On the Ambrosetti-Prodi problem for non-variational elliptic systems

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Abstract. We study the Ambrosetti-Prodi problem for nonlinear elliptic equations and systems, with uniformly elliptic operators in non-divergence form and non-smooth coefficients, and with nonlinearities with linear or power growth.

1 Introduction

In this paper we revisit an old and classical problem in the theory of elliptic partial differential equations, the so-called Ambrosetti-Prodi problem.

Although we deal with systems of equations, we start by some background on the scalar problem, for which we also have new results. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and $f(x,u), h(x)$ be real-valued Hölder continuous in $x \in \overline{\Omega}$, with f locally Lipschitz continuous in $u \in \mathbb{R}$. The issue here is the existence of classical solutions to the problem

$$\left\{ \begin{array}{rcll} -Lu & = & f(x,u) + t\varphi_1(x) + h(x) & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1}$$

where L is general second-order uniformly elliptic operator with Hölder continuous coefficients, $L = \displaystyle\sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x)\frac{\partial}{\partial x_i}$. Here $t \in \mathbb{R}$ is a parameter and φ_1 is the first (positive) eigenfunction of L, i.e. $-L\varphi_1 = \lambda_1\varphi_1$ in Ω, with $\varphi_1 = 0$ on $\partial\Omega$; we refer to [6] for properties of $\lambda_1 > 0$ and φ_1.

We obtain Ambrosetti-Prodi type results (see below) for the solutions of (1), both in the case of a single equation and in the case of a system. Our main concern will be obtaining a priori bounds for the solutions. The need of these comes from the fact that topological methods (degree theory) have to be used in order to obtain multiplicity of solutions.

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Problem (1) is said to be of Ambrosetti-Prodi type provided there exist constants a, b, C such that $b > \lambda_1 > a$, and for all $x \in \overline{\Omega}$,

$$f(x,s) \geq bs - C \quad \text{for } \; s \geq 0, \qquad f(x,s) \geq as - C \quad \text{for } \; s \leq 0. \tag{2}$$

This hypothesis is equivalent to the existence of constants a', b' such that $\limsup_{s\to-\infty} \dfrac{f(x,s)}{s} \leq a' < \lambda_1 < b' \leq \liminf_{s\to\infty} \dfrac{f(x,s)}{s}$.

A typical result in this setting states

(AP) There exists $t_0 \in \mathbb{R}$ such that problem (1) has at least two solutions for $t < t_0$, at least one solution for $t = t_0$, and no solutions for $t > t_0$.

The first result in this line was obtained by Ambrosetti and Prodi in [3], and this fact originated the present terminology for this sort of problems. In [3], $L = \Delta$, $f(x,u) = f(u)$ was a convex function of class C^2 such that $0 < \lim_{s\to-\infty} f'(s) < \lambda_1 < \lim_{s\to+\infty} f'(s) < \lambda_2$. With $t\varphi_1(x) + h(x) = g(x)$ they proved, using results on differentiable mappings with singularities, that there is a closed connected C^1 manifold M of codimension 1 in the space $C^{0,\alpha}(\overline{\Omega})$ which splits the space into two connected components S_0, S_2 with the property that, if $g \in S_0$ then (1) has no solution, if $g \in M$ then (1) has exactly one solution, and if $g \in S_2$ then (1) has exactly two solutions.

The result in [3] received immediately attention by several authors trying to obtain similar conclusions and relaxing the original hypotheses. In [7] Berger and Podolak used the Liapunov-Schmidt method, so for them it was natural to use the decomposition of the function g as it appears in (1) above. The result obtained there and in most of the subsequent works is precisely the statement in (AP).

In [24] Kazdan and Warner consider more general functions f and smooth differential operators of second order. However only one solution is obtained. In [15], Dancer extended the result in [24], for differential operators in the divergence form, by getting a second solution. A result for a general L with smooth coefficients and a nonlinearity f with linear growth is due to Hess [22]. In these problems, the existence of a second solution depends very heavily on the growth of the non-linearity at $+\infty$, that is: the existence of a $p \geq 1$ such that for all $x \in \overline{\Omega}$, $s \geq 0$, $|f(x,s)| \leq C(1 + s^p)$. The method used in these papers is topological and the a priori bound for the solutions of (1) either depends on the linear growth of f or is obtained using the Hardy-Sobolev inequality (this method of obtaining a priori bounds is due to Brezis-Turner [10]). The use of Hardy-Sobolev inequalities requires divergence form operators and restricts the growth of the non-linearity at infinity to $p < (N + 1)/(N - 1)$.

Later variational methods were used for divergence form operators, and fairly general results as far as the growth of the non-linearity at $+\infty$ is concerned were obtained in [17], [13]. In these papers the result holds for all sub-critical problems, that is, $p < (N+2)/(N-2)$, when an additional condition of Ambrosetti-Rabinowitz (mountain-pass) type is assumed in order to get a Palais-Smale condition for the associated functional. The critical case $p = p_c$ was studied in [19] for dimensions $N > 6$ and the other dimensions in [12]. Lately the Ambrosetti-Prodi problem for operators of the type $\mathrm{div}(A(x,u)Du)$ and a nonlinearity with pure power subcritical growth was considered in [4]. The problem for the m-Laplacian and f growing as $|u|^{m-2}u$ was studied in [5]. For further work on similar problems and for various multiplicity results see [9], [25], [11], [28], [30], as well as the references in these papers.

To our knowledge, there are no results on the Ambrosetti-Prodi problem with a superlinear subcritical nonlinearity, when the operator is in non-divergence form, that is, the problem does not admit a weak formulation in terms of integrals. It is this situation that we want to study here. We will even not suppose that the adjoint operator of L has a principal eigenfunction. As we shall explain later, the methods from the papers quoted above do not apply in this case, and some new ideas are needed. An overview of the method we use, and of the novelties in the approach is given in the beginning of the next section.

We are going to show that (AP) holds for any operator L, provided f has a precise subcritical power growth at $+\infty$. The following theorem is a consequence of more general results for systems of equations, which are the main results of this paper.

Theorem 1 *Suppose that (2) holds and that there exists a bounded function $a(x)$, positive on $\overline{\Omega}$, such that for all $x \in \overline{\Omega}$*

$$\lim_{s\to\infty} \frac{f(x,s)}{s^p} = a(x), \qquad \textit{for some} \;\; p \in \left(1\,, \frac{N+2}{N-2}\right). \tag{3}$$

Then (AP) *holds.*

Remark. We could weaken even further the regularity assumptions on the operator – instead of Hölder functions, we could consider operators with bounded (and continuous second-order) coefficients, and $h \in L^p(\Omega), p \geq N$. Then the solutions we obtain belong to $W^{2,p}(\Omega)$.

