SURGERY AND HARMONIC SPINORS

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Abstract. Let M be a compact manifold with a fixed spin structure χ. The Atiyah-Singer index theorem implies that for any metric g on M the dimension of the kernel of the Dirac operator is bounded from below by a topological quantity depending only on M and χ. We show that for generic metrics on M this bound is attained.

1. Introduction

We suppose that M is a compact spin manifold. By a *spin manifold* we will always mean a smooth manifold equipped with an orientation and a spin structure. After choosing a metric g on M, one can define the spinor bundle $\Sigma^g M$ and the Dirac operator $D^g : \Gamma(\Sigma^g M) \to \Gamma(\Sigma^g M)$ see [6, 12, 8].

Being a self-adjoint elliptic operator D^g shares many properties with the Hodge-Laplacian $\Delta_p^g : \Gamma(\Lambda^p T^* M) \to \Gamma(\Lambda^p T^* M)$. In particular, if M is compact, then the spectrum is discrete and real, and the kernels of Δ_p^g and D^g are finite-dimensional. Elements of $\ker \Delta_p^g$ resp. $\ker D^g$ are called *harmonic forms* resp. *harmonic spinors*.

However, the relation of Δ_p^g resp. D^g to topology is different. Hodge theory tells us that the Betti numbers $b_p := \dim \ker \Delta_p^g$ only depend on the topological type of M. The dimension of the kernel of D^g is invariant under conformal changes of the metric, however it does depend on the choice of conformal structure. The first examples of this phenomenon were constructed by Hitchin [9], and it was conjectured by several people including Bär and the second named author [2] that $\dim \ker D^g$ depends on the metric for any compact spin manifold of dimension ≥ 3.

On the other hand, $\dim \ker D^g$ is topologically obstructed. The Index Theorem by Atiyah and Singer gives a topological lower bound on the dimension of the kernel of the Dirac operator. For M a compact spin manifold of dimension n this bound is [12], [2, Section 3]

$$\dim \ker D^g \geq \begin{cases} |\hat{A}(M)|, & \text{if } n \equiv 0 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 2, & \text{if } n \equiv 2 \mod 8 \text{ and } \alpha(M) \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

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Here the $\hat{A}$-genus $\hat{A}(M) \in \mathbb{Z}$ and the α-genus $\alpha(M) \in \mathbb{Z}/2\mathbb{Z}$ are invariants of (the spin bordism class of) the differential spin manifold M, and g is any Riemannian metric on M.

It is hence natural to ask whether metrics exist, such that equality holds in (1). Such metrics will be called *D-minimal*. In [13] it is proved that a generic metric on a manifold of dimension ≤ 4 is D-minimal. In [2] the same result is proved for manifolds of dimension at least 5 which are simply connected or have certain fundamental groups. The argument in [2] utilizes the surgery-bordism method which has proven itself very powerful in the study of manifolds with positive scalar curvature metrics. In a similar fashion we will use surgery methods to prove the following.

Theorem 1.1. *Let M be a compact connected spin manifold. Then a generic metric on M is D-minimal.*

Our method also yields a new proof in dimensions 2, 3 and 4. Since $\dim \ker D$ behaves additively with respect to disjoint union of spin manifolds while the $\hat{A}$-genus/α-genus may cancel it is easy to find disconnected manifolds with no D-minimal metric.

The case $n = 2$ is special. If M is a compact Riemann surface of genus ≤ 2, then all metrics are D-minimal. The same holds for Riemann surface of genus 3 whose spin structure is not spin bordant 0. However if the genus is ≥ 4 (or equal to 3 with spin structures that are spin bordant 0), then there are also metrics with larger kernel [9], see also [3].

In order to explain the surgery-bordism method in the proof of Theorem 1.1 we have to fix some notation.

A smooth embedding $f : N \to M$ is called *spin preserving* if the pullback of the orientation and spin structure of M to N under f is the orientation and spin structure of N. If M is a spin manifold we denote by M^- the same manifold with the opposite orientation.

For $l \geq 1$ we denote by $B^l(R)$ the standard l-dimensional open ball of radius R and by $S^{l-1}(R)$ its boundary. We abbreviate $B^l = B^l(1)$ and $S^{l-1} = S^{l-1}(1)$. The standard Riemannian metrics on $B^l(R)$ and $S^{l-1}(R)$ are denoted by g^{flat} and g^{round}. We equip $S^{l-1}(R)$ with the *bounding spin structure*, i.e. the spin structure obtained by restricting the unique spin structure on $B^l(R)$ (if $l > 2$ the spin structure on $S^{l-1}(R)$ is unique, if $l = 2$ it is not).

Let $f : S^k \times \overline{B^{n-k}} \to M$ be a spin prerserving embedding, Then we define

$$\widetilde{M} = \left(M \setminus f(S^k \times B^{n-k})\right) \cup \left(\overline{B^{k+1}} \times S^{n-k-1}\right) / \sim$$

where $\sim$ identifies the boundary of $S^k \times S^{n-k-1}$ with $f(S^k \times S^{n-k-1})$. The toplogical space $\widetilde{M}$ carries a differential structure and a spin structure such that the inclusions $M \setminus f(S^k \times B^{n-k}) \hookrightarrow \widetilde{M}$ and $\overline{B^{k+1}} \times S^{n-k-1} \hookrightarrow \widetilde{M}$ are spin preserving smooth embeddings.

We say that $\widetilde{M}$ is obtained from M *by surgery of dimension k* or *by surgery of codimension $n-k$*.

The proof of Theorem of Theorem 1.1 relies on the following surgery theorem.

Theorem 1.2. *Let (M, g^M) be a compact n-dimensional Riemannian spin manifold. Let $\widetilde{M}$ be obtained from M by surgery in dimension k, $k \in \{0, 1, \ldots, n-2\}$. Then $\widetilde{M}$ carries a metric $g^{\widetilde{M}}$ such that*

$$\dim \ker D^{g^{\widetilde{M}}} \leq \dim \ker D^{g^M}.$$

2. Preliminaries

2.1. **Spinor bundles for different metrics.** Let M be a spin manifold of dimension n and let g, g' be Riemannian metrics on M. The goal of this paragraph is to identify the spinor bundles of (M, g) and (M, g') using the method of Bourguignon and Gauduchon introduced in [5].

