PERVERSE SHEAVES ON ARTIN STACKS

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Abstract. In this paper we develop the theory of perverse sheaves on Artin stacks continuing the study in [10] and [11].

1. Introduction

In this third paper in our series on Grothendieck's six operations for étale sheaves on stacks, we define the perverse t–structure on the derived category of étale sheaves (with either finite or adic coefficients). This generalizes the t–structure defined in the case of schemes in [5] (but note also that in this paper we consider unbounded schemes which is not covered in loc. cit.). By [5, 3.2.4] the perverse sheaves on a scheme form a stack with respect to the smooth topology. This enables one to define the notion of a perverse sheaf on any Artin stack (using also the unbounded version of the gluing lemma [5, 3.2.4] proven in [10, 2.3.3]). The main contribution of this paper is to define a t–structure on the derived category whose heart is this category of perverse sheaves. In fact it is shown in [5, 4.2.5] that pullback along smooth morphisms of schemes is an exact functor (up to a shift) with respect to the perverse t–structure. This implies that the definition of the functor ${}^{\mathrm{p}}\mathcal{H}^0$ ($\tau_{\geq 0}\tau_{\leq 0}$ with respect to the perverse t–structure) for stacks is forced upon us from the case of schemes and the gluing lemma. We verify in this paper that the resulting definitions of the subcategories ${}^{\mathrm{p}}\mathcal{D}^{\geq j}$ and ${}^{\mathrm{p}}\mathcal{D}^{\leq j}$ of the derived category of étale sheaves (in either finite coefficient or adic coefficient case) define a t–structure.

Remark 1.1. The reader should note that unlike the case of schemes (Beilinson's theorem [4]) the derived category of the abelian category of perverse sheaves is not equivalent to the derived category of sheaves on the stack. An explicit example suggested by D. Ben-Zvi is the following: Let $\mathcal{X} = \mathrm{B}\mathbb{G}_m$ over an algebraically closed field k. The category of perverse $\mathbb{Q}_l$–sheaves on $\mathcal{X}$ is the equivalent to the category of $\mathbb{G}_m$–equivariant perverse sheaves on $\mathrm{Spec}(k)$ as defined in [12, III.15]. In particular, this is a semisimple category. In particular, if $\mathcal{D}$ denotes the derived category of perverse sheaves we see that for two objects V and W the groups $\mathrm{Ext}^i_{\mathcal{D}}(\mathrm{V}, \mathrm{W})$ are zero for all $i > 0$. On the other hand, we have $\mathrm{H}^{2i}(\mathcal{X}, \mathbb{Q}_l) \neq 0$ for all $i \geq 0$.

Remark 1.2. The techniques used in this paper can also be used to define perverse sheaves of $\mathcal{D}$–modules on complex analytic stacks.

Throughout the paper we work over a ground field k and write $\mathrm{S} = \mathrm{Spec}(k)$.

2. Gluing of t–structures

For the convenience of the reader, we review the key result [5, 1.4.10].

Let $\mathcal{D}$, $\mathcal{D}_{\mathrm{U}}$, and $\mathcal{D}_{\mathrm{F}}$ be three triangulated categories with exact functors

$$\mathcal{D}_{\mathrm{F}} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}_{\mathrm{U}}.$$

Write also $i_! := i_*$ and $j^! := j^*$. Assume the following hold:

(i) The functor i_* has a left adjoint i^* and a right adjoint $i^!$.

(ii) The functor j^* has a left adjoint $j_!$ and a right adjoint j_*.

(iii) We have $i^! j_* = 0$.

(iv) For every object $\mathrm{K} \in \mathcal{D}$ there exists a morphism $d : i_* i^* \mathrm{K} \to j_! j^* \mathrm{K}[1]$ (resp. $d : j_* j^* \mathrm{K} \to i_* i^! \mathrm{K}[1]$) such that the induced triangle

$$j_! j^* \mathrm{K} \to \mathrm{K} \to i_* i^* \mathrm{K} \to j_! j^* \mathrm{K}[1]$$

$$(\text{resp. } i_* i^! \mathrm{K} \to \mathrm{K} \to j_* j^* \mathrm{K} \to i_* i^! \mathrm{K}[1])$$

is distinguished.

(v) The adjunction morphisms $i^* i_* \to \mathrm{id} \to i^! i_*$ and $j^* j_* \to \mathrm{id} \to j^* j_!$ are all isomorphisms.

The main example we will consider is the following:

Example 2.1. Let $\mathcal{X}$ be an algebraic stack locally of finite type over a field k, and let $i : \mathcal{F} \hookrightarrow \mathcal{X}$ be a closed substack with complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$. Let Λ be a complete discrete valuation ring of residue characteristic prime to $\mathrm{char}(k)$, and for an integer n let Λ_n denote $\Lambda/\mathfrak{m}^{n+1}$.