Now we come to the discussion of Ambrosetti-Prodi results for systems of elliptic equations. Let us have a system of d equations, written in matrix format

$$(\mathcal{P}_t) \qquad \begin{cases} -Lu &= f(x,u) + t\varphi_1(x) + h(x) \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

here $u = (u_1, \ldots, u_d)^T$, $h = (h_1, \ldots, h_d)^T$, $f = (f_1, \ldots, f_d)^T$, $t = (t_1, \ldots, t_d)^T$, $\varphi_1 = (\varphi_{1,1}, \ldots, \varphi_{1,d})^T$, $t\varphi_1 = (t_1\varphi_{1,1}, \ldots, t_d\varphi_{1,d})^T$, $L = $diag$(L_1, \ldots, L_d)$, with $L_k = \sum_{i,j=1}^N a_{ij}^{(k)}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i^{(k)}(x) \frac{\partial}{\partial x_i}$, where $a_{ij}^{(k)}(x)$,$b_i^{(k)}(x)$ are Hölder continuous (or, if one wants to have only $W^{2,p}$-solutions of the system we suppose that $a_{ij}^{(k)}(x)$ are continuous in $\overline{\Omega}$ and $b_i^{(k)}(x)$ are bounded), and $\varphi_{1,i}$ is the first eigenfunction of the operator L_i, normalized so that $\max_\Omega \varphi_{1,i} = 1$, see [6]. All (in)equalities between vectors will be understood to hold component-wise. Up to changing h we assume $f(x, 0, \ldots, 0) = 0$. For any $u \in \mathbb{R}^d$, we shall denote with u^+ (resp. u^-) the vector of the positive (resp negative) parts of the components of u. So $u = u^+ - u^-$. We set $e = (1, \ldots, 1) \in \mathbb{R}^d$.

In order to state an Ambrosetti-Prodi problem for a system, one needs to define a first eigenvalue for a matrix operator of the type $L + A(x)$, which has the essential property to be a dividing value for the maximum principle to hold. This was recently done in [8], provided $A(x)$ is a bounded *cooperative* matrix, that is, all off-diagonal entries of A are nonnegative (and examples were given showing that for noncooperative matrices this may not be possible), and *fully coupled*. Note that any matrix A can be written in block-triangular form $A = (A_{ij})_{i,j=1}^m$, for some $m \in \{1, \ldots, d\}$, with A_{ii} fully coupled and $A_{ij} = 0$ for $i < j$ (see Section 2.1).

We shall suppose all along this paper that the map $f(x, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is *quasi-monotone* for all $x \in \overline{\Omega}$, that is, $f_i(x, u)$ is nondecreasing in u_j, for any $i \neq j$ (this condition is of course void for a scalar equation). This is the usual condition to have a Maximum Principle for systems.

Condition (2) for systems will be written as : there exist bounded cooperative matrices $A_1(x), A_2(x)$, and constants b_1, b_2, such that ($L_{1,2}^{(i)}$ will denote the minor diag$(L_j)_{j\in J}$ where J contains the same indices as those in the fully coupled blocks $A_{1,ii}, A_{2,ii}$, for any $i \in \{1 \ldots, m\}$)

$$\lambda_1(L_1^{(i)} + A_{1,ii}) > 0 \quad \text{for all } i \; ; \qquad \lambda_1(L_2^{(i)} + A_{2,ii}) < 0 \quad \text{for all } i \; ; \tag{4}$$

$$f(x, s) \geq A_1(x)s - b_1 e \quad \text{in } \{s \in \mathbb{R}^d \; : \; s \leq 0\} \; ; \tag{5}$$

$$f(x, s) \geq A_2(x)s - b_2 e \quad \text{in } \{s \in \mathbb{R}^d \; : \; s \geq 0\}. \tag{6}$$

We will also need the following (mild) assumption : for any sequence $\{s_n\} \subset \mathbb{R}^d$ such that $\{\|s_n^-\|\}$ is bounded and $\|s_n^+\| \to \infty$ as $n \to \infty$,

$$\liminf_{n\to\infty} \frac{f(x,s_n) - f(x,s_n^+)}{\|s_n^+\|} \geq 0. \tag{7}$$

Note that (7) is void when $d = 1$ and is trivial if f is globally Lipschitz in s.

An Ambrosetti-Prodi result for systems should state

(APS) There exists a Lipschitz hypersurface $\Gamma \subset \mathbb{R}^d$ which divides $\mathbb{R}^d$ into two parts Y, N such that problem $(\mathcal{P}_t)$ has at least two solutions for $t \in Y$, at least one solution for $t \in \Gamma$, and no solutions for $t \in N$.

Theorem 2 (linear growth) *Suppose that (4)–(7) hold and, in addition,*

$$f(x,s) \leq C(1 + |s_1| + \ldots + |s_d|)\,, \qquad \text{for all } x \in \Omega,\ s \in \mathbb{R}^d. \tag{8}$$

Then (APS) holds.

This result extends a previous one of K.C. Chang [14]. Note that, contrary to [14], we do not suppose that the matrices A_1, A_2 are fully coupled, nor that f is globally Lipschitz, nor that inequalities (5), (6) hold for all $s \in \mathbb{R}^d$. Note also that in [14] a different notion of first eigenvalue was used, namely concerning problems with weight. That eigenvalue exists under stronger hypotheses.

The result given by the particular case of Theorem 2, when we have only one equation, i.e. $d = 1$, has appeared in several papers (see the remarks above), under stronger restrictions on the differential operator as well as in the non-linearities involved. We state this result here, since we are unaware of a reference where it appears in the present generality.

Corollary 1.1 *Let f, h, L, φ_1 be scalar ($d = 1$). Suppose*

$$\limsup_{s\to-\infty} \frac{f(x,s)}{s} \leq a_1(x) \in L^\infty(\Omega), \qquad \liminf_{s\to\infty} \frac{f(x,s)}{s} \geq a_2(x) \in L^\infty(\Omega),$$

and $f(x,s) \leq C(1+|s|)$. Assume that the first eigenvalue of $L + a_1(x)$ is positive, and the first eigenvalue of $L + a_2(x)$ is negative. Then (AP) holds.

Next we turn to the more difficult case of superlinear systems. We will only consider systems of two equations (this is due to the necessity of using Liouville type results for positive solutions of such systems). Also we will need the (technical) assumption that the second order coefficients of L_1 and L_2 coincide. So, let us have a system with two equations of the form

$$(\mathcal{P}_t) \qquad \begin{cases} -L_1 u &= f_1(x,u_1,u_2) + t_1\varphi_1(x) + h_1(x) \quad \text{in} \quad \Omega \\ -L_2 u &= f_2(x,u_1,u_2) + t_2\varphi_2(x) + h_2(x) \quad \text{in} \quad \Omega \\ u_1 &= u_2 = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\varphi_i > 0$ is the first eigenfunction of L_i. We suppose that the functions f_i, $i = 1, 2$, in $(\mathcal{P}_t)$ satisfy

$$\lim_{s_j\to\infty} \frac{f_i(x,s_1,s_2)}{s_j^{\alpha_{ij}}} = a_{ij}(x), \qquad i,j = 1,2, \tag{9}$$

where the exponents $\alpha_{ij} > 1$, and $a_{ij}(x) \geq 0$, $a_{ij} \in C(\overline{\Omega})$.