There exists a unique endomorphism $b^g_{g'}$ of TM which is positive, symmetric with respect to g, and satisfies $g(X, Y) = g'(b^g_{g'}X, b^g_{g'}Y)$ for all $X, Y \in TM$. This endomorphism maps g-orthonormal frames at a point to g'-orthonormal frames at the same point and we get a map $b^g_{g'} : \mathrm{SO}(M, g) \to \mathrm{SO}(M, g')$ of $\mathrm{SO}(n)$-principal bundles. If we assume that $\mathrm{Spin}(M, g)$ and $\mathrm{Spin}(M, g')$ are equivalent spin structures on M the map $b^g_{g'}$ lifts to a map $\beta^g_{g'}$ of $\mathrm{Spin}(n)$-principal bundles,

$$\begin{array}{ccc} \mathrm{Spin}(M,g) & \xrightarrow{\beta^g_{g'}} & \mathrm{Spin}(M,g') \\ \downarrow & & \downarrow \\ \mathrm{SO}(M,g) & \xrightarrow{b^g_{g'}} & \mathrm{SO}(M,g') \end{array}.$$

From this we get a map between the spinor bundles $\Sigma^g M$ and $\Sigma^{g'} M$ denoted by the same symbol and defined by

$$\begin{array}{rcl} \Sigma^g M = \mathrm{Spin}(M, g) \times_\sigma \Sigma_n & \to & \mathrm{Spin}(M, g') \times_\sigma \Sigma_n = \Sigma^{g'} M \\ \psi = [s, \varphi)] & \mapsto & [\beta^g_{g'} s, \varphi] = \beta^g_{g'} \psi \end{array} \tag{2}$$

where (σ, Σ_n) is the complex spinor representation, and where $[s, \varphi]$ denotes the equivalence class of $(s, \varphi) \in \mathrm{Spin}(M, g) \times_\sigma \Sigma_n$ for the equivalence relation given by the action of $\mathrm{Spin}(n)$. The map $\beta^g_{g'}$ preserves fiberwise length of spinors.

We define the Dirac operator $D^{g'}$ acting on sections of the spinor bundle for g by

$${}^g D^{g'} = (\beta^g_{g'})^{-1} \circ D^{g'} \circ \beta^g_{g'}$$

In [5, Thm. 20] the operator ${}^g D^{g'}$ is computed in terms of D^g and some extra terms which are small if g and g' are close. Formulated in a way convenient for us the relationship is

$${}^g D^{g'} \psi = D^g \psi + A^g_{g'}(\nabla^g \psi) + B^g_{g'}(\psi) \tag{3}$$

where $A^g_{g'} \in \hom(T^* M \otimes \Sigma^g M, \Sigma^g M)$ satisfies

$$|A^g_{g'}| \leq C|g - g'|_g \tag{4}$$

and $B^g_{g'} \in \hom(\Sigma^g M, \Sigma^g M)$ satisfies

$$|B^g_{g'}| \leq C(|g - g'|_g + |\nabla^g (g - g')|_g) \tag{5}$$

for some constant C.

In the special case that g' and g are conformal with $g' = F^2 g$ for a positive smooth function F we have

$$ {}^gD^{g'}(F^{-\frac{n-1}{2}}\psi) = F^{-\frac{n+1}{2}} D^g\psi \tag{6}$$

according to [9, 4, 8].

2.2. **Notations for spaces of spinors.** Throughout the article φ and ψ and its variants denote spinors, i.e. sections of the spinor bundle. If S is a closed or open subset of M, we write $C^k(S)$ both for the space of k times differentiable functions on S and for the space of k times differentiable spinors. As the bundle will be clear from the context, this will not lead to ambiguities. On $C^k(S)$ we define the norm

$$\|\varphi\|_{C^k(S)} := \sum_{l=0}^{k} \sup_{x\in S} |\nabla^l\varphi(x)|.$$

We sometimes write $\|\varphi\|_{C^k(S,g)}$ instead of $\|\varphi\|_{C^k(S)}$ to indicate that the spinor bundle and the norm depend on g. The analogous notation is used for Schauder spaces $C^{k,\alpha}$.

Similarly $L^2(S) = L^2(S,g)$ and $H_k^2(S) = H_k^2(S,g)$ denote the space of L^2-spinors and H_k^2-spinors. These spaces come with the norms

$$\|\varphi\|^2_{L^2(S,g)} := \int_S |\varphi|^2\, dv^g \qquad \|\varphi\|^2_{H_k^2(S,g)} := \sum_{l=0}^{k} \int_S |\nabla^l\varphi|^2\, dv^g.$$

Let U be an open set. The set of locally C^1-spinors $C^1_{\mathrm{loc}}(U)$ carries a topology such that $\varphi_i \to \varphi$ in $C^1_{\mathrm{loc}}(U)$ if and only if $\varphi_i \to \varphi$ in $C^1(K)$ for any compact subset $K \subset U$.

2.3. **Regularity and elliptic estimates.** In the following section M is not necessarily compact.

Lemma 2.1. *Let (M,g) be a Riemannian manifold, and let ψ be a spinor of regularity L^2. If ψ is* weakly harmonic, *i.e.*

$$\int_M \langle\psi, D\varphi\rangle\, dv^g = 0$$

for all compactly supported smooth spinors φ, then ψ is smooth.

Lemma 2.2. *Let (M,g) be a Riemannian manifold and let $K \subset M$ a compact subset. Then there is a constant $C = C(K,M,g)$ such that*

$$\|\psi\|_{C^2(K,g)} \le C\|\psi\|_{L^2(M,g)}$$

for all harmonic spinors ψ on (M,g).

