log |2\phi_c|$) is a Hölder function on J_0 because $(c, z) \mapsto \phi_c(z)$ is a holomorphic motion and thus for all $c \in [0, \frac{1}{4}[$, ϕ_c has a quasiconformal extension to $\mathbb{C}$. Ruelle theorem follows from the fact that the map $\varphi \mapsto P(\varphi)$ is real-analytic on the spaces of Hölder functions.
The key in Ruelle's result is to interpret the pressure in terms of an operator called

transfer operator or Ruelle operator.
If φ is continuous on J_0, the associated operator is defined as

$$\forall x \in \partial\mathbb{D} \qquad \mathcal{L}_\varphi(f)(x^2) = e^{\varphi(x)}f(x) + e^{\varphi(-x)}f(-x). \tag{2.6}$$

Perron-Frobenius-Ruelle theorem [Ru[1]] asserts that if φ is Hölder then $\beta = e^{P(\varphi)}$ is a single eigenvalue of $\mathcal{L}_\varphi$ associated with an eigenvector $h > 0$ which is Hölder with same exponent.
On the other hand there exists a unique probability measure ω on the circle such that $\mathcal{L}_\varphi^*(\omega) = \beta\omega$. When $\varphi = \varphi_c = -d(c)\log|2\phi_c|$ this measure corresponds on J_c to a measure ω_c which is The Hausdorff measure (normalized to be of mass 1) in dimension $d(c)$ on J_c.
It can be shown that it is the unique conformal measure on J_c, i.e. it is the unique probability measure for which there exists $t > 0$ such that

$$\omega_c(P_c(A)) = \int_A |P_c'|^t d\omega_c \tag{2.7}$$

for every $A \subset J_c$ on which P_c is injective.
For every $0 < c < \frac{1}{4}$, if $\widetilde{h}_c$ denote the eigenvector of $\mathcal{L}_{\varphi_c}$ we define $h_c = \widetilde{h}_c \circ \phi_c^{-1}$. We can choose $\widetilde{h}_c$ so that $\mu_c = h_c\omega_c$ is a probability measure. It is then the unique P_c-invariant measure equivalent to ω_c. In thermodynamic terms, it is an equilibrium state, that is an invariant probability measure maximizing some functional called free energy. There is no other equilibrium state (no phase transition). For $c = \frac{1}{4}$, there are exactly two equilibrium states [Ha[1]]; there is a phase transition similar to the transition water-ice.
This long introduction on thermodynamic formalism had for purpose the following theorem which will allow us to start the computation.

Proposition 2.1. *Let $0 < c < \frac{1}{4}$, we have*

$$d'(c) = -\frac{d(c)}{\int_{\partial\mathbb{D}} \log|2\phi_c| d\widetilde{\mu}_c} \int_{\partial\mathbb{D}} \frac{\partial}{\partial c}\left(\log|2\phi_c|\right) d\widetilde{\mu}_c. \tag{2.8}$$

Proof. Let $\varphi_c = -d(c)\log|2\phi_c|$: because of Bowen's formula, we know that the pressure of φ_c is 0. So $\mathcal{L}_{\varphi_c}$ and $\mathcal{L}_{\varphi_c}^*$ have an eigenvector associated with the eigenvalue 1. Denote by $\widetilde{h}_c$ and $\widetilde{\omega}_c$ those eigenvectors. We choose $\widetilde{\omega}_c$ so that it is a probability measure and $\widetilde{h}_c$ so that $\widetilde{\mu}_c = \widetilde{h}_c\widetilde{\omega}_c$ is a probability measure.
We have

$$\mathcal{L}_{\varphi_c}(\widetilde{h}_c) = \widetilde{h}_c.$$

Differentiating, we obtain

$$\left(\frac{\partial}{\partial c}\mathcal{L}_{\varphi_c}\right)(\widetilde{h}_c) + \mathcal{L}_{\varphi_c}\left(\frac{\partial}{\partial c}\widetilde{h}_c\right) = \frac{\partial}{\partial c}\widetilde{h}_c.$$

Now we integrate with respect to $\widetilde{\omega}_c$. Using the fact that it is an eigenvector for $\mathcal{L}_{\varphi_c}^*$ we get

$$\int_{\partial\mathbb{D}} \left(\frac{\partial}{\partial c}\mathcal{L}_{\varphi_c}\right)(\widetilde{h}_c) d\widetilde{\omega}_c = 0. \tag{2.9}$$

Using (2.6) we compute

$$\forall f \in C(\partial\mathbb{D}) \qquad \left(\frac{\partial}{\partial c}\mathcal{L}_{\varphi_c}\right)(f) = \mathcal{L}_{\varphi_c}\left(\left[-d'(c)\log|2\phi_c| - d(c)\frac{\partial}{\partial c}\left(\log|2\phi_c|\right)\right] f\right),$$

and (2.9) leads to

$$\int_{\partial\mathbb{D}} \left(-d'(c)\log|2\phi_c| - d(c)\frac{\partial}{\partial c}(\log|2\phi_c|) \right) d\widetilde{\mu}_c = 0$$

which is what we wanted.

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The result of [Bo,Zi] implies that $d(c) \to \delta$ as $c \to \frac{1}{4}$. On the other hand the denominator in (2.8) is the Lyapounov exponent of P_c. In the next paragraph we prove the convergence as $c \to \frac{1}{4}$ of this Lyapounov exponent towards the Lyapounov exponent of $P_{\frac{1}{4}}$ which is positive.

Finally the numerator will be estimated in section 4.

3. Convergence of equilibrium states and Lyapounov exponents

3.1. **Approximate Fatou coordinates.** We will denote $z_c$and ζ_c respectively the repelling and attracting fixed point of P_c :

$$z_c = \frac{1+\sqrt{1-4c}}{2} \qquad \zeta_c = \frac{1-\sqrt{1-4c}}{2}.$$

If $\varepsilon_c = \frac{1}{4} - c$, the approximate Fatou coordinates is defined as

$$Z_c(z) = \frac{1}{2\sqrt{\varepsilon_c}} \log\left(\frac{z-z_c}{z-\zeta_c}\right) \tag{3.1}$$

$$\mathbb{C}\backslash[\zeta_c, z_c] \quad \overset{\frac{z-z_c}{z-\zeta_c}}{\longrightarrow} \quad \mathbb{C}\backslash]-\infty, 0[\quad \overset{\frac{1}{2\sqrt{\varepsilon_c}}\log(.)}{\longrightarrow} \quad \{\mathcal{R}e\, z \in \mathbb{R}\} \cap \{\mathcal{I}m\, z \in]-i\frac{\pi}{2\sqrt{\varepsilon_c}}, i\frac{\pi}{2\sqrt{\varepsilon_c}}[\}.$$

Notice that z being fixed we have

$$Z_c \underset{\varepsilon_c \to 0}{\sim} -\frac{1}{z-\frac{1}{2}} :$$

which is the approximate Fatou coordinate for $c = \frac{1}{4}$.

