Synthesis of unilateral radiators

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Abstract. - A radiator is typically a parabolic mirror illuminated by an electromagnetic source, or a cylindrical transducer of resonant vibrations. Both of these devices are designed to radiate either a beam of parallel rays or a (focused) beam that converges to a point or a line. Consequently, at the worst, the radiation pattern is largely restricted to a *half space*, and at the best, to a cone or cylinder-like subspace of this half space. Such devices can therefore be termed *unilateral radiators*. This study is devoted to the synthesis of the sources that can give rise to such radiation, the underlying motivation being the removal of the material presence of the mirror or transducer casing from which waves coming from other boundaries could reflect or diffract.

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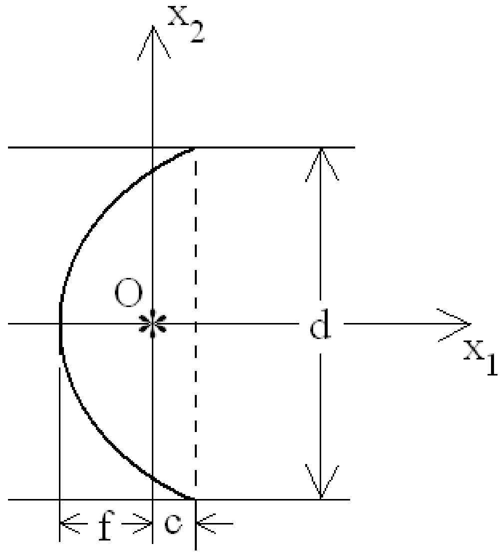

Figure 1: Cross section view of a parabolic cylinder radiator fed by a line source situated at the focal point O.

1 Introduction

Often one wants to predict the wavefield arising from sources radiating waves that are diffracted from various material objects (termed *obstacles*) in an otherwise homogeneous space. In real life, the so-called radiated wave is the result of a complex process involving conversion of an electrical signal into a (e.g., acoustic) wave which is formed in some manner within a so-called antenna (which sometimes reduces to a casing or radome). The radiated wave then propagates towards the obstacles present in the space and is diffracted by the latter. Some of the diffracted waves are redirected towards the antenna, and since the latter is also a material object, it can diffract these waves in its turn.

It is difficult to account for this multiple diffraction between the antenna and the objects, so the usual procedure is to assume that the radiator is only a collection of sources without material presence. In other words, one assumes that the antenna radiates waves, but does not diffract waves returning from the obstacles.

In order for this to be possible, at least in theory, one must reduce the radiator, which is composed of sources and an antenna (see for instance, fig. 1 for an example of a parabolic cylinder radiator), to a mere collection of sources. The latter, by definition, radiate outgoing waves, but do not diffract incoming waves. Moreover, in the present instance, one wants the radiator to radiate unilaterally, i.e., predominantly within a half space. Usually, this is not a simple task, as a point source radiates in all space, so that it is natural to think that a collection of point sources will also radiate in all space.

In the present investigation, we show that a particular combination of so-called single and double sources enables one to synthesize an essentially-unilateral radiator.

2 Diffraction of a wave by an impenetrable screen with a finite-size aperture

2.1 Preliminaries

This section is devoted to the study of the simplest of radiating devices: an impenetrable screen with an aperture.

Henceforth, we treat only 2D problems in which the support of the sources are cylinders parallel to the x_3 cartesian coordinate and the objects do not depend on this coordinate either.

The configuration, in the $x_1 - x_2$ plane, is depicted in fig. 2. The infinitely-thin impenetrable screen is denoted by the vertical dark black lines Γ^+ and Γ^- which are separated by a slit-like aperture Γ^0. This obstacle is illuminated by a wave radiated by sources contained within the domain Ω^i. The half space to the left of the screen (minus the support of the sources) is designated by Ω^- and the half space to the right of the screen by Ω^+. These half spaces can be thought of as being closed by the semi circles (of infinite radius) Γ_∞^- Γ_∞^+ respectively. The outward-pointing unit vector normal to Γ^-, Γ^0, Γ_∞^-, Γ_∞^+ is designated by $\boldsymbol{\nu}$, as in fig. 2.

The horizontal distance of the screen from the origin O is c and the width of the slit is d, with the x_1 axis being at equal vertical distance from the two extremities of the slit.

Henceforth, we shall be concerned with a scalar wave problem such as one that arises in acoustics (in fluids). The total scalar (pressure) wavefield will be represented by the function $u(\mathbf{x},\omega)$ in the space-frequency domain, with $\mathbf{x} = (x_1, x_2)$ the position vector in the cross-section plane and ω the angular frequency. The implicit temporal factor is $\exp(-i\omega t)$, with t the time variable. The wavefield does not depend on x_3 due to the fact that the sources and the screen+slit do not depend on this variable.

2.2 The field in the absence of the screen

In the absence of the screen, the total field is only the one radiated by the applied sources so that the problem reduces to determining $u(\mathbf{x},\omega)$ which satisfies

$$(\triangle + k^2)u(\mathbf{x},\omega) = -s(\mathbf{x},\omega) \quad , \quad \mathbf{x} \in \mathbb{R}^2 \ , \tag{1}$$

(wherein $s(\mathbf{x},\omega)$ is the source density, $k = \frac{\omega}{v}$ the wavenumber, and v the velocity in the fluid medium, assumed at present to occupy all of $\mathbb{R}^2$),

$$u(\mathbf{x},\omega) \sim \text{outgoing waves} \ ; \ \|\mathbf{x}\| \to \infty \quad , \quad \mathbf{x} \in \mathbb{R}^2 \ . \tag{2}$$

If $s(\mathbf{x},\omega) = \delta(\mathbf{x} - \mathbf{x}')$, wherein $\mathbf{x}' = (x_1', x_2')$ defines the position of the line source and $\delta(\ )$ is the Dirac delta distribution, then

$$u(\mathbf{x},\omega) := G(\mathbf{x},\mathbf{x}',\omega) = \frac{i}{4}H_0^{(1)}(k\|\mathbf{x}-\mathbf{x}'\|) \quad , \quad \mathbf{x} \in \mathbb{R}^2 \ , \tag{3}$$

with $H_0^{(1)}(\ )$ the zeroth order Hankel function of the first kind.