Proof of the lemmata.
The condition of the first lemma implies $\int\langle\psi, D^2\Phi\rangle\, dv^g = 0$ for any compactly supported smooth spinor Φ. Writing down the equation in local coordinates, one can use standard tools from partial differential equations (as for example [7, Theorem 8.13]) to derive via recursion that ψ is contained in $H_k^2(K_1)$ for any $k \in \mathbb{N}$ and any K_1 compact in M, and that

$$\|\psi\|_{H_k^2(K_1,g)} \le C\|\psi\|_{L^2(M,g)}. \tag{7}$$

Suppose that the boundary of K_1 is smooth. One then uses the Sobolev embedding $H_k^2(K_1,g) \to C^1(K_1,g)$ for $k > n/2+1$ (see [1, Theorem 6.2]), and we get $\psi \in C^1(K_1,g)$ and an estimate for $\|\psi\|_{C^1(K_1,g)}$ analogous to (7). Now one can use Schauder estimates as in [7, Theorem 6.6] to conclude that ψ is smooth on any compactum K contained in the interior of K_1, and in order to derive a C^2 estimate. □

Lemma 2.3 (Ascoli's theorem, [1, Theorem 1.30 and 1.31])**.** *Let φ_i be a sequence bounded in $C^{1,\alpha}(K)$. Then a subsequence converges in $C^1(K)$.*

2.4. **Removal of singularities lemma.** In the proof of Theorem 1.2 we will need the following lemma.

Lemma 2.4. *Let (M,g) be an n-dimensional Riemannian spin manifold and let $S \subset M$ be a compact submanifold of dimension $k \leq n-2$. Assume that φ is a spinor field such that $\|\varphi\|_{L^2(M)} < \infty$ and $D^g\varphi = 0$ weakly on $M \setminus S$. Then $D^g\varphi = 0$ holds weakly also on M.*

Proof. Let ψ be a smooth spinor compactly supported in M. We have to show that

$$\int_M \langle \varphi, D^g\psi\rangle \, dv^g = 0. \tag{8}$$

Let $U_S(\varepsilon)$ be the set of points of distance at most ε to S. For a small $\varepsilon > 0$ we choose a smooth function $\eta : M \to [0,1]$ such that $\eta = 1$ on $U_S(\varepsilon)$, $|\mathrm{grad}\eta| \leq 2/\varepsilon$ and $\eta = 0$ outside $U_S(2\varepsilon)$. We rewrite the left hand side of (8) as

$$\begin{aligned}\int_M \langle \varphi, D^g\psi\rangle \, dv^g &= \int_M \langle \varphi, D^g((1-\eta)\psi + \eta\psi)\rangle \, dv^g \\ &= \int_M \langle \varphi, D^g((1-\eta)\psi)\rangle \, dv^g \\ &+ \int_M \langle \varphi, \eta D^g\psi\rangle \, dv^g + \int_M \langle \varphi, \mathrm{grad}\eta \cdot \psi\rangle \, dv^g.\end{aligned}$$

As $D^g\varphi = 0$ weakly on $M \setminus S$ the first term vanishes. The absolute value of the second term is bounded by

$$\|\varphi\|_{L^2(U_S(2\varepsilon))}\|D^g\psi\|_{L^2(U_S(2\varepsilon))}$$

which tends to 0 as $\varepsilon \to 0$. Finally, the absolute value of the third term is bounded by

$$\begin{aligned}\frac{2}{\varepsilon}\|\varphi\|_{L^2(U_S(2\varepsilon))}\|\psi\|_{L^2(U_S(2\varepsilon))} &\leq \frac{C}{\varepsilon}\|\varphi\|_{L^2(U_S(2\varepsilon))}(\mathrm{Vol}(U_S(2\varepsilon)\cap \mathrm{supp}(\psi))^{\frac{1}{2}} \\ &\leq C\|\varphi\|_{L^2(U_S(2\varepsilon))}\varepsilon^{\frac{n-k}{2}-1}.\end{aligned}$$

Since $n-k \geq 2$, the third term also tends to 0 as $\varepsilon \to 0$. □

2.5. **Products with spheres.** The spectrum of $(D^{g^{\mathrm{round}}})^2$ is bounded from below by $l^2/4$.

If (M,g) and (N,h) are compact Riemannian spin manifolds then the squared Dirac operator $(D^{g+h})^2$ on $(M \times N, g+h)$ can be identified with $(D^g)^2 + (D^h)^2$. We conclude the following.

Proposition 2.5. *Let (M, g) be a compact spin manifold and $l \geq 1$. Then the spectrum of $(D^{g+g^{round}})^2$ on $M \times S^l$ is bounded from below by $l^2/4$.*

3. Proof of Theorem 1.2

Our standing assumptions are: (M, g) is a compact Riemannian spin manifold of dimension n together with a k-dimensional submanifold S of M diffeomorphic to S^k. We assume $n - k \geq 2$. The restriction of g to S is denoted by h. Let $\nu \to S$ be the normal bundle of S. We assume furthermore that a trivialization of the normal bundle is given, that is a vector bundle map $\iota : \mathbb{R}^{n-k} \times S \to \nu$. We assume that ι is fiberwise an isometry.

