

Canonical bases of higher-level q -deformed Fock spaces

Xavier YVONNE

Abstract

We show that the transition matrices between the standard and the canonical bases of infinitely many weight subspaces of the higher-level q -deformed Fock spaces are equal.

1 Introduction

The q -deformed higher-level Fock spaces were introduced in [JMMO] in order to compute the crystal graph of any irreducible integrable representation of level $l \geq 1$ of $U_q(\widehat{\mathfrak{sl}}_n)$. More precisely, the Fock representation $\mathbf{F}_q[\mathbf{s}_l]$ depends on a parameter $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$ called multi-charge. It contains as a submodule the irreducible integrable $U_q(\widehat{\mathfrak{sl}}_n)$ -module with highest weight $\Lambda_{s_1} + \dots + \Lambda_{s_l}$. The representation $\mathbf{F}_q[\mathbf{s}_l]$ is a generalization of the level-one Fock representation of $U_q(\widehat{\mathfrak{sl}}_n)$ ([H, MM], see also [LT1, LT2]).

The canonical bases are bases of the Fock representations that are invariant under a certain involution $\bar{}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ and that give at $q = 0$ and $q = \infty$ the crystal bases. They were constructed for $l = 1$ in [LT1, LT2] and for $l \geq 1$ by Uglov [U]. In [U], Uglov provides an algorithm for computing these canonical bases. He also gives an expression of the transition matrices between the standard and the canonical bases in terms of Kazhdan-Lusztig polynomials for affine Hecke algebras of type A .

In this paper we prove three theorems.

1. The first one (Theorem 3.9) is a generalization to $l \geq 1$ of a result of [LM]. It compares the transition matrices of the canonical bases of some weight subspaces inside a given Fock space $\mathbf{F}_q[\mathbf{s}_l]$. The weights involved are conjugated under the action of the Weyl group of $U_q(\widehat{\mathfrak{sl}}_n)$. This action leads to bijections σ_i that can be described in a combinatorial way by adding/removing as many i -nodes as possible to the l -multi-partitions indexing the canonical bases. These bijections are generalizations of the Scopes bijections introduced in [S] in order to study, when $n = p$ is a prime number, the p -blocks of symmetric groups of a given defect.
2. In a dual manner, our second result (Theorem 4.4) gives some sufficient conditions on multi-charges \mathbf{s}_l and \mathbf{t}_l with given residues modulo n that ensure that the transition matrices of the canonical bases of some weight subspaces of $\mathbf{F}_q[\mathbf{s}_l]$ and $\mathbf{F}_q[\mathbf{t}_l]$ coincide.
3. Our third result (Theorem 5.2) is an application of Theorem 4.4 to the case when the multi-charges $\mathbf{s}_l = (s_1, \dots, s_l)$ and $\mathbf{t}_l = (t_1, \dots, t_l)$ are dominant, that is $s_1 \gg \dots \gg s_l$ and $t_1 \gg \dots \gg t_l$. It shows that the transition matrices of the canonical bases of

the Fock spaces $\mathbf{F}_q[\mathbf{s}_l]$ stabilize when \mathbf{s}_l becomes dominant (with a given sequence of residues modulo n). This supports the following conjecture (see [Y2]). We conjecture that if $\mathbf{s}_l = (s_1, \dots, s_l)$ is dominant, then the transition matrix of the homogeneous component of degree m of the canonical basis of the Fock space $\mathbf{F}_q[\mathbf{s}_l]$ is equal to the decomposition matrix of the cyclotomic v -Schur algebra $\mathcal{S}_{\mathbb{C},m}(\zeta; \zeta^{s_1}, \dots, \zeta^{s_l})$ of [DJM], where ζ is a complex primitive n -th root of unity. This conjecture generalizes both Ariki's theorem for Ariki-Koike algebras (see [A]) and a result of Varagnolo and Vasserot (see [VV]) which relates the canonical basis of the level-one Fock space and the decomposition matrix of v -Schur algebras with parameter a complex n -th root of unity.

Acknowledgments. I thank Bernard Leclerc for his advice and his reading of the preliminary versions of this paper.

Notation. Let \mathbb{N} (*resp.* \mathbb{N}^*) denote the set of nonnegative (*resp.* positive) integers, and for $a, b \in \mathbb{R}$ denote by $\llbracket a; b \rrbracket$ the discrete interval $[a, b] \cap \mathbb{Z}$. For $X \subset \mathbb{R}$, $t \in \mathbb{R}$, $N \in \mathbb{N}^*$, put

$$(1) \quad X^N(t) := \{(s_1, \dots, s_N) \in X^N \mid s_1 + \dots + s_N = t\}.$$

Throughout this article, we fix 3 integers $n, l \geq 1$ and $s \in \mathbb{Z}$. Let Π denote the set of all integer partitions. If (W, S) is a Coxeter system, denote by $\ell : W \rightarrow \mathbb{N}$ the length function on W .

2 Higher-level q -deformed Fock spaces

In this section, we introduce the higher-level Fock spaces and their canonical bases. We follow here [U], to which we refer the reader for more details. All definitions and results given here are due to Uglov.

2.1 The quantum algebras $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$

In this section, we assume that $n \geq 2$ and $l \geq 2$. Let $\widehat{\mathfrak{sl}}_n$ be the Kac-Moody algebra of type $A_{n-1}^{(1)}$ defined over the field \mathbb{Q} [Kac]. Let \mathfrak{h}^* be the dual of the Cartan subalgebra of $\widehat{\mathfrak{sl}}_n$. Let $\Lambda_0, \dots, \Lambda_{n-1} \in \mathfrak{h}^*$ be the fundamental weights, $\alpha_0, \dots, \alpha_{n-1} \in \mathfrak{h}^*$ be the simple roots and $\delta := \alpha_0 + \dots + \alpha_{n-1}$ be the null root. It will be convenient to extend the index set of the fundamental weights by setting $\Lambda_i := \Lambda_{i \bmod n}$ for all $i \in \mathbb{Z}$. The simple roots are related to the fundamental weights by

$$(2) \quad \alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i,0}\delta \quad (0 \leq i \leq n-1).$$

For $0 \leq i, j \leq n-1$, let a_{ij} be the coefficient of Λ_j in α_i . The space

$$\mathfrak{h}^* = \bigoplus_{i=0}^{n-1} \mathbb{Q} \Lambda_i \oplus \mathbb{Q} \delta = \bigoplus_{i=0}^{n-1} \mathbb{Q} \alpha_i \oplus \mathbb{Q} \Lambda_0$$

is equipped with a non-degenerate bilinear symmetric form (\cdot, \cdot) defined by

$$(3) \quad (\alpha_i, \alpha_j) = a_{ij}, \quad (\Lambda_0, \alpha_i) = \delta_{i,0}, \quad (\Lambda_0, \Lambda_0) = 0 \quad (0 \leq i, j \leq n-1).$$

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the q -deformed universal enveloping algebra of $\widehat{\mathfrak{sl}}_n$. This is an algebra over $\mathbb{Q}(q)$ with generators $e_i, f_i, t_i^{\pm 1}$ ($0 \leq i \leq n-1$) and ∂ . Let $U'_q(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by $e_i, f_i, t_i^{\pm 1}$ ($0 \leq i \leq n-1$). The relations in $U'_q(\widehat{\mathfrak{sl}}_n)$ are standard and will be omitted (see *e.g.* [KMS]). The relations among the degree generator ∂ and the generators of $U'_q(\widehat{\mathfrak{sl}}_n)$ can be found in [U]. If M is a $U_q(\widehat{\mathfrak{sl}}_n)$ -module, denote by $\mathcal{P}(M)$ the set of weights of M and let $M\langle w \rangle$ denote the subspace of M of weight w . If $x \in M\langle w \rangle \setminus \{0\}$ is a weight vector, denote by

$$(4) \quad \text{wt}(x) := w$$

the weight of x . The *Weyl group* of $\widehat{\mathfrak{sl}}_n$ (or $U_q(\widehat{\mathfrak{sl}}_n)$), denoted by W_n , is the subgroup of $\text{GL}(\mathfrak{h}^*)$ generated by the simple reflections σ_i defined by

$$(5) \quad \sigma_i(\Lambda) = \Lambda - (\Lambda, \alpha_i) \alpha_i \quad (\Lambda \in \mathfrak{h}^*, 0 \leq i \leq n-1).$$

Note that W_n is isomorphic to $\widetilde{\mathfrak{S}}_n$, the affine symmetric group which is a Coxeter group of type $A_{n-1}^{(1)}$.

We also introduce the algebra $U_p(\widehat{\mathfrak{sl}}_l)$ with

$$(6) \quad p := -q^{-1}.$$

In order to distinguish the elements related to $U_q(\widehat{\mathfrak{sl}}_n)$ from those related to $U_p(\widehat{\mathfrak{sl}}_l)$, we put dots over the latter. For example, $\dot{e}_i, \dot{f}_i, \dot{t}_i^{\pm 1}$ ($0 \leq i \leq l-1$) and $\dot{\partial}$ are the generators of $U_p(\widehat{\mathfrak{sl}}_l)$, $\dot{\alpha}_i$ ($0 \leq i \leq l-1$) are the simple roots for $U_p(\widehat{\mathfrak{sl}}_l)$, $\dot{W}_l = \langle \dot{\sigma}_0, \dots, \dot{\sigma}_{l-1} \rangle$ is the Weyl group of $U_p(\widehat{\mathfrak{sl}}_l)$ and so on.

2.2 The space Λ^s

2.2.1 The vector space Λ^s and its standard basis

Following [U], we now recall the definition of Λ^s , the space of (semi-infinite) q -wedge products of charge s (this space is denoted by $\Lambda^{s+\frac{\infty}{2}}$ in [U]). Let $P(s)$ be the set of the sequences of integers (k_1, k_2, \dots) such that $k_i = s - i + 1$ for i large enough. As a vector space over $\mathbb{Q}(q)$, Λ^s is spanned by the *q -wedge products*

$$(7) \quad u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \dots, \quad \mathbf{k} = (k_1, k_2, \dots) \in P(s),$$

with relations given in [U, Prop. 3.16]. These relations are called *straightening rules*. Using them, one can express a q -wedge product as a linear combination of so-called *ordered q -wedge products*, namely q -wedge products $u_{\mathbf{k}}$ with $\mathbf{k} \in P^{++}(s)$, where

$$(8) \quad P^{++}(s) := \{(k_1, k_2, \dots) \in P(s) \mid k_1 > k_2 > \dots\}.$$

In fact, the ordered q -wedge products $\{u_{\mathbf{k}} \mid \mathbf{k} \in P^{++}(s)\}$ form a basis of Λ^s , called the *standard basis*. For the sequel, it will be convenient to use different indexations of this basis which we give now.

* Indexation $u_{\mathbf{k}}$. This is the indexation we have just described.

* Indexation λ . To the ordered q -wedge product $u_{\mathbf{k}}$ corresponds a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ defined by

$$(9) \quad \lambda_i := k_i - (s + 1 - i) \quad (i \geq 1).$$

If $u_{\mathbf{k}}$ and λ are related this way, write

$$(10) \quad |\lambda, s\rangle := u_{\mathbf{k}}.$$

* Indexation λ_n . Recall the definition of $\mathbb{Z}^n(s)$ from (1). Uglov constructed a bijection

$$(11) \quad \tau'_n : \Pi \rightarrow \Pi^n \times \mathbb{Z}^n(s), \quad \lambda \mapsto (\boldsymbol{\lambda}_n, \mathbf{s}_n)$$

(see [U, Sect. 4.1], where this map is denoted by τ_n^s). With the notation above, $\boldsymbol{\lambda}_n$ is the n -quotient of λ and \mathbf{s}_n is a variation of the n -core of λ (see *e.g.* [Mac, Ex.8, p.12]). Write

$$(12) \quad |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle^\bullet := |\lambda, s\rangle$$

if $(\boldsymbol{\lambda}_n, \mathbf{s}_n) = \tau'_n(\lambda)$. Note that this indexation coincides with the indexation λ if $n = 1$.

* Indexation λ_l . Uglov constructed a bijection

$$(13) \quad \tau_l : \Pi \rightarrow \Pi^l \times \mathbb{Z}^l(s), \quad \lambda \mapsto (\boldsymbol{\lambda}_l, \mathbf{s}_l)$$

(see again [U, Sect. 4.1], where this map is denoted by τ_l^s). The map τ_l is a variation of the map τ'_n defined above. Write

$$(14) \quad |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle := |\lambda, s\rangle$$

if $(\boldsymbol{\lambda}_l, \mathbf{s}_l) = \tau_l(\lambda)$. Note that this indexation coincides with the indexation λ if $l = 1$.

Example 2.1 Take $n = 2$ and $l = 3$. Then we have

$$|(4, 3, 3, 2, 1), -1\rangle = |((3, 3), \emptyset), (-1, 0)\rangle^\bullet = |((1, 1), (1, 1), (1)), (0, 0, -1)\rangle.$$

◇

2.2.2 3 actions on Λ^s

Following [JMMO],[FLOTW],[U], the vector space Λ^s can be made into an integrable representation of level l of the quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$. This representation can be described in a nice way if we use the indexation λ_l . In order to recall the explicit formulas, let us first introduce some notation. Fix $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi^l$ and $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$. Identify the multi-partition λ_l with its Young diagram $\{(i, j, b) \in \mathbb{N}^* \times \mathbb{N}^* \times \llbracket 1; l \rrbracket \mid 1 \leq j \leq \lambda_i^{(b)}\}$, whose elements are called *nodes* of λ_l . For each node $\gamma = (i, j, b)$ of λ_l , define its *residue modulo n* by

$$(15) \quad \text{res}_n(\gamma) = \text{res}_n(\gamma, \mathbf{s}_l) := (s_b + j - i) \bmod n \in \mathbb{Z}/n\mathbb{Z} \cong \llbracket 0; n-1 \rrbracket.$$

If $\text{res}_n(\gamma) = c$, we say that γ is a *c-node*. If $\mu_l \in \Pi^l$ is such that $\mu_l \supset \lambda_l$ and $\gamma := \mu_l \setminus \lambda_l$ is a *c-node* of μ_l , we say that γ is a *removable c-node* of μ_l or that γ is an *addable c-node* of λ_l . For $0 \leq c \leq n-1$, denote by

$$(16) \quad N_c(\lambda_l; \mathbf{s}_l; n) \quad (\text{resp. } A_c(\lambda_l; \mathbf{s}_l; n), \quad \text{resp. } R_c(\lambda_l; \mathbf{s}_l; n))$$

the number of *c-nodes* (*resp.* of *addable c-nodes*, *resp.* of *removable c-nodes*) of λ_l . Put

$$(17) \quad M_c(\lambda_l; \mathbf{s}_l; n) := A_c(\lambda_l; \mathbf{s}_l; n) - R_c(\lambda_l; \mathbf{s}_l; n).$$

For $\lambda_l, \mu_l \in \Pi^l$, $\mathbf{s}_l \in \mathbb{Z}^l$, $c \in \mathbb{Z}$ and $k \in \mathbb{N}^*$, write

$$(18) \quad \lambda_l \xrightarrow{c:k} \mu_l$$

if there exists a sequence of l -multi-partitions $\nu_l^{(0)} \subset \nu_l^{(1)} \subset \dots \subset \nu_l^{(k)}$ such that $\lambda_l = \nu_l^{(0)}$, $\mu_l = \nu_l^{(k)}$ and for all $1 \leq j \leq k$, $\nu_l^{(j)} \setminus \nu_l^{(j-1)}$ is an addable *c-node* of $\nu_l^{(j-1)}$. Given a multi-charge (s_1, \dots, s_l) and two nodes $\gamma = (i, j, b)$ and $\gamma' = (i', j', b')$, write

$$(19) \quad \gamma < \gamma'$$

if either $s_b + j - i < s_{b'} + j' - i'$ or $s_b + j - i = s_{b'} + j' - i'$ and $b < b'$. This defines a total ordering on the set of the addable and removable *c-nodes* of a given multi-partition. If $\lambda_l \xrightarrow{c:k} \mu_l$, put

$$(20) \quad M_c^>(\lambda_l; \mu_l; \mathbf{s}_l; n) = \sum_{\gamma \in \mu_l \setminus \lambda_l} \left(\#\{\beta \in \mathbb{N}^3 \mid \beta \text{ is an addable } c\text{-node of } \mu_l \text{ and } \beta > \gamma\} - \#\{\beta \in \mathbb{N}^3 \mid \beta \text{ is a removable } c\text{-node of } \lambda_l \text{ and } \beta > \gamma\} \right),$$

and define similarly $M_c^<(\lambda_l; \mu_l; \mathbf{s}_l; n)$.