We denote $\vec{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$, and consider the lines

$$\begin{aligned} l_{11} &= \left\{\vec{\beta} \mid \beta_1 + 2 - \beta_1\alpha_{11} = 0\right\}, \quad & l_{22} &= \left\{\vec{\beta} \mid \beta_2 + 2 - \beta_2\alpha_{22} = 0\right\}, \\ l_{12} &= \left\{\vec{\beta} \mid \beta_1 + 2 - \beta_2\alpha_{12} = 0\right\}, \quad & l_{21} &= \left\{\vec{\beta} \mid \beta_2 + 2 - \beta_1\alpha_{21} = 0\right\}. \end{aligned}$$

The expressions of the lines above appear quite naturally when applying the Blow-up Method in order to obtain a priori bounds for the solutions of $(\mathcal{P}_t)$ in the case of systems, see [18]. We call a pair (β_1^0, β_2^0) *a blow-up pair* if it has the property to be in the intersection of two of those lines, and further $\vec{\beta}^0$ is to the left of or on l_{11}, below or on l_{22}, below or on l_{12}, and above or on l_{21}. Suppose the $a_{ij}(x)$ corresponding to these two (or more) lines are positive on $\overline{\Omega}$. We recall the following idea from [18] : if the exponents α_{ij} are such that one can choose a blow-up pair, then this will lead to statements of Liouville type, and consequently to a priori bounds for the solutions.

Theorem 3 (superlinear growth) *Let the above hypotheses hold,and suppose that (4)–(7) are satisfied by the system of two equations $(\mathcal{P}_t)$, and, in addition,*

$$\min\left\{\beta_1^0, \beta_2^0\right\} > \frac{N-2}{2} \qquad \textit{or} \qquad \max\left\{\beta_1^0, \beta_2^0\right\} > N-2\,. \tag{10}$$

Then (APS) holds.

We remark here that Theorem 1 is a particular case of Theorem 3 (with $\alpha_{12} = \alpha_{21} = 0$, $\alpha_{11} = \alpha_{22} = p$, $\beta_1 = \beta_2 = 2/(p-1)$), and that (10) with $\vec{\beta} = l_{12} \cap l_{21}$ is the best hypothesis under which the Lane-Emden system $-\Delta u_1 = u_2^{\alpha_{12}}, -\Delta u_2 = u_1^{\alpha_{21}}$ is known not to have classical positive solutions in $\mathbb{R}^N$.

Previous works on Ambrosetti-Prodi problems for subcritical superlinear systems are [26] and [27]. In [26] variational systems are considered and a variant of the result in [17] is obtained. The paper [27] is devoted to nonvariational systems. In that paper the restriction to the exponents was considerably stronger than (10) due to the use of Hardy-Sobolev inequalities; it was supposed that $L_1 = L_2 = \Delta$, and (4)–(6) were replaced by a stronger hypothesis concerning the first eigenvalue of Δ. Recently, [20] has used variational methods to obtain results for variational systems with non-linearities of the Ambrosetti-Prodi type, both critical and subcritical, extending results of [19] proved before for the scalar case.

The next section is devoted to the proof of the main theorems. We start by an overview, then in Section 2.1 we give some preliminaries and results on the applicability of Perron's method to our case. The heart of the paper is Section 2.2, where we prove a priori bounds for solutions of $(\mathcal{P}_t)$, as well as nonexistence of solutions for large t. The proof is concluded in Section 2.3.

2 Proofs

Here are the steps in the proof of Ambrosetti-Prodi type results :

1. prove supersolutions exist for sufficiently small t, subsolutions of $(\mathcal{P}_t)$ exist for all t and can be chosen to be smaller than any solution of $(\mathcal{P}_t)$; deduce by Perron's method that solutions of $(\mathcal{P}_t)$ exist for $t \in (-\infty, t^*)$;

2. prove an a priori bound on the negative part of u, for $t \geq -C$;

3. prove an a priori upper bound on t, such that $(\mathcal{P}_t)$ has a solution ;

4. prove an a priori bound on u, for $t \geq -C$. There are two general ways to do this in the superlinear case :

   - use the Brezis-Turner technique; this restricts the growth of f to $(N+1)(N-1)$;

   - use the Gidas-Spruck blow-up technique; this requires exact power growth of f at ∞ ;

5. use fixed point and degree theory to conclude.

This scheme is well-known since the 1970's and has been used many times ever since, when Ambrosetti-Prodi results were to be established. We have followed this scheme too.

It is known how to prove Step 1 when Perron's method is applicable and one has solvability of the Dirichlet problem and a maximum principle for L. Consequently, in Section 2.1, we recall some notations and results, essentially from [8], where these question were studied for systems of equations, and prove some easy results on the application of the Method of Monotone Iteration to our case.

The main difficulty is in proving steps 2, 3, and 4. In case the operator L is in divergence form, there are well-known techniques for proving steps 2 and 3, which consist in testing the equation with u^- and φ_1 respectively (this could easily be checked in the model case $L = \Delta$, $f(u) = (\lambda_1 - \varepsilon)u + (u^+)^p$). In particular, step 3 follows directly from testing with φ_1 and $\exists\, C, \delta > 0\ :\ f(u) \geq (\lambda_1 + \delta)u - C, \forall\, u \geq 0$. Then, once one has the uniform upper bound in t, one can prove Step 4 supposing t is in a compact interval, that is, the $t\varphi_1$ term in the equation is a L^∞ right-hand side, and so trivially disappears after a blow-up.

None of the above can be done when the operator is in non-divergence form. We will now explain how we deal with the problem. First, the bound on u^- is obtained by showing that the (nonsmooth) function u^- satisfies a linear inequality in the viscosity sense, and then by applying Caffarelli's ABP inequality for such solutions, and its extensions to systems, proved in [8]. Second, the proofs of Steps 3 and 4 are carried out jointly. We perform a simultaneous blow-up argument in $\|u\|_{L^\infty(\Omega)}$ and t. This argument gives a bound neither on u nor on t, but rather leads to the inequality

$$\|u\|_{L^\infty(\Omega)} \leq Ct^{1/p}, \tag{11}$$

which is interesting in its own right. Then through a maximum principle argument we show that

$$t \leq C(1 + \|u\|_{L^\infty(\Omega)}).$$

These two inequalities together yield bounds both on u and t. To our knowledge, no similar approach has been used in other works.

We remark that the implementation of the blow-up method follows the standard blow-up procedure except that the term coming from $t\varphi_1$ does not necessarily disappear at the limit, since t is unbounded as well. Actually we find its disappearing is equivalent to the failure of (11), which permits to conclude the contradiction argument.

Finally, it is not difficult to prove Step 5, when one has an uniform upper bound for t and $\|u\|$, such that u is a solution of $\mathcal{P}_t$.