For $R > 0$ we denote by $\nu(R)$ the disk bundle of vectors of length $\leq R$ in ν. For sufficiently small R the normal exponential map $\exp^\nu$ of S defines a diffeomorphism of $\nu(R)$ onto a neighborhood of S. For such small $R > 0$ one has

$$U_S(R) = (\exp^\nu \circ \iota)(\overline{B^{n-k}(R)} \times S) = \exp^\nu(\nu(R)).$$

Lemma 3.1. *Let $n \geq 3$. Let χ be the canonical spin structure on $\mathbb{R}^{n-1}$, let χ_b be the bounding spin structure on S^1 and χ_{nb} the non-bounding spin structure on S^1. There is a diffeomorphism from $F : \mathbb{R}^{n-1} \times S^1$ to itself preserving the linear structure of $\mathbb{R}^{n-1}$ with*

$$F^*(\chi \times \chi_b) = \chi \times \chi_{nb}.$$

Proof. Let $\gamma : S^1 \to \mathrm{SO}(n-1)$ be a generator of $\pi_1(\mathrm{SO}(n-1))$. Then the map $(X, x) \mapsto (\gamma(x)X, x)$ is a diffeomorphism as desired. □

Let $\exp^\nu : \nu \to M$ be the restriction of the exponential map to ν. Close to the zero section of ν, exp^ν is a diffeomorphism onto its image, and hence for small $\varepsilon > 0$ the map

$$I_\iota : \mathbb{R}^{n-k} \times S, \qquad (X, x) \mapsto \exp\left(R \frac{\iota(X, x)}{\sqrt{1 + \|X\|^2}}\right)$$

is a diffeomorpism onto the interior of $U_S(R)$. The spin structure on M induces a spin structure on $\mathbb{R}^{n-k} \times S$. If $k \geq 2$, then the spin structure on $\mathbb{R}^{n-k} \times S$ is unique. However, in the case $k = 1$, the induced spin structure might be $\chi \times \chi_b$ or $\chi \times \chi_{nb}$. If the induced spin structure is $\chi \times \chi_{nb}$, we replace ι by $\iota' = \iota \circ F$, and the spin structure induced by $I_{\iota'}$ is $\chi \times \chi_b$. Hence, we can assume from now on without loss of generality that the trivialization ι induces the spin structure $\chi \times \chi_b$.

3.1. Approximation by a metric of product form near S.

In the following $r(x)$ denotes the distance from the point x to S with respect to the metric g.

Lemma 3.2. *For sufficiently small $R > 0$ there is a constant $C > 0$ so that*

$$G = g - ((\exp^\nu \circ \iota)^{-1})^*(g^{flat} + h)$$

satisfies

$$|G(x)| \leq C r(x), \quad |\nabla G(x)| \leq C$$

on $U_S(R)$.

Note that in this lemma the function $r(x)$ is by definition the distance of x to S with respect to g but it coincides with the distance of x to S with respect to the metric $((\exp^\nu \circ\iota)^{-1})^*(g^{\mathrm{flat}} + h)$

Proof. Since $x \mapsto \nabla G(x)$ is continuous on a neighborhood of S we can find a constant C such that $|\nabla G(x)| \le C$ for sufficiently small $R > 0$. Now, let $x \in S$. At first the spaces T_xS and ν_x are orthogonal with respect to the two scalar products $g(x)$ and $((\exp^\nu \circ\iota)^{-1})^*(g^{\mathrm{flat}} + h)(x)$. It is also clear that these two scalar products coincide on T_xS. Since the differential $d(\exp^\nu \circ\iota)$ is an isometry, they coincide also on ν_x. This implies that $g(x) = ((\exp^\nu \circ\iota)^{-1})^*(g^{\mathrm{flat}} + h)(x)$ and hence that $G(x) = 0$. We obtain that G vanishes on S. Since G is C^1, $|G|$ is 1-lipschitzian and thus there exists $C > 0$ such that $|G(x)| \le Cr(x)$. □

The following proposition allows us to assume that the metric g has product form close to the surgery sphere S.

Proposition 3.3. *Let (M, g) and S be as above. Then there is a metric $\tilde g$ on M and $\varepsilon > 0$ such that $d^g(x, S) = d^{\tilde g}(x, S)$, $\tilde g$ has product form on $U_S(\varepsilon)$ and*

$$\dim\ker D^{\tilde g} \le \dim\ker D^g.$$

For $\delta > 0$ let η be a smooth cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ on $U_S(\delta)$, $\eta = 0$ on $M \setminus U_S(2\delta)$, and $|d\eta|_g \le 2/\delta$. We set

$$g_\delta = \eta((\exp^\nu \circ\iota)^{-1})^*(g^{\mathrm{flat}} + h) + (1-\eta)g.$$

Then $d^g(x, S) = d^{g_\delta}(x, S) = r(x)$. Through a series of lemmas we will prove the proposition for $\tilde g = g_\delta$ for δ sufficiently small.

In the following estimates C denotes a constant whose values might vary from one line to another, which is independent of δ and η but might depend on M, g, S. Terms denoted by $o_i(1)$ tend to zero when $i \to \infty$.

Lemma 3.4. *Let δ_i be a sequence with $\delta_i \to 0$ as $i \to \infty$. Let φ_i be a sequence of spinors on (M, g_{δ_i}) such that $D^{g_{\delta_i}}\varphi_i = 0$ and $\int_M |\varphi_i|^2\, dv^{g_{\delta_i}} = 1$. Then the sequence $\beta_g^{g_{\delta_i}}\varphi_i$ is bounded in $H_1^2(M, g)$.*

Proof. As $\int |\beta_g^{g_{\delta_i}}\varphi_i|^2\, dv^g = 1 + o_i(1)$ we have to show that $\alpha_i = \sqrt{\int_M |\nabla^g(\beta_g^{g_{\delta_i}}\varphi_i)|_g^2\, dv^g}$ is bounded. We assume the opposite, that is $\alpha_i \to \infty$, and set $\psi_i = \alpha_i^{-1}\beta_g^{g_{\delta_i}}\varphi_i$. Then we have ${}^gD^{g_{\delta_i}}\psi_i = 0$ since $\beta^g_{g_{\delta_i}} \circ \beta_g^{g_{\delta_i}} = \mathrm{Id}$, so formula (3) gives us

$$\begin{aligned}
1 &= \int_M |\nabla^g\psi_i|_g^2\, dv^g\\
&= \int_M (|D^g\psi_i|^2 - \frac14 \mathrm{scal}^g|\psi_i|^2)\, dv^g\\
&= \int_M (|A^g_{g_{\delta_i}}(\nabla^g\psi_i) + B^g_{g_{\delta_i}}(\psi_i)|^2 - \frac14 \mathrm{scal}^g|\psi_i|^2)\, dv^g\\
&\le \int_M (2|A^g_{g_{\delta_i}}(\nabla^g\psi_i)|^2 + 2|B^g_{g_{\delta_i}}(\psi_i)|^2 - \frac14 \mathrm{scal}^g|\psi_i|^2)\, dv^g.
\end{aligned}$$