Example 2.2 Take $\mathbf{s}_l = (5, 0, 2, 1)$, $\lambda_l = ((5, 3, 3, 1), (3, 2), (4, 3, 1), (2, 2, 2, 1))$, $n = 3$ and $c = 0$. Then we have

$$N_c(\lambda_l; \mathbf{s}_l; n) = 11, \quad A_c(\lambda_l; \mathbf{s}_l; n) = R_c(\lambda_l; \mathbf{s}_l; n) = 5 \quad \text{and} \quad M_c(\lambda_l; \mathbf{s}_l; n) = 0.$$

The addable c -nodes of λ_l are $(5, 1, 4)$, $(4, 2, 1)$, $(1, 4, 2)$, $(1, 3, 4)$ and $(1, 5, 3)$. The removable c -nodes of λ_l are $(2, 2, 2)$, $(3, 1, 3)$, $(3, 2, 4)$, $(2, 3, 3)$ and $(1, 5, 1)$. The list of all these nodes arranged with respect to the ordering described above is

$$(5, 1, 4) < (2, 2, 2) < (3, 1, 3) < (3, 2, 4) < (4, 2, 1) < (1, 4, 2) < (2, 3, 3) < (1, 3, 4) < (1, 5, 3) < (1, 5, 1).$$

Take also $\mu_l = ((5, 3, 3, 1), (3, 2), (5, 3, 1), (2, 2, 2, 1))$, so that $\mu_l \setminus \lambda_l = \{(1, 5, 3)\}$ is a single c -node. Then $M_c^>(\lambda_l; \mu_l; \mathbf{s}_l; n) = 0 - 1 = -1$ and $M_c^<(\lambda_l; \mu_l; \mathbf{s}_l; n) = 4 - 4 = 0$. \diamond

For $a \in \mathbb{Z}$, denote by $a \bmod n \in \{0, \dots, n-1\}$ the residue of a modulo n . Finally, for $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$, define

$$(21) \quad \Delta(\mathbf{s}_l, n) := \frac{1}{2} \sum_{b=1}^l \left(\frac{s_b^2}{n} - s_b \right) - \left(\frac{(s_b \bmod n)^2}{n} - (s_b \bmod n) \right).$$

Now we can state the following result.

Theorem 2.3 ([JMMO],[FLOTW],[U]) *The following formulas define on Λ^s a structure of an integrable representation of level l of the quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$.*

$$(22) \quad e_i \cdot |\nu_l, \mathbf{s}_l\rangle = \sum_{\lambda_l \xrightarrow{i:1} \nu_l} q^{-M_i^<(\lambda_l; \nu_l; \mathbf{s}_l; n)} |\lambda_l, \mathbf{s}_l\rangle,$$

$$(23) \quad f_i \cdot |\nu_l, \mathbf{s}_l\rangle = \sum_{\nu_l \xrightarrow{i:1} \mu_l} q^{M_i^>(\nu_l; \mu_l; \mathbf{s}_l; n)} |\mu_l, \mathbf{s}_l\rangle,$$

$$(24) \quad t_i \cdot |\nu_l, \mathbf{s}_l\rangle = q^{M_i(\nu_l; \mathbf{s}_l; n)} |\nu_l, \mathbf{s}_l\rangle,$$

$$(25) \quad \partial \cdot |\nu_l, \mathbf{s}_l\rangle = -(\Delta(\mathbf{s}_l, n) + N_0(\nu_l; \mathbf{s}_l; n)) |\nu_l, \mathbf{s}_l\rangle.$$

□

Note that these formulas do not involve the straightening of q -wedge products. They are therefore handy to use for computations.

In a completely similar way, Λ^s can be made into an integrable representation of level n of the quantum algebra $U_p(\widehat{\mathfrak{sl}}_l)$. This action can be described using the indexation λ_n . Namely, we have (with obvious notation) the following result.

Theorem 2.4 ([JMMO],[FLOTW],[U]) *The following formulas define on Λ^s a structure of an integrable representation of level n of the quantum algebra $U_p(\widehat{\mathfrak{sl}}_l)$.*

$$(26) \quad \dot{e}_i \cdot |\nu_n, \mathbf{s}_n\rangle^\bullet = \sum_{\lambda_n \xrightarrow{i:1} \nu_n} p^{-M_i^<(\lambda_n; \nu_n; \mathbf{s}_n; l)} |\lambda_n, \mathbf{s}_n\rangle^\bullet,$$

$$(27) \quad \dot{f}_i \cdot |\boldsymbol{\nu}_n, \mathbf{s}_n\rangle^\bullet = \sum_{\boldsymbol{\nu}_n \xrightarrow{i:1} \boldsymbol{\mu}_n} p^{M_i^\triangleright(\boldsymbol{\nu}_n; \boldsymbol{\mu}_n; \mathbf{s}_n; l)} |\boldsymbol{\mu}_n, \mathbf{s}_n\rangle^\bullet,$$

$$(28) \quad \dot{t}_i \cdot |\boldsymbol{\nu}_n, \mathbf{s}_n\rangle^\bullet = p^{M_i(\boldsymbol{\nu}_n; \mathbf{s}_n; l)} |\boldsymbol{\nu}_n, \mathbf{s}_n\rangle^\bullet,$$

$$(29) \quad \dot{\partial} \cdot |\boldsymbol{\nu}_n, \mathbf{s}_n\rangle^\bullet = -(\Delta(\mathbf{s}_n, l) + N_0(\boldsymbol{\nu}_n; \mathbf{s}_n; l)) |\boldsymbol{\nu}_n, \mathbf{s}_n\rangle^\bullet.$$

□

Theorems 2.3 and 2.4 show in particular that the vectors of the standard basis of Λ^s are weight vectors for the actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$, and the weights are given by:

Corollary 2.5 ([U]) *With obvious notation, we have*

$$(30) \quad \text{wt}(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle) = -\Delta(\mathbf{s}_l, n)\delta + \Lambda_{s_1} + \cdots + \Lambda_{s_l} - \sum_{i=0}^{n-1} N_i(\boldsymbol{\lambda}_l; \mathbf{s}_l; n) \alpha_i,$$

$$(31) \quad \dot{\text{wt}}(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle) = -(\Delta(\mathbf{s}_l, n) + N_0(\boldsymbol{\lambda}_l; \mathbf{s}_l; n))\dot{\delta} + (n - s_1 + s_l)\dot{\Lambda}_0 + \sum_{i=1}^{l-1} (s_i - s_{i+1})\dot{\Lambda}_i,$$

$$(32) \quad \dot{\text{wt}}(|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle^\bullet) = -\Delta(\mathbf{s}_n, l)\dot{\delta} + \dot{\Lambda}_{s_1} + \cdots + \dot{\Lambda}_{s_n} - \sum_{i=0}^{l-1} N_i(\boldsymbol{\lambda}_n; \mathbf{s}_n; l) \dot{\alpha}_i,$$

$$(33) \quad \text{wt}(|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle^\bullet) = -(\Delta(\mathbf{s}_n, l) + N_0(\boldsymbol{\lambda}_n; \mathbf{s}_n; l))\delta + (l - s_1 + s_n)\Lambda_0 + \sum_{i=1}^{n-1} (s_i - s_{i+1})\Lambda_i.$$

□

Definition 2.6 For $m \in \mathbb{Z}^*$, define an endomorphism B_m of Λ^s by

$$(34) \quad B_m(u_{k_1} \wedge u_{k_2} \wedge \cdots) := \sum_{j=1}^{+\infty} u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge u_{k_j - nlm} \wedge u_{k_{j+1}} \wedge \cdots \quad ((k_1, k_2, \dots) \in P^{++}(s)).$$

◇

Note that by [U, Lemma 3.18], the sum above involves only finitely many nonzero terms, hence B_m is well-defined. This definition comes from a passage to the limit $r \rightarrow \infty$ in the action of the center of the Hecke algebra of $\widehat{\mathfrak{S}}_r$ on q -wedge products of r factors. However, the operators B_m do not commute, but by [U, Prop. 4.4 & 4.5], they span a Heisenberg algebra

$$(35) \quad \mathcal{H} := \langle B_m \mid m \in \mathbb{Z}^* \rangle.$$

We now recall some results concerning the actions of $U_q(\widehat{\mathfrak{sl}}_n)$, $U_p(\widehat{\mathfrak{sl}}_l)$ and \mathcal{H} on Λ^s .

Proposition 2.7 ([U]) *The actions of $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ and \mathcal{H} on Λ^s pairwise commute. \square*

For $L, N \in \mathbb{N}^*$, introduce the finite set

$$(36) \quad A_{L,N}(s) := \{(r_1, \dots, r_L) \in \mathbb{Z}^L(s) \mid r_1 \geq \dots \geq r_L, r_1 - r_L \leq N\}.$$

It is not hard to see that if $\mathbf{r}_l \in \mathbb{Z}^l(s)$ and $\mathbf{r}_n \in \mathbb{Z}^n(s)$ are such that $|\emptyset_l, \mathbf{r}_l\rangle = |\emptyset_n, \mathbf{r}_n\rangle^\bullet$, then $\mathbf{r}_l \in A_{l,n}(s)$ and $\mathbf{r}_n \in A_{n,l}(s)$. Conversely, if $\mathbf{r}_l \in A_{l,n}(s)$, then there exists a unique $\mathbf{r}_n \in A_{n,l}(s)$ such that $|\emptyset_n, \mathbf{r}_n\rangle^\bullet = |\emptyset_l, \mathbf{r}_l\rangle$, and if $\mathbf{r}_n \in A_{n,l}(s)$, then there exists a unique $\mathbf{r}_l \in A_{l,n}(s)$ such that $|\emptyset_l, \mathbf{r}_l\rangle = |\emptyset_n, \mathbf{r}_n\rangle^\bullet$. Therefore,

$$\{|\emptyset_l, \mathbf{r}_l\rangle \mid \mathbf{r}_l \in A_{l,n}(s)\} = \{|\emptyset_n, \mathbf{r}_n\rangle^\bullet \mid \mathbf{r}_n \in A_{n,l}(s)\}$$

is a set of highest weight vectors simultaneously for the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$ and $U'_p(\widehat{\mathfrak{sl}}_l)$. It is easy to see that these vectors are also singular for the action of \mathcal{H} , that is, they are annihilated by the B_m , $m > 0$. It turns out that these vectors are the only singular vectors simultaneously for the actions of $U'_q(\widehat{\mathfrak{sl}}_n)$, $U'_p(\widehat{\mathfrak{sl}}_l)$ and \mathcal{H} , and we have the following theorem.

Theorem 2.8 ([U], **Thm. 4.8**) *We have*

$$\Lambda^s = \bigoplus_{\mathbf{r}_l \in A_{l,n}(s)} U'_q(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{H} \otimes U'_p(\widehat{\mathfrak{sl}}_l) \cdot |\emptyset_l, \mathbf{r}_l\rangle = \bigoplus_{\mathbf{r}_n \in A_{n,l}(s)} U'_q(\widehat{\mathfrak{sl}}_n) \otimes \mathcal{H} \otimes U'_p(\widehat{\mathfrak{sl}}_l) \cdot |\emptyset_n, \mathbf{r}_n\rangle^\bullet.$$

\square

2.2.3 The involution $\bar{}$ of Λ^s

Following [U], the space Λ^s can be endowed with an involution $\bar{}$. Instead of recalling the definition of this involution, we give its main properties (they turn out to characterize it completely).

Proposition 2.9 ([U]) *There exists an involution $\bar{}$ of Λ^s such that:*

- (i) $\bar{}$ is a \mathbb{Q} -linear map of Λ^s such that for all $u \in \Lambda^s$, $k \in \mathbb{Z}$, we have $\overline{q^k u} = q^{-k} \bar{u}$.
- (ii) (Unitriangularity property). For all $\lambda \in \Pi$, we have

$$\overline{|\lambda, s\rangle} \in |\lambda, s\rangle + \bigoplus_{\mu \triangleleft \lambda} \mathbb{Z}[q, q^{-1}] |\mu, s\rangle,$$

where \triangleleft stands for the dominance ordering on partitions.

- (iii) For all $\lambda \in \Pi$, we have $\text{wt}(\overline{|\lambda, s\rangle}) = \text{wt}(|\lambda, s\rangle)$ and $\dot{\text{wt}}(\overline{|\lambda, s\rangle}) = \dot{\text{wt}}(|\lambda, s\rangle)$.
- (iv) For all $0 \leq i \leq n-1$, $0 \leq j \leq l-1$, $m < 0$, $v \in \Lambda^s$, we have

$$\overline{f_i \cdot v} = f_i \cdot \bar{v}, \quad \overline{f_j \cdot v} = f_j \cdot \bar{v} \quad \text{and} \quad \overline{B_m \cdot v} = B_m \cdot \bar{v}.$$

Proof. Let $\bar{}$ be the involution of Λ^s defined in [U, Prop. 3.23 & Eq. (39)]. By construction, (i) holds. The other statements come from [U, Prop. 4.11 & 4.12] and Corollary 2.5. \square

2.3 q -deformed higher-level Fock spaces

2.3.1 Definition

By Theorem 2.3, the space

$$(37) \quad \mathbf{F}_q[\mathbf{s}_l] := \bigoplus_{\lambda_l \in \Pi^l} \mathbb{Q}(q)|\lambda_l, \mathbf{s}_l\rangle \subset \Lambda^s \quad (\mathbf{s}_l \in \mathbb{Z}^l(s))$$

is a $U_q(\widehat{\mathfrak{sl}}_l)$ -submodule of Λ^s . The reader should be aware that $\mathbf{F}_q[\mathbf{s}_l]$ is *not* a $U_p(\widehat{\mathfrak{sl}}_l)$ -submodule of Λ^s . In a similar way, by Theorem 2.4, the space

$$(38) \quad \mathbf{F}_p[\mathbf{s}_n]^\bullet := \bigoplus_{\lambda_n \in \Pi^n} \mathbb{Q}(q)|\lambda_n, \mathbf{s}_n\rangle^\bullet \subset \Lambda^s \quad (\mathbf{s}_n \in \mathbb{Z}^n(s))$$

is a $U_p(\widehat{\mathfrak{sl}}_l)$ -submodule of Λ^s .

Definition 2.10 ([U]) The representations $\mathbf{F}_q[\mathbf{s}_l]$ and $\mathbf{F}_p[\mathbf{s}_n]^\bullet$ ($\mathbf{s}_l \in \mathbb{Z}^l(s)$, $\mathbf{s}_n \in \mathbb{Z}^n(s)$) are called (q -deformed) Fock spaces. When $l > 1$ and $n > 1$, we speak of higher-level Fock spaces. \diamond

Since the maps τ_l and τ'_n are bijections, we have the following decompositions:

$$(39) \quad \Lambda^s = \bigoplus_{\mathbf{s}_l \in \mathbb{Z}^l(s)} \mathbf{F}_q[\mathbf{s}_l] = \bigoplus_{\mathbf{s}_n \in \mathbb{Z}^n(s)} \mathbf{F}_p[\mathbf{s}_n]^\bullet.$$

Neither of these decompositions is compatible with the decompositions of Λ^s given in Theorem 2.8.

2.3.2 Fock spaces as weight subspaces of Λ^s . Actions of the Weyl groups.

Let $N, L \in \mathbb{N}^*$. Recall the definition of $\mathbb{Q}^L(s)$ and $\mathbb{Q}^L(N)$ from (1) and define a map

$$(40) \quad \theta_{L,N} : \mathbb{Q}^L(s) \rightarrow \mathbb{Q}^L(N), \quad (s_1, \dots, s_L) \mapsto (N - s_1 + s_L, s_1 - s_2, \dots, s_{L-1} - s_L).$$

It is easy to see that $\theta_{L,N}$ is bijective. Moreover, for $(a_1, \dots, a_L) \in \mathbb{Q}^L(N)$, the l -tuple $(s_1, \dots, s_L) := \theta_{L,N}^{-1}(a_1, \dots, a_L)$ is given by

$$(41) \quad s_i = \frac{1}{L} \left(s - \sum_{j=1}^{L-1} j a_j \right) + \sum_{j=i+1}^L a_j \quad (1 \leq i \leq L).$$

The next result shows that the Fock spaces are sums of certain weight subspaces of Λ^s . The proof follows easily from Corollary 2.5.