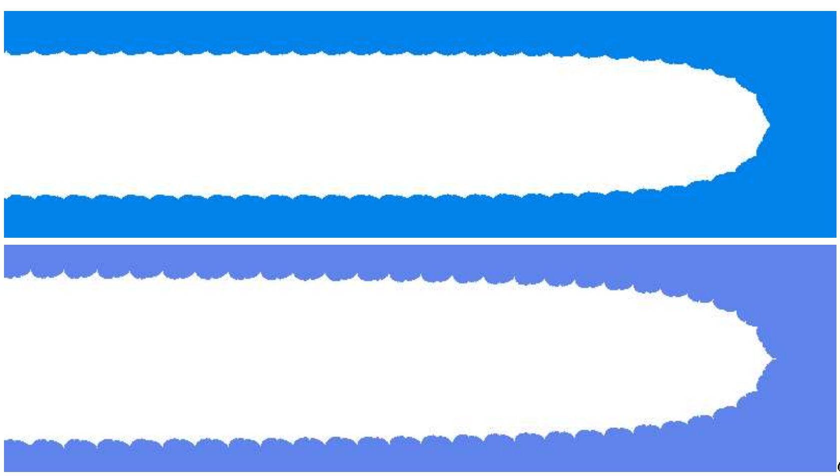

Figure 3. Filled-in Julia sets in Fatou coordinates
(*Pictures made with Arnaud Cheritat's program).*

Under this change of coordinates P_c is conjugated to F_c.
Let $E_c(Z) = e^{2\sqrt{\varepsilon_c}Z}$: one can compute using $c - \zeta_c = \zeta_c^2$ and $c - z_c = z_c^2$,

(3.2)

$$F_c(Z) = Z + \frac{1}{2\sqrt{\varepsilon_c}} \log(1 + 2\sqrt{\varepsilon_c}) + \frac{1}{2\sqrt{\varepsilon_c}} \log\left(1 + \frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right).$$

Next lemma gives an estimation of how far from translation by one is F_c when $|Z|$ is big.

Lemma 3.1. *There exist $K > 0$ and $\varepsilon_0 > 0$ such that for $\mathcal{R}e\ Z \leq -K$ and $\varepsilon_c \leq \varepsilon_0$,*

$$|F_c(Z) - (Z+1)| \leq \frac{1}{4}.$$

Proof. By formula (3.2) this boils down to proving that

$$\left|\frac{1}{2\sqrt{\varepsilon_c}} \log\left(1 + \frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right)\right| \leq \frac{1}{10},$$

for $\mathcal{R}e\ Z \leq -K$ and $\varepsilon_c \leq \varepsilon_0$.
Put $K \geq 100$, then if $\mathcal{R}e\ Z \leq -K$,

$$\begin{aligned} |(1+2\sqrt{\varepsilon_c}) - (1-4\varepsilon_c)E_c(Z)| &\geq (1+2\sqrt{\varepsilon_c}) - (1-4\varepsilon_c)e^{-100\sqrt{\varepsilon_c}} \\ &\geq 50\sqrt{\varepsilon_c} \qquad \text{if } \varepsilon_c \leq \varepsilon_0. \end{aligned}$$

For $\mathcal{R}e\ Z \leq -K < -100$ and $\varepsilon_c \leq \varepsilon_0$ this leads to

$$\left|\frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right| \leq \frac{4\varepsilon_c e^{-K\sqrt{\varepsilon_c}}}{50\sqrt{\varepsilon_c}}.$$

So we can conclude that

$$\left|\frac{1}{2\sqrt{\varepsilon_c}} \log\left(1 + \frac{4E_c(Z)\varepsilon_c}{(1-4\varepsilon_c)E_c(Z) - (1+2\sqrt{\varepsilon_c})}\right)\right| \leq \frac{1}{10}.$$

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We want now to estimate the size of the cylinders near z_c. For $n \geq 1$ let $C_n(c)$ be the set of points of J_c with external angle belonging to $[2^{-(n+1)}, 2^{-n}]$. In symbolic dynamics it corresponds to the cylinder

$$C_n(c) = \underbrace{\mathbf{1}\cdots\mathbf{1}}_{n}\ \mathbf{0}.$$

Lemma 3.2. *There exist $c_0 < \frac{1}{4}$ and $K > 0$ such that*

$$\forall c \in [c_0, \frac{1}{4}] \qquad \forall z_n \in C_n(c) \qquad \sum_{n \geq 0} |\mathcal{I}m\ z_n| \leq K.$$

Proof. The proof will be done by showing relations between $\mathcal{I}m\ z_n = y_n$ and $\mathcal{I}m\ Z_c(z_n) = \mathcal{I}m\ Z_n = Y_n$.
By construction,

$$z_n = \frac{1}{2} + \sqrt{\varepsilon_c} + 2\sqrt{\varepsilon_c}\frac{e^{2\sqrt{\varepsilon_c}Z_n}}{1 - e^{2\sqrt{\varepsilon_c}Z_n}}.$$

This leads to

$$y_n = \frac{\sqrt{\varepsilon_c}}{2} \frac{\sin(\sqrt{\varepsilon_c}Y_n)}{(\sinh(\sqrt{\varepsilon_c}X_n))^2 + (\sin(\sqrt{\varepsilon_c}Y_n))^2}.$$

This can also be written

$$y_n = \frac{\sin(\sqrt{\varepsilon_c} Y_n)}{2\sqrt{\varepsilon_c}} \frac{1}{\left(\frac{\sinh(\sqrt{\varepsilon_c} X_n)}{\sqrt{\varepsilon_c} X_n}\right)^2 X_n^2 + \left(\frac{\sin(\sqrt{\varepsilon_c} Y_n)}{\sqrt{\varepsilon_c} Y_n}\right)^2 Y_n^2}. \tag{3.3}$$

Using this relation and the fact that $\sqrt{\varepsilon_c} Y_n \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we easily get

$$|y_n| \leq \frac{\pi^2}{8} \frac{|Y_n|}{|Z_n|^2}. \tag{3.4}$$

With the lemma 3.1 this leads to

$$|y_n| \leq \lambda \frac{|Y_n|}{n^2} \leq \frac{\lambda}{n}. \tag{3.5}$$

On the other hand, $|Y_n| \leq |Y_0| + \sum_{k=0}^{n-1} |Y_{k+1} - Y_k|$, and

$$|Y_{k+1} - Y_k| = \frac{1}{2\sqrt{\varepsilon_c}} \left| \mathcal{I}m \ \log\left(\frac{z_{k+1} - z_c}{z_k - z_c} \frac{z_k - \zeta_c}{z_{k+1} - \zeta_c} \right) \right|.$$