Using (4), (5), Lemma 3.2, and the fact that g and g_{δ_i} coincide outside $U_S(2\delta_i)$ we get

$$\begin{aligned} 1 &\leq C\delta_i^2 \int_{U_S(2\delta_i)} |\nabla^g \psi_i|_g^2 \, dv^g + C \int_{U_S(2\delta_i)} |\psi_i|^2 \, dv^g + C \int_M |\psi_i|^2 \, dv^g \\ &\leq C\delta_i^2 + C \int_{U_S(2\delta_i)} |\psi_i|^2 \, dv^g + \alpha_i^{-2}(1 + o_i(1)) \\ &\leq C \int_{U_S(2\delta_i)} |\psi_i|^2 \, dv^g + o_i(1) \end{aligned}$$

As ψ_i is bounded in $H_1^2(M,g)$, a subsequence converges weakly in $H_1^2(M,g)$ and strongly in $L^2(M,g)$ to a limit spinor $\psi \in H_1^2(M,g)$. Hence for this subsequence

$$\int_{U_S(2\delta_i)} |\psi_i|_g^2 \, dv^g \to 0$$

which implies a contradiction. ☐

Lemma 3.5. *Again let δ_i be a sequence with $\delta_i \to 0$ as $i \to \infty$ and let φ_i be a sequence of spinors on (M, g_{δ_i}) such that $D^{g_{\delta_i}} \varphi_i = 0$ and $\int_M |\varphi_i|^2 \, dv^{g_{\delta_i}} = 1$. Then, after passing to a subsequence, $\beta_g^{g_{\delta_i}} \varphi_i$ converges weakly in $H_1^2(M,g)$ and strongly in $L^2(M,g)$ to a harmonic spinor on (M,g).*

Proof. According to the previous Lemma the sequence $\beta_g^{g_{\delta_i}} \varphi_i$ is bounded in $H_1^2(M,g)$ and hence a subsequence converges weakly in $H_1^2(M,g)$. After passing to a subsequence once again we obtain strong convergence in $L^2(M,g)$. Denote the limit spinor by φ.

For any $\varepsilon > 0$ Lemma 2.2 implies that $\beta_g^{g_{\delta_i}} \varphi_i$ is bounded in $C^2(M \setminus U_S(\varepsilon))$, and Lemma 2.3 then implies that a subsequence converges in $C^1(M \setminus U_S(\varepsilon))$. Hence the limit φ is in $C^1_{\mathrm{loc}}(M \setminus S)$ and satisfies $D^g \varphi = 0$ on $M \setminus U(S)$. Since φ is in $L^2(M,g)$ it follows from Lemma 2.4 that φ is a weak solution of $D\psi = 0$ on (M,g). By elliptic regularity theory φ is a strong solution and a harmonic spinor on (M,g). ☐

Proof of Proposition 3.3. Let $m = \liminf_{\delta \to 0} \dim \ker D^{g_\delta}$. For sufficiently small δ let $\varphi_\delta^1, \ldots, \varphi_\delta^m \in \ker D^{g_\delta}$ be spinors such that

$$\int_M \langle \varphi_\delta^j, \varphi_\delta^k \rangle \, dv^{g_\delta} = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases} \tag{9}$$

According to Lemma 3.5 there are spinors $\varphi^1, \ldots, \varphi^m \in \ker D^g$ and a sequence $\delta_i \to 0$ such that $\beta_g^{g_{\delta_i}} \varphi_{\delta_i}^j$ converges to φ^j weakly in $H_1^2(M,g)$ and strongly in $L^2(M,g)$ for $j = 1, \ldots, m$. Because of strong L^2-convergence the orthogonality relation (9) is preserved in the limit so $\dim \ker D^g \geq m$. Hence there is a $\delta_0 > 0$ so that $\dim \ker D^{g_{\delta_0}} = m \leq \dim \ker D^g$ and the Proposition is proved with $\tilde{g} = g_{\delta_0}$. ☐

3.2. **Proof for metrics of product form near S.** We assume that g is a product metric on $U_S(R_{\max})$ for some $R_{\max} > 0$, as we may from Proposition 3.3. In polar coordinates $(r, \Theta) \in (0, R_{\max}) \times S^{n-k-1}$ on $B^{n-k}(R_{\max})$ we get

$$g = g^{\mathrm{flat}} + h = dr^2 + r^2 g^{\mathrm{round}} + h.$$

$$0 < \rho << r_0 < r_1/2 << R_{\max}$$

FIGURE 1. Hierachy of variables

Let $\rho > 0$ be a small number which we will finally let tend to 0. We decompose M into three parts

(1) $M \setminus U_S(R_{\max})$,
(2) $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$,
(3) $U_S(\rho/2) = B^{n-k}(\rho/2) \times S^k$.

The manifold $\widetilde{M}$ is obtained by removing part (3) and by gluing in $S^{n-k-1} \times B^{k+1}$, that is $\widetilde{M}$ is the union of

(1) $M \setminus U_S(R_{\max})$,
(2) $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$,
(3') $S^{n-k-1} \times B^{k+1}$.

We now define a sequence of metrics g_ρ on $\widetilde{M}$ such that the theorem holds for small $\rho > 0$. The metrics g_ρ will coincide with g on part (1), but will be modified in part (2) in order to close up nicely in part (3') (see Figure 3.2).