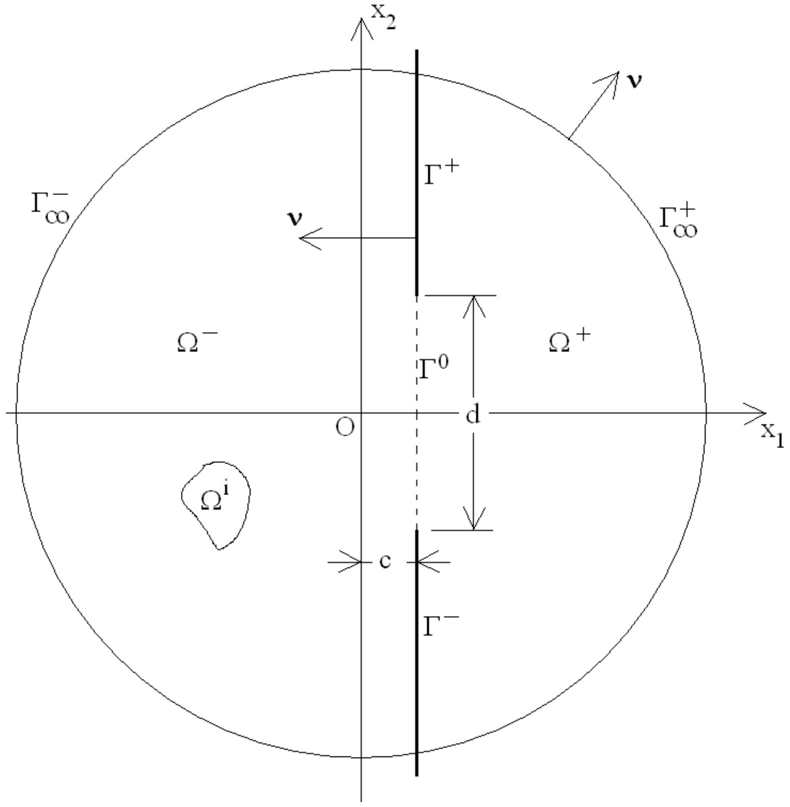

Figure 2: Cross section view of the configuration in which the wave radiated by a cylindrical source is diffracted by a slit aperture in an impenetrable screen.

For other source distributions, one finds

$$u(\mathbf{x},\omega)=\int_{\mathbb{R}^2} G(\mathbf{x},\mathbf{x}',\omega)s(\mathbf{x}',\omega)d\varpi(\mathbf{x}') \ \ ; \ \ \mathbf{x}\in\mathbb{R}^2 \ , \tag{4}$$

wherein $d\varpi(\mathbf{x}')$ is the infinitesimal element of area in the $x_1 - x_2$ plane.

2.3 The field in the presence of the screen

In the presence of the screen, the field of (4) no longer constitutes the total field. In fact, to distinguish it from the latter, we call it the incident field and designate it by u^i such that

$$u^i(\mathbf{x},\omega)=\int_{\Omega^i} G(\mathbf{x},\mathbf{x}',\omega)s(\mathbf{x}',\omega)d\varpi(\mathbf{x}') \ \ ; \ \ \mathbf{x}\in\mathbb{R}^2 \ , \tag{5}$$

wherein we have employed the fact that the support of the sources is $\Omega^i \subset \mathbb{R}^2$.

The total field is then $u(\mathbf{x},\omega) = u^i(\mathbf{x},\omega) + u^d(\mathbf{x},\omega)$ to the left of the screen, and $u(\mathbf{x},\omega) = u^d(\mathbf{x},\omega)$ to the right of the screen (other definitions of the diffracted field $u^d(\mathbf{x},\omega)$ are, of course, possible, but we choose this one). The diffracted field satisfies:

$$(\triangle + k^2)u^d(\mathbf{x},\omega) = 0 \ \ ; \ \ \mathbf{x}\in\Omega^- \cup \Omega^+ \cup \Gamma^0 \ , \tag{6}$$

$$u^d(\mathbf{x},\omega) \sim \text{outgoing waves} \ ; \ \|\mathbf{x}\| \to \infty \ \ , \ \ \mathbf{x}\in\Omega^- \ , \ \ \mathbf{x}\in \ \Omega^+ \ . \tag{7}$$

$$u(c^-,x_2,\omega) = u(c^+,x_2,\omega) \ \ ; \ \ x_2 \in [-d/2, d/2] \ , \tag{8}$$

$$u_{,2}(c^-,x_2,\omega) = u_{,2}(c^+,x_2,\omega) \ \ ; \ \ x_2 \in [-d/2, d/2] \ , \tag{9}$$

wherein $c^{\pm} := \lim_{\epsilon\to 0} c \pm \epsilon$. The notion of impenetrability of the screen implies either

$$u(c^-,x_2,\omega) = u(c^+,x_2,\omega) = 0 \ \ ; \ \ x_2 \in \mathbb{R} - [-d/2, d/2] \ , \tag{10}$$

(for a so-called acoustically-soft screen) or

$$u_{,2}(c^-,x_2,\omega) = u_{,2}(c^+,x_2,\omega) = 0 \ \ ; \ \ x_2 \in \mathbb{R} - [-d/2, d/2] \ , \tag{11}$$

(for a so-called acoustically-hard screen). This notion of impenetrability will be generalized further on.