Since $\frac{z_{k+1}-z_c}{z_k - z_c} = \frac{z_{k+1}-z_c}{P_c(z_{k+1}) - P_c(z_c)} = \frac{1}{z_{k+1}+z_c}$, we have

$$\begin{aligned} |Y_{k+1} - Y_k| &\leq \frac{1}{2\sqrt{\varepsilon_c}} \left| \mathcal{I}m \ \log\left(\frac{z_{k+1}+\zeta_c}{z_{k+1}+z_c} \right) \right| \\ &\leq \frac{1}{2\sqrt{\varepsilon_c}} \left| \mathcal{A}rg \ \left(1 - \frac{2\sqrt{\varepsilon_c}(x_{k+1}+z_c)}{|z_{k+1}+z_c|^2} + i \frac{2\sqrt{\varepsilon_c} y_k}{|z_{k+1}+z_c|^2} \right) \right|. \end{aligned}$$

We can conclude that there is $\lambda > 0$ such that

$$|Y_{k+1} - Y_k| \leq \lambda |y_k| \qquad \text{and also } |Y_n| \leq |Y_0| + \lambda \sum_{k=0}^{n-1} |y_k|. \tag{3.6}$$

We use (3.5) and obtain

$$|y_n| \leq \frac{\lambda |Y_0|}{n^2} + \frac{\lambda}{n^2} \sum_{k=1}^{n} \frac{1}{k} \leq \lambda \frac{\log n}{n^2}. \tag{3.7}$$

The proof of the lemma is finished since $\sum \frac{\log n}{n^2} < +\infty$.

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Using (3.6) and the lemma 3.2 one obtains

Corollary 3.3. *There exists $c_0 < \frac{1}{4}$ and $K > 0$ such that*

$$\forall c \in [c_0, \frac{1}{4}] \qquad Z_c(J_c) \subset \{\mathcal{I}m \ Z \in [-K, K]\}.$$

We are now in position to prove the main technical result of this section.

Lemma 3.4. *There exist $\alpha > 0$, $c_0 < \frac{1}{4}$ and $K > 0$ such that for all $c \in [c_0, \frac{1}{4}]$, all $n \in \mathbb{N}$ and all $Z_n = X_n + iY_n \in C_n(c)$ we have*

$$n\sqrt{\varepsilon_c} < \alpha \Longrightarrow \quad \frac{1}{Kn^2} \leq diam\ (C_n(c)) \leq \frac{K}{n^2}$$

$$n\sqrt{\varepsilon_c} \geq \alpha \Longrightarrow \quad \frac{\varepsilon_c e^{2X_n\sqrt{\varepsilon_c}}}{K} \leq diam\ (C_n(c)) \leq K\varepsilon_c e^{2X_n\sqrt{\varepsilon_c}}.$$

Remark 3.5. There is K such that inequalities

$$\frac{\varepsilon_c e^{-Kn\sqrt{\varepsilon_c}}}{K} \leq \text{diam } (C_n(c)) \leq \frac{K}{n^2}$$

are true for all integer n and all $c \in [c_0, \frac{1}{4}]$.

Proof. Let Z_n be a point in $Z_c(C_n(c))$, by lemma 3.1

$$|Z_n| \geq A + Bn.$$

Then by bounded distortion

$$\text{diam } Z_c(C_n(c)) \leq K|Z_n - Z_{n+1}| \leq K,$$

and

$$\frac{1}{K}|(Z_c^{-1})'(Z_n)| \leq \text{diam } C_n(c) \leq K|(Z_c^{-1})'(Z_n)|.$$

But

$$Z_c^{-1}(Z) = \frac{\zeta_c E_c(Z) - z_c}{E_c(Z) - 1} \qquad \text{and} \qquad (Z_c^{-1})'(Z) = \frac{4\varepsilon_c E_c(Z)}{(E_c(Z)-1)^2} = \frac{\varepsilon_c}{(\sinh(\sqrt{\varepsilon_c} Z))^2}.$$

So we have

$$\begin{aligned} |(Z_c^{-1})'(Z)| &= \frac{\varepsilon_c}{(\sinh(\sqrt{\varepsilon_c} X))^2 + (\sin(\sqrt{\varepsilon_c} Y))^2} \\ &= \frac{1}{\left(\frac{\sinh(\sqrt{\varepsilon_c} X)}{\sqrt{\varepsilon_c} X}\right)^2 X^2 + \left(\frac{\sin(\sqrt{\varepsilon_c} Y)}{\sqrt{\varepsilon_c} Y}\right)^2 Y^2}. \end{aligned}$$

But $\sqrt{\varepsilon_c} Y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ thus $\frac{4}{\pi^2} \leq \left(\frac{\sinh(\sqrt{\varepsilon_c} Y)}{\sqrt{\varepsilon_c} Y}\right)^2 \leq 1$. And if $-\alpha \leq \sqrt{\varepsilon_c} X \leq 2$ then $1 \leq \left(\frac{\sin(\sqrt{\varepsilon_c} X)}{\sqrt{\varepsilon_c} X}\right)^2 \leq \left(\frac{\sinh(\alpha)}{\alpha}\right)^2$. So if $Z \in Z_c(J_c)$ and if $-\alpha \leq \sqrt{\varepsilon_c} X \leq 2$ then

$$\left(\frac{\alpha}{\sinh(\alpha)}\right)^2 \frac{1}{|Z|^2} \leq |(Z_c^{-1})'(Z)| \leq \frac{\pi^2}{4} \frac{1}{|Z|^2}. \tag{3.8}$$

On the other hand if Z lies in $Z_c(J_c)$, corollary 3.3 implies $|\mathcal{I}m\ Z| \leq K$. If $\sqrt{\varepsilon_c} X \leq -\alpha$ and if ε_c is small enough then

$$\left(\frac{1 - e^{-2\alpha}}{2}\right)^2 e^{-2\sqrt{\varepsilon_c} X} \leq (\sinh(\sqrt{\varepsilon_c} X))^2 + (\sin(\sqrt{\varepsilon_c} Y))^2 \leq e^{-2\sqrt{\varepsilon_c} X}.$$

We can conclude that for $Z \in Z_c(J_c)$ and $\sqrt{\varepsilon_c} X \leq -\alpha$ we have

$$\varepsilon_c e^{2\sqrt{\varepsilon_c} X} \leq |(Z_c^{-1})'(Z)| \leq \frac{4}{(1 - e^{-2\alpha})^2} \varepsilon_c e^{2\sqrt{\varepsilon_c} X} \tag{3.9}$$

Applying estimations (3.8) and (3.9) with $Z = Z_n$ gives the result.