Fix an integer n, and let $\mathcal{D}$ (resp. $\mathcal{D}_{\mathcal{U}}$, $\mathcal{D}_{\mathcal{F}}$) denote the bounded derived category $\mathcal{D}^b_c(\mathcal{X}, \Lambda_n)$ (resp. $\mathcal{D}^b_c(\mathcal{U}, \Lambda_n)$, $\mathcal{D}^b_c(\mathcal{F}, \Lambda_n)$), and let $i_* : \mathcal{D}_{\mathcal{F}} \to \mathcal{D}$ and $j^* : \mathcal{D} \to \mathcal{D}_{\mathcal{U}}$ be the usual pushforward and pullback functors. By the theory developed in [10] conditions (i)-(v) hold.

We can also consider adic sheaves. Let $\mathcal{D}$ (resp. $\mathcal{D}_{\mathcal{U}}$, $\mathcal{D}_{\mathcal{F}}$) denote the bounded derived category $\mathbf{D}^b_c(\mathcal{X}, \Lambda)$ (resp. $\mathbf{D}^b_c(\mathcal{U}, \Lambda)$, $\mathbf{D}^b_c(\mathcal{F}, \Lambda)$) of Λ–modules on $\mathcal{X}$ (resp. $\mathcal{U}$, $\mathcal{F}$) constructed in [11]. We then again have functors

$$\mathcal{D}_{\mathcal{F}} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}_{\mathcal{U}}.$$

Conditions (i)-(iii) hold by the results of [11], and condition (v) holds by base change to a smooth cover of $\mathcal{X}$ and the case of schemes.

To construct the distinguished triangles in (iv) recall that $\mathbf{D}_c^b(\mathcal{X}, \Lambda)$ is constructed as a quotient of the category $\mathcal{D}_c^b(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$ (the derived category of projective systems of Λ_n–modules), and similarly for $\mathbf{D}_c^b(\mathcal{U}, \Lambda)$ and $\mathbf{D}_c^b(\mathcal{F}, \Lambda)$. All the functors in (i)-(v) are then obtained from functors defined already on the level of the categories $\mathcal{D}_c^b(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)$, $\mathcal{D}_c^b(\mathcal{U}^{\mathbb{N}}, \Lambda_\bullet)$, and $\mathcal{D}_c^b(\mathcal{F}^{\mathbb{N}}, \Lambda_\bullet)$. In this case the first distinguished triangle in (iv) is constructed by the same reasoning as in [10, 4.9] for the finite case, and the second distinguished triangle is obtained by duality.

Returning to the general setup of the beginning of this section, suppose given t–structures $(\mathcal{D}_{\mathrm{F}}^{\leq 0}, \mathcal{D}_{\mathrm{F}}^{\geq 0})$ and $(\mathcal{D}_{\mathrm{U}}^{\leq 0}, \mathcal{D}_{\mathrm{U}}^{\geq 0})$ on $\mathcal{D}_{\mathrm{F}}$ and $\mathcal{D}_{\mathrm{U}}$ respectively and define

$$\mathcal{D}^{\leq 0} := \{\mathrm{K} \in \mathcal{D} | j^*\mathrm{K} \in \mathcal{D}_{\mathrm{U}}^{\leq 0} \text{ and } i^*\mathrm{K} \in \mathcal{D}_{\mathrm{F}}^{\leq 0}\}$$

$$\mathcal{D}^{\geq 0} := \{\mathrm{K} \in \mathcal{D} | j^*\mathrm{K} \in \mathcal{D}_{\mathrm{U}}^{\geq 0} \text{ and } i^!\mathrm{K} \in \mathcal{D}_{\mathrm{F}}^{\geq 0}\}.$$

Theorem 2.2 ([5, 1.4.10])**.** *The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defines a t–structure on $\mathcal{D}$.*

3. Review of the perverse t–structure for schemes

Let k be a field and X/k a scheme of finite type. Let Λ be a complete discrete valuation ring and for every n let Λ_n denote the quotient $\Lambda/\mathfrak{m}^{n+1}$ so that $\Lambda = \varprojlim \Lambda_n$. Assume that the characteristic l of Λ_0 is invertible in k.

For every n, we can define the perverse t–structure $({}^p\mathcal{D}^{\leq 0}(\mathrm{X}, \Lambda_n), {}^p\mathcal{D}^{\geq 0}(\mathrm{X}, \Lambda_n))$ on $\mathcal{D}_c^b(\mathrm{X}, \Lambda_n)$ (in this paper we consider only the middle perversity) as follows:

> A complex $\mathrm{K} \in \mathcal{D}_c^b(\mathrm{X}, \Lambda_n)$ is in ${}^p\mathcal{D}^{\leq 0}(\mathrm{X}, \Lambda_n)$ (resp. ${}^p\mathcal{D}^{\geq 0}(\mathrm{X}, \Lambda_n)$) if for every point $x \in \mathrm{X}$ with inclusion $i_x : \mathrm{Spec}(k(x)) \to \mathrm{X}$ and $j > -\dim(x)$ (resp. $j < -\dim(x)$) we have $\mathcal{H}^j(i_x^*\mathrm{K}) = 0$ (resp. $\mathcal{H}^j(i_x^!\mathrm{K}) = 0$)[1].