We now apply Green's theorem to the Green's function G and u^d in Ω^+ so as to obtain

$$\mathcal{H}_{\Omega^+}(\mathbf{x})u^d(\mathbf{x},\omega) = \int_{\partial\Omega^+} \Big[G(\mathbf{x},\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' u^d(\mathbf{x}',\omega) - u^d(\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}',\omega)\Big]\, d\gamma(\mathbf{x}') \ . \tag{12}$$

wherein $d\gamma(\mathbf{x}')$ is the infinitesimal element of arc length, $\partial\Omega^+ := \Gamma^+_\infty + \Gamma^- + \Gamma^0 + \Gamma^+$, and

$$\mathcal{H}_{\Omega^+}(\mathbf{x}) = \begin{cases} 1 & ; \ \mathbf{x}\in\Omega^+ \\ 0 & ; \ \mathbf{x}\notin(\Omega^+ + \partial\Omega^+) \end{cases} \ . \tag{13}$$

On account of the radiation conditions satisfied by G and u^d, the integral along Γ_∞^+ vanishes and since $u^d = u$ on $\partial\Omega^+$ and in Ω^+, we have

$$\mathcal{H}_{\Omega^+}(\mathbf{x})u(\mathbf{x},\omega) = \int_{\Gamma^-+\Gamma^0+\Gamma^+} \left[G(\mathbf{x},\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' u(\mathbf{x}',\omega) - u(\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}',\omega)\right] d\gamma(\mathbf{x}') \ , \tag{14}$$

from which we extract the two results:

$$u(\mathbf{x},\omega) = \int_{\Gamma^-+\Gamma^0+\Gamma^+} \left[G(\mathbf{x},\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' u(\mathbf{x}',\omega) - u(\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}',\omega)\right] d\gamma(\mathbf{x}') \ ; \ \mathbf{x}\in\Omega^+ \ , \tag{15}$$

$$0 = \int_{\Gamma^-+\Gamma^0+\Gamma^+} \left[G(\mathbf{x},\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' u(\mathbf{x}',\omega) - u(\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}',\omega)\right] d\gamma(\mathbf{x}') \ ; \ \mathbf{x}\in\Omega^- \ . \tag{16}$$

Eq. (15) is a bona fide boundary integral representation of the field in the right hand half space. If the integral in (16) were equal to $u(\mathbf{x},\omega)$ in the left hand half space, then (16) would seem to imply that $u(\mathbf{x},\omega) = 0 \ ; \ \mathbf{x}\in\Omega^-$ which would mean that, by some miracle, we had devised a unilateral radiator by simply placing a screen with a slit in front of an arbitrary source distribution. This, of course, cannot be true, but it is a result that we are aiming for.

To proceed further in rigorous manner would require solving an integral equation, a procedure we wish to avoid. Thus, we adopt the approximation method employed since more than a hundred years by many researchers in the acoustics, electromagnetics, and optics communities (Baker and Copson, 1950). To begin, this involves the generalization of the notion of screen impenetrability, which, simply stated, requires that *the screen is simultaneously acoustically-hard and acoustically-soft*. The consequence of this (mathematically-impossible) requirement is that

$$\mathcal{H}_{\Omega^+}(\mathbf{x})u(\mathbf{x},\omega) \approx \int_{\Gamma^0} \left[G(\mathbf{x},\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' u(\mathbf{x}',\omega) - u(\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}',\omega)\right] d\gamma(\mathbf{x}') \ , \tag{17}$$

wherein we have replaced the previous $=$ sign by the $\approx$ sign to stress the fact that we are violating a mathematical constraint (which is generally the case when some approximate boundary conditions are invoked).

A further aspect of the procedure adopted by the above-mentioned researchers is the introduction of the so-called *Kirchhoff approximation* of the field in the slit. This ansatz (similar in some respects to the Born approximation in other contexts) is expressed by

$$u(\mathbf{x},\omega) \approx u^i(\mathbf{x},\omega) \quad , \quad \boldsymbol{\nu}(\mathbf{x})\cdot\nabla u(\mathbf{x},\omega) \approx \boldsymbol{\nu}(\mathbf{x})\cdot\nabla u^i(\mathbf{x},\omega) \quad ; \quad \mathbf{x}\in\Gamma^0 \ , \tag{18}$$

which has been shown to be reasonable as soon as the width of the slit exceeds several wavelengths (Facq and Robin, 1972; Colombeau et al., 1973) and results in the even-stronger approximation

$$\mathcal{H}_{\Omega^+}(\mathbf{x})u(\mathbf{x},\omega) \approx \int_{\Gamma^0} \left[G(\mathbf{x},\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' u^i(\mathbf{x}',\omega) - u^i(\mathbf{x}',\omega)\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' G(\mathbf{x},\mathbf{x}',\omega)\right] d\gamma(\mathbf{x}') \ , \tag{19}$$

or, on account of the fact that $\boldsymbol{\nu}(\mathbf{x}')\cdot\nabla' = -\frac{\partial}{\partial x_1'}$ and $d\gamma(\mathbf{x}') = dx_2'$ along Γ^0:

$$\mathcal{H}_{\Omega^+}(\mathbf{x})u(\mathbf{x},\omega) \approx -\int_{-d/2}^{d/2} \left[G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega) - u^i(c,x_2',\omega)G_{,1'}(\mathbf{x},c,x_2',\omega)\right] dx_2' \ . \tag{20}$$

Once again, we extract two results from this expression:

$$u(\mathbf{x},\omega) \approx -\int_{-d/2}^{d/2} \left[G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega) - u^i(c,x_2',\omega)G_{,1'}(\mathbf{x},c,x_2',\omega)\right] dx_2' \;\; ; \;\; \mathbf{x} \in \Omega^+ \; , \quad (21)$$

$$0 \approx -\int_{-d/2}^{d/2} \left[G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega) - u^i(c,x_2',\omega)G_{,1'}(\mathbf{x},c,x_2',\omega)\right] dx_2' \;\; ; \;\; \mathbf{x} \in \Omega^- \; . \quad (22)$$

The previous remarks apply even more forcefully here.