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3.2. **Equicontinuity of the ϕ_c's.** The function $c \mapsto \phi_c(1)$ converges to $\phi_{\frac{1}{4}}(1)$ when c grows to $\frac{1}{4}$. It follows that ϕ_c converges to $\phi_{\frac{1}{4}}$ on the dense set of points of the unit circle with dyadic argument. To show that ϕ_c converges uniformly on $\partial\mathbb{D}$ to $\phi_{\frac{1}{4}}$ it thus suffices to show that the family $\{\phi_c\}$ is equicontinuous.
In order to prove this it suffices to find a sequence ε_n decreasing to 0 such that

$$\forall \gamma_c \in \Gamma_n(c) \qquad \text{diam } \gamma_c < \varepsilon_n \tag{3.10}$$

Among the cylinders $\Gamma_n(c)$ we will distinguish the "good" cylinders as the ones for which the symbolic writing ends with **1 0** or **0 1**. These cylinders are the cylinders for which the diameter may be computed "uniformly" by Kœbe distortion theorem. More precisely there exists $K > 0$ independent of c such that if $\gamma_n(c)$ is a good cylinders of order n then

$$\frac{1}{K}\frac{1}{|(P_c^{\circ n})'(\gamma_n(c))|} \le \text{diam } \gamma_n(c) \le \frac{K}{|(P_c^{\circ n})'(\gamma_n(c))|} \tag{3.11}$$

Lemma 3.6. *There exist $c_0 \in]0, \frac{1}{4}[$, $n_0 > 0$ and a constant $K > 0$ such that if $\gamma_n(c)$ is a good cylinder of order $n \ge n_0$ and $c \in [c_0, \frac{1}{4}[$ then*

$$|(P_c^{\circ n})'(\gamma_n(c))| \ge \lambda n^2 \tag{3.12}$$

Proof. First of all let us consider a point in the cylinder **0 0 1**. Its image by P_c is in the cylinder **0 1** which is easily seen to be included in $E = \{|z| \ge \frac{1}{2}\} \cap \{\mathcal{R}e\ z \le 0\} \cap \{\mathcal{I}m\ z \ge 0\}$. So $z \in$ **0 0 1** can be written $\sqrt{\zeta - c}$ with $\zeta \in E$. It follows that

$$|z| = \sqrt{|\zeta - c|} \ge (\frac{1}{4} + c^2)^{\frac{1}{4}} = m_c.$$

Since m_c converges to $\frac{5^{\frac{1}{4}}}{2}$ as c converges to $\frac{1}{4}$ we conclude that there exists $c_0 \in [0, \frac{1}{4}[$ and $\varepsilon_0 > 0$ such that

$$\forall c \ge c_0 \qquad \forall z \in \mathbf{0\ 0\ 1} \qquad |2z| \ge 1 + \varepsilon_0.$$

By symmetry it is also true for **0 1 0**, **1 0 1** and **1 1 0**.
In other words if z belongs to a good cylinder of order 3 then $|2z| \ge 1 + \varepsilon_0$ if $c \in [c_0, \frac{1}{4}[$.
Let now $\gamma_n(c)$ be any good cylinder of order $n \ge 3$, $\gamma_n(c) = x_1 \cdots x_n$ ($x_i \in \{\mathbf{0}, \mathbf{1}\}$). Let p be the number of indices $i \le n-3$ such that $x_i x_{i+2} x_{i+2}$ is a good cylinder and let $k_1, \cdots, k_q$ be the lengths of the long (≥ 2) sequences of **0** or **1** that appear in $\gamma_n(c)$. Clearly $p \ge q$ and by the result of the preceding paragraph

$$|(P_c^{\circ n})'(M_n\ \mathbf{0})| \ge \max(\lambda n^2, 1)$$

where M_n denotes the cylinder of length n with n **1**. The above estimation implies

$$|(P_c^{\circ n})'(\gamma_n(c))| \ge (1+\varepsilon_0)^p (\max(\lambda k_1^2, 1)) \cdots (\max(\lambda k_q^2, 1)) \tag{3.13}$$

If $(1+\varepsilon_0)^p \ge n^2$ there is nothing to prove. If $(1+\varepsilon_0)^p \le n^2$ then $p \le K \log n$. On the other hand the largest k_j is greater than $\frac{k_1 + \cdots + k_q}{q}$. It follows that

$$|(P_c^{\circ n})'(\gamma_n(c))| \ge \lambda (1+\varepsilon_0)^p \left(\frac{k_1 + \cdots + k_q}{q}\right)^2.$$

Since $p \le \log n$, necessarily $k_1 + \cdots + k_q \ge \frac{n}{2}$ if n is greater than some n_0, and

$$|(P_c^{\circ n})'(\gamma_n(c))| \ge \frac{\lambda}{4}\frac{(1+\varepsilon_0)^p}{q^2} n^2 \ge K n^2,$$

since $\frac{(1+\varepsilon_0)^p}{q^2} \geq \frac{(1+\varepsilon_0)^p}{p^2} \geq K$.

∎

If $\gamma_n(c)$ is a good cylinder, the lemma 3.6 and (3.11) immediately imply

$$\text{diam } \gamma_n(c) \leq \frac{K}{n^2} \tag{3.14}$$

We recall that M_n is the cylinder of length n with n $\mathbf{1}$, and let $\overline{M_n}$ be the cylinder of length n with n $\mathbf{0}$.
M_n can be written as $\cup_{k\geq n} M_k\,\mathbf{0}$, thus

$$\text{diam } M_n \leq \sum_{k\geq n} \text{diam } M_k\,\mathbf{0}.$$

But $M_k\,\mathbf{0}$ ends with a good cylinder so that

$$\text{diam } M_n \leq \frac{K}{n}. \tag{3.15}$$

One can now conclude. Let X be any cylinder of order $n \geq n_0$. If it is good we already know that diam $X \leq \frac{K}{n^2}$. If not then we may assume that $X = YM_k$ (or $X = Y\overline{M_k}$) with $k \geq 2$ and Y good. If $k \geq \frac{n}{2}$ then diam $X \leq K$diam $M_k \leq \frac{K}{n}$. If $k < \frac{n}{2}$ then Y is a good cylinder of order $n - k \geq \frac{n}{2}$ and

$$\text{diam } X \leq \text{diam } Y \leq \frac{K}{n^2}.$$

In any case we have proven that there exists $K > 0$ such that

$$\forall n \geq n_0 \qquad \forall \gamma_n(c) \in \Gamma_n(c) \qquad \text{diam } \gamma_n(c) \leq \frac{K}{n} \tag{3.16}$$

3.3. **Weak convergence of equilibrium states.** As we already know that the ϕ_c's converge uniformly, and in order to prove that the Lyapounov exponents converge to a non zero limit, it suffices to prove that the equilibrium states weakly converge to a measure $\widetilde{\mu}$ such that

$$\int_{\partial\mathbb{D}} \log|2\phi_{\frac{1}{4}}|d\widetilde{\mu} > 0. \tag{3.17}$$