2.1 Preliminaries

In this section we state some results, and consequences of results from [8] (see in particular Sections 8, 13 and 14 in that paper).

Let us consider d uniformly elliptic operators in the general non-divergence form

$$L_k = \sum_{i,j=1}^{N} a_{ij}^{(k)}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i^{(k)}(x) \frac{\partial}{\partial x_i},$$

where $a_{ij}^{(k)}(x)$ are continuous in $\overline{\Omega}$ and $b_i^{(k)}(x)$ are bounded. Let $c_{ij}(x)$ be bounded functions and set $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^{n}$. Let $f_i(x) \in L^N(\Omega)$. We will consider systems in the form

$$Lu + \mathcal{C}u = f, \tag{12}$$

where $L = \operatorname{diag}(L_1, \ldots, L_d)$, $\mathcal{C}(x) = (c_{ij}(x))_{i,j=1}^{d}$, $u = (u_1, \ldots, u_d)^T$, and $f = (f_1, \ldots, f_d)^T$.

We shall need to consider solutions of this system in the viscosity sense, whose definition we recall next. A function $u \in C(\overline{\Omega}, \mathbb{R}^d)$ is called a *viscosity subsolution* of (12) provided for each i, each $x_0 \in \Omega$, and each $\varphi \in C^2(\Omega)$ such that $\varphi(x_0) = u_i(x_0), \varphi \geq u_i$ in Ω, we have $L_i\varphi(x_0) \geq f_i(x_0) - \sum_k c_{ik}(x_0)u_k(x_0)$. A function is a *viscosity supersolution* if this definition with reverse inequalities holds, and u is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution. Note that this definition is valid if all functions in (12) are continuous in x ; if this is not the case one needs to use the so-called L^N-viscosity solutions, see [8]. Any classical solution is of course a viscosity solution. It is very simple to check that, in the viscosity sense, the maximum of two subsolutions is a subsolution, and the minimum of two supersolutions is a supersolution (note that even if two functions are smooth their maximum is merely continuous, that is why we need this weaker notion). We shall work with solutions in the viscosity sense, without necessarily specifying each time.

Viewing to use Alexandrov-Bakelman-Pucci estimates, Harnack inequalities and Maximum Principles we consider cooperative systems. System (12) is called *cooperative* (or *quasi-monotone*) if $c_{ij} \geq 0$ for all $i \neq j$.

We recall that a system of this type is called *fully coupled* (and the matrix $\mathcal{C}$ is called *irreducible*) provided for any non-empty sets $I, J \subset \{1, \ldots, d\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, d\}$, there exist $i_0 \in I$ and $j_0 \in J$ for which

$$\operatorname{meas}\{x \in \Omega \mid c_{i_0 j_0}(x) > 0\} > 0. \tag{13}$$

For simplicity, when (13) holds we write $c_{i_0j_0} \not\equiv 0$ in Ω. Simply speaking, a system is fully coupled provided it cannot be split into two subsystems, one of which does not depend on the other.

As explained in [8], any matrix can have its lines and columns renumbered in such a way that it is in block triangular form, with each block on the main diagonal being fully coupled. More precisely, $\mathcal{C} = (\mathcal{C}_{kl})_{k,l=1}^m$, where $1 \le m \le d$, $\mathcal{C}_{kl}$ are $t_k \times t_l$ matrices for some $t_k \le d$ with $\sum_{k=1}^m t_k = d, \mathcal{C}_{kk}$ is an *irreducible* matrix for all $k = 1, \dots, m$, and $\mathcal{C}_{kl} \equiv 0$ in Ω, for all $k, l \in \{1, \dots, m\}$ with $k < l$. Note that $m = 1$ means $\mathcal{C}$ itself is irreducible, while $m = d$ means $\mathcal{C}$ is in triangular form. We set $s_0 = 0$, $s_k = \displaystyle\sum_{j=1}^k t_j$, and $S_k = \{s_{k-1}+1, \dots, s_k\}$.

It was proved in Theorem 13.1 in [8] that the matrix operator $L+\mathcal{C}$ admits a principal eigenvalue with all the usual properties of the principal eigenvalue of a scalar operator (see [6]), provided $\mathcal{C}$ is cooperative and irreducible. We recall that the *principal eigenvalue* of $L + \mathcal{C}$ is defined by:

$$\begin{aligned} \lambda_1 &= \lambda_1(L+\mathcal{C}) \\ &= \sup\{\lambda \in \mathbb{R} : \exists \psi \in W^{2,N}_{loc}(\Omega, R^d), \text{ s.t. } \psi > 0,\ (L + \mathcal{C} + \lambda I)\psi \le 0 \,\text{in}\, \Omega\} \end{aligned}$$

Hence, using the above explained block triangular representation of the cooperative matrix $\mathcal{C}$, we can associate to $\mathcal{C}$ a set of eigenvalues $\lambda_1^{(1)}, \dots, \lambda_1^{(m)}$, where $\lambda_1^{(k)}$ is the principal eigenvalue of $L^{(k)} + \mathcal{C}_{kk}$. Here we have denoted $L^{(k)} = \text{diag}(L_{s_{k-1}+1}, \dots, L_{s_k})$ (see above for the notations).

By combining Theorems 8.1, 12.1, 13.1, 13.2, 14.1 and Lemma 14.1 in [8] we obtain the following result.

Theorem 4 **(i)** *The following are equivalent :*

(a) $\lambda_1^{(k)} > 0$ *for all* $k = 1, \dots, m$ *;*

(b) there exists a vector $\psi(x) \in C^2(\Omega)$ *(or* $W^{2,p}(\Omega) \cap C(\overline{\Omega})$*) such that* $\psi \ge e$ *and* $L\psi + \mathcal{C}\psi \le 0$ *in* Ω *;*

(c) for any $f \in L^N(\Omega)$ *and any viscosity subsolution of (12) there holds*

$$\sup_\Omega \max\{u_1, \dots, u_d\} \le C \left(\sup_{\partial\Omega} \max\{u_1, \dots, u_d\} + \|f\|_{L^N(\Omega)} \right),$$

where C *depends only on* Ω *and on the coefficients of* L *and* $\mathcal{C}$*. Respectively, for any supersolution we have*

$$\inf_\Omega \min\{u_1, \dots, u_d\} \ge C \left(\inf_{\partial\Omega} \min\{u_1, \dots, u_d\} - \|f\|_{L^N(\Omega)} \right).$$

(d) the operator $L + \mathcal{C}$ satisfies the maximum principle in Ω, that is, if $Lu + \mathcal{C}u \le 0$ in Ω and $u \ge 0$ on $\partial\Omega$, then $u \ge 0$ in Ω.