3 A radiation problem with a particular type and distribution of sources

The previous analysis showed that employing the generalized impenetrability conditions and the Kirchoff approximation enables one to obtain an approximate solution for the field in the right hand half space and what appears like a null field in the left hand half space around a slit in an impenetrable screen. Since this result is approximate, it does not satisfy the governing equations of the original problem. As concerns the space-frequency wave equation expressed in (6), this fact is easy to demonstrate by simply taking the spatial derivatives of (20) whereupon one finds that the right hand side of the equation in (6) is no longer nil, i.e.,

$$(\triangle + k^2)u(\mathbf{x},\omega) = -S(\mathbf{x},\omega) \neq 0 \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; . \quad (23)$$

Rather than do these operations, we will give the result for S and then go the other way around by showing that u for this source distribution has the desired properties, notably of producing a null field in Ω^-.

The source density ansatz is

$$S(\mathbf{x},\omega) = -2\delta(x_1 - c)[H(x_2 - d/2) + H(-x_2 - d/2)]u^i_{,1}(\mathbf{x},\omega) - \\ u^i(\mathbf{x},\omega)\delta_{,1}(x_1 - c)[H(x_2 - d/2) + H(-x_2 - d/2)] \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; , \quad (24)$$

(wherein H is the Heaviside function defined by $H(\zeta > 0) = 1$ and $H\zeta < 0) = 0$) which will be recognized to be a distribution of single and double sources on a strip. The latter is none other than the slit of the previous problem.

To ensure uniqueness of this radiation problem, we must specify that the wave radiated by this distribution of sources is outgoing far from the support of the sources.

As previously, we can show that the radiated field is of the form

$$u(\mathbf{x},\omega) = \int_{\mathbb{R}^2} G(\mathbf{x},\mathbf{x}',\omega)S(\mathbf{x}',\omega)d\varpi(\mathbf{x}') \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; , \quad (25)$$

or

$$u(\mathbf{x},\omega) = \int_{-\infty}^{\infty} dx_2' \int_{-\infty}^{\infty} dx_1' G(\mathbf{x},x_1',x_2',\omega)S(x_1',x_2',\omega) \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; . \quad (26)$$

The introduction of (24) therein gives

$$u(\mathbf{x},\omega)=\int_{-\infty}^{\infty}dx_2'[H(x_2'-d/2)+H(-x_2'-d/2)]\times$$
$$\int_{-\infty}^{\infty}dx_1'G(\mathbf{x},x_1',x_2',\omega)[-2\delta(x_1'-c)u^i_{,1'}(\mathbf{x}',\omega)-u^i(\mathbf{x}',\omega)\delta_{,1'}(x_1'-c)] \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; , \quad (27)$$

or

$$u(\mathbf{x},\omega)=\int_{-d/2}^{d/2}dx_2'\int_{-\infty}^{\infty}dx_1'G(\mathbf{x},x_1',x_2',\omega)[-2\delta(x_1'-c)u^i_{,1'}(\mathbf{x}',\omega)-u^i(\mathbf{x}',\omega)\delta_{,1'}(x_1'-c)] \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; . \quad (28)$$

But

$$\int_{-\infty}^{\infty}dx_1'G(\mathbf{x},x_1',x_2',\omega)\delta(x_1'-c)u^i_{,1'}(\mathbf{x}',\omega)=G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega) \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; . \quad (29)$$

Furthermore:

$$\int_{-\infty}^{\infty}dx_1'G(\mathbf{x},x_1',x_2',\omega)u^i(\mathbf{x}',\omega)\delta_{,1'}(x_1'-c)=G(\mathbf{x},x_1',x_2',\omega)\delta(x_1'-c)u^i(x_1',x_2',\omega)\Big|_{x_1'=-\infty}^{\infty}-$$
$$\int_{-\infty}^{\infty}[G(\mathbf{x},x_1',x_2',\omega)u^i(\mathbf{x}',\omega)]_{,1'}\delta(x_1'-c)dx_1' \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; . \quad (30)$$

But $\delta(\pm\infty-c)=0$ and $G(\mathbf{x},\pm\infty,x_2',\omega)u^i(\pm\infty,x_2',\omega)$ is bounded, so that

$$\int_{-\infty}^{\infty}dx_1'G(\mathbf{x},x_1',x_2',\omega)u^i(\mathbf{x}',\omega)\delta_{,1'}(x_1'-c)=$$
$$-\int_{-\infty}^{\infty}[G(\mathbf{x},x_1',x_2',\omega)u^i_{,1'}(\mathbf{x}',\omega)+G_{,1'}(\mathbf{x},x_1',x_2',\omega)u^i(\mathbf{x}',\omega)]\delta(x_1'-c)dx_1'=$$
$$-G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega)-G_{,1'}(\mathbf{x},c,x_2',\omega)u^i(c,x_2',\omega) \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; . \quad (31)$$