The first prove of this fact is in [Ha²]. The strategy there is as follow : we write $\widetilde{\mu}_c = \widetilde{h}_c\widetilde{\omega}_c$ and we use the fact that $\widetilde{\omega}_c$ converges weakly to $\widetilde{\omega}_{\frac{1}{4}}$, the normalized Hausdorff measure on $J_{\frac{1}{4}}$ ([Bo,Zi]). Unfortunately $(\widetilde{h}_c)_{c\in[c_0,\frac{1}{4}[}$ is not an equicontinuous family because $\widetilde{h}_c(\phi_c(1))$ tends to $+\infty$ as c grows to $\frac{1}{4}$. The difficulty is overcome by showing that $(\frac{1}{\widetilde{h}_c})_{c\in[c_0,\frac{1}{4}[}$ actually is equicontinuous. Every weak limit of $\widetilde{\mu}_{c_k}$ is then seen to be absolutely continuous with respect to $\widetilde{\omega}_{\frac{1}{4}}$. The author then uses Aaronson-Denker-Urbański theorem [Aa,De,Ur] saying that there is only one measure and that it satisfies (3.17).
In the purpose of being self contained we give another proof. It uses the same idea as in [Do,Se,Zi]; we use a renormalization of P_c which will appear to be uniformly hyperbolic.
We denote by C_0 the set of points in J_0 with external angles between $\frac{1}{3}$ and $\frac{2}{3}$ and by C_j, $j \geq 1$ the successive inverse images of C_0 that are in the upper-half plane. The associated Markov partition is then $\{C_j,\ j \geq 0\} \cup \{\overline{C_j},\ j \geq 1\}$. On each C_j or $\overline{C_j}$ we replace P_c by $P_c^{\circ j}$ sending C_j into C_0. The advantage is that this new

dynamical system is uniformly hyperbolic, the drawback being that the "alphabet" becomes infinite. But the results of Walters or Mauldin-Urbański [Wa^2], [Ma,Ur] show that usual features go through in this case. In particular for every $c \in [0, \frac{1}{4}]$ there exists a unique invariant measure μ_c^r (r for renormalized) for the new dynamics which is equivalent to the conformal measure ω_c, which is independent of the renormalization. Moreover, we claim that $(\mu_c^r)_c \in [0, \frac{1}{4}[$ weakly converges to $\mu_{\frac{1}{4}}^r$. Indeed $\mu_c^r = h_c^r \omega_c$ and in this case, because of thermodynamic formalism, the family $(h_c^r)_c \in [0, \frac{1}{4}]$ is equicontinuous.
It remains to elucidate the link between μ_c^r and μ_c. We first observe that, if continuous on $C(\partial\mathbb{D})$, the linear form

$$\varphi \longmapsto \int_{\partial\mathbb{D}} S_{N(x)}(\varphi)(x) d\widetilde{\mu_c^r}(x),$$

where $N(x) = j$ if $x \in C_j \cup \overline{C_j}$, defines a P_0-invariant measure. This form is indeed continuous since

$$\begin{aligned}
\int_{\partial\mathbb{D}} |S_{N(x),c}(\varphi)(x)| d\widetilde{\mu_c^r}(x) &\leq ||\varphi||_\infty \int_{\partial\mathbb{D}} N(x) d\widetilde{\mu_c^r}(x) \\
&\leq K||\varphi||_\infty \left(\sum_{j \geq 0} (1+j) \widetilde{\omega_c}(C_j) \right) \\
&\leq K||\varphi||_\infty \left(\sum_{j \geq 0} \frac{1}{j^{2d(c)-1}} \right) \\
&\leq K||\varphi||_\infty.
\end{aligned}$$

Since $d(c) \geq 1 + \alpha$ for some $\alpha > 0$ on $[\frac{1}{8}, \frac{1}{4}]$.
One can now conclude. If $\varphi \in C(\partial\mathbb{D})$ then

$$\begin{aligned}
\int_{\partial\mathbb{D}} \varphi d\widetilde{\mu_c} &= \int_{\partial\mathbb{D}} S_{N(x)}(\varphi)(x) d\widetilde{\mu_c^r}(x) \\
&\underset{c \to \frac{1}{4}}{\longrightarrow} \int_{\partial\mathbb{D}} S_{N(x)}(\varphi)(x) d\widetilde{\mu_c^r}(x) = \int_{\partial\mathbb{D}} \varphi d\widetilde{\mu}_{\frac{1}{4}}.
\end{aligned}$$

This approach also allows us to have estimates on $\mu_c(C_j)$ which will be used later. What precedes indeed also proves that μ_c is uniformly equivalent to ω_c on C_0 and

Lemma 3.7. *There exist $c_0 < \frac{1}{4}$, $\lambda > 1$ such that for all $c \in [c_0, \frac{1}{4}]$ and all $n \geq 1$,*

$$\frac{1}{\lambda} \varepsilon_c^{d(c)-\frac{1}{2}} e^{-\lambda n \sqrt{\varepsilon_c}} \leq \widetilde{\mu_c}(C_n(c)) \leq \frac{\lambda}{n^{2d(c)-1}}.$$

Notice that this approach gives a new proof in this particular case of the theorem in [Aa,De,Ur].

Proof. By construction,

$$\widetilde{\mu_c}(C_n(c)) = \sum_{k=n}^{+\infty} \widetilde{\mu_c^r}(C_n(c)).$$

But $\frac{1}{\lambda}\widetilde{\mu_c^r} \leq \widetilde{\omega_c} \leq \lambda \widetilde{\mu_c^r}$. On the other hand ω_c is normalized Hausdorff measure and bounded distortion theorem implies

$$\frac{1}{\lambda} \widetilde{\omega_c}(C_n(c)) \leq \text{diam } (C_n(c))^{d(c)} \leq \lambda \widetilde{\omega_c}(C_n(c)).$$

Using lemma 3.4 and remark 3.5 we get

$$\frac{\varepsilon_c^{d(c)}}{\lambda}\sum_{k=n}^{+\infty} e^{-\lambda k\sqrt{\varepsilon_c}} \leq \widetilde{\mu_c}(C_n(c)) \leq \lambda \sum_{k=n}^{+\infty}\frac{1}{k^{2d(c)}}.$$

And this leads to

$$\frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{1}{2}} e^{-\lambda n\sqrt{\varepsilon_c}} \leq \widetilde{\mu_c}(C_n(c)) \leq \frac{\lambda}{n^{2d(c)-1}}.$$

∎

4. Proof of the main theorem

First of all if we want to give an estimation of $d'(c)$ we need (see 2.8) an explicit expression of $\frac{\partial}{\partial c}(\log|2\phi_c|)$.
Differentiating the following functional equation

$$\phi_c(z^2) = \phi_c(z)^2 + c \tag{4.1}$$

we obtain, denoting $\dot{\phi_c}$ for $\frac{\partial}{\partial c}(\phi_c)$,

$$\dot{\phi_c}\,(z^2) = 2\phi_c(z)\,\dot{\phi_c}\,(z) + 1. \tag{4.2}$$

This can also be written

$$\dot{\phi_c}\,(z) = -\frac{1}{2\phi_c(z)} - \frac{1}{2\phi_c(z)}\,\dot{\phi_c}\,(z^2). \tag{4.3}$$

Reinjecting this formula one obtains

(4.4)

$$\dot{\phi_c}\,(z) = -\sum_{k=0}^{n-1}\frac{1}{2\phi_c(z)2\phi_c(z^2)\cdots 2\phi_c(z^{2^k})} - \frac{1}{2\phi_c(z)2\phi_c(z^2)\cdots 2\phi_c(z^{2^{n-1}})}\,\dot{\phi_c}\,(z^{2^n})$$

