

# On the Principal eigenvalues and the Dirichlet problem for fully nonlinear operators

## Sur les valeurs propres et le problème de Dirichlet pour des opérateurs complètement non-linéaires

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### Abstract

We study uniformly elliptic fully nonlinear equations of the type  $F(D^2u, Du, u, x) = f(x)$ . We

- show that convex positively 1-homogeneous operators possess two principal eigenvalues and eigenfunctions, and study these objects ;
- obtain existence and uniqueness results for non-proper operators whose principal eigenvalues (in some cases, only one of them) are positive ;
- obtain an existence result for non-proper Isaac's equations.

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### Résumé

On étudie des équations complètement non-linéaires, uniformément elliptiques, du type  $F(D^2u, Du, u, x) = f(x)$ . On

- montre que les opérateurs convexes et positivement homogènes de degré 1 possèdent deux valeurs propres et deux fonctions propres principales. On étudie les propriétés de ces objets ;
- obtient des résultats d'existence et d'unicité pour des équations qui ne sont pas "propres", mais dont les valeurs propres (l'une ou les deux) sont positives ;
- obtient un résultat d'existence pour une équation de Isaac.

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## Version française abrégée

Dans cette note on étudie des équations uniformément elliptiques complètement nonlinéaires  $F(D^2u, Du, u, x) = f(x)$  dans un domaine borné de  $\mathbb{R}^N$ . On montre que les opérateurs convexes possèdent deux valeurs propres principales, et on étudie leur propriétés. On montre que le problème de Dirichlet a une solution pour toute donnée  $f$  si et seulement si les valeurs propres principales de  $F$  sont positives. On obtient des résultats d'existence pour des opérateurs qui ne sont pas *propres*, dont l'une des valeurs propres est positive.

In this note we study uniformly elliptic fully nonlinear equations

$$F(D^2u, Du, u, x) = f(x) \tag{1}$$

in a bounded domain  $\Omega \subset \mathbb{R}^N$ . We show that positively homogeneous operators which are convex (or concave) possess two principal eigenvalues and study their properties. We also show that existence and uniqueness theory for the Dirichlet problem can be developed for coercive *non-proper* operators (for example, Isaac's equations).

A starting point for our work is the paper by Lions [L1]. In this paper he proved the existence of principal eigenvalues for operators which are the supremum of linear operators with  $C^{1,1}$ -coefficients, and obtained results about the solvability of related Dirichlet problems. We note that the first to observe the phenomenon of appearance of two "half"-eigenvalues was Berestycki in [Be]. Very recently existence of principal eigenvalues was proven in another particular case, namely when  $F$  is a Pucci extremal operator, by Felmer and Quaas [FQ] (see also [BEQ], [Q]). It is our aim here to bring the eigentheory of fully nonlinear equations closer to the level of the well studied linear case, see the paper by Berestycki, Nirenberg and Varadhan [BNV]. The results we obtain extend most of the main results in [BNV] to nonlinear operators, and exhibit the particularities due to the nonlinear nature of the operators we consider. Further, we show that a great deal of recent results on existence, uniqueness and regularity properties of proper equations of type (1) (see [CKLS], where a good list of references is given) can be extended to operators with positive eigenvalues. The proofs of our results combine viscosity results and techniques with some ideas of [BNV]. The details of the proofs and examples can be found in the forthcoming paper [QS].

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For the definition of eigenvalues to make sense, we have to assume that the operator is positively homogeneous of order 1, that is,

$$(H_0) \quad F(tM, tp, tu, x) = t F(M, p, u, x), \quad \text{for all } t \geq 0.$$

We consider operators which satisfy the following hypothesis (H1) : for some  $\gamma, \delta > 0$ , all  $M, N \in \mathcal{S}_N$ ,  $p, q \in \mathbb{R}^N$ , and almost all  $x \in \Omega$

$$\begin{aligned} \mathcal{M}_{\lambda, \Lambda}^-(M - N) - \gamma|p - q| - \delta|u - v| &\leq F(M, p, u, x) - F(N, q, u, x) \\ &\leq \mathcal{M}_{\lambda, \Lambda}^+(M - N) + \gamma|p - q| + \delta|u - v|, \end{aligned}$$

and  $F(M, 0, 0, x)$  is continuous in  $\mathcal{S}_N \times \overline{\Omega}$ .

Note that when  $F$  is linear ( $H_1$ ) means  $F$  is uniformly elliptic, with bounded coefficients, and continuous second-order coefficients.

We denote  $G(M, p, u, x) = -F(-M, -p, -u, x)$ . An important role is played by the following definition. We say that an operator  $H(M, p, u, x)$  satisfies condition ( $D_F$ ) provided

$$\begin{aligned} G(M - N, p - q, u - v, x) &\leq H(M, p, u, x) - H(N, q, v, x) \\ &\leq F(M - N, p - q, u - v, x). \end{aligned}$$

It is easy to see that under ( $H_0$ ) the following are equivalent : (i)  $F$  is convex in  $(M, p, u)$  ; (ii)  $F$  satisfies ( $D_F$ ) ; (iii)  $F$  satisfies one of the two inequalities in ( $D_F$ ).