Let r_0, r_1 be fixed such that $2\rho < r_0 < r_1/2 < R_{\max}/2$. Define g_ρ on $\widetilde{M}$ by

(1) $g_\rho = g$ on $M \setminus U_S(R_{\max})$,
(2) $g_\rho = F^2(dr^2 + r^2 g^{\text{round}} + f_\rho^2 h)$ on $(\rho/2, R_{\max}) \times S^{n-k-1} \times S^k$, where F and f_ρ satisfy

$$F(r) = \begin{cases} 1, & \text{if } r_1 < r < R_{\max}; \\ 1/r, & \text{if } r < r_0, \end{cases} \quad \text{and} \quad f_\rho(r) = \begin{cases} 1, & \text{if } r > 2\rho; \\ r, & \text{if } r < \rho\ . \end{cases}$$

(3') $g_\rho = g^{\text{round}} + \gamma_\rho$ on $S^{n-k-1} \times B^{k+1}$ where γ_ρ is some metric so that g_ρ is smooth.

We are going to prove that

$$\dim \ker D^{g_\rho} \le \dim \ker D^g \tag{10}$$

for small $\rho > 0$. Before proving (10), we need some estimates.

For $\alpha \in (0, \rho/2)$, let $\widetilde{U}(\alpha) = \widetilde{M} \setminus (M \setminus U_S(\alpha))$ so that $M \setminus U_S(\alpha) = \widetilde{M} \setminus \widetilde{U}(\alpha)$.

Proposition 3.6. *Let $s \in (0, r_1/2)$. Let ψ_ρ be a harmonic spinor on $(\widetilde{M}, g_\rho)$. Then for $\rho \in (0, s)$ it holds that*

$$\frac{(n-k-1)^2}{32} \int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |F^{\frac{n-1}{2}} \psi_\rho|^2 \, dv^g \le \int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} |F^{\frac{n-1}{2}} \psi_\rho|^2 \, dv^g.$$

Proof. Let $\eta \in C^\infty(\widetilde{M})$ be a cut-off function with $0 \le \eta \le 1$, $\eta = 1$ on $\widetilde{U}(s)$, $\eta = 0$ on $\widetilde{M} \setminus \widetilde{U}(2s)$, and

$$|d\eta|_g \le \frac{2}{s}. \tag{11}$$

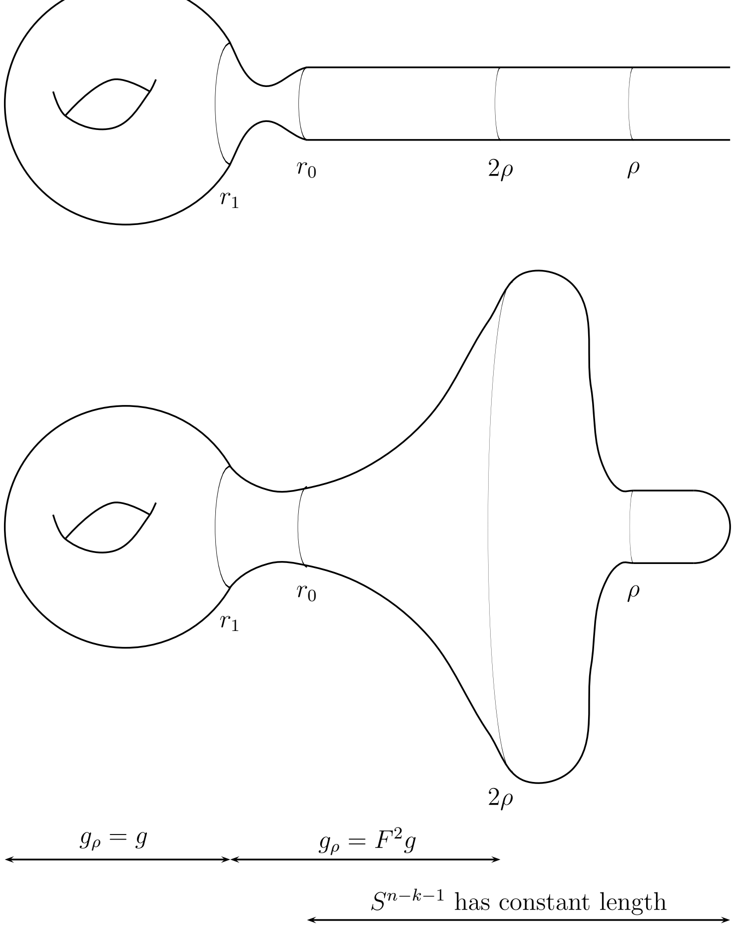

FIGURE 2. The metric g_ρ.

The spinor $\eta\psi_\rho$ is compactly supported in $\widetilde{U}(2s)$. Moreover, the metric g_ρ can be written as $g_\rho = g^{\mathrm{round}} + h_\rho$ on $\widetilde{U}(2s)$ where the metric h_ρ is equal to $r^{-2}dr^2 + r^{-2} f_\rho^2 h$ on $\widetilde{U}(2s) \setminus \widetilde{U}(\rho/2)$ and is equal to γ_ρ on $S^{n-k-1} \times B^{k+1} = \widetilde{U}(\rho/2)$. Hence $(\widetilde{U}(2s), g_\rho)$ is isometric to an open subset of a manifold of the form $S^{n-k-1} \times N$ equipped with a product metric $g^{\mathrm{round}} + g_N$, where N is compact. By Proposition 2.5 the squared eigenvalues of the Dirac operator on this product manifold are greater than or equal to $(n-k-1)^2/4$. Writing the Rayleigh quotient of $\eta\psi_\rho$ we obtain

$$\frac{(n-k-1)^2}{4} \leq \frac{\int_{\widetilde{U}(2s)} |D^{g_\rho}(\eta\psi_\rho)|^2 \, dv^{g_\rho}}{\int_{\widetilde{U}(2s)} |\eta\psi_\rho|^2 \, dv^{g_\rho}}. \tag{12}$$