For $c<\frac{1}{4}$ the rest in (4.4) tends to 0 as n grows to infinity. So $\dot{\phi_c}$ can be written as an infinite sum

$$\dot{\phi_c}\,(z) = -\sum_{k=0}^{+\infty}\frac{1}{2\phi_c(z)2\phi_c(z^2)\cdots 2\phi_c(z^{2^k})}. \tag{4.5}$$

In particular we will use later the following functional equality

$$\dot{\phi_c}\,(z) = \frac{1}{(P_c^{\circ n})'(\phi_c(z))}\,\dot{\phi_c}\,(z^{2^n}) + \sum_{k=1}^{n}\frac{1}{(P_c^{\circ k})'(\phi_c(z))}. \tag{4.6}$$

Notice finally that

$$\frac{\partial}{\partial c}(\log|2\phi_c|) = \mathcal{R}e\,\left(\frac{\dot{\phi_c}}{\phi_c}\right). \tag{4.7}$$

In order to prove that the main contribution in $\int_{\partial\mathbb{D}}\mathcal{R}e\,\left(\frac{\dot{\phi_c}}{\phi_c}\right)d\widetilde{\mu}_c$ comes from the integral near z_c, we must show that as long as we keep far from z_c, the integral of the modulus is bounded by a constant independent of $c_0 < c < \frac{1}{4}$.
Let N be an integer which will be chosen later. We define $\mathcal{B}_N$ as $\mathbb{D}\setminus\mathcal{M}_N$, where

$\mathcal{M}_N = M_N \cup \overline{M_N}$. Because of bounded distortion theorem and our estimation in section 3.3 we know that there is $\lambda(N) > 0$ independent of $c_0 < c < \frac{1}{4}$ such that

$$\forall z \in \partial\mathbb{D} \qquad P_c^{\circ n}(\phi_c(z)) \in \mathcal{B}_N \Longrightarrow \frac{1}{|(P_c^{\circ n})'(\phi_c(z))|} \leq \frac{\lambda(N)}{n^2} \tag{4.8}$$

and also

$$\forall A \subset \mathcal{B}_N \qquad \widetilde{\mu}_c(A) \leq \lambda(N)\widetilde{\omega}_c(A) \tag{4.9}$$

We prove

Proposition 4.1. *For all $N \in \mathbb{N}$ there exists $\lambda(N) > 0$ such that*

$$\forall c \in [c_0, \frac{1}{4}[\qquad \int_{\mathcal{B}_N} |\frac{\dot{\phi}_c}{\phi_c}| d\widetilde{\mu}_c \leq \lambda(N).$$

Proof. Notice that $|\phi_c| \geq \frac{1}{2}$ so we just have to bound $\int_{\mathcal{B}_N} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c$.
Let U_n $(n \geq 1)$ denotes the set of points in $\partial\mathbb{D}$ that come back or arrived for the first time in $\mathcal{B}_N$ after exactly n iterations of $T : z \mapsto z^2$.
Let N_0 be an integer and set $A_{N_0,n} = T^{-N_0}(U_n) \cap \mathcal{B}_N$. The set $\{A_{N_0,n}\}_{n\geq 1}$ is a $\widetilde{\mu}_c$-partition of $\mathcal{B}_N$. So we can write

$$\int_{\mathcal{B}_N} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c = \sum_{n=1}^{+\infty} \int_{A_{N_0,n}} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c. \tag{4.10}$$

Now we study $\int_{A_{N_0,n}} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c$ and using (4.6) with $n + N_0$ we get

$$\int_{A_{N_0,n}} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c \leq \int_{A_{N_0,n}} \frac{1}{|(P_c^{\circ n+N_0})'(\phi_c)|} |\ \dot{\phi}_c \circ T^{\circ n+N_0} | d\widetilde{\mu}_c + (n + N_0)\widetilde{\mu}_c(A_{N_0,n})$$

But if $z \in A_{N_0,n}$ then $P_c^{\circ(n+N_0)}(\phi_c(z)) \in \mathcal{B}_N$ so using (4.8) we obtain

$$\int_{A_{N_0,n}} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c \leq \frac{\lambda(N)}{(n + N_0)^2} \int_{A_{N_0,n}} |\ \dot{\phi}_c \circ T^{\circ n+N_0} | d\widetilde{\mu}_c + (n + N_0)\widetilde{\mu}_c(A_{N_0,n}).$$

We recall that $\widetilde{\mu}_c$ is T-invariant, and also that $1\!\!1_{A_{N_0,n}} \leq 1\!\!1_{\mathcal{B}_N} \circ T^{\circ n+N_0}$ so we have

$$\int_{A_{N_0,n}} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c \leq \frac{\lambda(N)}{(n + N_0)^2} \int_{\mathcal{B}_N} |\ \dot{\phi}_c\ | d\widetilde{\mu}_c + (n + N_0)\widetilde{\mu}_c(A_{N_0,n}). \tag{4.11}$$

We need an estimation of $\widetilde{\mu}_c(A_{N_0,n})$. But $A_{N_0,n}$ can be written as

$$A_{N_0,n} = \bigcup_{i=2}^{N_0} \bigcup_{X \in \mathbb{B}_i} XC_{n+N+N_0-i},$$

where $\mathbb{B}_i$ is the set of good cylinders of length i (see section 1). We use (4.9) in order to obtain

$$\widetilde{\mu}_c(A_{N_0,n}) \leq \lambda(N) \sum_{i=2}^{N_0} \sum_{X \in \mathbb{B}_i} \widetilde{\omega}_c(XC_{n+N+N_0-i}). \tag{4.12}$$

We know that $\widetilde{\omega}_c$ is a kind of quasi-Bernoulli measure on good cylinders. More precisely there exists $\lambda > 1$ independent of $c \in [c_0, \frac{1}{4}]$ such that :

$$\frac{1}{\lambda}\widetilde{\omega_c}(X)\widetilde{\omega_c}(Y) \leq \widetilde{\omega_c}(XY) \leq \lambda\widetilde{\omega_c}(X)\widetilde{\omega_c}(Y)$$

for all good cylinders X. This leads to

$$\widetilde{\mu}_c(A_{N_0,n}) \leq \lambda(N) \sum_{i=2}^{N_0} \widetilde{\omega}_c(C_{n+N+N_0-i}) \sum_{X\in\mathbb{B}_i} \widetilde{\omega}_c(X).$$

But $\widetilde{\omega}_c(C_{n+N+N_0-i}) \leq \frac{\lambda}{(n+N+N_0-i)^{2d(c)}}$, and $\sum_{X\in\mathbb{B}_i} \widetilde{\omega}_c(X) \leq 1$. So we have shown that

$$\widetilde{\mu}_c(A_{N_0,n}) \leq \lambda(N) \frac{N_0}{(n+N)^{2d(c)}}.$$

This estimation in (4.11) gives

$$\int_{A_{N_0,n}} |\dot{\phi}_c| d\widetilde{\mu}_c \leq \frac{\lambda(N)}{(n+N_0)^2} \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c + \lambda(N) N_0 \frac{n+N_0}{(n+N)^{2d(c)}}.$$