We assume that the domain  $\Omega$  is smooth. We stress however that most results can be extended to arbitrary bounded domains, by using an approximation argument, as in [BNV]. We make the convention that all (in)equations are satisfied in the  $L^N$ -viscosity sense - see for example [CCKS] for definitions and properties of these solutions.

For any  $\lambda \in \mathbb{R}$  we define the sets

$$\begin{aligned} \Psi^\pm(F, \Omega, \lambda) &= \{\psi \in C(\overline{\Omega}) \mid \psi > 0 \text{ (resp. } < 0) \text{ in } \Omega, \quad \text{and} \\ &\quad F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ (resp. } \geq 0) \text{ in } \Omega\}, \end{aligned}$$

and the following quantities (depending on  $F$  and  $\Omega$ )

$$\lambda_1^+ = \sup \{\lambda \mid \Psi^+(F, \Omega, \lambda) \neq \emptyset\}, \quad \lambda_1^- = \sup \{\lambda \mid \Psi^-(F, \Omega, \lambda) \neq \emptyset\}.$$

The following theorem asserts the existence of two couples of principal eigen-

functions and eigenvalues of a nonlinear operator. Set  $E_p = W_{\text{loc}}^{2,p}(\Omega) \cap C(\bar{\Omega})$ .

**Theorem 0.1** *Suppose  $F$  satisfies  $(H_0)$ ,  $(H_1)$ , and  $(D_F)$ . Then there exist functions  $\varphi_1^+, \varphi_1^- \in E_p$  for each  $p < \infty$ , such that*

$$F(D^2\varphi_1^+, D\varphi_1^+, \varphi_1^+, x) = -\lambda_1^+ \varphi_1^+, \quad \varphi_1^+ > 0 \text{ in } \Omega, \quad \varphi_1^+ = 0 \text{ on } \partial\Omega,$$

resp.  $F(D^2\varphi_1^-, D\varphi_1^-, \varphi_1^-, x) = -\lambda_1^- \varphi_1^-$  in  $\Omega$ ,  $\varphi_1^- < 0$  in  $\Omega$ ,  $\varphi_1^- = 0$  on  $\partial\Omega$ .

If  $\varphi_1^+$  (or  $\varphi_1^-$ ) is normalized so that  $\varphi_1^+(x_0) = 1$  (resp.  $\varphi_1^-(x_0) = -1$ ) for a fixed point  $x_0 \in \Omega$ , then  $\varphi_1^+ \leq C$  (resp.  $\varphi_1^- \geq -C$ ) in  $\Omega$ , where  $C$  depends only on  $x_0, \Omega, \lambda, \Lambda, \gamma$  and  $\delta$ . In addition,  $\lambda_1^+$  (resp.  $\lambda_1^-$ ) is the only eigenvalue corresponding to a positive (resp. negative) eigenfunction.

The next result implies that the principal eigenfunctions are simple in a strong sense, even in the set of viscosity solutions.

**Theorem 0.2** *Assume there exists a viscosity solution  $u \in C(\bar{\Omega})$  of*

$$F(D^2u, Du, u, x) \geq -\lambda_1^+ u \text{ in } \Omega, \quad u(x_0) > 0, \quad u \leq 0 \text{ on } \Omega, \quad (2)$$

for some  $x_0 \in \Omega$ , or of  $F(D^2u, Du, u, x) = -\lambda_1^+ u$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ . Then  $u \equiv t\varphi_1^+$ , for some  $t \in \mathbb{R}$ . If a function  $v \in C(\bar{\Omega})$  satisfies these equalities or the inverse inequalities in (2), with  $\lambda_1^+$  replaced by  $\lambda_1^-$ , then  $v \equiv t\varphi_1^-$  for some  $t \in \mathbb{R}$ .

The next theorem gives a necessary and sufficient condition for the positivity of the principal eigenvalues. It also shows that the existence of a nontrivial positive viscosity supersolution implies the existence of a positive uniformly bounded (below, and in the global  $W^{2,p}$ -norm) strong supersolution.

**Theorem 0.3 (a)** *Assume there is a function  $u \in C(\bar{\Omega})$  such that we have  $F(D^2u, Du, u, x) \leq 0$  in  $\Omega$ ,  $u > 0$  in  $\Omega$ , (resp.  $F(D^2u, Du, u, x) \geq 0$  in  $\Omega$ ,  $u < 0$  in  $\Omega$ ). Then either  $\lambda_1^+ > 0$  or  $\lambda_1^+ = 0$  and  $u \equiv t\varphi_1^+$ , for some  $t > 0$  (resp.  $\lambda_1^- > 0$  or  $\lambda_1^- = 0$  and  $u \equiv t\varphi_1^-$ , for some  $t > 0$ ).*

**(b)** *Conversely, if  $\lambda_1^+ > 0$  then there exists a function  $u \in W^{2,p}(\Omega)$ ,  $p < \infty$ , such that  $F(D^2u, Du, u, x) \leq 0$ ,  $u \geq 1$  in  $\Omega$ , and  $\|u\|_{W^{2,p}(\Omega)} \leq C$ , where  $C$  depends on  $p, N, \lambda, \Lambda, \gamma, \delta$ , and  $\lambda_1^+$ .*

*Remark 1.* When  $F$  is proper,  $u \equiv 1$  satisfies the condition of Theorem 0.3. Hence proper operators have positive eigenvalues. Another consequence from Theorem 0.3 is that the eigenvalues are *strictly* decreasing and continuous with respect to the domain.