Proposition 2.11 ([U])

(i) Let $\mathbf{s}_n \in \mathbb{Z}^n(s)$. Let $(a_0, \dots, a_{n-1}) := \theta_{n,l}(\mathbf{s}_n)$ and $w := \sum_{i=0}^{n-1} a_i \Lambda_i$. Then

$$\mathbf{F}_p[\mathbf{s}_n]^\bullet = \bigoplus_{d \in \mathbb{Z}} \Lambda^s \langle w + d\delta \rangle.$$

(ii) Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$. Let $(a_0, \dots, a_{l-1}) := \theta_{l,n}(\mathbf{s}_l)$ and $\dot{w} := \sum_{i=0}^{l-1} a_i \dot{\Lambda}_i$. Then

$$\mathbf{F}_q[\mathbf{s}_l] = \bigoplus_{d \in \mathbb{Z}} \Lambda^s \langle \dot{w} + d\dot{\delta} \rangle.$$

□

Note that the operator B_m ($m \in \mathbb{Z}^*$) maps the weight subspace $\Lambda^s \langle w \rangle$ (resp. $\Lambda^s \langle \dot{w} \rangle$) into $\Lambda^s \langle w + m\delta \rangle$ (resp. $\Lambda^s \langle \dot{w} + m\dot{\delta} \rangle$). Therefore, by Proposition 2.11, the Fock spaces $\mathbf{F}_q[\mathbf{s}_l]$ and $\mathbf{F}_p[\mathbf{s}_n]^\bullet$ ($\mathbf{s}_l \in \mathbb{Z}^l(s)$, $\mathbf{s}_n \in \mathbb{Z}^n(s)$) are stable under the action of \mathcal{H} .

We now compare some weight subspaces of the Fock spaces. The proof follows again from Corollary 2.5.

Proposition 2.12 Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$ and w be a weight of $\mathbf{F}_q[\mathbf{s}_l]$. Then there exists a unique pair (\mathbf{s}_n, \dot{w}) such that $\mathbf{s}_n \in \mathbb{Z}^n(s)$, \dot{w} is a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$ and $\mathbf{F}_q[\mathbf{s}_l] \langle w \rangle = \mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle$. More precisely, write $w = d\delta + \sum_{i=0}^{n-1} a_i \Lambda_i$ with $a_0, \dots, a_{n-1}, d \in \mathbb{Z}$, \mathbf{s}_l , and put $s_0 := n + s_l$. Then we have $\mathbf{s}_n = \theta_{n,l}^{-1}(a_0, \dots, a_{n-1})$ and $\dot{w} = d\dot{\delta} + \sum_{i=0}^{l-1} (s_i - s_{i+1}) \dot{\Lambda}_i$. □

Example 2.13 Take $n = 3$, $l = 2$, $\mathbf{s}_l = (1, 0)$ and $w = -2\Lambda_0 + \Lambda_1 + 3\Lambda_2 - 2\delta$. Then by (30), we have $\text{wt}(|((1, 1), (1)), \mathbf{s}_l\rangle) = w$, so w is a weight of $\mathbf{F}_q[\mathbf{s}_l]$. By Proposition 2.12, we have $\mathbf{F}_q[\mathbf{s}_l] \langle w \rangle = \mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle$ with $\mathbf{s}_n = (2, 1, -2)$ and $\dot{w} = 2\dot{\Lambda}_0 + \dot{\Lambda}_1 - 2\dot{\delta}$. Moreover, using (30) and (32), we see that for all $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle^\bullet \in \mathbf{F}_q[\mathbf{s}_l] \langle w \rangle = \mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle$, we have $N_0(\boldsymbol{\lambda}_l; \mathbf{s}_l; n) = 2$, $N_1(\boldsymbol{\lambda}_l; \mathbf{s}_l; n) = 1$, $N_2(\boldsymbol{\lambda}_l; \mathbf{s}_l; n) = 0$ and $N_0(\boldsymbol{\lambda}_n; \mathbf{s}_n; l) = N_1(\boldsymbol{\lambda}_n; \mathbf{s}_n; l) = 0$ (this shows *a posteriori* that $\dim \mathbf{F}_q[\mathbf{s}_l] \langle w \rangle = 1$). ◊

We now deal with the actions of the Weyl groups of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$ on the set of the weight subspaces of Λ^s . Recall that $W_n = \langle \sigma_0, \dots, \sigma_{n-1} \rangle$ is the Weyl group of $U_q(\widehat{\mathfrak{sl}}_n)$. By (5), W_n acts on the weight lattice $\bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ by

$$(42) \quad \sigma_i \cdot \delta = \delta \quad \text{and} \quad \sigma_i \cdot \Lambda_j = \begin{cases} \Lambda_j & \text{if } j \neq i, \\ \Lambda_{i-1} + \Lambda_{i+1} - \Lambda_i - \delta_{i,0} \delta & \text{if } j = i \end{cases} \quad (0 \leq i, j \leq n-1).$$

Moreover, it is easy to see that W_n acts faithfully on $\mathbb{Z}^n(s)$ by

$$(43) \quad \begin{cases} \sigma_0 \cdot (s_1, \dots, s_n) &= (s_n + l, s_2, \dots, s_{n-1}, s_1 - l), \\ \sigma_i \cdot (s_1, \dots, s_n) &= (s_1, \dots, s_{i+1}, s_i, \dots, s_n) \quad (1 \leq i \leq n-1), \end{cases}$$

and the set $A_{n,l}(s)$ defined by (36) is a fundamental domain for this action. In a similar way, one can define two actions of the Weyl group W_l of $U_p(\widehat{\mathfrak{sl}}_l)$, one on the weight lattice $\bigoplus_{i=0}^{l-1} \mathbb{Z}\dot{\Lambda}_i \oplus \mathbb{Z}\dot{\delta}$ and one on $\mathbb{Z}^l(s)$. The following lemma will be useful later.

Lemma 2.14 *Let $\mathbf{s}_n \in \mathbb{Z}^n(s)$ and \dot{w} be a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$. Let $(\mathbf{s}_l, w) \in \mathbb{Z}^l(s) \times \mathcal{P}(\Lambda^s)$ be the unique pair such that $\mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle = \mathbf{F}_q[\mathbf{s}_l] \langle w \rangle$ (see Proposition 2.12). Let $\dot{\sigma} \in \dot{W}_l$. In the same way, let $(\mathbf{t}_l, w') \in \mathbb{Z}^l(s) \times \mathcal{P}(\Lambda^s)$ be the unique pair such that $\mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{\sigma} \cdot \dot{w} \rangle = \mathbf{F}_q[\mathbf{t}_l] \langle w' \rangle$. Then we have*

$$\mathbf{t}_l = \dot{\sigma} \cdot \mathbf{s}_l \quad \text{and} \quad w' = w + \text{wt}(|\emptyset_l, \mathbf{t}_l\rangle) - \text{wt}(|\emptyset_l, \mathbf{s}_l\rangle).$$

Proof. The proof follows immediately from the formulas given in Proposition 2.12. \square

2.3.3 The lower crystal basis $(\mathcal{L}[\mathbf{s}_l], \mathcal{B}[\mathbf{s}_l])$ of $\mathbf{F}_q[\mathbf{s}_l]$ at $q = 0$

Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$. Following [Kas1], let $\mathbb{A} \subset \mathbb{Q}(q)$ be the ring of rational functions which are regular at $q = 0$, $\mathcal{L}[\mathbf{s}_l] := \bigoplus_{\lambda_l \in \Pi^l} \mathbb{A} |\lambda_l, \mathbf{s}_l\rangle$ and for $0 \leq i \leq n-1$, let \tilde{e}_i^{low} , \tilde{f}_i^{low} , \tilde{e}_i^{up} and \tilde{f}_i^{up} denote Kashiwara's operators acting on $\mathcal{L}[\mathbf{s}_l]$. The following lemma shows that sometimes, certain powers of the operators \tilde{e}_i^{low} and \tilde{e}_i^{up} coincide (one has an analogous result for \tilde{f}_i^{low} and \tilde{f}_i^{up}).

Lemma 2.15 *Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$, $u \in (\text{Ker } e_i) \cap \mathbf{F}_q[\mathbf{s}_l] \langle w \rangle$ and $k := (w, \alpha_i)$. Then we have*

$$(\tilde{e}_i^{\text{up}})^k \cdot (f_i^{(k)} \cdot u) = (\tilde{e}_i^{\text{low}})^k \cdot (f_i^{(k)} \cdot u) = u.$$

Proof. The second equality follows easily by induction on k from the definition of \tilde{e}_i^{low} . Let us now show that $(\tilde{e}_i^{\text{up}})^k \cdot (f_i^{(k)} \cdot u) = u$. Note that for $0 \leq j \leq k$, we have

$$(\text{wt}(f_i^{(k-j)} \cdot u), \alpha_i) = (\text{wt}(u), \alpha_i) - (k-j)(\alpha_i, \alpha_i) = 2j - k.$$

By induction on $0 \leq k' \leq k$, we get therefore, by definition of \tilde{e}_i^{up} ,

$$(\tilde{e}_i^{\text{up}})^{k'} \cdot (f_i^{(k)} \cdot u) = \left(\prod_{j=0}^{k'-1} \frac{[(2j-k) + (k-j) + 1]}{[k-j]} \right) f_i^{(k-k')} \cdot u \quad (0 \leq k' \leq k).$$

As a consequence, we have $(\tilde{e}_i^{\text{up}})^k \cdot (f_i^{(k)} \cdot u) = \left(\prod_{j=0}^{k-1} \frac{[j+1]}{[k-j]} \right) u = u$. \square

Put

$$(44) \quad \mathcal{B}[\mathbf{s}_l] := \{|\lambda_l, \mathbf{s}_l\rangle \bmod q\mathcal{L}[\mathbf{s}_l] \mid \lambda_l \in \Pi^l\}$$

In the sequel, we shall write more briefly λ_l for the element in $\mathcal{B}[\mathbf{s}_l]$ indexed by the corresponding multi-partition. By [JMMO],[FLOTW],[U], the pair $(\mathcal{L}[\mathbf{s}_l], \mathcal{B}[\mathbf{s}_l])$ is a lower crystal basis of $\mathbf{F}_q[\mathbf{s}_l]$ at $q = 0$ in the sense of [Kas1], and the crystal graph $\mathcal{B}[\mathbf{s}_l]$ contains the arrow $\lambda_l \xrightarrow{i} \mu_l$ if and only if the multi-partition μ_l is obtained from λ_l by adding a good i -node in the sense of [U, Thm. 2.4]. We shall still denote by \tilde{e}_i^{low} and \tilde{f}_i^{low} Kashiwara's operators acting on $\mathcal{B}[\mathbf{s}_l] \cup \{0\}$.

2.3.4 Uglov's canonical bases of the Fock spaces

By Propositions 2.9 (iii) and 2.11, the Fock spaces $\mathbf{F}_q[\mathbf{s}_l]$ and $\mathbf{F}_p[\mathbf{s}_n]^\bullet$ ($\mathbf{s}_l \in \mathbb{Z}^l(s)$, $\mathbf{s}_n \in \mathbb{Z}^n(s)$) are stable under the involution $-$. The involution induced on these spaces will still be denoted by $-$. Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$. For $\boldsymbol{\mu}_l \in \Pi^l$, write

$$(45) \quad \overline{|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle} = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}(q) |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$$

with $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}(q) \in \mathbb{Z}[q, q^{-1}]$, and let

$$(46) \quad A_{\mathbf{s}_l}(q) := (a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}(q))_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l}$$

denote the matrix of the involution $-$ of $\mathbf{F}_q[\mathbf{s}_l]$. Since the weight subspaces of $\mathbf{F}_q[\mathbf{s}_l]$ are stable under the involution $-$, (30) implies that $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}(q)$ is zero unless $|\boldsymbol{\lambda}_l| = |\boldsymbol{\mu}_l|$, where $|\boldsymbol{\lambda}_l|$ (*resp.* $|\boldsymbol{\mu}_l|$) denotes the number of boxes contained in the Young diagram of $|\boldsymbol{\lambda}_l|$ (*resp.* $|\boldsymbol{\mu}_l|$). By Proposition 2.9 (ii), the matrix $A_{\mathbf{s}_l}(q)$ is unitriangular. One can therefore define, by a classical argument, canonical bases of the Fock space $\mathbf{F}_q[\mathbf{s}_l]$ as follows.

Theorem 2.16 ([U]) *Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$. Then there exists a unique base*

$$\{G^+(\boldsymbol{\lambda}_l, \mathbf{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\} \quad \left(\text{resp. } \{G^-(\boldsymbol{\lambda}_l, \mathbf{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\} \right)$$

of $\mathbf{F}_q[\mathbf{s}_l]$ such that:

- (i) $\overline{G^+(\boldsymbol{\lambda}_l, \mathbf{s}_l)} = G^+(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ (resp. $\overline{G^-(\boldsymbol{\lambda}_l, \mathbf{s}_l)} = G^-(\boldsymbol{\lambda}_l, \mathbf{s}_l)$),
- (ii) $G^+(\boldsymbol{\lambda}_l, \mathbf{s}_l) \equiv |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \pmod{q\mathcal{L}^+[\mathbf{s}_l]}$ (resp. $G^-(\boldsymbol{\lambda}_l, \mathbf{s}_l) \equiv |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \pmod{q^{-1}\mathcal{L}^-[\mathbf{s}_l]}$),

where $\mathcal{L}^\epsilon[\mathbf{s}_l] := \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbb{Z}[q^\epsilon] |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ ($\epsilon = \pm 1$). □

Definition 2.17 The bases $\{G^+(\boldsymbol{\lambda}_l, \mathbf{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ and $\{G^-(\boldsymbol{\lambda}_l, \mathbf{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ are called *Uglov's canonical bases* of $\mathbf{F}_q[\mathbf{s}_l]$. Define entries $\Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^+(q)$, $\Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^-(q) \in \mathbb{Z}[q, q^{-1}]$ ($\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l$) by

$$(47) \quad G^+(\boldsymbol{\mu}_l, \mathbf{s}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^+(q) |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle, \quad G^-(\boldsymbol{\mu}_l, \mathbf{s}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi^l} \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^-(q) |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle,$$

and denote by

$$(48) \quad \Delta_{\mathbf{s}_l}^\epsilon(q) := (\Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^\epsilon(q))_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l} \quad (\epsilon = \pm 1)$$

the transition matrices between the standard and the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]$. ◇

By [U], the entries of $\Delta_{\mathfrak{s}_l}^+(q)$ (*resp.* $\Delta_{\mathfrak{s}_l}^-(q)$) are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type A , so by [KT], these polynomials are in $\mathbb{N}[q]$ (*resp.* $\mathbb{N}[p]$). Moreover, both canonical bases of $\mathbf{F}_q[\mathfrak{s}_l]$ are dual to each other with respect to a certain bilinear form, which gives an inversion formula for Kazhdan-Lusztig polynomials; see [U, Thm. 5.15]. By [U], the basis $\{G^+(\boldsymbol{\lambda}_l, \mathfrak{s}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ is a lower global crystal basis (in the sense of [Kas1]) of the integrable $U_q(\widehat{\mathfrak{sl}}_n)$ -module $\mathbf{F}_q[\mathfrak{s}_l]$.