(ii) *if $\lambda_1^{(k)} > 0$ for all $k = 1, \dots, m$, then for any $f \in C^\alpha(\Omega)$ (or $f \in L^p(\Omega), p \ge N$) there exists a unique classical (resp. in $W^{2,p}(\Omega) \cap C(\overline{\Omega})$) solution of (12), such that $u = 0$ on $\partial\Omega$;*

(iii) *Suppose $\psi \in C(\overline{\Omega}, \mathbb{R}^d)$ is such that $\psi \ge 0$ and $L\psi + \mathcal{C}\psi \le 0$ in Ω. If $\psi_j \not\equiv 0$ in $\overline{\Omega}$ for some $j \in S_k$ and some $k \in \{1 \dots, m\}$, then $\lambda_1^{(k)} \ge 0$.*

Proof. (i) Theorem 14.1 and Lemma 14.1 in [8] give $(a) \Leftrightarrow (b) \Leftrightarrow (d)$. Theorem 8.1 in [8] gives $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$ is obvious.
(ii) If $m = 1$ this is Theorem 13.2 in [8] (due to Sweers [29]). If $m > 1$ we apply this theorem m times : using the block-diagonal structure of $\mathcal{C}$, first we solve $(L^{(1)} + \mathcal{C}_{11})u^{(1)} = f^{(1)}$, then $(L^{(2)} + \mathcal{C}_2)u^{(2)} = f^{(2)} - \mathcal{C}_{21}u^{(1)}$, etc.
(iii) This follows from the cooperativeness of $\mathcal{C}$ and the definition of the first eigenvalue, together with Theorem 14.1 from [8]. □

Using this theorem, it is easy to prove the following two lemmas, classical in the Ambrosetti-Prodi setting. From now on, any time we write a norm of a function, we are going to mean the $L^\infty(\Omega)$-norm.

Lemma 2.1 *Under the hypotheses of either Theorem 2 or Theorem 3, for each $t \in \mathbb{R}^d$ there exists a classical subsolution $\underline{u} \le 0$ of system $(\mathcal{P}_t)$.*

Proof. Set $K = 2\max_{i=1,\dots,d}\{\|h_i\| + |t_i|\} + b_1$ (b_1 is the constant from hypothesis (5)). By the previous theorem and (4) we can find a solution of the system

$$Lu + A_1(x)u = Ke - h(x) - t\varphi_1(x)$$

with Dirichlet boundary condition. Clearly the solution of this problem is nonpositive (by the maximum principle, Theorem 4(d)) and can be taken as the subsolution we are searching for. □
Remark. We will show in the proof of Proposition 2.1 in the next section that $\underline{u}$ is smaller than any supersolution of $(\mathcal{P}_t)$.

Lemma 2.2 *Under the hypotheses of either Theorem 2 or Theorem 3, there exists $t_0 \in \mathbb{R}$ such that for each $t \le t_0\, e$ there exists a classical supersolution $\overline{u} \ge 0$ of system $(\mathcal{P}_t)$.*

Proof. By the hypotheses, there exist constants C_1 and $p_i \ge 1$ such that for all $u \ge 0$

$$f(x, u) \le C_1\,(1 + u_1^{p_1} + \ldots + u_d^{p_d})\; e.$$

Let $\overline{u}$ be the solution of the following d equations

$$Lu + h^+ + C_1 e = 0$$

with $\overline{u} = 0$ on $\partial\Omega$. By the maximum principle $\overline{u} \geq 0$ in Ω. By the well known properties of φ_1 and Hopf's lemma we can choose $t_0 \in \mathbb{R}$ such that

$$-t_0\varphi_1 \geq C_1 \left(\overline{u}_1^{p_1} + \ldots + \overline{u}_d^{p_d}\right).$$

Using the two inequalities above in the equation for $\overline{u}$ we get the result. $\square$

Once the results of the two previous lemmas are available, the Method of Monotone Iteration (see [1]) can be applied to get a minimal solution of the problem for sufficiently small t, see Proposition 2.1 below. Observe that this method applies for cooperative elliptic systems, which is the case here, since the functions f are quasi-monotone. For the reader's convenience we state the following result, which will be sufficient for our purposes (see for example [23] for more general statement).

Theorem 5 *Suppose $f(x,u)$ is a quasi-monotone map, which is Hölder continuous in x and locally Lipschitz continuous in u. Suppose $\underline{u}, \overline{u} \in C(\overline{\Omega}, \mathbb{R}^d)$ are respectively a subsolution and a supersolution of the system*

$$Lu + f(x,u) = 0 \tag{14}$$

in Ω, such that $\underline{u} \leq \overline{u}$ in Ω, $\underline{u} \leq 0 \leq \overline{u}$ on $\partial\Omega$. Then there exists a classical solution u of (14) such that $\underline{u} \leq u \leq \overline{u}$ in Ω, $u = 0$ on $\partial\Omega$.

Sketch of the proof of Theorem 5. Set $u_0 = \underline{u}$, $m = \inf_\Omega \min_i \underline{u}_i$, $M = \sup_\Omega \max_i \overline{u}_i$ and let

$$k = \max_{1\leq i\leq n} \left|\left|\frac{\partial f_i}{\partial u_i}\right|\right|_{L^\infty(\Omega\times[m,M]^n)}.$$

By Theorem 4 we can solve the hierarchy of problems

$$\begin{array}{rcll} -Lu^{(n+1)} + ku^{(n+1)} & = & f(x,u^{(n)}) + ku^{(n)} & \text{in} \quad \Omega \\ u^{(n+1)} & = & 0 & \text{on} \quad \partial\Omega. \end{array}$$

It is then easy to check, with the help of the maximum principle, that we have $\underline{u} \leq u^{(n)} \leq u^{(n+1)} \leq \overline{u}$ for all n, so $u^{(n)}$ converges to a solution of (14).

2.2 A priori bounds

We start with a lemma which shows that the negative parts of the solutions of $(\mathcal{P}_t)$ are uniformly bounded, provided t is above some fixed level. From now on, all constants we write may change from line to line and depend only on the data in $(\mathcal{P}_t)$ - that is, on L, f, h, Ω (and on other quantities, if stated). We will also make the convention that any norm of a vector is the maximum of the corresponding norms of its components.

Lemma 2.3 *Under the hypotheses of either Theorem 2 or Theorem 3, for each $C_0 \in \mathbb{R}_+$ there exists a constant M such that for any $t \geq -C_0 e$ and any solution u of $(\mathcal{P}_t)$ with this t we have*

$$\|u^-\| \leq M.$$

Proof. Set $m = \max_i \{\|h_i\|_{L^\infty(\Omega)} + C_0\}$. So $(\mathcal{P}_t)$ yields

$$Lu + f(x,u) \leq m\, e \qquad \text{in } \Omega.$$

Since f is quasi-monotone this implies

$$L_i u_i + f_i(x, -u_1^-, \dots, -u_{i-1}^-, u_i, -u_{i+1}^-, \dots, -u_d^-) \leq m, \tag{15}$$

for each $i \in \{1, \dots, d\}$. On the other hand, again by the quasi-monotonicity of f,

$$f_i(x, -u_1^-, \dots, -u_{i-1}^-, 0, -u_{i+1}^-, \dots, -u_d^-) \leq f_i(x, 0, \dots, 0) = 0.$$

This means that $u_i \equiv 0$ is also a solution of (15), seen as a scalar equation in u_i. As explained in the previous section, the minimum of two supersolutions is a viscosity supersolution, hence (15) continues to hold if we replace u_i by $-u_i^-$, which gives,

$$L(-u^-) + f(x, -u^-) \leq m\, e,$$

in the viscosity sense. By hypothesis (5)

$$L(-u^-) + A_1(x)(-u^-) \leq (m + b_1)e,$$

Since (4) holds, Theorem 4 (c) implies the lemma. □.