The introduction of (29) and (31) into (28) then gives

$$u(\mathbf{x},\omega)=\int_{-d/2}^{d/2}dx_2'\Big[-2G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega)+$$
$$G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega)+G_{,1'}(\mathbf{x},c,x_2',\omega)u^i(c,x_2',\omega)\Big] \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; , \quad (32)$$

or

$$u(\mathbf{x},\omega)=-\int_{-d/2}^{d/2}\Big[G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega)-G_{,1'}(\mathbf{x},c,x_2',\omega)u^i(c,x_2',\omega)\Big]dx_2' \;\; ; \;\; \mathbf{x}\in\mathbb{R}^2 \; , \quad (33)$$

from which we extract the two (*rigorous*) results:

$$u(\mathbf{x},\omega)=-\int_{-d/2}^{d/2}\Big[G(\mathbf{x},c,x_2',\omega)u^i_{,1'}(c,x_2',\omega)-G_{,1'}(\mathbf{x},c,x_2',\omega)u^i(c,x_2',\omega)\Big]dx_2' \;\; ; \;\; \mathbf{x}\in\Omega^+ \; , \quad (34)$$

$$u(\mathbf{x},\omega) = -\int_{-d/2}^{d/2} \Big[G(\mathbf{x},c,x_2',\omega) u^i_{,1'}(c,x_2',\omega) - G_{,1'}(\mathbf{x},c,x_2',\omega) u^i(c,x_2',\omega) \Big] dx_2' \;\; ; \;\; \mathbf{x} \in \Omega^- \; . \quad (35)$$

We showed previously (see (22)) that

$$-\int_{-d/2}^{d/2} \left[G(\mathbf{x},c,x_2',\omega) u^i_{,1'}(c,x_2',\omega) - u^i(c,x_2',\omega) G_{,1'}(\mathbf{x},c,x_2',\omega) \right] dx_2' \approx 0 \;\; ; \;\; \mathbf{x} \in \Omega^- \; . \quad (36)$$

so that we can conclude that

$$u(\mathbf{x},\omega) \approx 0 \;\; ; \;\; \mathbf{x} \in \Omega^- \; , \quad (37)$$

where it is understood that the field $u(\mathbf{x},\omega)$ in (34) and (37) is the field radiated by the distribution of applied sources given in (24). *This means that the source function (24) radiates in the sought-for unilateral manner.* Of course, this is only an approximation, but we shall discover further on that it is a good approximation. Thus, we have shown how to synthesize the sources that give rise to unilateral radiator. Replacing the physical radiator (e.g., parabolic antenna, cylindrical transducer) by this source distribution enables the elimination of undesirable multiple diffraction effects between the radiator and obstacles.

4 Radiation from a parabolic cylinder radiator

The cross section view of the parabolic cylinder radiator is given in fig. 1. An infinitely-thin impenetrable curved sheet (Γ) reflector, in the form of a portion of a parabola, is illuminated by a line source located at the origin O. The equation of the sheet is

$$x_1 = F(x_2) := -f + \frac{x_2^2}{4f} \;\; ; \;\; x_2 \in [-d/2, d/2] \; , \quad (38)$$

wherein f is the focal length and d the width of the reflector aperture. The left hand extremity of the reflector is at $x_1 = -f$ and the right hand extremity at $x_1 = c$. Thus, the slit aperture $\mathcal{A}$ is located at $x_1 = c = -f + \frac{d^2}{16f}$.

The problem is once again to determine the total field $U = U^i + U^d$ such that

$$(\triangle + k^2) U^i(\mathbf{x},\omega) = -\delta(\mathbf{x}) \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; , \quad (39)$$

$$(\triangle + k^2) U^d(\mathbf{x},\omega) = 0 \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \cap \Gamma \; , \quad (40)$$

$$U^d(\mathbf{x},\omega) \sim \text{outgoing waves} \; ; \; \|\mathbf{x}\| \to \infty \;\; , \;\; \mathbf{x} \in \mathbb{R}^2 \; . \quad (41)$$

$$U(\mathbf{x},\omega) = 0 \;\; ; \;\; \mathbf{x} \in \Gamma \; . \quad (42)$$

Note that the Dirichlet boundary condition (42) could just as well be replaced by the Neumann condition

$$\boldsymbol{\nu} \cdot \nabla U(\mathbf{x},\omega) = 0 \;\; ; \;\; \mathbf{x} \in \Gamma \; , \quad (43)$$

or, for that matter, by an impedance boundary condition, since, in the high frequency situation of interest herein, the precise nature of the boundary condition is not important.

There exists a variety of exact and approximate methods for predicting the radiation produced by this device (Tanteri and Wirgin, 1975); we shall choose a so-called *aperture method* (Silver, 1949) which is approximate in nature and based on the following two hypotheses:

1. geometrical optics (or acoustics) governs the propagation of the field from the source to the reflector and from the latter to the aperture,

2. the Huyghens-Fresnel principle (equivalent to the result of the analysis in sect. 2) governs the propagation of the field from the aperture of the mirror to points within Ω^+ (i.e., the half space to the right of the aperture).

A necessary (although not necessarily-sufficient) condition for the validity of the first hypothesis is that $kf >> 1$ (Tanteri and Wirgin, 1975). If it is recalled that the field radiated by the line source located at $\mathbf{x}' = 0$ is $\frac{i}{4}H_0^{(1)}(k\|\mathbf{x}\|)$ and that the asymptotic (large-argument) form of the Hankel function is (Abramowitz and Stegun, 1965)

$$H_n^{(1)}(\zeta) \sim \left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}} e^{i\left(\zeta-\frac{n\pi}{2}-\frac{\pi}{4}\right)} \quad ; \quad \zeta \to \infty \quad , \quad n = 0, 1, 2, ... \ , \tag{44}$$

then the incident field at the reflector can be replaced by the asymptotic expression

$$U^i(\mathbf{x}, \omega) \sim \tilde{U}^i(\mathbf{x}, \omega) = \frac{i\xi}{4}\frac{e^{ikfE(x_2)}}{[kfE(x_2)]^{\frac{1}{2}}} \quad ; \quad \mathbf{x} \in \Gamma \ , \tag{45}$$