Let $\mathfrak{s}_n \in \mathbb{Z}^n(s)$. In a similar way, one can define canonical bases $\{G^\epsilon(\boldsymbol{\lambda}_n, \mathfrak{s}_n)^\bullet \mid \boldsymbol{\lambda}_n \in \Pi^n\}$ ($\epsilon = \pm 1$) of the Fock space $\mathbf{F}_p[\mathfrak{s}_n]^\bullet$. By [U], the basis $\{G^-(\boldsymbol{\lambda}_n, \mathfrak{s}_n)^\bullet \mid \boldsymbol{\lambda}_n \in \Pi^n\}$ is a lower global crystal basis of the integrable $U_p(\widehat{\mathfrak{sl}}_l)$ -module $\mathbf{F}_p[\mathfrak{s}_n]^\bullet$. For $\boldsymbol{\mu}_n \in \Pi^n$, $\epsilon = \pm 1$, write

$$(49) \quad G^\epsilon(\boldsymbol{\mu}_n, \mathfrak{s}_n)^\bullet = \sum_{\boldsymbol{\lambda}_n \in \Pi^n} \dot{\Delta}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \mathfrak{s}_n}^\epsilon(q) |\boldsymbol{\lambda}_n, \mathfrak{s}_n\rangle^\bullet,$$

where the entries $\dot{\Delta}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \mathfrak{s}_n}^\epsilon(q)$ are in $\mathbb{Z}[q, q^{-1}]$. Since the weight subspaces of Λ^s are stable under the involution $-$, we have $\dot{\Delta}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \mathfrak{s}_n}^\epsilon(q) = 0$ unless $\text{wt}(|\boldsymbol{\lambda}_n, \mathfrak{s}_n\rangle^\bullet) = \text{wt}(|\boldsymbol{\mu}_n, \mathfrak{s}_n\rangle^\bullet)$ and $\text{wt}(|\boldsymbol{\lambda}_n, \mathfrak{s}_n\rangle^\bullet) = \text{wt}(|\boldsymbol{\mu}_n, \mathfrak{s}_n\rangle^\bullet)$. In this case, by Corollary 2.5, there exist $\mathfrak{s}_l \in \mathbb{Z}^l(s)$ and $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l$ such that $|\boldsymbol{\lambda}_l, \mathfrak{s}_l\rangle = |\boldsymbol{\lambda}_n, \mathfrak{s}_n\rangle^\bullet$ and $|\boldsymbol{\mu}_l, \mathfrak{s}_l\rangle = |\boldsymbol{\mu}_n, \mathfrak{s}_n\rangle^\bullet$. It is then not hard to see that

$$(50) \quad \dot{\Delta}_{\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n; \mathfrak{s}_n}^\epsilon(q) = \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathfrak{s}_l}^\epsilon(q).$$

3 Comparison of canonical bases of weight subspaces of $\mathbf{F}_q[\mathfrak{s}_l]$ ($\mathfrak{s}_l \in \mathbb{Z}^l(s)$)

In the sequel we use the following notation.

Notation 3.1 For $\mathfrak{s}_l \in \mathbb{Z}^l(s)$ and $w \in \mathcal{P}(\mathbf{F}_q[\mathfrak{s}_l])$, put

$$(51) \quad \Pi^l(\mathfrak{s}_l; w) := \{\boldsymbol{\lambda}_l \in \Pi^l \mid |\boldsymbol{\lambda}_l, \mathfrak{s}_l\rangle \in \Lambda^s\langle w\rangle\},$$

and define similarly $\Pi^n(\mathfrak{s}_n; \dot{w})$ for $\mathfrak{s}_n \in \mathbb{Z}^n(s)$ and $\dot{w} \in \dot{\mathcal{P}}(\mathbf{F}_p[\mathfrak{s}_n]^\bullet)$. \diamond

Definition 3.2 Let $\mathfrak{s}_l, \mathfrak{t}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\mathfrak{s}_l])$ and $w' \in \mathcal{P}(\mathbf{F}_q[\mathfrak{t}_l])$. We say that the canonical bases of $\mathbf{F}_q[\mathfrak{s}_l]\langle w\rangle$ and $\mathbf{F}_q[\mathfrak{t}_l]\langle w'\rangle$ are *similar* if there exists a bijection

$$\sigma : \Pi^l(\mathfrak{s}_l; w) \rightarrow \Pi^l(\mathfrak{t}_l; w')$$

such that for all $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l(\mathfrak{s}_l; w)$, $\epsilon = \pm 1$, we have

$$\Delta_{\sigma(\boldsymbol{\lambda}_l), \sigma(\boldsymbol{\mu}_l); \mathfrak{s}_l}^\epsilon(q) = \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathfrak{s}_l}^\epsilon(q).$$

In other words, the canonical bases of $\mathbf{F}_q[\mathfrak{s}_l]\langle w\rangle$ and $\mathbf{F}_q[\mathfrak{t}_l]\langle w'\rangle$ are similar if the transition matrices between the standard bases and the canonical bases are equal up to a reindexing of rows and columns. \diamond

Notation 3.3 Throughout this section we fix a multi-charge $\mathbf{s}_l \in \mathbb{Z}^l(s)$ and an integer $i \in \llbracket 0; n-1 \rrbracket$. To simplify, we drop the multi-charge \mathbf{s}_l in the notation of this section, that is we denote by $\boldsymbol{\lambda}_l$ (*resp.* $G^\pm(\boldsymbol{\lambda}_l)$) the vector of the standard (*resp.* canonical) basis of $\mathbf{F}_q[\mathbf{s}_l]$ indexed by the corresponding multi-partition and so on. In particular, we use the notation $\boldsymbol{\lambda}_l$ either for a vector of the standard basis of $\mathbf{F}_q[\mathbf{s}_l]$ or for a vertex in the crystal graph $\mathcal{B} := \mathcal{B}[\mathbf{s}_l]$. \diamond

3.1 The bijection σ_i

We recall here the definition of the involution σ_i of the crystal graph \mathcal{B} . We sometimes view σ_i as a bijection of Π^l .

Definition 3.4 Let $\boldsymbol{\lambda}_l \in \mathcal{B} \cong \Pi^l$ and $i \in \llbracket 0; n-1 \rrbracket$. Let \mathcal{C} be the i -chain in \mathcal{B} containing $\boldsymbol{\lambda}_l$. Let $\sigma_i(\boldsymbol{\lambda}_l) \in \mathcal{B} \cong \Pi^l$ be the unique element in \mathcal{C} such that $\text{wt}(\sigma_i(\boldsymbol{\lambda}_l)) = \sigma_i(\text{wt}(\boldsymbol{\lambda}_l))$. In other words, $\sigma_i(\boldsymbol{\lambda}_l)$ is obtained from $\boldsymbol{\lambda}_l$ via a central symmetry in the middle of \mathcal{C} . This defines an involution σ_i of \mathcal{B} . This map induces, for $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$, a bijection

$$(52) \quad \sigma_i : \Pi^l(\mathbf{s}_l; w) \xrightarrow{\sim} \Pi^l(\mathbf{s}_l; \sigma_i.w).$$

\diamond

By [Kas2], the definition of $\sigma_0, \dots, \sigma_{n-1}$ as bijections of \mathcal{B} gives actually rise to an action of the Weyl group W_n on \mathcal{B} , but we do not need this fact in the sequel.

Proposition 3.5 Let $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$ and $i \in \llbracket 0; n-1 \rrbracket$ be such that $w + \alpha_i \notin \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$. Let $\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)$ and $\boldsymbol{\mu}_l := \sigma_i(\boldsymbol{\lambda}_l)$. Then we have the following:

- (i) $\boldsymbol{\mu}_l$ is the multi-partition obtained by adding to $\boldsymbol{\lambda}_l$ all its addable i -nodes, and we have $|\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l| = k_i := (w, \alpha_i)$.
- (ii) In $\mathbf{F}_q[\mathbf{s}_l]$, we have $\boldsymbol{\mu}_l = f_i^{(k_i)} \cdot \boldsymbol{\lambda}_l$ and $\boldsymbol{\lambda}_l = e_i^{(k_i)} \cdot \boldsymbol{\mu}_l$.
- (iii) In \mathcal{B} , we have $\boldsymbol{\mu}_l = (\tilde{f}_i^{\text{low}})^{k_i} \cdot \boldsymbol{\lambda}_l$ and $\boldsymbol{\lambda}_l = (\tilde{e}_i^{\text{low}})^{k_i} \cdot \boldsymbol{\mu}_l$.

Proof. Let \mathcal{C} be the i -chain in \mathcal{B} containing $\boldsymbol{\lambda}_l$ and $\boldsymbol{\mu}_l$. Since $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\mathbf{s}_l]$, $\boldsymbol{\lambda}_l$ has no removable i -node. This implies, by [U, Thm. 2.4], that $\boldsymbol{\lambda}_l$ is the head of the chain \mathcal{C} , and by symmetry $\boldsymbol{\mu}_l$ is the tail of \mathcal{C} . Note that since $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\mathbf{s}_l]$, $\sigma_i(w + \alpha_i) = (\sigma_i.w) - \alpha_i$ is also not a weight of $\mathbf{F}_q[\mathbf{s}_l]$, so $\boldsymbol{\mu}_l$ has no addable i -node. By [U, Thm. 2.4], $\boldsymbol{\mu}_l$ is obtained by adding some i -nodes (let us say k of them) to $\boldsymbol{\lambda}_l$. The integer $k = |\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l|$ is none other than the length of the chain \mathcal{C} . By a well-known formula for crystal graphs relating weights and positions in the i -chains, since $\boldsymbol{\lambda}_l$ is the head of \mathcal{C} , we have $k_i = (\text{wt}(\boldsymbol{\lambda}_l), \alpha_i) = k$. Thus \mathcal{C} is a i -chain of length k_i with head $\boldsymbol{\lambda}_l$ and tail $\boldsymbol{\mu}_l$, which proves (iii). The divided powers $e_i^{(k)}, f_i^{(k)} \in U_q(\widehat{\mathfrak{sl}}_n)$ ($k \in \mathbb{N}^*$) act on $\boldsymbol{\nu}_l \in \mathbf{F}_q[\mathbf{s}_l]$ as follows, with notation of Section 2.2.2:

$$(*) \quad e_i^{(k)} \cdot \boldsymbol{\nu}_l = \sum_{\boldsymbol{\kappa}_l \xrightarrow{i,k} \boldsymbol{\nu}_l} q^{-M_i^<(\boldsymbol{\kappa}_l; \boldsymbol{\nu}_l; \mathbf{s}_l; n)} \boldsymbol{\kappa}_l \quad \text{and} \quad f_i^{(k)} \cdot \boldsymbol{\nu}_l = \sum_{\boldsymbol{\nu}_l \xrightarrow{i,k} \boldsymbol{\xi}_l} q^{M_i^>(\boldsymbol{\nu}_l; \boldsymbol{\xi}_l; \mathbf{s}_l; n)} \boldsymbol{\xi}_l.$$

(For $k = 1$, this is a part of Theorem 2.3. The general case follows by induction on k ; see *e.g.* [J] for a detailed proof.) Since $\boldsymbol{\mu}_l$ has no addable i -node and $\boldsymbol{\lambda}_l$ has no removable i -node, $\boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l$ is the set of the addable i -nodes of $\boldsymbol{\lambda}_l$ and also the set of the removable i -nodes of $\boldsymbol{\mu}_l$, and this set has exactly k_i elements. This proves (i), and this together with $(*)$ proves (ii). \square

Example 3.6 Take $n=3$, $l=2$, $\mathbf{s}_l=(1,2)$, $i=0$, $\boldsymbol{\lambda}_l=((2,2,1),(3,2))$ and $w = \text{wt}(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle)$. Then we have $\sigma_i(\boldsymbol{\lambda}_l) = ((3,2,2),(3,3,1))$. \diamond

Remark 3.7 The proof of Proposition 3.5 shows that (with the assumptions and notation of this proposition) $\boldsymbol{\lambda}_l = \sigma_i^{-1}(\boldsymbol{\mu}_l)$ is the multi-partition obtained by removing to $\boldsymbol{\mu}_l$ all its removable i -nodes. \diamond

3.2 A first theorem of comparison

Define a symmetric bilinear non-degenerate form (\cdot, \cdot) on $\mathbf{F}_q[\mathbf{s}_l]$ by

$$(53) \quad (\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l) = q^{|\boldsymbol{\lambda}_l|} \delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l} \quad (\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l),$$

where we put

$$(54) \quad \|\boldsymbol{\lambda}_l\| := (\text{wt}(\boldsymbol{\lambda}_l), \text{wt}(\boldsymbol{\lambda}_l))/2 \quad (\boldsymbol{\lambda}_l \in \Pi^l).$$

This form enjoys the following property:

Lemma 3.8 For $u, v \in \mathbf{F}_q[\mathbf{s}_l]$, $0 \leq i \leq n-1$, we have

$$(t_i.u, v) = (u, t_i.v) \quad \text{and} \quad (e_i.u, v) = (u, f_i.v).$$

Proof. Identical to the proof of [LLT, Prop. 8.1]. \square

Let $\{G^*(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ denote the adjoint basis of $\{G^+(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ with respect to the form (\cdot, \cdot) . Since the basis $\{G^+(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ is a lower global crystal basis of $\mathbf{F}_q[\mathbf{s}_l]$ in the sense of [Kas1], it follows by Lemma 3.8 and [Kas1, Prop. 3.2.2] that $\{G^*(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ is an upper global crystal basis of $\mathbf{F}_q[\mathbf{s}_l]$.

We are now ready to prove the following result, which is a generalization to higher-level Fock spaces of [LM, Thm. 20].

Theorem 3.9 Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$ and $i \in \llbracket 0; n-1 \rrbracket$ be such that $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\mathbf{s}_l]$. Let $\sigma_i : \Pi^l(\mathbf{s}_l; w) \rightarrow \Pi^l(\mathbf{s}_l; \sigma_i.w)$ be the bijection defined by (52). Then we have, for $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l(\mathbf{s}_l; w)$,

$$(i) \quad \Delta_{\sigma_i(\boldsymbol{\lambda}_l), \sigma_i(\boldsymbol{\mu}_l); \mathbf{s}_l}^+(q) = \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^+(q) \quad \text{and} \quad (ii) \quad \Delta_{\sigma_i(\boldsymbol{\lambda}_l), \sigma_i(\boldsymbol{\mu}_l); \mathbf{s}_l}^-(q) = \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^-(q).$$

As a consequence, the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle$ and $\mathbf{F}_q[\mathbf{s}_l]\langle \sigma_i.w \rangle$ are similar in the sense of Definition 3.2.

Proof. Let us prove (i). Let $\boldsymbol{\mu}_l \in \Pi^l(\mathbf{s}_l; w)$. Taking adjoint bases in (47) yields

$$q^{-\|\sigma_i(\boldsymbol{\mu}_l)\|} \sigma_i(\boldsymbol{\mu}_l) = \sum_{\boldsymbol{\nu}_l \in \Pi^l(\mathbf{s}_l; \sigma_i.w)} \Delta_{\sigma_i(\boldsymbol{\mu}_l), \boldsymbol{\nu}_l; \mathbf{s}_l}^+(q) G^*(\boldsymbol{\nu}_l).$$

Since $\sigma_i : \Pi^l(\mathbf{s}_l; w) \rightarrow \Pi^l(\mathbf{s}_l; \sigma_i.w)$ is a bijection, we can make in the sum above the reindexing $\boldsymbol{\nu}_l = \sigma_i(\boldsymbol{\lambda}_l)$. If we now apply $e_i^{(k_i)}$ with $k_i := (w, \alpha_i)$ to both hand-sides of this equality, we get

$$(*) \quad q^{-\|\sigma_i(\boldsymbol{\mu}_l)\|} e_i^{(k_i)} \cdot \sigma_i(\boldsymbol{\mu}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)} \Delta_{\sigma_i(\boldsymbol{\mu}_l), \sigma_i(\boldsymbol{\lambda}_l); \mathbf{s}_l}^+(q) e_i^{(k_i)} \cdot G^*(\sigma_i(\boldsymbol{\lambda}_l)).$$

Note that

$$(**) \quad \|\sigma_i(\boldsymbol{\mu}_l)\| = (\sigma_i.\text{wt}(\boldsymbol{\mu}_l), \sigma_i.\text{wt}(\boldsymbol{\mu}_l))/2 = (\text{wt}(\boldsymbol{\mu}_l), \text{wt}(\boldsymbol{\mu}_l))/2 = \|\boldsymbol{\mu}_l\|.$$

By Proposition 3.5 (ii), we have $e_i^{(k_i)} \cdot \sigma_i(\boldsymbol{\mu}_l) = \boldsymbol{\mu}_l$. Now let $\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)$. Since $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\mathbf{s}_l]$, we have $e_i \cdot \boldsymbol{\lambda}_l = 0$, whence $\tilde{e}_i^{\text{up}} \cdot \boldsymbol{\lambda}_l = 0$. Moreover, again by Proposition 3.5, we have $\sigma_i(\boldsymbol{\lambda}_l) = f_i^{(k_i)} \cdot \boldsymbol{\lambda}_l$. Therefore, by Lemma 2.15, we have

$$\boldsymbol{\lambda}_l = (\tilde{e}_i^{\text{low}})^{k_i} \cdot (f_i^{(k_i)} \cdot \boldsymbol{\lambda}_l) = (\tilde{e}_i^{\text{up}})^{k_i} \cdot (f_i^{(k_i)} \cdot \boldsymbol{\lambda}_l) = (\tilde{e}_i^{\text{up}})^{k_i} \cdot (\sigma_i(\boldsymbol{\lambda}_l)),$$

whence $(\tilde{e}_i^{\text{up}})^{k_i+1} \cdot \sigma_i(\boldsymbol{\lambda}_l) = 0$. Since $\{G^*(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ is an upper global crystal basis of $\mathbf{F}_q[\mathbf{s}_l]$, [Kas1, Lemma 5.1.1 (ii)] then implies

$$(***) \quad e_i^{(k_i)} \cdot G^*(\sigma_i(\boldsymbol{\lambda}_l)) = G^*((\tilde{e}_i^{\text{up}})^{k_i} \cdot (\sigma_i(\boldsymbol{\lambda}_l))) = G^*(\boldsymbol{\lambda}_l).$$

Combining (*), (**) and (***) we get

$$q^{-\|\boldsymbol{\mu}_l\|} \boldsymbol{\mu}_l = \sum_{\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)} \Delta_{\sigma_i(\boldsymbol{\mu}_l), \sigma_i(\boldsymbol{\lambda}_l); \mathbf{s}_l}^+(q) G^*(\boldsymbol{\lambda}_l).$$

Since this is valid for any $\boldsymbol{\mu}_l \in \Pi^l(\mathbf{s}_l; w)$, we get the claimed formula by taking again adjoint bases.