Proposition 2.1 *Under the hypotheses of either Theorem 2 or Theorem 3, there exists $t_0 \in \mathbb{R}$ such that for each $t \leq t_0\, e$ there exists a minimal solution of system $(\mathcal{P}_t)$.*

Proof. Using Theorem 5 and Lemmas 2.1 and 2.2 we show the existence of a minimal solution of $(\mathcal{P}_t)$, for any fixed $t \le t_0\, e$, where the t_0 is the one of 2.2. For that matter we first claim that any supersolution $\overline{u}$ of $(\mathcal{P}_t)$ satisfies $\overline{u} \ge \underline{u}$ in Ω, where $\underline{u}$ is the subsolution constructed in Lemma 2.1. Once this is done, we use Theorem 5 and finish. In order to prove the claim, we proceed as in the proof of Lemma 2.3 we have $L(-\overline{u}^-) + f(x, -\overline{u}^-) \le me$ provided $\overline{u}$ is a supersolution of $(\mathcal{P}_t)$, so, by the way $\underline{u}$ is chosen,

$$L(-\overline{u}^- - \underline{u}) + f(x, -\overline{u}^-) - A_1(x)\underline{u} \le 0.$$

Hence, by (5),

$$(L + A_1(x))(-\overline{u}^- - \underline{u}) \le 0,$$

and so the maximum principle implies $-\overline{u}^- - \underline{u} \ge 0$, which implies $\overline{u} \ge -\overline{u}^- \ge \underline{u}$ in Ω. $\square$.

Remark. The above proof shows that, if for some t problem $(\mathcal{P}_t)$ has a solution u, then $u \ge \underline{u}$, where $\underline{u}$ is the subsolution in Lemma 2.1. So as soon as $(\mathcal{P}_t)$ is solvable for some t, then a minimal solution exists.

Next, we show that for any unbounded sequence $\{t_n\}$, the growth of the corresponding solutions – if such solutions exist – controls the growth of $\{t_n\}$.

Lemma 2.4 *Under the hypotheses of either Theorem 2 or Theorem 3, for each $C_0 \in \mathbb{R}_+$ there exists a constant C_1 such that for any $t \ge -C_0 e$ and any solution u of $(\mathcal{P}_t)$ with this t we have*

$$t_i^+ \le C_1(1 + \|u_i^+\|) \le C_1(1 + \|u\|), \quad i = 1, \ldots, d.$$

Proof. By using successively the quasi-monotonicity of f, property (8) and finally property (7), we get

$$f_i(x, u) \ge -C(1 + u_i^+), \tag{16}$$

for all $i \in \{1, \ldots, n\}$, $x \in \Omega, u \ge -M\, e$ (M is the constant from Lemma 2.3). Suppose now for some $t \in \mathbb{R}^d$ and some function u we have

$$Lu + f(x, u) + t\varphi_1 + h = 0,$$

and $u = 0$ on $\partial\Omega$. This implies, by (16), that we have the following d scalar inequalities

$$L_i(u_i - t_i\lambda_{1,i}^{-1}\varphi_{1,i}) \le C(1 + u_i^+),$$

and $u_i - t_i\lambda_{1,i}^{-1}\varphi_{1,i} = 0$ on $\partial\Omega$, where the constant C absorbs the norm of h. By the Alexandrov-Bakelman-Pucci inequality (Theorem 4 (c) for $d = 1$) we get

$$u_i - t_i\lambda_{1,i}^{-1}\varphi_{1,i} \ge -C(1 + \|u_i^+\|), \quad i = 1, \ldots, d.$$

from which the result follows (we recall that the first eigenvectors $\varphi_{1,i}$ are normalized so that $\|\varphi_{1,i}\| = 1$). $\square$

We can now deduce an a priori bound in the linear growth case.

Proposition 2.2 *Under the hypotheses of Theorem 2, for each* $C_0 \in \mathbb{R}_+$ *there exists a constant* M *such that for any* $t \geq -C_0 e$ *and any solution* u *of* $(\mathcal{P}_t)$ *with this* t *we have*

$$\|u\| \leq M.$$

Proof. Suppose for contradiction that there exists sequences $\{u_n\}$, $\{t_n\}$, such that $t_n \geq -C_0\, e$, and

$$Lu_n + f(x, u_n) + t_n \varphi_1 + h = 0, \qquad \|u_n\| \to \infty,$$

as $n \to \infty$. Using Lemma 2.3, we get $\lim \|u_n^+\| = \infty$, $\|u_n^+\| = \|u_n\|$ for n large. By dividing the equation by $\|u_n^+\|$ and by using $t_n \geq -C_0\, e$ and (16), we see that $Lv_n \leq C$, where $v_n = u_n / \|u_n^+\|$. By using the linear growth (9) and Lemma 2.4 we have $Lv_n \geq -C$. Hence, by elliptic theory, v_n converges (up to a subsequence) to a function v in $W^{2,p}(\Omega)$. Note that $v \geq 0$, by Lemma 2.3. Of course $\|v_n\| = 1$ for n large, so $\|v\| = 1$.

By using the fact that (7) implies $f(x, u_n) \geq f(x, u_n^+) - o(1)\|u_n\|$, we obtain

$$Lv_n + \frac{f(x, u_n^+)}{\|u_n^+\|} \leq o(1),$$

Hence, by (6) and passage to the limit we see that $v \geq 0$ is a nontrivial solution to $Lv + A_2(x)v \leq 0$, which contradicts (4) and Theorem 4 (iii). $\square$

We now turn to the superlinear case. The following bound plays an essential role.

Proposition 2.3 *Under the hypotheses of Theorem 3, there exists a constant* C *such that for every vector* $t \geq e$ *and every solution* $u = (u_1, u_2)$ *of* $(\mathcal{P}_t)$ *corresponding to this* t*, the following inequalities hold*

$$\|u_1\|^{1+\frac{2}{\beta_1^0}} \leq Ct_1 \qquad \textit{and} \qquad \|u_2\|^{1+\frac{2}{\beta_2^0}} \leq Ct_2.$$

Here $(\beta_1^0, \beta_2^0) > 0$ *is the vector which appears in Theorem 3.*

For clarity, before proving Proposition 2.3, we state the particular case of it when only one equation is considered, that is, when we are in the framework of Theorem 1.