And with (4.10) we get

$$\begin{aligned}\int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c &\leq \lambda(N) \left(\sum_{n=N_0}^{+\infty} \frac{1}{n^2} \right) \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c + N_0 \lambda(N) \sum_{n=N}^{+\infty} \frac{1}{n^{2d(c)-1}} + N_0^2 \lambda(N) \sum_{n=N}^{+\infty} \frac{1}{n^{2d(c)}} \\ &\leq \frac{\lambda(N)}{N_0} \int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c + \frac{\lambda(N) N_0}{N^{2d(c)-2}} + \frac{\lambda(N) N_0^2}{N^{2d(c)-1}}.\end{aligned}$$

For $\frac{\lambda(N)}{2} < N_0 \leq \frac{\lambda(N)}{2} + 1$ we get

$$\int_{\mathcal{B}_N} |\dot{\phi}_c| d\widetilde{\mu}_c \leq \frac{\lambda(N)}{N^{2d(c)-1}}.$$

∎

We are now in position to give a proof theorem 1.1.

Proof. By (2.8) and (4.7) we know that

$$d'(c) = \frac{d(c)}{\int_{\partial\mathbb{D}} \log |2\phi_c| d\widetilde{\mu}_c} \int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c. \tag{4.13}$$

Let N be an integer, we have

$$\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c \geq \int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c - \int_{\mathcal{B}_N} |\frac{\dot{\phi}_c}{\phi_c}| d\widetilde{\mu}_c.$$

So using proposition 4.1 we get

$$\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c \geq \int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c - \lambda(N).$$

We study $\int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c$. We know that

$$\int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c = \sum_{n=N}^{+\infty} \int_{C_n} \mathcal{R}e \left(\sum_{k=1}^{+\infty} \frac{1}{(P_c^k)'(\phi_c(z))} \frac{1}{\phi_c(z)} \right) d\widetilde{\mu}_c(z).$$

This leads to

$$\int_{\mathcal{M}_N} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c} \right) d\widetilde{\mu}_c \geq \sum_{n=N}^{+\infty} \sum_{k=1}^{n} \int_{C_n} \mathcal{R}e \left(\frac{1}{(P_c^k)'(\phi_c(z))} \right) d\widetilde{\mu}_c(z) -$$

$$\sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right|d\widetilde{\mu}_c(z).$$

But

$$\begin{aligned}\mathbb{1}_{C_n}(z)\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right| &= \mathbb{1}_{C_n}(z)\left|\frac{1}{(P_c^n)'(\phi_c(z))}\right|\left|\dot{\phi_c}\circ T^n(z)\right| \\ &\leq \frac{\lambda(N)}{n^2}\mathbb{1}_{\mathcal{B}_N}\circ T^{\circ n}(z)|\,\dot{\phi_c}\circ T^n(z)|.\end{aligned}$$

Using the T-invariance we get

$$\sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right|d\widetilde{\mu}_c(z)\leq\lambda(N)\int_{\mathcal{B}_N}|\,\dot{\phi_c}\,|d\widetilde{\mu}_c\sum_{n=N}^{+\infty}\frac{1}{n^2}.$$

With proposition 4.1 this leads to

$$\sum_{n=N}^{+\infty}\int_{C_n}\left|\sum_{k=n+1}^{+\infty}\frac{1}{(P_c^k)'(\phi_c(z))}\right|d\widetilde{\mu}_c(z)\leq\lambda(N). \tag{4.14}$$

Concerning $\int_{\mathcal{M}_N}\mathcal{R}e\,\left(-\frac{\dot{\phi_c}}{\phi_c}\right)d\widetilde{\mu}_c$ we have

$$\int_{\mathcal{M}_N}\mathcal{R}e\,\left(-\frac{\dot{\phi_c}}{\phi_c}\right)d\widetilde{\mu}_c\geq\sum_{n=N}^{+\infty}\sum_{k=1}^{n}\int_{C_n}\mathcal{R}e\,\left(\frac{1}{(P_c^k)'(\phi_c(z))}\right)d\widetilde{\mu}_c(z)-\lambda(N).$$

Now we fix N large enough so that $|\mathcal{A}rg\ (P_c^k)'(\phi_c(z))|\leq\frac{\pi}{6}$ for all $z\in C_n$ with $n\geq N$ and $k\leq n$. Thanks to lemma 3.2 we know that it is possible and we obtain

$$\forall n\geq N\qquad\forall k\geq n\qquad\forall z\in C_n\qquad\mathcal{R}e\,\left(\frac{1}{(P_c^k)'(\phi_c(z))}\right)\geq\frac{1}{\sqrt{3}}\frac{1}{|(P_c^k)'(\phi_c(z))|}.$$

So we have shown that

$$\int_{\partial\mathbb{D}}\mathcal{R}e\,\left(-\frac{\dot{\phi_c}}{\phi_c}\right)d\widetilde{\mu}_c\geq\frac{1}{\sqrt{3}}\sum_{n=N}^{+\infty}\sum_{k=1}^{n}\int_{C_n}\frac{1}{|(P_c^k)'(\phi_c(z))|}d\widetilde{\mu}_c(z)-\lambda(N).$$

The estimation

$$\forall z\in C_n\qquad|(P_c^k)'(\phi_c(z))|\leq\lambda\frac{n^2}{(n-k)^2}, \tag{4.15}$$

which is a consequence of lemma 3.4, is true for $n\sqrt{\varepsilon_c}<\alpha$, $(\varepsilon_c=\frac{1}{4}-c)$.
Using (4.15) we write

$$\int_{\partial\mathbb{D}}\mathcal{R}e\,\left(-\frac{\dot{\phi_c}}{\phi_c}\right)d\widetilde{\mu}_c\geq\lambda\sum_{n=0}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}\sum_{k=1}^{n}\widetilde{\mu_c}(C_n)\left(\frac{n-k}{n}\right)^2-\lambda(N).$$

We know (see lemma 3.7) that $\widetilde{\mu}_c(C_n)\geq\lambda\varepsilon_c^{d(c)-\frac{1}{2}}e^{-\lambda n\sqrt{\varepsilon_c}}$. This leads to

$$\begin{aligned}\int_{\partial\mathbb{D}}\mathcal{R}e\,\left(-\frac{\dot{\phi_c}}{\phi_c}\right)d\widetilde{\mu}_c &\geq \frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{1}{2}}\sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}\frac{e^{-\lambda n\sqrt{\varepsilon_c}}}{n^2}\sum_{k=1}^{n}k^2-\lambda(N) \\ &\geq \frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{1}{2}}\sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}ne^{-\lambda n\sqrt{\varepsilon_c}}-\lambda(N) \\ &\geq \frac{1}{\lambda}\varepsilon_c^{d(c)-\frac{3}{2}}-\lambda(N).\end{aligned}$$