Since $D^{g_\rho}\psi_\rho = 0$ we have $D^{g_\rho}(\eta\psi_\rho) = \mathrm{grad}^{g_\rho}\eta \cdot \psi_\rho$ so

$$|D^{g_\rho}(\eta\psi_\rho)|^2 = |\mathrm{grad}^{g_\rho}\eta \cdot \psi_\rho|^2 = |d\eta|^2_{g_\rho} |\psi_\rho|^2_{g_\rho}. \tag{13}$$

By definition $d\eta$ is supported in $\widetilde{U}(2s) \setminus \widetilde{U}(s)$. On $\widetilde{M} \setminus \widetilde{U}(2\rho)$ we have $g_\rho = F^2 g$. Moreover, by Relation (11) and since $F = 1/r$ on the support of $d\eta$, we have

$$|d\eta|_{g_\rho}^2 = r^2 |d\eta|_g^2 \leq \frac{4r^2}{s^2}$$

and hence

$$|D^{g_\rho}(\eta\psi_\rho)|^2 \leq \frac{4r^2}{s^2}|\psi_\rho|^2,$$

Since $g_\rho = r^{-2}g$ on $\widetilde{U}(2s) \setminus \widetilde{U}(s)$ we have $dv^{g_\rho} = r^{-n}\, dv^g$. Using equation (13) it follows that

$$\begin{aligned} \int_{\widetilde{U}(2s)} |D^{g_\rho}(\eta\psi_\rho)|^2 \, dv^{g_\rho} &\leq \frac{4}{s^2} \int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} r^{2+(n-1)-n} |r^{-\frac{n-1}{2}}\psi_\rho|^2 \, dv^g \\ &\leq \frac{8}{s} \int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} |F^{\frac{n-1}{2}}\psi_\rho|^2 \, dv^g, \end{aligned} \tag{14}$$

where we also use that $r \leq 2s$ on the domain of integration. Since $\eta \in [0,1]$ on $\widetilde{U}(2s) \setminus \widetilde{U}(s)$, since $\eta = 1$ on $\widetilde{U}(s)$ and since $g_\rho = r^{-2}g$ on $\widetilde{U}(s) \setminus \widetilde{U}(2\rho)$, we have

$$\begin{aligned} \int_{\widetilde{U}(2s)} |\eta\psi_\rho|^2 \, dv^{g_\rho} &\geq \int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |\psi_\rho|^2 \, dv^{g_\rho} \\ &= \int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} r^{(n-1)-n} |r^{-\frac{n-1}{2}}\psi_\rho|^2 \, dv^g \\ &\geq \frac{1}{s} \int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |F^{\frac{n-1}{2}}\psi_\rho|_g^2 \, dv^g, \end{aligned} \tag{15}$$

where we use that $r \leq s$ in the last inequality. Plugging (14) and (15) into (12) we get

$$\frac{(n-k-1)^2}{4} \leq \frac{\frac{8}{s}\int_{\widetilde{U}(2s)\setminus\widetilde{U}(s)} |F^{\frac{n-1}{2}}\psi_\rho|^2 \, dv^g}{\frac{1}{s}\int_{\widetilde{U}(s)\setminus\widetilde{U}(2\rho)} |F^{\frac{n-1}{2}}\psi_\rho|_g^2 \, dv^g}$$

and hence Proposition 3.6 follows. □

Proof of Theorem 1.2. As explained above we need to prove Relation (10), for a contradiction assume that it is false. Then there is a strictly decreasing sequence $\rho_i \to 0$ such that $\dim\ker D^g < \dim\ker D^{g_{\rho_i}}$ for all i. To simplify the notation for subsequences we define $E = \{\rho_i : i \in \mathbb{N}\}$. We have $0 \in \overline{E}$ and passing to a subsequence of ρ_i means passing to a subset $E' \subset E$ of with $0 \in \overline{E'}$.

Let $m = \dim\ker D^g + 1$. For all $\rho \in E$ we can find D^{g_ρ}-harmonic spinors $\psi_\rho^1, \ldots, \psi_\rho^m$ on $(\widetilde{M}, g_\rho)$ such that

$$\int_{M\setminus U(s)} \langle\psi_\rho^j, \psi_\rho^k\rangle \, dv^g = \int_{\widetilde{M}\setminus\widetilde{U}(s)} \langle\psi_\rho^j, \psi_\rho^k\rangle \, dv^g = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k, \end{cases} \tag{16}$$

where $s \leq r_0 < r_1/2$ is fixed as above. Let $\varphi_\rho^j = F^{\frac{n-1}{2}}\psi_\rho^j$. These spinor fields are defined on $M \setminus U(2\rho)$ and by (6) they are D^g-harmonic.

Step 1. *Let $\delta \in (0, R_{max})$. For $\rho > 0$ small enough we have*

$$\int_{M\setminus U(\delta)} |\varphi_\rho^j|^2 \, dv^g \leq \frac{(n-k-1)^2 + 32}{(n-k-1)^2}. \tag{17}$$

By Proposition 3.6 we have

$$\int_{U(s)\setminus U(2\rho)} |\varphi_\rho^j|^2\, dv^g \leq \frac{32}{(n-k-1)^2} \int_{U(2s)\setminus U(s)} |\varphi_\rho^j|^2\, dv^g.$$

and hence if $2\rho \leq \delta$ it follows that

$$\int_{U(s)\setminus U(\delta)} |\varphi_\rho^j|^2\, dv^g \leq \frac{32}{(n-k-1)^2} \int_{M\setminus U(s)} |\varphi_\rho^j|^2\, dv^g.$$

It follows that

$$\begin{aligned}\int_{M\setminus U(\delta)} |\varphi_\rho^j|^2\, dv^g &= \int_{M\setminus U(s)} |\varphi_\rho^j|^2\, dv^g + \int_{U(s)\setminus U(\delta)} |\varphi_\rho^j|^2\, dv^g \\ &\leq (1 + \frac{32}{(n-k-1)^2}) \int_{M\setminus U(s)} |\varphi_\rho^j|^2\, dv^g.\end{aligned}$$

From (16) we now obtain Inequality (17).