wherein $\xi := \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{4}}$ and $E(x_2) := 1 + \frac{x_2^2}{4f^2}$. The standard geometrical optics (acoustics) ray analysis (Sletten, 1969; Holt, 1969; Hansen, 1964; Silver, 1949) then shows that

$$U^d(\mathbf{x}, \omega) \sim \tilde{U}^d(\mathbf{x}, \omega) = \begin{cases} A(x_2)\exp[i(kx_1 + \psi)] & ; \ \mathbf{x} \in \tilde{\Omega} \\ 0 & ; \ \mathbf{x} \in (\mathbb{R}^2 - \Gamma - \tilde{\Omega}) \end{cases} \ , \tag{46}$$

wherein

$$\tilde{\Omega} = \{F(x_2) < x_1 < \infty \quad ; \quad \forall x_2 \in [-d/2, d/2]\} \ , \tag{47}$$

$$A(x_2) = \|\tilde{U}^i(F(x_2), x_2, \omega)\| = \frac{1}{4}\sqrt{\frac{2}{\pi kf}}\frac{1}{\sqrt{E(x_2)}} \ , \tag{48}$$

$$\psi = \arg[-\tilde{U}^i(F(x_2), x_2, \omega)] + kf = -\frac{3\pi}{4} + 2kf \ . \tag{49}$$

Thus, the first hypothesis of this aperture method, is equivalent to the statement that the phase $kc + \psi$ is a constant, and the amplitude $A(x_2)$ is a tapered function of x_2, in the aperture $\mathcal{A}$ of the parabolic reflector. Note that for a plane body wave normally-incident on the screen+slit system, both the amplitude and phase are constant in $\mathcal{A}$.

Naturally, (46) does not account for diffraction effects (assumed to be produced only in the half space to the right of the aperture) due to the encounter of u^i with the edges of the mirror. The contribution of these effects to points within Ω^+ is introduced by means of the second hypothesis whose mathematical expression is given either by the first or second Rayleigh-Sommerfeld formulae (Siver, 1949), or, as herein, by (21). The choice of one or another of these formulae constitutes the essential difference between the three types of aperture methods. Experience shows that the two Rayleigh-Sommerfled formulae yield similar results in both the near and far field regions of Ω^+ when $kf > 2.5\pi$. We prefer the third aperture method, embodied in (21), because it has the unique property of leading to unilateral radiation if abstraction is made of the reflector antenna once the field it generates attains the aperture.

The question that arises is what should be taken for u^i and/or $u^i_{,1'}$ in (21). In other words, should one take $u^i \approx \tilde{U}$ and $u^i_{,1'} \approx \tilde{U}_{,1'}$, or, on the contrary, $u^i \approx \tilde{U}^d$ and $u^i_{,1'} \approx \tilde{U}^d_{,1'}$? Due to the fact that, in practice, the source of reflector antennas is generally masked so as not to radiate in directions other than towards the reflector, it seems reasonable to choose the second solution.

Thus, we take

$$u(\mathbf{x}, \omega) \approx -\int_{-d/2}^{d/2} \Big[G(\mathbf{x}, c, x_2', \omega) \tilde{U}^d_{,1'}(c, x_2', \omega) - \tilde{U}^d(c, x_2', \omega) G_{,1'}(\mathbf{x}, c, x_2', \omega) \Big] \, dx_2' \quad ; \quad \mathbf{x} \in \Omega^+ \ , \tag{50}$$

wherein

$$\tilde{U}^d(c, x_2', \omega) = A(x_2') \exp[i(kc + \psi)] \quad ; \quad x_2 \in [-d/2, d/2] \ , \tag{51}$$

$$\tilde{U}^d_{,1'}(c, x_2', \omega) = ikA(x_2') \exp[i(kc + \psi)] \quad ; \quad x_2 \in [-d/2, d/2] \ . \tag{52}$$

Although we are now in a position to compute the field radiated into Ω^+, we shall not accomplish this task since what we are really interested in is the prediction of the radiation from the aperture sources that synthesize the action of a parabolic cylinder unilateral radiator.

5 The quasi-unilateral radiation from the synthesized sources of a parabolic cylinder antenna

By taking into the final results of sects. 3 and 4, we find that the field radiated by the sources which synthesize the action of a parabolic cylinder antenna is

$$u(\mathbf{x}, \omega) \approx -\int_{-d/2}^{d/2} \Big[G(\mathbf{x}, c, x_2', \omega) \tilde{U}^d_{,1'}(c, x_2', \omega) - \tilde{U}^d(c, x_2', \omega) G_{,1'}(\mathbf{x}, c, x_2', \omega) \Big] \, dx_2' \quad ; \quad \mathbf{x} \in \mathbb{R}^2 \ , \tag{53}$$

wherein $\tilde{U}^d$ and $\tilde{U}^d_{,1'}$ are given in (50)-(51) and

$$G(\mathbf{x}, \mathbf{x}', \omega) = \frac{i}{4} H_0^{(1)}(k|\sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}|) \ , \tag{54}$$

so that

$$G(\mathbf{x}, c, x_2', \omega) = \frac{i}{4} H_0^{(1)}(k|R|) \ , \tag{55}$$

wherein $R = \sqrt{(x_1 - c)^2 + (x_2 - x_2')^2}$, and

$$G_{,1'}(\mathbf{x}, c, x_2', \omega) = \frac{ik}{4} \frac{(x_1 - c)}{|R|} H_1^{(1)}(k|R|) \ . \tag{56}$$

It follows that

$$u(\mathbf{x},\omega) \approx \frac{k}{4} e^{i(kc+\psi)} \int_{-d/2}^{d/2} A(x_2') \left[H_0^{(1)}(k|R|) + i\frac{(x_1-c)}{|R|} H_1^{(1)}(k|R|) \right] dx_2' \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; . \tag{57}$$

This expression can be computed by any (e.g., rectangle) numerical quadrature scheme.