Let us now prove (ii). Let $w' \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$. If we know the basis $\{G^+(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w')\}$, then we can compute the involution of $\mathbf{F}_q[\mathbf{s}_l]\langle w' \rangle$ by solving a unitriangular system. Since the canonical basis $\{G^-(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w')\}$ is uniquely determined by the involution of $\mathbf{F}_q[\mathbf{s}_l]\langle w' \rangle$, the basis $\{G^-(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w')\}$ is uniquely determined by the basis $\{G^+(\boldsymbol{\lambda}_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w')\}$ (and conversely). Thus (i) implies (ii). (For a different proof of this fact, one can also apply [U, Thm 5.15].) \square

Example 3.10 Take $n = 3$, $l = 2$, $\mathbf{s}_l = (1, 0)$, $w = \text{wt}(|\emptyset_l, \mathbf{s}_l\rangle) - (2\alpha_0 + 3\alpha_1 + \alpha_2)$ and $i = 2$. One can easily check that $w + \alpha_i$ is not a weight of $\mathbf{F}_q[\mathbf{s}_l]$. The elements of $\Pi^l(\mathbf{s}_l; w)$ are

$$((1), (5)), ((4), (2)), ((4, 2), \emptyset), ((1), (2, 2, 1)), ((2, 2), (2)), ((1, 1), (2, 1, 1)), ((1, 1, 1, 1), (2)), ((1), (2, 1, 1, 1)),$$

and their respective images by the map σ_i are

$$((2), (6, 1)), ((5), (3, 1)), ((5, 3, 1), \emptyset), ((2), (3, 2, 2)), ((2, 2, 1), (3, 1)), ((2, 1, 1), (3, 1, 1)), ((2, 1, 1, 1), (3, 1)), ((2), (3, 1, 1, 1, 1)).$$

With obvious notation, the transition matrices of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle$ are

$$\Delta_{\mathbf{s}_l}^+\langle w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ q & q^2 & q & 0 & 1 & \cdot & \cdot & \cdot \\ q^2 & 0 & 0 & q & q & 1 & \cdot & \cdot \\ 0 & 0 & q & 0 & q^2 & q & 1 & \cdot \\ 0 & 0 & 0 & q^2 & 0 & q & 0 & 1 \end{pmatrix} \begin{matrix} ((1), (5)) \\ ((4), (2)) \\ ((4, 2), \emptyset) \\ ((1), (2, 2, 1)) \\ ((2, 2), (2)) \\ ((1, 1), (2, 1, 1)) \\ ((1, 1, 1, 1), (2)) \\ ((1), (2, 1, 1, 1)) \end{matrix} \quad \text{and}$$

$$\Delta_{\mathbf{s}_l}^-\langle w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^{-2} & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -q^{-1} & 0 & 1 & \cdot & \cdot & \cdot \\ q^{-2} & -q^{-1} & q^{-2} & -q^{-1} & -q^{-1} & 1 & \cdot & \cdot \\ -q^{-3} & q^{-2} & 0 & q^{-2} & 0 & -q^{-1} & 1 & \cdot \\ 0 & q^{-2} & -q^{-3} & 0 & q^{-2} & -q^{-1} & 0 & 1 \end{pmatrix} \begin{matrix} ((1), (5)) \\ ((4), (2)) \\ ((4, 2), \emptyset) \\ ((1), (2, 2, 1)) \\ ((2, 2), (2)) \\ ((1, 1), (2, 1, 1)) \\ ((1, 1, 1, 1), (2)) \\ ((1), (2, 1, 1, 1)) \end{matrix}.$$

In the same way, the transition matrices of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle \sigma_i \cdot w \rangle$ are

$$\Delta_{\mathbf{s}_l}^+\langle \sigma_i \cdot w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ q & q^2 & q & 0 & 1 & \cdot & \cdot & \cdot \\ q^2 & 0 & 0 & q & q & 1 & \cdot & \cdot \\ 0 & 0 & q & 0 & q^2 & q & 1 & \cdot \\ 0 & 0 & 0 & q^2 & 0 & q & 0 & 1 \end{pmatrix} \begin{matrix} ((2), (6, 1)) \\ ((5), (3, 1)) \\ ((5, 3, 1), \emptyset) \\ ((2), (3, 2, 2)) \\ ((2, 2, 1), (3, 1)) \\ ((2, 1, 1), (3, 1, 1)) \\ ((2, 1, 1, 1), (3, 1)) \\ ((2), (3, 1, 1, 1, 1)) \end{matrix} \quad \text{and}$$

$$\Delta_{\mathbf{s}_l}^-\langle \sigma_i \cdot w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^{-2} & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -q^{-1} & 0 & 1 & \cdot & \cdot & \cdot \\ q^{-2} & -q^{-1} & q^{-2} & -q^{-1} & -q^{-1} & 1 & \cdot & \cdot \\ -q^{-3} & q^{-2} & 0 & q^{-2} & 0 & -q^{-1} & 1 & \cdot \\ 0 & q^{-2} & -q^{-3} & 0 & q^{-2} & -q^{-1} & 0 & 1 \end{pmatrix} \begin{matrix} ((2), (6, 1)) \\ ((5), (3, 1)) \\ ((5, 3, 1), \emptyset) \\ ((2), (3, 2, 2)) \\ ((2, 2, 1), (3, 1)) \\ ((2, 1, 1), (3, 1, 1)) \\ ((2, 1, 1, 1), (3, 1)) \\ ((2), (3, 1, 1, 1, 1)) \end{matrix},$$

in agreement with Theorem 3.9. \diamond

4 Comparison of canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle$ for w in a given coset in $\mathcal{P}(\Lambda^s)/\mathbb{Z}\delta$ and different multi-charges \mathbf{s}_l

Notation 4.1 For $\mathbf{a}_l, \mathbf{b}_l \in \mathbb{Z}^l(s)$, introduce the shorter notation

$$(55) \quad d(\mathbf{a}_l, \mathbf{b}_l) := \Delta(\mathbf{a}_l, n) - \Delta(\mathbf{b}_l, n) \in \mathbb{Z}.$$

Throughout this section we fix $\mathbf{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$ and $i \in \llbracket 0; l-1 \rrbracket$. Finally, let $(\mathbf{s}_n, \dot{w}) \in \mathbb{Z}^n(s) \times \dot{\mathcal{P}}(\Lambda^s)$ be the pair such that $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle = \mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle$ (see Proposition 2.12). \diamond

4.1 A second theorem of comparison

Recall that $\dot{W}_l = \langle \dot{\sigma}_0, \dots, \dot{\sigma}_{l-1} \rangle \cong \tilde{\mathfrak{S}}_l$ is the Weyl group of $U_p(\widehat{\mathfrak{sl}}_l)$.

Definition 4.2 Keep Notation 4.1. By analogy with Definition 3.4, define a bijection

$$(56) \quad \dot{\sigma}_i : \Pi^n(\mathbf{s}_n; \dot{w}) \rightarrow \Pi^n(\mathbf{s}_n; \dot{\sigma}_i \cdot \dot{w})$$

which enjoys similar properties as the bijections σ_j from (52). Since $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle = \mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle$, we have a bijection between the standard basis of $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle$ (as a subspace of $\mathbf{F}_q[\mathbf{s}_l]$) and the standard basis of $\mathbf{F}_p[\mathbf{s}_n]^\bullet \langle \dot{w} \rangle$ (as a subspace of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$). We thus have a bijection $\Pi^l(\mathbf{s}_l; w) \xrightarrow{\sim} \Pi^n(\mathbf{s}_n; \dot{w})$. Put $\mathbf{t}_l := \dot{\sigma}_i \cdot \mathbf{s}_l$. The same argument gives, by Lemma 2.14, a bijection $\Pi^l(\mathbf{t}_l; w + d(\mathbf{s}_l, \mathbf{t}_l)\delta) \xrightarrow{\sim} \Pi^n(\mathbf{s}_n; \dot{\sigma}_i \cdot \dot{w})$. We therefore have the following commutative diagram of bijections, in which the dashed arrow will still be denoted by $\dot{\sigma}_i$.

$$\begin{array}{ccc} \Pi^n(\mathbf{s}_n; \dot{w}) & \xrightarrow{\dot{\sigma}_i} & \Pi^n(\mathbf{s}_n; \dot{\sigma}_i \cdot \dot{w}) \\ \uparrow & & \uparrow \\ \Pi^l(\mathbf{s}_l; w) & \xrightarrow{\sim} & \Pi^l(\dot{\sigma}_i \cdot \mathbf{s}_l; w + d(\mathbf{s}_l, \dot{\sigma}_i \cdot \mathbf{s}_l)\delta). \end{array}$$

\diamond

Example 4.3 Take $n = 2$, $l = 3$, $\mathbf{s}_l = (0, 2, -1)$ and $i = 2$. Note that $d(\mathbf{s}_l, \dot{\sigma}_i \cdot \mathbf{s}_l) = 0$ in this case. Take $\boldsymbol{\lambda}_l = (\emptyset, (2), \emptyset) \in \Pi^l(\mathbf{s}_l; w)$, where $w := \text{wt}(|\emptyset_l, \mathbf{s}_l\rangle) - (\alpha_0 + \alpha_1)$. By Proposition 2.12, we have $\dot{w} = \text{wt}(|\emptyset_n, \mathbf{s}_n\rangle^\bullet) - (2\dot{\alpha}_0 + 3\dot{\alpha}_1 + \dot{\alpha}_2)$ with $\mathbf{s}_n := (1, 0)$. We have $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle^\bullet$ with $\boldsymbol{\lambda}_n := ((1), (5))$. One computes $\boldsymbol{\mu}_n := \dot{\sigma}_i(\boldsymbol{\lambda}_n) = ((2), (6, 1))$ (see Example 3.10). Let $\mathbf{t}_l := \dot{\sigma}_i \cdot \mathbf{s}_l = (0, -1, 2)$. Then $\dot{\sigma}_i(\boldsymbol{\lambda}_l)$ is the l -multi-partition such that $|\dot{\sigma}_i(\boldsymbol{\lambda}_l), \mathbf{t}_l\rangle = |\boldsymbol{\mu}_n, \mathbf{s}_n\rangle^\bullet$, namely $\dot{\sigma}_i(\boldsymbol{\lambda}_l) = (\emptyset, \emptyset, (2))$. \diamond

Since $\{G^-(\boldsymbol{\lambda}_n, \mathbf{s}_n)^\bullet \mid \boldsymbol{\lambda}_n \in \Pi^n\}$ is a lower global crystal basis of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$, one can prove for the Fock space $\mathbf{F}_p[\mathbf{s}_n]^\bullet$ an analogue of Theorem 3.9. Rephrasing this result in terms of the indexation $\boldsymbol{\lambda}_l$ leads to the following result.

Theorem 4.4 Keep Notation 4.1 and assume that $\dot{w} + \dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$. Let $\dot{\sigma}_i : \Pi^l(\mathbf{s}_l; w) \rightarrow \Pi^l(\dot{\sigma}_i \cdot \mathbf{s}_l; w + d(\mathbf{s}_l, \dot{\sigma}_i \cdot \mathbf{s}_l)\delta)$ be the bijection from Definition 4.2. Then we have, for $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l(\mathbf{s}_l; w)$, $\epsilon = \pm 1$:

$$\Delta_{\dot{\sigma}_i(\boldsymbol{\lambda}_l), \dot{\sigma}_i(\boldsymbol{\mu}_l); \dot{\sigma}_i \cdot \mathbf{s}_l}^\epsilon(q) = \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l; \mathbf{s}_l}^\epsilon(q).$$

Proof. Apply the analogue for $\mathbf{F}_p[\mathbf{s}_n]^\bullet$ of Theorem 3.9 mentioned above, then use Lemma 2.14 and (50). \square

Example 4.5 (see Examples 3.10 and 4.3) Take n, l, \mathbf{s}_l, w and i as in Example 4.3 (namely, $n := 2, l := 3, \mathbf{s}_l := (0, 2, -1), w := \text{wt}(\langle \emptyset_l, \mathbf{s}_l \rangle) - (\alpha_0 + \alpha_1)$ and $i := 2$). Note that $\dot{w} + \dot{\alpha}_i \notin \dot{\mathcal{P}}(\mathbf{F}_p[\mathbf{s}_n]^\bullet)$ (see Example 3.10). The elements of $\Pi^l(\mathbf{s}_l; w)$ are

$$(\emptyset, (2), \emptyset), (\emptyset, \emptyset, (1, 1)), (\emptyset, (1), (1)), (\emptyset, \emptyset, (2)), ((2), \emptyset, \emptyset), ((1), \emptyset, (1)), ((1, 1), \emptyset, \emptyset), (\emptyset, \emptyset, (1, 1)),$$

and their respective images by the map $\dot{\sigma}_i$ are

$$(\emptyset, \emptyset, (2)), (\emptyset, \emptyset, (1, 1)), (\emptyset, (1), (1)), ((2), \emptyset, \emptyset), (\emptyset, (2), \emptyset), ((1), (1), \emptyset), ((1, 1), \emptyset, \emptyset), (\emptyset, (1, 1), \emptyset),$$

On the one hand, the transition matrices of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle$ are

$$\Delta_{\mathbf{s}_l}^+\langle w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^2 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ q^2 & q & q^2 & q & q & 1 & \cdot & \cdot & \cdot \\ q^3 & q^2 & 0 & 0 & q^2 & q & 1 & \cdot & \cdot \\ 0 & q^2 & q^3 & q^2 & 0 & q & 0 & 1 & \cdot \end{pmatrix} \begin{matrix} (\emptyset, (2), \emptyset) \\ (\emptyset, \emptyset, (1, 1)) \\ (\emptyset, (1), (1)) \\ (\emptyset, \emptyset, (2)) \\ ((2), \emptyset, \emptyset) \\ ((1), \emptyset, (1)) \\ ((1, 1), \emptyset, \emptyset) \\ (\emptyset, \emptyset, (1, 1)) \end{matrix} \quad \text{and}$$

$$\Delta_{\mathbf{s}_l}^-\langle w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & q^{-2} & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ q^{-2} & 0 & 0 & -q^{-1} & -q^{-1} & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & -q^{-1} & q^{-2} & 0 & -q^{-1} & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & q^{-2} & -q^{-1} & 0 & 1 & \cdot \end{pmatrix} \begin{matrix} (\emptyset, (2), \emptyset) \\ (\emptyset, \emptyset, (1, 1)) \\ (\emptyset, (1), (1)) \\ (\emptyset, \emptyset, (2)) \\ ((2), \emptyset, \emptyset) \\ ((1), \emptyset, (1)) \\ ((1, 1), \emptyset, \emptyset) \\ (\emptyset, \emptyset, (1, 1)) \end{matrix}.$$