Step 2. *There exists $E' \subset E$ with $0 \in \overline{E'}$ and spinors $\Phi^1, \ldots, \Phi^m \in C^1(M \setminus S)$, D^g-harmonic on $(M\setminus S, g)$ such that φ_ρ^j tend to Φ^j in $C^1_{\mathrm{loc}}(M\setminus S)$ as $\rho \to 0$, $\rho \in E'$.*

Let $Z \in \mathbb{N}$ be an integer, $Z > 1/s$. By (17) the sequence $\{\varphi_\rho^j\}_{\rho\in E}$ is bounded in $L^2(M \setminus U(1/Z))$. By Lemma 2.2 it follows that $\{\varphi_\rho^j\}_{\rho\in E}$ is bounded in $C^2(M \setminus U(2/Z))$ for all sufficiently large Z. For a fixed $Z_0 > 1/s$ we apply Lemma 2.3 and conclude that for any j there is a subsequence $\{\varphi_\rho^j\}_{\rho\in E_0}$ of $\{\varphi_\rho^j\}_{\rho\in E}$ that converges in $C^1(M \setminus U(2/Z_0))$ to a spinor Φ_0^j. Similarly we construct further and further subsequences $\{\varphi_\rho^j\}_{\rho\in E_i}$ converging to Φ_i^j in $C^1(M \setminus U(2/(Z_0 + i)))$ with $E_i \subset E_{i-1} \subset \cdots \subset E_0 \subset E$, $0 \in \overline{E_i}$. Obviously Φ_i^j extends Φ_{i-1}^j. Define $E' \subset E$ as consisting of one ρ_i from each E_i chosen so that $\rho_i \to 0$ as $i \to \infty$. Then the sequence $\{\varphi_\rho^j\}_{\rho\in E'}$ converges in $C^1_{\mathrm{loc}}(M \setminus S)$ to a spinor Φ^j. As ψ_ρ^j is D^g-harmonic on $(M \setminus U(2\rho))$ the $C^1_{\mathrm{loc}}(M \setminus S)$-convergence implies that $D^g\Phi^j = 0$ on $M \setminus S$. We have proved Step 2.

Step 3. *Conclusion.*

Let $j \in \{1, \ldots, m\}$. By (17) we conclude that

$$\int_{M\setminus S} |\Phi^j|^2\, dv^g \leq \frac{(n-k-1)^2 + 32}{(n-k-1)^2}$$

and hence $\Phi^j \in L^2(M)$. By Lemma 2.4 and elliptic regularity Φ^j is harmonic and smooth on all of (M, g). Since $M \setminus U(s)$ is a relatively compact subset of $M \setminus S$ the normalization (16) is preserved in the limit $\rho \to 0$ and hence

$$\int_{M\setminus U(s)} \langle \Phi^j, \Phi^k\rangle\, dv^g = \begin{cases} 1, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

This proves that $\Phi^1, \ldots, \Phi^m$ are linearly independent harmonic spinors on (M, g) and hence $\dim\ker D^g \geq m$ which contradicts the definition of m. This proves Relation (10) and Theorem 1.2. □

4. Proof of Theorem 1.1

The proof will follow the argument of [2] so we introduce notation in accordance to that paper. For a compact spin manifold M the space of smooth Riemannian metrics on M is denoted by $\mathcal{R}(M)$ and the subset of D-minimal metrics is denoted by $\mathcal{R}_{\min}(M)$.

From standard results in perturbation theory it follows that $\mathcal{R}_{\min}(M)$ is open in the C^1-topology on $\mathcal{R}(M)$ and if $\mathcal{R}_{\min}(M)$ is not empty then it is dense in $\mathcal{R}(M)$ in all C^k-topologies, $k \geq 1$, see for example [13, Prop. 3.1]. We define the word generic to mean these open and dense properties satisfied by $\mathcal{R}_{\min}(M)$ if non-empty. Theorem 1.1 is then equivalent to the following.

Theorem 4.1. *Let M be a compact connected spin manifold. Then there is a D-minimal metric on M.*

Before we start the proof we note the following consequence of Theorem 1.2.

Proposition 4.2. *Let N be a compact spin manifold which has a D-minimal metric and suppose that M is obtained from N by surgery of codimension ≥ 2. Then M has a D-minimal metric.*

Proof. This follows from Theorem 1.2 since the left hand side of (1) is the same for M and N while the right hand side may only decrease. □

From the proof of handle decompositions of bordisms we have the following.

Proposition 4.3. *Suppose that M is connected, $\dim M \geq 3$, and that M is spin bordant to a manifold N. Then M can be obtained from N by a sequence of surgeries of codimension ≥ 2.*

Proof. The statement follows from [10, VII Theorem 3] if $\dim M = 3$. If $\dim M \geq 4$, then we can do surgery in dimension 0 and 1 at a given spin cobordism between M and N, and obtain a connected, simply connected spin cobordism W between M and N. It then follows from [11, VIII 3.1] that one can obtain M from N by surgeries of dimension $0, \ldots, n-2$. □

Proof of Theorem 4.1. From the solution of the Gromov-Lawson conjecture by Stolz [14] together with knowledge of some explicit manifolds with D-minimal metrics one can show that any compact spin manifold is spin bordant to a manifold with a D-minimal metric, this is worked out in detail in [2, Prop. 3.9]. We may thus assume that the given manifold M is spin bordant to a manifold N equipped with a D-minimal metric. The Theorem now follows from Propositions 4.2 and 4.3 if $\dim M \geq 3$.

Now, let $\dim M = 2$. If $\alpha(M) = 0$, then M can be obtained by adding handles to S^2, i.e. by 0-dimensional surgery. If $\alpha(M) \neq 0$, then M can be obtained by adding handles to T^2 where T^2 carries the spin structure with $\alpha \neq 0$. Any metric on T^2 with that spin structure has a 2-dimensional kernel, and is thus D-minimal. With Proposition 4.2 we get Theorem 4.1 in the 2-dimensional case. □

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