5.1 Far field zone radiation

Let r', ϕ' be the polar coordinates of the integration point subtended by the vector $\mathbf{x}'$ such that $r'\cos\phi' = c$, and r, ϕ the polar coordinates of the observation point subtended by the vector $\mathbf{x}$ such that $r\cos\phi = x_1$. Then

$$R = \sqrt{r^2 + r'^2 - 2rr'\cos(\phi - \phi')} \; . \tag{58}$$

In the far field (Fraunhofer) zone, $r >> 1$, and we assume also that $kr >> 1$, from which it follows that $k|R| >> 1$; $\forall x_2' \in [-d/2, d/2]$, so that we can make use of the asymptotic forms

$$H_0^{(1)}(k|R|) \sim \left(\frac{2}{\pi k|R|}\right)^{\frac{1}{2}} e^{i\left(k|R|-\frac{\pi}{4}\right)} \;\; , \;\; H_1^{(1)}(k|R|) \sim \left(\frac{2}{\pi k|R|}\right)^{\frac{1}{2}} e^{i\left(k|R|-\frac{\pi}{2}-\frac{\pi}{4}\right)} \;\; ; \;\; k|R| \to \infty \; , \tag{59}$$

so as to obtain

$$u(\phi) \approx \frac{k}{4} e^{i(kc+\psi)} \int_{-d/2}^{d/2} A(x_2') \sqrt{\frac{2}{\pi k|R|}} e^{i\left(k|R|-\frac{\pi}{4}\right)} \left[1 + \frac{(x_1-c)}{|R|}\right] dx_2' \;\; ; \;\; \mathbf{x} \in \mathbb{R}^2 \; . \tag{60}$$

Furthermore:

$$R \approx r - r'\cos(\phi - \phi') \;\; ; \;\; \frac{r'}{r} << 1 \; , \tag{61}$$

so that we can make the approximations

$$\frac{1}{\sqrt{|R|}} \approx \frac{1}{\sqrt{r}} \;\; , \;\; \frac{x_1 - c}{|R|\sqrt{|R|}} \approx \cos\phi \;\; , \;\; e^{ik|R|} \approx e^{ik[r - r'\cos(\phi-\phi')]} \; , \tag{62}$$

Then

$$u(\mathbf{x},\omega) \sim \hat{u}(\phi) \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{i\left(kr - \frac{\pi}{4}\right)} \;\; ; \;\; kr \to \infty \; . \tag{63}$$

and $\hat{u}(\phi)$ is the so-called *far field radiation pattern* given by

$$\hat{u}(\phi) \approx \frac{k}{4}(1+\cos\phi) e^{i[kc(1-\cos\phi)+\psi]} \int_{-d/2}^{d/2} A(x_2') e^{-ikx_2'\sin\phi} dx_2' \;\; ; \;\; \phi \in [0, 2\pi[\; . \tag{64}$$

We note that in observation angles close to $\phi = \pi$ (the backraditon angle), $\cos\phi \approx -1$, so that $(1+\cos\phi) \approx 0$, which is the reason why this synthesized source gives rise (approximately) to unilateral radiation.

We can define the power radiation pattern in the Fraunhofer zone by

$$\sigma(\phi) := 10\log_{10}(\|\hat{u}(\phi)\|^2)(\text{in dB}) \; . \tag{65}$$

We plot this function (dotted curve) in fig. 3 for a source distribution which synthesizes the radiation of a parabolic cylinder antenna for which $kd = 20\pi$ and $c = 0$. The full line curve therein is the result of a rigorous computation of the field radiated by this antenna on which a Dirichlet boundary condition is imposed, and the other two curves represent the predictions resulting from the other two aperture methods. We note that our synthesized sources indeed give rise to very weak radiation in the left hand half space (i.e., for $\phi > 90^o$, there being symmetry around $\phi = 0$. We also not that the other three radiation patterns do not possess this property.

6 Final comments on the use of the synthesized sources of a unilateral radiator in a scattering problem

A typical problem of the scattering of a wave u^i radiated from synthesized sources s of support Ω^i by some boundary Γ on which the field is nil (i.e., Dirichlet boundary condition) is expressed as follows: determine the total field $u = u^i + u^d$ in a domain Ω such that:

$$(\triangle + k^2)u(\mathbf{x}, \omega) = -s(\mathbf{x}) \;\;;\;\; \mathbf{x} \in \Omega \;, \tag{66}$$

$$u^d(\mathbf{x}, \omega) \sim \text{outgoing waves} \;;\; \|\mathbf{x}\| \to \infty \;\;,\;\; \mathbf{x} \in \Omega \;, \tag{67}$$

$$u(\mathbf{x}, \omega) = 0 \;\;;\;\; \mathbf{x} \in \Gamma \;, \tag{68}$$

wherein

$$s(\mathbf{x}, \omega) = -2\delta(x_1 - c)[H(x_2 - d/2) + H(-x_2 - d/2)]\tilde{U}^i_{,1}(\mathbf{x}, \omega) - \\ \tilde{U}^i(\mathbf{x}, \omega)\delta_{,1}(x_1 - c)[H(x_2 - d/2) + H(-x_2 - d/2)] \;\;;\;\; \mathbf{x} \in \mathbb{R}^2 \;. \tag{69}$$

and

$$\tilde{U}^i(\mathbf{x}, \omega) = A(x_2)e^{i(kx_1 + \psi)} \;. \tag{70}$$

Actually, it is possible to assume other expressions for $\tilde{U}^i$ as long as they are connected in some plausible way with an actual physically-realizable unilateral radiator.