On the other hand, the transition matrices of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w \rangle$ are

$$\Delta_{\dot{\sigma}_i, \mathbf{s}_l}^+\langle w \rangle(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^2 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ q^2 & q & q^2 & q & q & 1 & \cdot & \cdot & \cdot \\ q^3 & q^2 & 0 & 0 & q^2 & q & 1 & \cdot & \cdot \\ 0 & q^2 & q^3 & q^2 & 0 & q & 0 & 1 & \cdot \end{pmatrix} \begin{matrix} (\emptyset, \emptyset, (2)) \\ (\emptyset, \emptyset, (1, 1)) \\ (\emptyset, (1), (1)) \\ ((2), \emptyset, \emptyset) \\ (\emptyset, (2), \emptyset) \\ ((1), (1), \emptyset) \\ ((1, 1), \emptyset, \emptyset) \\ (\emptyset, (1, 1), \emptyset) \end{matrix} \quad \text{and}$$

$$\Delta_{\dot{\sigma}_i, \mathbf{s}_l}^-(w)(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & q^{-2} & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ q^{-2} & 0 & 0 & -q^{-1} & -q^{-1} & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & -q^{-1} & q^{-2} & 0 & -q^{-1} & 1 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & q^{-2} & -q^{-1} & 0 & 1 & \cdot \end{pmatrix} \begin{matrix} (\emptyset, \emptyset, (2)) \\ (\emptyset, \emptyset, (1, 1)) \\ (\emptyset, (1), (1)) \\ ((2), \emptyset, \emptyset) \\ (\emptyset, (2), \emptyset) \\ ((1), (1), \emptyset) \\ ((1, 1), \emptyset, \emptyset) \\ (\emptyset, (1, 1), \emptyset) \end{matrix},$$

in agreement with Theorem 4.4 (note that $d(\mathbf{s}_l, \dot{\sigma}_i \cdot \mathbf{s}_l) = 0$ in this case). \diamond

4.2 A sufficient condition for Theorem 4.4

Keep again Notation 4.1. We can apply Theorem 4.4 if $\dot{w} + \dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$. By [Kac, Prop. 3.6 (iv)], this condition holds if and only if $\dot{e}_i \cdot (|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle) = 0$ for all $\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)$. We therefore have to check, for all $\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)$, whether $\boldsymbol{\lambda}_n$ has a removable i -node, where $\boldsymbol{\lambda}_n \in \Pi^n$ is related to $\boldsymbol{\lambda}_l$ by $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle^\bullet$. It is not very convenient to make such tests in practice when the cardinality of $\Pi^l(\mathbf{s}_l; w)$ is large. We shall therefore give a sufficient condition on \mathbf{s}_l and w that ensures, without further computation, that $\dot{w} + \dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$.

Notation 4.6 Let $\mathbf{s}_l \in \mathbb{Z}^l(s)$, $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$ and $i \in \llbracket 0; l-1 \rrbracket$. By (30), the integer $N_i(\boldsymbol{\lambda}_l; \mathbf{s}_l; n)$ only depends on \mathbf{s}_l and w , but not on $\boldsymbol{\lambda}_l \in \Pi^l(\mathbf{s}_l; w)$. In the sequel this number will be denoted by $N_i(w; \mathbf{s}_l)$. \diamond

Lemma 4.7 *Keep Notation 4.1. Then for all $\dot{\sigma} \in \dot{W}_l$, $0 \leq i \leq n-1$, $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{s}_l])$, we have*

$$N_i(w + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta; \dot{\sigma} \cdot \mathbf{s}_l) = N_i(w; \mathbf{s}_l).$$

Proof. By (30) and the definition of the integers $N_j(w + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta; \dot{\sigma} \cdot \mathbf{s}_l)$ and $d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l)$, we have

$$\begin{aligned} w + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta &= \text{wt}(|\emptyset_l, \dot{\sigma} \cdot \mathbf{s}_l\rangle) - \sum_{j=0}^{n-1} N_j(w + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta; \dot{\sigma} \cdot \mathbf{s}_l) \alpha_j \\ &= \text{wt}(|\emptyset_l, \mathbf{s}_l\rangle) + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta - \sum_{j=0}^{n-1} N_j(w + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta; \dot{\sigma} \cdot \mathbf{s}_l) \alpha_j, \end{aligned}$$

whence $w = \text{wt}(|\emptyset_l, \mathbf{s}_l\rangle) - \sum_{j=0}^{n-1} N_j(w + d(\mathbf{s}_l, \dot{\sigma} \cdot \mathbf{s}_l) \delta; \dot{\sigma} \cdot \mathbf{s}_l) \alpha_j$. The lemma follows. \square

Lemma 4.8 *Keep Notation 4.1. Assume that*

$$s_i - s_{i+1} \geq n(N_0(w; \mathbf{s}_l) + 1)$$

(where we put $s_0 := n + s_l$ if $i = 0$). Then $\dot{w} + \dot{\alpha}_i$ is not a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$.

Proof. We only sketch this proof and leave the details to the reader. Assume on the contrary that $\dot{w} + \dot{\alpha}_i$ is a weight of $\mathbf{F}_p[\mathbf{s}_n]^\bullet$. Let $(\boldsymbol{\lambda}_l, \mathbf{t}_l) \in \Pi^l \times \mathbb{Z}^l(s)$ be such that $\text{wt}(|\boldsymbol{\lambda}_l, \mathbf{t}_l\rangle) = \dot{w} + \dot{\alpha}_i$. By (31) we get

$$N_0(\boldsymbol{\lambda}_l; \mathbf{t}_l; n) = \Delta(\mathbf{s}_l, n) - \Delta(\mathbf{t}_l, n) + N_0(w; \mathbf{s}_l) - \delta_{i,0}$$

and $\mathbf{t}_l = (t_1, \dots, t_l)$, where t_0, \dots, t_l are defined by

$$t_0 = t_l + n, \quad t_i := s_i + 1, \quad t_{i+1} := s_{i+1} - 1 \quad \text{and} \quad t_j := s_j \text{ if } j \notin \{i, i+1, i+l\}.$$

Moreover, we have by assumption

$$N_0(w; \mathbf{s}_l) - \frac{1}{n}(s_i - s_{i+1}) \leq -1.$$

Combining these facts gives, after a little computation: $N_0(\boldsymbol{\lambda}_l; \mathbf{t}_l; n) < 0$, which is absurd since $N_0(\boldsymbol{\lambda}_l; \mathbf{t}_l; n)$ is the number of 0-nodes of the multi-partition $\boldsymbol{\lambda}_l$. \square

Remark 4.9 The lower bound $s_i - s_{i+1} \geq n(N_0(w; \mathbf{s}_l) + 1)$ from Lemma 4.8 is certainly not the best to ensure that $\dot{w} + \dot{\alpha}_i \notin \mathcal{P}(\mathbf{F}_p[\mathbf{s}_n]^\bullet)$. We actually conjecture that the latter statement holds if

$$s_i - s_{i+1} \geq N_0(w; \mathbf{s}_l) + \dots + N_{n-1}(w; \mathbf{s}_l)$$

(this lower bound is in general better). \diamond

4.3 A graph containing multi-charges conjugated under the action of \dot{W}_l

Definition 4.10 Fix $\mathbf{r}_l \in A_{l,n}(s)$ and $M \in \mathbb{N}^*$. (Recall that the set $A_{l,n}(s)$ defined by (36) is a fundamental domain for the action of \dot{W}_l .) For $\mathbf{s}_l = (s_1, \dots, s_l) \in \dot{W}_l \cdot \mathbf{r}_l$, $\mathbf{t}_l \in \dot{W}_l \cdot \mathbf{r}_l$, $0 \leq i \leq l-1$, write $\mathbf{s}_l \xrightarrow{i} \mathbf{t}_l$ if $\mathbf{t}_l = \dot{\sigma}_i \cdot \mathbf{s}_l$ and $s_i - s_{i+1} \geq M$ (if $i=0$, we put $s_0 := n + s_l$). Let $\Gamma(M)$ be the graph containing $\dot{W}_l \cdot \mathbf{r}_l$ as set of vertices and the arrows $\mathbf{s}_l \xrightarrow{i} \mathbf{t}_l$ ($\mathbf{s}_l, \mathbf{t}_l \in \dot{W}_l \cdot \mathbf{r}_l$, $0 \leq i \leq l-1$). \diamond

Remark 4.11 Note that $\Gamma(1)$ is connected. More generally, we claim that $\Gamma(M)$ has finitely many connected components. For $\mathbf{s}_l \in \dot{W}_l \cdot \mathbf{r}_l$, let $\dot{\sigma}(\mathbf{s}_l) \in \dot{W}_l$ be the element of minimal length such that $\mathbf{s}_l = \dot{\sigma}(\mathbf{s}_l) \cdot \mathbf{r}_l$. Now, for each connected component C in $\Gamma(M)$, choose $\mathbf{s}_l(C) \in C$ such that $\ell(\mathbf{s}_l(C))$ is minimal in the set $\{\ell(\mathbf{t}_l) \mid \mathbf{t}_l \in C\}$. (We think that this determines $\mathbf{s}_l(C)$ in a unique way, but we do not have the proof for this fact.) One can then show easily that $\mathbf{s}_l(C)$ lies in the finite set

$$\{(t_1, \dots, t_l) \in \mathbb{Z}^l(s) \mid \forall 0 \leq i \leq l-1, t_{i+1} - t_i \leq M-1\},$$

which proves the claim. \diamond

We now give a relation between the (connected components of) $\Gamma(M)$ and the comparison of canonical bases. We shall give an important application of this result in Section 5.

Proposition 4.12 *Let $\mathbf{r}_l \in A_{l,n}(s)$ and $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{r}_l])$. Put $M := n(N_0(w; \mathbf{r}_l) + 1)$. Let \mathbf{s}_l and \mathbf{t}_l be two multi-charges in the same connected component of $\Gamma(M)$ (in particular, \mathbf{s}_l and \mathbf{t}_l are in the \dot{W}_l -orbit of \mathbf{r}_l). Then the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w + d(\mathbf{r}_l, \mathbf{s}_l)\delta \rangle$ and $\mathbf{F}_q[\mathbf{t}_l]\langle w + d(\mathbf{r}_l, \mathbf{t}_l)\delta \rangle$ are similar in the sense of Definition 3.2.*

Proof. We may assume that $\mathbf{s}_l \xrightarrow{i} \mathbf{t}_l$ with $i \in \llbracket 0; l-1 \rrbracket$. With obvious notation, we have by Lemma 4.7 : $s_i - s_{i+1} \geq n(N_0(w; \mathbf{r}_l) + 1) = n(N_0(w + d(\mathbf{r}_l, \mathbf{s}_l)\delta; \mathbf{s}_l) + 1)$. We can therefore apply Lemma 4.8 and then Theorem 4.4 to conclude. \square

With the notation above, Proposition 4.12 and Remark 4.11 show that there are only finitely many similarity classes of canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w + d(\mathbf{r}_l, \mathbf{s}_l)\delta \rangle$, where \mathbf{s}_l ranges over the \dot{W}_l -orbit of \mathbf{r}_l and (\mathbf{r}_l, w) is fixed.

5 Comparison of canonical bases for dominant multi-charges

Definition 5.1 Let $M \in \mathbb{N}$. We say that $(x_1, \dots, x_N) \in \mathbb{R}^N$ is M -dominant if for all $1 \leq i \leq N-1$, we have

$$x_i - x_{i+1} \geq M.$$

\diamond

Throughout Section 5, we keep the following notation.

5.1 Notation

* Recall that $\mathbb{R}^l(s) = \{(x_1, \dots, x_l) \in \mathbb{R}^l \mid x_1 + \dots + x_l = s\}$. The subset of $\mathbb{R}^l(s)$ formed by the M -dominant elements will be denoted by \mathcal{C}_M .

* The group $\dot{W}_l \cong \tilde{\mathfrak{S}}_l$ is a semidirect product of the finite symmetric group \mathfrak{S}_l and an abelian group $\dot{\mathcal{Q}}$ which is free of rank $l-1$. More precisely, $\dot{\mathcal{Q}}$ is spanned by $\dot{\tau}_1, \dots, \dot{\tau}_{l-1}$, where $\dot{\tau}_i$ ($1 \leq i \leq l-1$) acts on $\mathbb{Z}^l(s)$ by

$$(57) \quad \dot{\tau}_i.(s_1, \dots, s_l) = (s_1, \dots, s_{i-1}, s_i + n, s_{i+1} - n, s_{i+2}, \dots, s_l) \quad ((s_1, \dots, s_l) \in \mathbb{Z}^l(s)).$$

(since \dot{W}_l acts faithfully on $\mathbb{Z}^l(s)$, this determines $\dot{\tau}_i$ completely).

* For $\mathbf{a}_l = (a_1, \dots, a_l) \in \mathbb{Z}^l$, put

$$(58) \quad \mathcal{L}_{\mathbf{a}_l} := \{(s_1, \dots, s_l) \in \mathbb{Z}^l(s) \mid \forall 1 \leq i \leq l, s_i \equiv a_i \pmod{n}\}.$$

Note that $\dot{\mathcal{Q}}$ acts transitively on $\mathcal{L}_{\mathbf{a}_l}$; in particular, two elements of $\mathcal{L}_{\mathbf{a}_l}$ lie in a same \dot{W}_l -orbit.

* For $\mathbf{s}_l, \mathbf{t}_l \in \mathbb{Z}^l(s)$ and $M \in \mathbb{N}$, write

$$(59) \quad \mathbf{s}_l \stackrel{M}{\equiv} \mathbf{t}_l$$

if $\mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$, and there exist $\mathbf{s}_l^{(0)}, \dots, \mathbf{s}_l^{(r)} \in \mathcal{L}_{\mathbf{s}_l} \cap \mathcal{C}_M$ such that $\mathbf{s}_l^{(0)} = \mathbf{s}_l, \mathbf{s}_l^{(r)} = \mathbf{t}_l$ and for all $1 \leq i \leq r$, we have $\mathbf{s}_l^{(i)} \in \{\dot{\tau}_j^{\pm 1} \cdot \mathbf{s}_l^{(i-1)} \mid 1 \leq j \leq l-1\}$. In other words, put into a non-oriented graph all the elements of $\mathbb{Z}^l(s)$ and draw an edge between two vertices if they are \dot{W}_l -conjugated to each other by a generator of \dot{Q} or its inverse. Then $\mathbf{s}_l \stackrel{M}{\equiv} \mathbf{t}_l$ if and only if there exists a path in this graph connecting \mathbf{s}_l and \mathbf{t}_l through M -dominant vertices (including \mathbf{s}_l and \mathbf{t}_l).

* Let $\mathbf{r}_l = (r_1, \dots, r_l) \in A_{l,n}(s)$ and $w \in \mathcal{P}(\mathbf{F}_q[\mathbf{r}_l])$. Recall the definition of the integers $d(\mathbf{r}_l, \mathbf{s}_l)$ ($\mathbf{s}_l \in \mathbb{Z}^l(s)$) from (55) and $N_i(w; \mathbf{r}_l)$ ($0 \leq i \leq n-1$) from Notation 4.6.

5.2 A third theorem of comparison

The goal of Section 5 is to prove the following theorem.