Proposition 2.4 *Under the hypotheses of Theorem 1, there exists a constant C such that for every number $t \geq 1$ and every solution u of $(\mathcal{P}_t)$, corresponding to this t, we have*

$$\|u\| \leq C t^{\frac{1}{p}}.$$

More precisely, to get Proposition 2.4 from Proposition 2.3, we take $L_1 = L_2 = L$, $a_{11} = a_{22} = a$, $\alpha_{11} = \alpha_{22} = p$, $a_{12} = a_{21} = 0$, so $\beta_1^0 = \beta_2^0 = 2/(p-1)$.

Proof of Proposition 2.3. It follows from (9) that we can write, for $i = 1, 2$,

$$f_i(x, s_1, s_2) = a_{i1}(x) s_1^{\alpha_{i1}} + a_{i2}(x) s_2^{\alpha_{i2}} + g_i(x, s_1, s_2),$$

where

$$\lim_{|(s_1,s_2)|\to\infty} [a_{i1} s_1^{\alpha_{i1}} + a_{i2} s_2^{\alpha_{i2}}]^{-1} g_i(x, s_1, s_2) = 0.$$

Now suppose that the result of the proposition is false, that is, there exist sequences $\{u_n\}$, $\{t_n\}$, such that $t_n \geq e$,

$$\begin{aligned} -L_1 u_{1,n} &= a_{11}(u_{1,n}^+)^{\alpha_{11}} + a_{12}(u_{2,n}^+)^{\alpha_{12}} + g_1(x, u_{1,n}, u_{2,n}) + t_{1,n}\varphi_{1,1} + \tilde{h}_1 \\ -L_1 u_{2,n} &= a_{21}(u_{1,n}^+)^{\alpha_{21}} + a_{22}(u_{2,n}^+)^{\alpha_{22}} + g_2(x, u_{1,n}, u_{2,n}) + t_{2,n}\varphi_{1,2} + \tilde{h}_2, \end{aligned}$$

(here $\tilde{h}_i = h_i + d_i$, where d_i is some bounded function, that corresponds to the negative part of $u_{i,n}$, which is bounded by Lemma 2.3) and

$$\|u_{1,n}\|^{1+\frac{2}{\beta_1^0}} \geq n t_{1,n} \qquad \text{or} \qquad \|u_{2,n}\|^{1+\frac{2}{\beta_2^0}} \geq n t_{2,n}.$$

Taking a subsequence if necessary, we may suppose that one of these inequalities (say the first) holds for each n.

Assume first that

$$\lim_{n\to\infty} \frac{\|u_{1,n}\|^{1+\frac{2}{\beta_1^0}}}{t_{1,n}} = \lim_{n\to\infty} \frac{\|u_{2,n}\|^{1+\frac{2}{\beta_2^0}}}{t_{2,n}} = \infty. \tag{17}$$

We will use a blow-up type argument, originally due to Gidas and Spruck [21] in the case of a scalar equation, and developed for our type of systems in [18] - we refer to that paper for details. Set

$$\lambda_n = \|u_{1,n}\|^{-1/\beta_1^0} \qquad \text{if} \qquad \|u_{1,n}\|^{\beta_2^0} \geq \|u_{2,n}\|^{\beta_1^0},$$

and $\lambda_n = \|u_{2,n}\|^{-1/\beta_2^0}$ otherwise (say we are in the first of these situations). Then $\lambda_n \to 0$ and the functions

$$v_{i,n} = \lambda_n^{\beta_1^0} u_{i,n}(\lambda_n x + x_n)$$

are such that $v_{1,n}(0)=1, 0\le v_{i,n}\le 1$ in Ω (here x_n is the point in Ω where $u_{1,n}$ attains its maximum). Then

$$\begin{aligned} -L_{1,n}v_{1,n} &= a_{11}(\cdot)\lambda_n^{\gamma_{11}}(v_{1,n}^+)^{\alpha_{11}}+a_{12}(\cdot)\lambda_n^{\gamma_{12}}(v_{2,n}^+)^{\alpha_{12}}+\lambda_n^{\beta_1^0+2}t_{1,n}\varphi_{1,1} \\ -L_{2,n}v_{2,n} &= a_{21}(\cdot)\lambda_n^{\gamma_{21}}(v_{1,n}^+)^{\alpha_{21}}+a_{22}(\cdot)\lambda_n^{\gamma_{22}}(v_{2,n}^+)^{\alpha_{22}}+\lambda_n^{\beta_2^0+2}t_{2,n}\varphi_{1,2}, \end{aligned} \tag{18}$$

(we have omitted the terms coming from g_i, h_i, d_i, since they tend to zero as $n\to\infty$) ; here

$$\gamma_{ij}=\beta_i^0+2-\beta_j^0\alpha_{ij},$$

(recall the equations of the lines $l_{ij}=\{\vec{\beta}\mid \beta_i+2-\beta_j\alpha_{ij}=0\}$), and

$$L_{k,n}=\sum_{i,j=1}^N a_{ij}(\cdot)\frac{\partial^2}{\partial x_i\partial x_j}+\lambda_n\sum_{i=1}^N b_i^{(k)}(\cdot)\frac{\partial}{\partial x_i},$$

the dot stands for $\lambda_n x+x_n$, and the equations are given in the domain $\lambda_n^{-1}(\Omega-x_n)$.

We can pass to the limit in (18) as in Lemma 2.1 in [18], and conclude that $v_{i,n}$ converges (up to a subsequence) to a bounded function v_i (note that v_i are nonnegative, since the negative parts of $u_{i,n}$ are bounded, by Lemma 2.3). The difference with [18] is in the last terms in the right hand side of (18). However, these terms turn out to vanish as $n\to\infty$, under the hypothesis that Proposition 2.3 is false. Indeed, under (17),

$$\lambda_n^{\beta_1^0+2}t_{1,n}=\|u_{1,n}\|^{-\frac{\beta_1^0+2}{\beta_1^0}}t_{1,n}\longrightarrow 0 \quad \text{as} \quad n\to\infty,$$

while, by the choice of λ_n,

$$\lambda_n^{\beta_2^0+2}t_{2,n}\le\|u_{2,n}\|^{-\frac{\beta_2^0+2}{\beta_2^0}}t_{2,n}\longrightarrow 0 \quad \text{as} \quad n\to\infty,$$

by (17).

Hence, after the passage to the limit, we obtain a system in $\mathbb{R}^N$ or in a half-space, which, by the results in [18] (see also the references there), has only the trivial solution, which contradicts $v_1(0)=1$. Note that the differential operator in the limiting system has constant coefficients and can be transformed into the Laplacian through an orthogonal change of variables.