The previous analysis showed that

$$u^i(\mathbf{x}, \omega) = \int_{\Omega^i} G(\mathbf{x}, \mathbf{x}', \omega)s(\mathbf{x}', \omega)d\varpi(\mathbf{x}') \;\;;\;\; \mathbf{x} \in \Omega \;, \tag{71}$$

or, on account of (68) (and whatever the expression for $\tilde{U}^i$),

$$u^i(\mathbf{x}, \omega) = -\int_{-d/2}^{d/2} \left[G(\mathbf{x}, c, x_2', \omega)\tilde{U}^i_{,1'}(c, x_2', \omega) - G_{,1'}(\mathbf{x}, c, x_2', \omega)\tilde{U}^i(c, x_2', \omega)\right] dx_2' \;\;;\;\; \mathbf{x} \in \Omega^- \;. \tag{72}$$

So much for the incident field on the boundary.

The next step is to find an appropriate (boundary integral, domain integral, partial wave, etc.) representation of the scattered field u^d that incorporates the radiation condition (67). This involves some unknown functions that are determined in the final step by application of the boundary condition (68).

It will be noted that the use of our synthesized source distribution s: 1) enables us to simulate a unilateral radiated incident field, and 2) obviates multiple diffraction between the material boundaries of the radiator (of which abstraction is made in this method) and those of the scatterer.

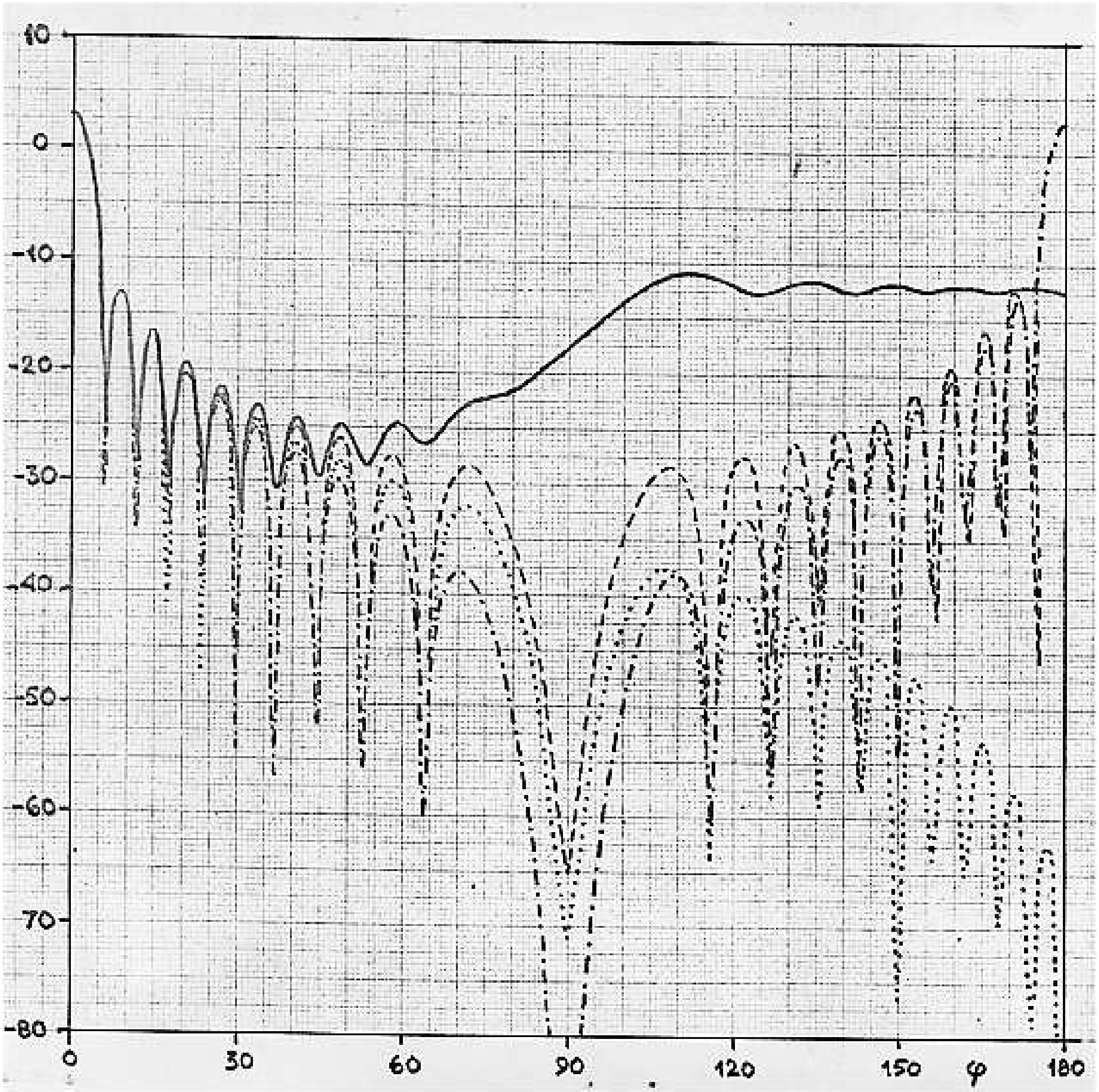

Figure 3: Graphs of the far-field radiated power pattern $\sigma(\phi)$ for a parabolic cylinder radiator. The full line curve stems from the exact theory. The other three curves stem from the various aperture method descriptions of the action of the antenna. The dotted curve results from the aperture method relying on the synthesized sources for unilateral radiation.

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