Theorem 5.2 *Keep Notation 5.1. Then there exists $N \in \mathbb{N}$ (which only depends on n, l and $N_0(w; \mathbf{r}_l)$) such that for all N -dominant multi-charges $\mathbf{s}_l, \mathbf{t}_l \in \dot{W}_l \cdot \mathbf{r}_l$ with $\mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$, the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w + d(\mathbf{r}_l, \mathbf{s}_l)\delta \rangle$ and $\mathbf{F}_q[\mathbf{t}_l]\langle w + d(\mathbf{r}_l, \mathbf{t}_l)\delta \rangle$ are similar. \square*

Remark 5.3 The proof of Theorem 5.1 will provide an integer N of the form

$$N = nN_0(w; \mathbf{r}_l) + c,$$

where c is a constant that can be explicitly calculated. More precisely, by Remark 5.13 we can take $c \geq n(l^2 + l + 3)$. However, this value of N is probably not optimal. Indeed, according to Remark 4.9 and explicit calculations of canonical bases, we conjecture that Theorem 5.2 holds if we replace this N by

$$N' := N_0(w; \mathbf{r}_l) + \dots + N_{n-1}(w; \mathbf{r}_l)$$

(the latter lower bound is in general better). \diamond

Example 5.4 Take $n = 3, l = 2, \mathbf{r}_l = (1, 0)$ and $w = \text{wt}(|\emptyset_l, \mathbf{r}_l\rangle) - (\alpha_0 + \alpha_1 + \alpha_2)$. With notation from Theorem 5.2 and Remark 5.3, we can take $N = 30$ and $N' = 3$. Note that all the multi-charges $\mathbf{s}_l^{(k)} := (3k+1, -3k)$ ($k \in \mathbb{Z}$) are in the \dot{W}_l -orbit of \mathbf{r}_l and they have the same pair of residues modulo n . Put $w_k := w + d(\mathbf{r}_l, \mathbf{s}_l^{(k)})\delta$ ($k \in \mathbb{Z}$). By Theorem 5.2, the canonical bases of $\mathbf{F}_q[\mathbf{s}_l^{(k)}]\langle w_k \rangle$, $k \geq 5$ are pairwise similar. By Remark 5.3, the canonical bases of $\mathbf{F}_q[\mathbf{s}_l^{(k)}]\langle w_k \rangle$, $k \geq 1$ should be pairwise similar, which can be actually checked by an explicit calculus. Namely, the transition matrices $\Delta_k^\epsilon(q)$ of the canonical bases of $\mathbf{F}_q[\mathbf{s}_l^{(k)}]\langle w_k \rangle$ ($k \geq 1, \epsilon = \pm 1$) are

$$\Delta_k^+(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^2 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & q^2 & q & q & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & q & 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & q^2 & 0 & q & q & 1 & \cdot \\ 0 & 0 & 0 & q & q^2 & 0 & q & 1 \end{pmatrix} \begin{matrix} ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix} \quad \text{and}$$

$$\Delta_k^-(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ q^{-2} & -q^{-1} & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & q^{-2} & -q^{-1} & -q^{-1} & 1 & \cdot & \cdot & \cdot \\ -q^{-1} & q^{-2} & -q^{-1} & 0 & 0 & 1 & \cdot & \cdot \\ q^{-2} & -q^{-3} & q^{-2} & q^{-2} & -q^{-1} & -q^{-1} & 1 & \cdot \\ -q^{-3} & 0 & 0 & 0 & 0 & q^{-2} & -q^{-1} & 1 \end{pmatrix} \begin{matrix} ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

One can check moreover that

$$\Delta_0^+(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ q & q & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & q^2 & q & q & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & q^2 & 0 & q & 1 & \cdot & \cdot \\ q^2 & 0 & 0 & q & 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & q^2 & q & 0 & q & 1 \end{pmatrix} \begin{matrix} ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

Note that $\Delta_1^+(q)$ has 22 nonzero entries, whereas $\Delta_0^+(q)$ has only 21 nonzero entries. As a consequence, the canonical bases of $\mathbf{F}_q[\mathbf{s}_l^{(k)}]\langle w_k \rangle$ for $k = 0$ and $k = 1$ are not similar. \diamond

The proof of Theorem 5.2 relies on the two following propositions.

Proposition 5.5 *Keep the notation from Section 5.1. Let $\mathbf{s}_l \in \dot{W}_l \cdot \mathbf{r}_l$ be an M -dominant multi-charge with $M := n(N_0(w; \mathbf{r}_l) + 2)$. Let $1 \leq i \leq l-1$ and $\mathbf{t}_l := \dot{\tau}_i \cdot \mathbf{s}_l$. Then the canonical bases of $\mathbf{F}_q[\mathbf{s}_l]\langle w + d(\mathbf{r}_l, \mathbf{s}_l) \delta \rangle$ and $\mathbf{F}_q[\mathbf{t}_l]\langle w + d(\mathbf{r}_l, \mathbf{t}_l) \delta \rangle$ are similar.*

Proof. We give the proof for $2 \leq i \leq l-3$ (the proof for $i = 1$, $i = l-2$ and $i = l-1$ is similar). One easily checks that

$$\dot{\tau}_i = \dot{\sigma}_{i-1} \dot{\sigma}_{i-2} \cdots \dot{\sigma}_1 \dot{\sigma}_0 \dot{\sigma}_{l-1} \cdots \dot{\sigma}_{i+2} \dot{\sigma}_{i+1} \dot{\sigma}_{i+2} \cdots \dot{\sigma}_{l-1} \dot{\sigma}_0 \dot{\sigma}_1 \cdots \dot{\sigma}_i.$$

Let $0 \leq k \leq 2l - 2$. Denote by $\dot{\tau}_i[k]$ the right factor of length k in this word (we thus have $\dot{\tau}_i[0] = \text{id}$, $\dot{\tau}_i[1] = \dot{\sigma}_i$, $\dot{\tau}_i[2] = \dot{\sigma}_{i-1}\dot{\sigma}_i$ and so on). Put

$$\mathbf{s}_l^{(k)} = (s_1^{(k)}, \dots, s_l^{(k)}) := \dot{\tau}_i[k].\mathbf{s}_l.$$

For $1 \leq k \leq 2l - 2$, let $i_k \in \llbracket 0; l - 1 \rrbracket$ be such that $\dot{\tau}_i[k] = \dot{\sigma}_{i_k}\dot{\tau}_i[k - 1]$. Let now $0 \leq k \leq 2l - 3$. By computing the action of $\dot{\tau}_i[k]$ on \mathbf{s}_l , one can show the following:

(i) $s_{i_{k+1}}^{(k)}, s_{i_{k+1}+1}^{(k)} \in \{s_1, \dots, s_l, s_l + n, s_i + n, s_{i+1} - n\}$.

(ii) If $s_{i_{k+1}}^{(k)} = s_a + \epsilon n$ and $s_{i_{k+1}+1}^{(k)} = s_{a'} + \epsilon' n$ with $a, a' \in \llbracket 1; l \rrbracket$, $\epsilon, \epsilon' \in \{-1, 0, 1\}$, then $a \neq a'$ and $\epsilon\epsilon' = 0$.

Note moreover that by assumption on \mathbf{s}_l , we have for all $a, b \in \llbracket 1; l \rrbracket$ such that $a \neq b$, $|s_b - s_a| \geq M|b - a| \geq M$. This together with (i), (ii) imply that

$$|s_{i_{k+1}}^{(k)} - s_{i_{k+1}+1}^{(k)}| \geq M - n.$$

As a consequence, $\Gamma(M - n)$ contains the arrow $\mathbf{s}_l^{(k)} \xrightarrow{i_k} \mathbf{s}_l^{(k+1)}$ or $\mathbf{s}_l^{(k+1)} \xrightarrow{i_k} \mathbf{s}_l^{(k)}$. It follows that \mathbf{s}_l and \mathbf{t}_l are in the same connected component of $\Gamma(M - n) = \Gamma(n(N_0(w; \mathbf{r}_l) + 1))$. We can therefore apply Proposition 4.12 to conclude. \square

Proposition 5.6 *Let $M \in \mathbb{N}$. Then there exists $c \in \mathbb{Z}$ (which only depends on l and n , but not on M) such that for all $(M + c)$ -dominant multi-charges $\mathbf{s}_l, \mathbf{t}_l \in \mathbb{Z}^l(s)$ with $\mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$, we have $\mathbf{s}_l \stackrel{M}{\equiv} \mathbf{t}_l$.*

Proof. We shall prove this proposition in Section 5.3. \square

Proof of Theorem 5.2 from Propositions 5.5 and 5.6.

Let $M := n(N_0(w; \mathbf{r}_l) + 2)$. Let $c \in \mathbb{Z}$ be the integer given by Proposition 5.6 and put $N := M + c$. Let $\mathbf{s}_l, \mathbf{t}_l \in \dot{W}_l.\mathbf{r}_l$ be two N -dominant multi-charges such that $\mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$. Put $\mathcal{L} := \mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$. By Proposition 5.6, there exist $\mathbf{s}_l^{(0)}, \dots, \mathbf{s}_l^{(r)} \in \mathcal{L} \cap \mathcal{C}_M$ such that $\mathbf{s}_l^{(0)} = \mathbf{s}_l$, $\mathbf{s}_l^{(r)} = \mathbf{t}_l$ and for all $1 \leq i \leq r$, we have $\mathbf{s}_l^{(i)} \in \{\dot{\tau}_j^{\pm 1}.\mathbf{s}_l^{(i-1)} \mid 1 \leq j \leq l - 1\}$. Let $1 \leq i \leq r$. Since $\mathcal{L}_{\mathbf{s}_l^{(i-1)}} = \mathcal{L}_{\mathbf{s}_l^{(i)}} = \mathcal{L}$, we have $\mathbf{s}_l^{(i-1)}, \mathbf{s}_l^{(i)} \in \dot{W}_l.\mathbf{r}_l$. By Proposition 5.5, the canonical bases of $\mathbf{F}_q[\mathbf{s}_l^{(i-1)}]\langle w + d(\mathbf{r}_l, \mathbf{s}_l^{(i-1)})\delta \rangle$ and $\mathbf{F}_q[\mathbf{s}_l^{(i)}]\langle w + d(\mathbf{r}_l, \mathbf{s}_l^{(i)})\delta \rangle$ are similar. Theorem 5.2 follows. \square

5.3 Proof of Proposition 5.6

The idea of the proof is the following. Let \mathbf{s}_l and \mathbf{t}_l be two multi-charges as in Proposition 5.6 and put $\mathcal{L} := \mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$. First, we introduce a suitable change of coordinates φ that

maps \mathcal{L} to \mathbb{Z}^{l-1} and such that $\mathbf{a}_l \stackrel{M}{\equiv} \mathbf{b}_l$ if and only if $\varphi(\mathbf{a}_l)$ is \mathbb{Z} -connected to $\varphi(\mathbf{b}_l)$, that is there exists a piecewise affine path connecting $\varphi(\mathbf{a}_l)$ to $\varphi(\mathbf{b}_l)$ with edges parallel to the axes of coordinates of \mathbb{Z}^{l-1} (see Lemma 5.8). Roughly speaking, the aim is to replace the lattice \mathcal{L} by \mathbb{Z}^{l-1} and the action of $\dot{\mathcal{Q}}$ by the obvious action of \mathbb{Z}^{l-1} by translations. Doing this, we replace the set of M -dominance \mathcal{C}_M by a certain cone (that is, an intersection of half-spaces), temporarily denoted by C_M . Note that two arbitrary points in $C_M \cap \mathbb{Z}^{l-1}$ are not necessarily \mathbb{Z} -connected in C_M . However, and this is the second step of the proof, we shall construct an integer c such that $C_{M+c} \subset C_M$ and any two points in $C_{M+c} \cap \mathbb{Z}^{l-1}$ are \mathbb{Z} -connected in C_M (see Proposition 5.12).

Notation 5.7 In addition to the notation from Section 5.1 and Proposition 5.6, we shall use for the proof the following notation.

* For $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ ($N \in \mathbb{N}^*$), put

$$(60) \quad \lfloor \mathbf{x} \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_N \rfloor) \in \mathbb{Z}^N$$

and define $\lceil \mathbf{x} \rceil$ in a similar way.

* Let

$$(61) \quad A := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

denote the Cartan matrix of \mathfrak{sl}_l . In particular, A has $l-1$ rows and $l-1$ columns.

* For $1 \leq j \leq l-1$, let $\epsilon_j := (\delta_{i,j})_{1 \leq i \leq l-1}$ be the j -th vector of the natural basis of \mathbb{R}^{l-1} . Put also $\mathbf{1} := \epsilon_1 + \dots + \epsilon_{l-1}$.

* Define a partial ordering on the set of matrices (or vectors) of a given size with entries in \mathbb{R} by writing $\mathbf{x} = (x_i)_{i \in I} \leq \mathbf{y} = (y_i)_{i \in I}$ if $x_i \leq y_i$ for all $i \in I$. We may speak of the maximum of \mathbf{x} and \mathbf{y} (with respect to this ordering). Now, for $\mathbf{b} \in \mathbb{R}^{l-1}$ define the cones

$$(62) \quad C_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}^{l-1} \mid A \cdot \mathbf{x} \geq \mathbf{b}\} \quad \text{and} \quad C'_{\mathbf{b}} := \{\mathbf{x} \in \mathbb{R}^{l-1} \mid A \cdot \mathbf{x} \leq \mathbf{b}\}.$$

The unique vector $\boldsymbol{\omega} := \boldsymbol{\omega}(\mathbf{b}) \in \mathbb{R}^{l-1}$ such that $A \cdot \boldsymbol{\omega} = \mathbf{b}$ is called the *vertex* of $C_{\mathbf{b}}$ (or $C'_{\mathbf{b}}$).

* For $M \in \mathbb{N}$, let $\mathbf{b}(M) = \mathbf{b}(M; n, l, \mathbf{r}_l) = (b_1^{(M)}, \dots, b_{l-1}^{(M)}) \in \mathbb{R}^{l-1}$ denote the vector defined by

$$(63) \quad b_i^{(M)} := (M + r_{i+1} - r_i)/n \quad (1 \leq i \leq l-1).$$

* Define a map $\varphi : (s_1, \dots, s_l) \in \mathbb{R}^l(s) \mapsto (x_1, \dots, x_{l-1}) = \varphi(s_1, \dots, s_l) \in \mathbb{R}^{l-1}$ by

$$(64) \quad x_i := \frac{1}{n} \sum_{j=1}^i (s_j - r_j) \quad (1 \leq i \leq l-1).$$

Conversely, let $\psi : (x_1, \dots, x_{l-1}) \in \mathbb{R}^{l-1} \mapsto (s_1, \dots, s_l) = \psi(x_1, \dots, x_{l-1}) \in \mathbb{R}^l(s)$ be the map defined by

$$(65) \quad s_i := n(x_i - x_{i-1}) + r_i \quad (1 \leq i \leq l),$$

where we put $x_0 = x_l := 0$. ◇

Lemma 5.8

(i) The maps $\varphi : \mathbb{R}^l(s) \rightarrow \mathbb{R}^{l-1}$ and $\psi : \mathbb{R}^{l-1} \rightarrow \mathbb{R}^l(s)$ are bijections inverse to each other.

(ii) We have $\psi(\mathbb{Z}^{l-1}) = \mathcal{L}_{\mathbf{r}_l}$, and for $1 \leq i \leq l-1$, $(x_1, \dots, x_{l-1}) \in \mathbb{Z}^{l-1}$, we have

$$\psi(x_1, \dots, x_i + 1, \dots, x_{l-1}) = \dot{\tau}_i \cdot \psi(x_1, \dots, x_{l-1}).$$

(iii) For $M \in \mathbb{N}$, we have $\varphi(\mathcal{C}_M) = C_{\mathbf{b}(M)}$.