Next, suppose (17) does not hold, that is

$$\|u_{2,n}\|^{1+\frac{2}{\beta_2^0}}\le Kt_{2,n},$$

for some constant K. Combining this inequality with Lemma 2.4 (used twice), we see that both sequences $\{\|u_{2,n}\|\}$ and $\{t_{2,n}\}$ are bounded. Then we can repeat exactly the same argument as above to get a contradiction with $\|u_{1,n}\|^{1+\frac{2}{\beta_1^0}} \geq nt_{1,n} \geq n$ - note that at the end $\lambda_n^{\beta_2^0+2} t_{2,n} \to 0$ trivially follows from $\lambda_n \to 0$. □

Actually, looking at the proof of Proposition 2.3, we see that it implies the following stronger statement.

Proposition 2.5 *Under the hypotheses of Theorem 3, for each $C_0 \in \mathbb{R}_+$ there exists a constant M such that for any $t \geq -C_0 e$ and any solution $u = (u_1, u_2)$ of $(\mathcal{P}_t)$ with this t we have*

$$\|u_1\|^{1+\frac{2}{\beta_1^0}} \leq M \max\{1, t_1\} \qquad \textit{and} \qquad \|u_2\|^{1+\frac{2}{\beta_2^0}} \leq M \max\{1, t_2\}.$$

Proof. First, since $t\varphi_1 = t^+\varphi_1 - t^-\varphi_1$, we can think of the bounded term $t^-\varphi_1$ as being part of the function $h(x)$, and assume $t > 0$. Then we simply repeat the proof of Proposition 2.3, replacing t by $\max\{t, e\}$ in it. □

We can now conclude that solutions of our system admit a priori bounds and that the system does not have solutions if t is large.

Proposition 2.6 *Under the hypotheses of Theorem 2 or Theorem 3, there exists a constant M such that if for some $t \geq 0$ there exists a solution u of system $(\mathcal{P}_t)$, then*

$$t \leq Me \qquad \textit{and} \qquad \|u\| \leq M.$$

More generally, for each $t \in \mathbb{R}^d$ there exists a constant M with this property, and M depends only on t^-.

Proof. Combine Proposition 2.2 or Proposition 2.5 with Lemma 2.3 and Lemma 2.4. □

2.3 Conclusion

In the previous sections we have established the following facts, which will now be used to complete the proofs of Theorems 2 or Theorem 3 :

(i) if C is sufficiently large, $(\mathcal{P}_t)$ has a minimal solution for $t \leq -Ce$;

(ii) if C is sufficiently large, $(\mathcal{P}_t)$ does not have a solution for $\|t\| \geq C$;

(iii) a priori bound : given $t_0 \in \mathbb{R}^d$, the (eventual) solutions of $(\mathcal{P}_t)$ for all $t \geq t_0$ are bounded by the same constant.

Next, the surface Γ will be defined by parametrization with respect to the hyperplane $H = \{t \in \mathbb{R}^d \mid t_1 + \ldots + t_d = 0\}$. Let us define, for each $t_0 \in H$

$$A(t_0) = \{k \in \mathbb{R} \ : \ (\mathcal{P}_{t_0+ke}) \text{ has a solution } \}$$

By (i) above this set is not empty.

On the other hand we know ((ii) above) that for each $t_0 \in H$ there is a $k_0 \in \mathbb{R}$ such that problem $(\mathcal{P}_{t_0+ke})$ does not have a solution for all $t_0 + ke$ with $k \geq k_0$. So the function $K : H \to \mathbb{R}$, $K(t) = \sup A(t)$ is well defined. Further, if $k \in A(t)$ for some t then any $k' \leq k$ also belongs to $A(t)$. Indeed, a solution of $(\mathcal{P}_{t_0+ke})$ is a supersolution for $(\mathcal{P}_{t_0+k'e})$ and by Lemma 2.1 and Theorem 5 we have a solution of $(\mathcal{P}_{t_0+k'e})$. So $A(t)$ is an interval.

Next, the function $K(t) : H \to \mathbb{R}$ is Lipschitz continuous, with Lipschitz constant 1. Indeed, to show this one can use the following argument from [14] : given $t^1, t^2 \in H$ it follows from what we just saw that

$$t^1 + K(t^1)e \not< t^2 + K(t^2)e, \qquad t^2 + K(t^2)e \not< t^1 + K(t^1)e.$$

Hence there exist indices $i, j \in \{1, \ldots, d\}$ such that

$$t_i^2 + K(t^2) \leq t_i^1 + K(t^1), \qquad t_j^2 + K(t^2) \geq t_j^1 + K(t^1),$$

so

$$-|t^1 - t^2| \leq t_i^2 - t_i^1 \leq K(t^1) - K(t^2) \leq t_j^2 - t_j^1 \leq |t^1 - t^2|.$$

This yields that the hypersurface $\Gamma = \{t + K(t)e \ : \ t \in H\}$ is Lipschitz.

Next we prove that problem $(\mathcal{P}_{t_0+ke})$ has at least two solutions for $k < K(t_0)$. Viewing to use topological degree arguments, let us define the mapping $S_t : C^{1,\alpha}(\Omega)^d \to C^{1,\alpha}(\Omega)^d$ by $u = S_t v$, where

$$\begin{cases} -Lu &= f(x,v) + t\varphi_1(x) + h(x) & \text{in} \quad \Omega \\ \quad u &= 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Finding a solution of $(\mathcal{P}_t)$ is equivalent to obtaining a fixed point of S_t. The search for fixed point will be done with the help of degree considerations. We will be brief here, we refer for example to [16], where a simple and thorough account of this type of argument is given.

Fix $t_0 \in H$ and $k_0 < K(t_0)$. By Proposition 2.1 problem $(\mathcal{P}_{t_0+k_0e})$ has a minimal solution. It is classical that the Leray-Schauder degree of this minimal solution is one, so there exists an open set $\mathcal{O}$ in $C^{1,\alpha}(\Omega)^d$ which contains the minimal solution and

$$\deg (I - S_{t_0+k_0e}, \mathcal{O}, 0) = 1.$$

On the other hand, by (ii) above there exists $\overline{k} \in \mathbb{R}$ such that problem $(\mathcal{P}_{t_0+ke})$ has no solution for $k \geq \overline{k}$. This implies

$$\deg\left(I - S_{t_0+\overline{k}e}, B_R, 0\right) = 0.$$

for any ball $B_R \subset C^{1,\alpha}(\Omega)^d$.

However, the a priori bound (iii) implies that there exists R sufficiently large, such that

$$\deg\left(I - S_{t_0+ke}, B_R, 0\right) \text{ is constant in } k \geq k_0.$$

This means that $\deg\left(I - S_{t_0+k_0e}, B_R, 0\right) = 0$, and by the excision property of the degree there exists a solution of $(\mathcal{P}_{t_0+k_0e})$ in $B_R \setminus \mathcal{O}$.

Finally, given $t \in H$ we take a sequence $k_n \nearrow K(t)$ and a sequence of solutions of $(\mathcal{P}_{t+k_ne})$. Thanks to the a priori bounds this sequence is bounded in $L^\infty(\Omega)$ and elliptic theory permits us to pass in the limit in $(\mathcal{P}_{t+k_ne})$, which gives one solution of $(\mathcal{P}_{t+K(t)e})$.

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