Further, we show that the positivity of the principal eigenvalues is a neces-

sary and sufficient condition for the operator to satisfy a comparison principle. We say that a second order operator  $H$  satisfies a *comparison principle* (CP), provided for any  $u, v \in C(\overline{\Omega})$ , one of which is in  $E_N$ , such that  $H(D^2u, Du, u, x) \geq H(D^2v, Dv, v, x)$  in  $\Omega$ ,  $u \leq v$  on  $\partial\Omega$ , we have  $u \leq v$  in  $\Omega$ . A particular case of (CP) is *the maximum principle*, when one of  $u, v$  is zero.

**Theorem 0.4** *Suppose a second-order operator  $F$  satisfies  $(H_0), (H_1)$ , and  $(D_F)$ . Then  $\lambda_1^+(F) > 0$  is necessary and sufficient for  $F$  to satisfy (CP). Hence, if a second-order operator  $H$  satisfies  $(D_F)$ , then  $\lambda_1^+(F) > 0$  is sufficient for  $H$  to satisfy (CP).*

If  $\lambda_1^- > 0$ , the comparison and even the maximum principle do not necessarily hold. However, it can be shown that  $\lambda_1^- > 0$  is necessary and sufficient for a *one-sided* maximum principle, see [QS].

We prove the following Alexandrov-Bakelman-Pucci inequality for nonproper second order operators.

**Theorem 0.5** *Suppose the operator  $F$  satisfies  $(H_0), (H_1)$ , and  $(D_F)$ . Then for any  $u \in C(\overline{\Omega})$ ,  $f \in L^N(\Omega)$  the inequalities  $F(D^2u, Du, u, x) \geq f$ ,  $\lambda_1^+(F) > 0$  (resp.  $F(D^2u, Du, u, x) \leq f$ ,  $\lambda_1^-(F) > 0$ ) imply*

$$\sup_{\Omega} u \leq C(\sup_{\partial\Omega} u^+ + \|f^-\|_{L^N(\Omega)}), \left( \text{resp. } \sup_{\Omega} u^- \leq C(\sup_{\partial\Omega} u^- + \|f^+\|_{L^N(\Omega)}) \right).$$

where  $C$  depends on  $\Omega$ ,  $N$ ,  $\lambda$ ,  $\Lambda$ ,  $\gamma$ ,  $\delta$ , and  $\lambda_1^+(F)$  (resp.  $\lambda_1^-(F)$ ).

We show that the Dirichlet problem is solvable for any right-hand side if and only if the eigenvalues of the operator are positive.

**Theorem 0.6** *Suppose  $F$  satisfies  $(H_0), (H_1)$ , and  $(D_F)$ . If  $\lambda_1^+(F) > 0$  then for any  $f \in L^p(\Omega)$ ,  $p \geq N$ , there exists a unique solution  $u \in E_p$  of the problem  $F(D^2u, Du, u, x) = f$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ . In addition, for any compact set  $\omega \subset\subset \Omega$  there holds  $\|u\|_{W^{2,p}(\omega)} \leq C\|f\|_{L^p(\Omega)}$ , where  $C$  depends on  $p, \omega, \Omega, \lambda, \Lambda, \gamma, \delta$ , and  $\lambda_1^+(F)$ .*

*This existence result is sharp : if  $\lambda_1^+(F) = 0$  then the Dirichlet problem does not possess a viscosity solution in  $C(\overline{\Omega})$ , provided  $f \leq 0$ ,  $f \not\equiv 0$  in  $\Omega$ .*

Finally, we have the following existence result, applicable to non-convex operator, like Isaac's operators, which completes (and uses) some recent results for proper operators, obtained in [CKLS] (see also [JS]).

**Theorem 0.7** *Assume  $F$  satisfies  $(H_0), (H_1)$ ,  $(D_F)$ , and  $H$  satisfies  $(D_F)$*

and  $(H_0)$ . If  $\lambda_1^+(F) > 0$  then the problem

$$\begin{cases} H(D^2u, Du, u, x) = f \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega \end{cases}$$

is solvable in the viscosity sense for any  $f \in L^p(\Omega)$ ,  $p \geq N$ . If  $H(M, p, u, x)$  is convex in  $M$  then  $u \in E_p$ , and  $u$  is unique.

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