Proof. The proof of (i) and (ii) is straightforward. With obvious notation, we have the equivalence

$$s_i - s_{i+1} \geq M \iff -x_{i-1} + 2x_i - x_{i+1} \geq (M + r_{i+1} - r_i)/n;$$

Statement (iii) follows. □

Definition 5.9 Let $D \subset \mathbb{R}^{l-1}$. We say that $\mathbf{x}, \mathbf{y} \in D$ are \mathbb{Z} -connected in D if there exist vectors $\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(N)} \in D \cap \mathbb{Z}^{l-1}$ such that $\mathbf{x}^{(0)} = \mathbf{x}$, $\mathbf{x}^{(N)} = \mathbf{y}$ and for all $0 \leq i \leq N-1$, we have $\mathbf{x}^{(i+1)} - \mathbf{x}^{(i)} \in \{\pm \epsilon_j \mid 1 \leq j \leq l-1\}$; in particular, we have $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{l-1}$. In this case, write $\mathbf{x} \underset{D}{\sim} \mathbf{y}$. ◇

In order to prove Proposition 5.6, we have to deal with \mathbb{Z} -connected points in cones $C_{\mathbf{b}}$ ($\mathbf{b} \in \mathbb{R}^{l-1}$). Note that two points in $C_{\mathbf{b}} \cap \mathbb{Z}^{l-1}$ are not necessary \mathbb{Z} -connected in $C_{\mathbf{b}}$. For example, take $l = 3$ (so $l-1 = 2$) and $\mathbf{b} = (0, 0)$. Then $\mathbf{x} := (1, 1)$ and $\mathbf{y} := (0, 0)$ are two points in $C_{\mathbf{b}}$ which are not \mathbb{Z} -connected in $C_{\mathbf{b}}$, because none of the points $(0, \pm 1)$ and $(\pm 1, 0)$ lies in $C_{\mathbf{b}}$. However, given $\mathbf{b} \in \mathbb{R}^{l-1}$, one can construct $\mathbf{c} \leq \mathbf{b}$ such that any two points in $C_{\mathbf{b}}$ with integer coordinates are \mathbb{Z} -connected in $C_{\mathbf{c}}$. This leads to the introduction of the following set. For $\mathbf{b} \in \mathbb{R}^{l-1}$, put

$$(66) \quad \mathcal{A}(\mathbf{b}) := \{\mathbf{c} \in \mathbb{R}^{l-1} \mid \mathbf{c} \leq \mathbf{b} \text{ and } \forall \mathbf{x}, \mathbf{y} \in C_{\mathbf{b}}, \mathbf{x} \underset{C_{\mathbf{c}}}{\sim} \mathbf{y}\}.$$

Proposition 5.10 *Let $\mathbf{b} \in \mathbb{R}^{l-1}$. Then we have the following.*

- (i) *The set $\mathcal{A}(\mathbf{b})$ is nonempty.*
- (ii) *For all $\mathbf{c} \in \mathcal{A}(\mathbf{b})$, we have $\mathbf{c}' \leq \mathbf{c} \Rightarrow \mathbf{c}' \in \mathcal{A}(\mathbf{b})$.*
- (iii) *The map $\mathbf{b} \mapsto \mathcal{A}(\mathbf{b})$ is increasing.*
- (iv) *For all $\mathbf{b}' \in \mathbb{R}^{l-1}$ such that $A^{-1} \cdot (\mathbf{b} - \mathbf{b}') \in \mathbb{Z}^{l-1}$, we have $\mathcal{A}(\mathbf{b}) = \mathcal{A}(\mathbf{b}') + (\mathbf{b} - \mathbf{b}')$.*

Proof. We prove (i) and leave the other statements to the reader. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{l-1})$ be the vertex of $C_{\mathbf{b}}$, $\mathbf{b}' := \mathbf{b} + 2\mathbf{1}$ and $\boldsymbol{\omega}'$ be the vertex of $C_{\mathbf{b}'}$. Let $\boldsymbol{\omega}'' = (\omega''_1, \dots, \omega''_{l-1})$ be equal to $\max(\boldsymbol{\omega}', \lceil \boldsymbol{\omega} \rceil)$ and let $\omega''_0 = \omega''_l := 0$. Now let $\mathbf{c} = (c_1, \dots, c_{l-1}) \in \mathbb{R}^{l-1}$ be the vector defined by

$$c_i := -\omega''_{i-1} + 2\omega_i - \omega''_{i+1} \quad (1 \leq i \leq l-1).$$

We shall show that $\mathbf{c} \in \mathcal{A}(\mathbf{b})$. It is well-known that $A^{-1} \geq 0$, whence $\mathbf{x} \in C_{\mathbf{b}} \Rightarrow \mathbf{x} \geq \boldsymbol{\omega}$ and $\mathbf{y} \in C_{\mathbf{b}'} \Rightarrow \mathbf{y} \leq \boldsymbol{\omega}'$. Since $\mathbf{b} \leq \mathbf{b}'$, we get $\boldsymbol{\omega} \in C_{\mathbf{b}'}$ and therefore $\boldsymbol{\omega} \leq \boldsymbol{\omega}' \leq \boldsymbol{\omega}''$. Consider the set

$$\mathbf{P} := \{\mathbf{x} \in \mathbb{R}^{l-1} \mid \boldsymbol{\omega} \leq \mathbf{x} \leq \boldsymbol{\omega}''\}.$$

The argument above shows that $C_{\mathbf{b}} \cap C_{\mathbf{b}'} \subset \mathbf{P}$, and by construction we also have $\lceil \boldsymbol{\omega} \rceil \in \mathbf{P}$. By definition of \mathbf{c} , we have $\boldsymbol{\omega} \in \mathbf{P} \subset C_{\mathbf{c}}$, whence $\mathbf{c} \leq \mathbf{b}$. Let us now show that any $\mathbf{x} \in C_{\mathbf{b}} \cap \mathbb{Z}^{l-1}$ is \mathbb{Z} -connected to $\lceil \boldsymbol{\omega} \rceil$ in $C_{\mathbf{c}}$. Note that for $\mathbf{x} = (x_1, \dots, x_{l-1}) \in C_{\mathbf{b}} \cap \mathbb{Z}^{l-1}$, we have $\mathbf{x} \geq \boldsymbol{\omega}$ (because $\mathbf{x} \in C_{\mathbf{b}}$) and therefore $\mathbf{x} \geq \lceil \boldsymbol{\omega} \rceil$. We can thus argue by induction on

$$N(\mathbf{x}) := \sum_{i=1}^{l-1} (x_i - \lceil \omega_i \rceil) \in \mathbb{N}.$$

If $N(\mathbf{x}) = 0$, we have $\mathbf{x} = \lceil \boldsymbol{\omega} \rceil \in \mathbf{P} \subset C_{\mathbf{c}}$ and we are done. Assume now that $\mathbf{x} \in C_{\mathbf{b}} \cap \mathbb{Z}^{l-1}$, $N(\mathbf{x}) > 0$, and consider two cases. Assume first that $\mathbf{x} \in C_{\mathbf{b}'}$. Then we have $\mathbf{x} \in C_{\mathbf{b}} \cap C_{\mathbf{b}'} \subset \mathbf{P}$; moreover, we have $\lceil \boldsymbol{\omega} \rceil \in \mathbf{P}$, so $\mathbf{x} \underset{\mathbf{P}}{\sim} \lceil \boldsymbol{\omega} \rceil$ because any two points in $\mathbf{P} \cap \mathbb{Z}^{l-1}$ are \mathbb{Z} -connected. Since $\mathbf{P} \subset C_{\mathbf{c}}$, we can conclude in this case. Assume now that $\mathbf{x} \notin C_{\mathbf{b}'}$. Let $1 \leq i \leq l-1$ be such that $-x_{i-1} + 2x_i - x_{i+1} > b_i + 2$ (where we put $x_0 = x_l := 0$). Consider the vector

$$\mathbf{y} = (y_1, \dots, y_{l-1}) := (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_{l-1}) \in \mathbb{Z}^{l-1}$$

and put $y_0 = y_l := 0$. Note that for $1 \leq j \leq l-1$, we have

$$-y_{j-1} + 2y_j - y_{j+1} \geq -x_{j-1} + 2x_j - x_{j+1} - 2\delta_{i,j}.$$

Using this fact together with the definition of i and the assumption $\mathbf{x} \in C_{\mathbf{b}}$, we get $\mathbf{y} \in C_{\mathbf{b}}$. Moreover, since $N(\mathbf{y}) = N(\mathbf{x}) - 1$, we have by induction $\mathbf{y} \underset{C_{\mathbf{c}}}{\sim} \lceil \boldsymbol{\omega} \rceil$. Note also that $\mathbf{x} \underset{C_{\mathbf{c}}}{\sim} \mathbf{y}$ because $\mathbf{c} \leq \mathbf{b}$, hence $\mathbf{x} \underset{C_{\mathbf{c}}}{\sim} \lceil \boldsymbol{\omega} \rceil$. \square

Given $M \in \mathbb{N}$, we shall construct $c \in \mathbb{Z}$ such that $\mathbf{b}(M) \in \mathcal{A}(\mathbf{b}(M+c))$. Let $\mathbf{b} \in \mathbb{R}^{l-1}$. By Proposition 5.10, we can define

$$(67) \quad m_M = m_M(n, l, \mathbf{r}_l) := \max_{\mathbf{c}} \min_{1 \leq i \leq l-1} (c_i) \in \mathbb{Z},$$

where $\mathbf{c} = (c_1, \dots, c_{l-1})$ ranges over the set $\mathcal{A}(\mathbf{b}(M; n, l, \mathbf{r}_l)) \cap \mathbb{Z}^{l-1}$.

Lemma 5.11 *The sequence $(m_M)_{M \in \mathbb{N}}$ is increasing. Moreover, we have $m_{nlM} = lM + m_0$ for all $M \in \mathbb{N}$.*

Proof. The first statement follows from the fact that the maps $M \rightarrow \mathbf{b}(M)$ and $\mathbf{b} \mapsto \mathcal{A}(\mathbf{b})$ are increasing (the latter by Proposition 5.10). Let $M \in \mathbb{N}$, $\mathbf{b} := \mathbf{b}(nlM)$ and $\mathbf{b}' := \mathbf{b}(0)$. Since $\det(A) = l$, the Cramer formula shows that $A^{-1} \cdot (\mathbf{b} - \mathbf{b}') \in \mathbb{Z}^{l-1}$. Applying Proposition 5.10 (iv) to the pair $(\mathbf{b}, \mathbf{b}')$ yields $\mathcal{A}(\mathbf{b}(nlM)) = \mathcal{A}(\mathbf{b}(0)) + (lM, \dots, lM)$. The second statement follows. \square

Proposition 5.12 *Denote by $m_0^{(\min)}$ the real number $\min_{\mathbf{a}_l \in A_{l,n}(s)} m_0(n, l; \mathbf{a}_l)$ and define $c := n + \lceil -m_0^{(\min)} n \rceil + nl \in \mathbb{Z}$. Then for all $M \in \mathbb{N}$, we have $\mathbf{b}(M) \in \mathcal{A}(\mathbf{b}(M+c))$.*

Proof. Let $\mathbf{c} = (c_1, \dots, c_{l-1}) \in \mathcal{A}(\mathbf{b}(M+c)) \cap \mathbb{Z}^{l-1}$ be such that $\min_{1 \leq i \leq l-1} (c_i) = m_{M+c}$. By proposition 5.10 (ii), it is enough to show that $\mathbf{b}(M) \leq \mathbf{c}$. We have

$$M+c \geq nl \left(\frac{M+n-m_0n}{nl} + 1 \right) \geq nl \left\lceil \frac{M+n-m_0n}{nl} \right\rceil = nl \left\lceil \frac{\left(\frac{M}{n} + 1\right) - m_0}{l} \right\rceil.$$

The previous lemma implies

$$m_{M+c} \geq l \left\lceil \frac{\left(\frac{M}{n} + 1\right) - m_0}{l} \right\rceil + m_0 \geq l \left(\frac{\left(\frac{M}{n} + 1\right) - m_0}{l} \right) + m_0 \geq \left\lceil \frac{M}{n} \right\rceil.$$

Let $1 \leq i \leq l-1$. Since $\mathbf{r}_l \in A_{l,n}(s)$, we have $r_{i+1} - r_i \leq 0$, whence

$$c_i \geq m_{M+c} \geq \left\lceil \frac{M}{n} \right\rceil \geq \frac{M}{n} + \frac{r_{i+1} - r_i}{n} = b_i^{(M)}.$$

The result follows. \square

Remark 5.13 The author proved in [Y1, Section 4.4.3] that $m_0^{(\min)} \geq -l^2$, which gives an explicit lower bound for the integer c from Propositions 5.12 and 5.6. \diamond

Proof of Proposition 5.6.

Let $c \in \mathbb{Z}$ be the integer defined in Proposition 5.12. Let $\mathbf{s}_l, \mathbf{t}_l \in \mathbb{Z}^l(s)$ be two $(M+c)$ -dominant multi-charges such that $\mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$. Let $\mathbf{r}_l \in A_{l,n}(s)$ be such that $\mathcal{L}_{\mathbf{r}_l} = \mathcal{L}_{\mathbf{s}_l} = \mathcal{L}_{\mathbf{t}_l}$. By Lemma 5.8, we have $\varphi(\mathbf{s}_l), \varphi(\mathbf{t}_l) \in C_{\mathbf{b}(M+c)} \cap \mathbb{Z}^{l-1}$. Proposition 5.12 now implies that $\mathbf{b}(M)$ is in $\mathcal{A}(\mathbf{b}(M+c))$, whence $\varphi(\mathbf{s}_l) \underset{C_{\mathbf{b}(M)}}{\equiv} \varphi(\mathbf{t}_l) \underset{M}{\equiv} \mathbf{t}_l$. Applying again Lemma 5.8 yields $\mathbf{s}_l \underset{M}{\equiv} \mathbf{t}_l$. \square

References

- [A] S. ARIKI, *On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$* , J. Math. Kyoto Univ. **36**, no. 4 (1996), 789-808.
- [DJM] R. DIPPER, G. JAMES, A. MATHAS, *Cyclotomic q -Schur algebras*, Math. Z. **229** (1998), 385-416.
- [FLOTW] O. FODA, B. LECLERC, M. OKADO, J.-Y. THIBON, T. WELSH, *Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras*, Adv. Math. **141** No.2 (1999), 322-365.
- [H] T. HAYASHI, *q -analogues of Clifford and Weyl algebras - spinor and oscillator representations of quantum enveloping algebras*, Commun. Math. Phys. **127** No.1 (1990), 129-144.
- [J] N. JACON, *On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity*, J. Math. Kyoto Univ. **44** No.4 (2004), 729-767.
- [JMMO] M. JIMBO, K. MISRA, T. MIWA, M. OKADO, *Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}_n)$ at $q = 0$* , Commun. Math. Phys. **136** No.3 (1991), 543-566.
- [Kac] V. G. KAC, *Infinite dimensional Lie algebras*, 3rd Ed. Cambridge University Press, 1990.
- [Kas1] M. KASHIWARA, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), 455-485.
- [Kas2] M. KASHIWARA, *The crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993), 839-958.
- [KMS] M. KASHIWARA, T. MIWA, E. STERN, *Decomposition of q -deformed Fock spaces*, Selecta Math. **1** (1995), 787-805.
- [KT] M. KASHIWARA, T. TANISAKI, *Parabolic Kazhdan-Lusztig polynomials and Schubert varieties*, J. Algebra **249** (2002), 306-325.
- [LLT] A. LASCoux, B. LECLERC, J.-Y. THIBON, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Commun. Math. Phys. **181** No.1 (1996), 205-263.
- [LM] B. LECLERC, H. MIYACHI, *Some closed formulas for canonical bases of Fock spaces*, Represent. Theory **6** (2002), 290-312.
- [LT1] B. LECLERC, J.-Y. THIBON, *Canonical bases of q -deformed Fock spaces*, Internat. Math. Res. Notices **9** (1996), 447-456.
- [LT2] B. LECLERC, J.-Y. THIBON, *Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials*, Combinatorial Methods in Representation Theory, Advanced Studies in Pure Mathematics **28** (2000), 155-220.

- [Mac] I. G. MACDONALD, *Symmetric functions and Hall polynomials*, 2nd Ed. Oxford Science Publications, Oxford University Press, 1995.
- [MM] K.C. MISRA, T. MIWA, *Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}_n)$* , Commun. Math. Phys. **134** No.1 (1990), 79-88.
- [S] J. SCOPES, *Cartan matrices and Morita equivalence for blocks of the symmetric groups*, J. Algebra **142** No.2 (1991), 441-455.
- [U] D. UGLOV, *Canonical bases of higher-level q -deformed Fock spaces and Kazhdan-Lusztig polynomials*, in Physical Combinatorics ed. M. Kashiwara, T. Miwa, Progress in Math. **191**, Birkhäuser (2000), arXiv math.QA/9905196 (1999).
- [VV] M. VARAGNOLO, E. VASSEROT, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), 267-297.
- [Y1] X. YVONNE, *Bases canoniques d'espaces de Fock de niveau supérieur*, Thèse de l'Université de Caen (2005).
- [Y2] X. YVONNE, *A conjecture for q -decomposition matrices of cyclotomic v -Schur algebras*, J. Algebra (2006), arXiv math.RT/0505379.

Xavier YVONNE, Laboratoire de Mathématiques Nicolas Oresme, Université de Caen, BP 5186, 14032 Caen Cedex, France.

E-mail address: xyvonne@math.unicaen.fr