Finally we obtain

$$\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c \geq \frac{\lambda}{(\frac{1}{4}-c)^{\frac{3}{2}-d(c)}} - \lambda(N).$$

In particular $d'(c) \to +\infty$ as $c \to \frac{1}{4}$ and so if $c \in [c_0, \frac{1}{4}]$, $d(c) < \delta$ and

$$\forall c \in [c_0, \frac{1}{4}] \qquad d'(c) \geq \frac{K}{(\frac{1}{4}-c)^{\frac{3}{2}-\delta}}.$$

In order to get the opposite inequality we note that

$$\left|\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c\right| \leq \int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c + \int_{\mathcal{B}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c. \tag{4.16}$$

Since the proposition 4.1 asserts that $|\int_{\mathcal{B}_N}|$ is bounded by a constant $\lambda(N)$ which only depends on N, we have

$$\left|\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c\right| \leq \int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c + \lambda(N). \tag{4.17}$$

For $\int_{\mathcal{M}_N}$ we write

$$\int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c \leq \sum_{n=N}^{+\infty} \sum_{k=1}^{n} \int_{C_n} \frac{1}{|(P_c^{\circ k})'(\phi_c)|} d\widetilde{\mu}_c + \sum_{n=N}^{+\infty} \int_{C_n} \left| \sum_{k=n+1}^{+\infty} \frac{1}{(P_c^{\circ k})'(\phi_c)} \right| d\widetilde{\mu}_c.$$

We use (4.14) and we obtain

$$\int_{\mathcal{M}_N} \left|\frac{\dot{\phi}_c}{\phi_c}\right| d\widetilde{\mu}_c \leq \sum_{n=N}^{+\infty} \sum_{k=1}^{n} \int_{C_n} \frac{1}{|(P_c^{\circ k})'(\phi_c)|} d\widetilde{\mu}_c + \lambda(N). \tag{4.18}$$

Note that bounded distortion theorem implies

$$\frac{1}{|(P_c^{\circ k})'(\phi_c(C_n))|} \sim \frac{|(P_c^{\circ(n-k)})'(\phi_c(C_{n-k}))|}{|(P_c^{\circ n})'(\phi_c(C_n))|} \leq \lambda \frac{\text{diam } C_n(c)}{\text{diam } C_{n-k}(c)}.$$

Using this inequality, the lemma 3.7 and inequality (4.18), (4.17) becomes

$$\left|\int_{\partial\mathbb{D}} \mathcal{R}e \left(-\frac{\dot{\phi}_c}{\phi_c}\right) d\widetilde{\mu}_c\right| \leq \lambda(N) + \lambda \sum_{n=N}^{+\infty} \frac{1}{n^{2d(c)-1}} \sum_{k=1}^{n} \frac{\text{diam } C_n(c)}{\text{diam } C_k(c)}.$$

Let $S_1 = \sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} \sum_{k=1}^{n}$, $S_2 = \sum_{\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \sum_{k=1}^{\frac{\alpha}{\sqrt{\varepsilon_c}}}$ and $S_3 = \sum_{\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \sum_{k=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{n}$, we study those three pieces showing that they are all bounded by $\lambda(\frac{1}{4}-c)^{d(c)-\frac{3}{2}}$.

For $n \leq \frac{\alpha}{\sqrt{\varepsilon_c}}$ and $k \leq n$, lemma 3.4 implies

$$\frac{\text{diam } C_n(c)}{\text{diam } C_k(c)} \leq \lambda \left(\frac{k}{n}\right)^2.$$

So we have

$$\begin{aligned} |S_1| &\leq \lambda \sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} \frac{1}{n^{2d(c)+1}} \sum_{k=1}^{n} k^2 \\ &\leq \lambda \sum_{n=N}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} \frac{1}{n^{2d(c)-2}} \\ &\leq \lambda \varepsilon_c^{d(c)-\frac{3}{2}}. \end{aligned}$$

For $n \geq \frac{\alpha}{\sqrt{\varepsilon_c}}$ and $1 \leq k \leq \frac{\alpha}{\sqrt{\varepsilon_c}}$ lemma 3.4 implies

$$\frac{\text{diam } C_n(c)}{\text{diam } C_k(c)} \leq \lambda k^2 \varepsilon_c e^{-\lambda n \sqrt{\varepsilon_c}}.$$

So

$$\begin{aligned} |S_2| &\leq \lambda \varepsilon_c \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{e^{-\lambda n \sqrt{\varepsilon_c}}}{n^{2d(c)-1}} \sum_{k=1}^{\frac{\alpha}{\sqrt{\varepsilon_c}}} k^2 \\ &\leq \lambda \frac{1}{\sqrt{\varepsilon_c}} \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \\ &\leq \lambda \varepsilon_c^{d(c)-\frac{3}{2}}. \end{aligned}$$

And for $n \geq \frac{\alpha}{\sqrt{\varepsilon_c}}$ and $\frac{\alpha}{\sqrt{\varepsilon_c}} \leq k \leq n$, we have

$$\frac{\text{diam } C_n(c)}{\text{diam } C_k(c)} \leq \lambda e^{2(X_n - X_k)\sqrt{\varepsilon_c}}$$

where X_n and X_k are real part respectively of $Z_n \in C_n(c)$ and $Z_k \in C_k(c)$. Bounded distortion theorem and lemma 3.1 imply

$$(X_n - X_k) \leq -\lambda(n-k)$$

and we obtain

$$\begin{aligned} |S_3| &\leq \lambda \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \sum_{k=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{n} e^{-\lambda(n-k)\sqrt{\varepsilon_c}} \\ &\leq \lambda \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \sum_{k=0}^{n-\frac{\alpha}{\sqrt{\varepsilon_c}}} e^{-\lambda k \sqrt{\varepsilon_c}} \\ &\leq \lambda \frac{1}{\sqrt{\varepsilon_c}} \sum_{n=\frac{\alpha}{\sqrt{\varepsilon_c}}}^{+\infty} \frac{1}{n^{2d(c)-1}} \\ &\leq \lambda \varepsilon_c^{d(c)-\frac{3}{2}} \end{aligned}.$$

We have shown

$$d'(c) \leq \lambda(\frac{1}{4} - c)^{d(c)-\frac{3}{2}}. \tag{4.19}$$

To finish the proof it remains to see why inequality (4.19) is actually equivalent with the same one with $d(c)$ replaced by δ. We would like to thank Pierrette Sentenac for showing us the following proof.

Putting $t = \frac{1}{4} - c$ and $y(t) = \delta - d(\frac{1}{4} - t)$, the question boils down to proving that $t \mapsto t^{-y(t)}$ is bounded at the origin. But we know that $y(t)$ converges to 0 at 0, so

$y(t) \leq \frac{1}{2}$ for t less than some $\eta > 0$. Integrating the analogue of (4.19) for y we then see that $|y(t)| \leq Kt^{\delta-1}$, $t < \eta$, and the result follows.

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