As explained in [5, 2.2.11] this defines a t–structure on $\mathcal{D}_c^b(\mathrm{X}, \Lambda_n)$: The *perverse* t–structure.

The same technique can be used in the adic case. We explain this in more detail since it is not covered in detail in the literature. As before let $\mathbf{D}_c^b(\mathrm{X}, \Lambda)$ denote the bounded derived category of Λ–modules constructed in [11]. Let $\mathrm{Mod}_c(\mathrm{X}, \Lambda)$ denote the heart of the standard t–structure on $\mathbf{D}_c(\mathrm{X}, \Lambda)$. In the language of [11, 3.1] the category $\mathrm{Mod}_c(\mathrm{X}, \Lambda)$ is the quotient

[1]As usual, $i_x^!\mathrm{K}, i_x^*\mathrm{K}$ denotes $(i_{\bar{x}}^!\mathrm{K})_x, (i_{\bar{x}}^*\mathrm{K})_x$ where $i_{\bar{x}}$ is the closed immersion $\bar{x}_{red} \hookrightarrow \mathrm{X}$.

of the category of λ–modules on X by almost zero systems. For every integer j there is then a natural functor

$$\mathcal{H}^j : \mathbf{D}_c^b(\mathrm{X}, \Lambda) \to \mathrm{Mod}_c(\mathrm{X}, \Lambda).$$

We then define categories $({}^{\mathrm{p}}\mathbf{D}^{\leq 0}(\mathrm{X}, \Lambda), {}^{\mathrm{p}}\mathbf{D}^{\geq 0}(\mathrm{X}, \Lambda))$ by the following condition:

> A complex $\mathrm{K} \in \mathbf{D}_c^b(\mathrm{X}, \Lambda)$ is in ${}^{\mathrm{p}}\mathbf{D}^{\leq 0}(\mathrm{X}, \Lambda)$ (resp. ${}^{\mathrm{p}}\mathbf{D}^{\geq 0}(\mathrm{X}, \Lambda)$) if for every point $x \in \mathrm{X}$ with inclusion $i_x : \mathrm{Spec}(k(x)) \to \mathrm{X}$ and $j > -\dim(x)$ (resp. $j < -\dim(x)$) we have $\mathcal{H}^j(i_x^*\mathrm{K}) = 0$ (resp. $\mathcal{H}^j(i_x^!\mathrm{K}) = 0$).

Proposition 3.1. *This defines a t–structure on* $\mathbf{D}_c^b(\mathrm{X}, \Lambda)$.

Proof. The only problem comes from perverse truncation. Recall that an adic sheaf (M_n) is *smooth* if all M_n are locally constant, or, what's amount to the same, if M_1 is locally constant. We say that a complex $\mathrm{K} \in \mathbf{D}_c^b(\mathrm{X}, \Lambda)$ is *smooth* if $\mathcal{H}^j(\mathrm{K})$ is represented by a smooth adic sheaf and is zero for almost all j.

We say that X is *essentially smooth* if $(\mathrm{X} \otimes_k \bar{k})_{\mathrm{red}}$ is smooth over $\bar{k}$. If X is essentially smooth of dimension d and $\mathrm{K} \in \mathbf{D}_c^b(\mathrm{X}, \Lambda)$ is smooth, then for any point $x \in \mathrm{X}$ of codimension s we have $i_x^!\mathrm{K} = i_x^*\mathrm{K}(s-d)[2(s-d)]$ (see for example [3, p. 62]).

We now prove by induction on $\dim(\mathrm{X})$ that the third axiom for a t–structure holds. Namely, that for any $\mathrm{K} \in \mathbf{D}_c^b(\mathrm{X}, \Lambda)$ there exists a distinguished triangle

$$ {}^{p}\tau_{\leq 0}\mathrm{K} \to \mathrm{K} \to {}^{p}\tau_{>0}\mathrm{K} \tag{3.1.1}$$

with ${}^{p}\tau_{\leq 0}\mathrm{K} \in {}^{\mathrm{p}}\mathbf{D}^{\leq 0}(\mathrm{K}, \Lambda)$ and ${}^{p}\tau_{>0}\mathrm{K} \in {}^{\mathrm{p}}\mathbf{D}^{>0}(\mathrm{K}, \Lambda)$.

For $\dim(\mathrm{X}) = 0$, it is clearly true. For the inductive step let d be the dimension of X and assume the result holds for schemes of dimension $< d$. Let $\mathrm{K} \in \mathbf{D}_c^b(\mathrm{X}, \Lambda)$ be a complex and choose some essentially smooth dense open subset U of X on which K is smooth. Then, the class of $\tau_{\leq -\dim(\mathrm{U})}\mathrm{K}_{|\mathrm{U}}$ (resp. $\tau_{>-\dim(\mathrm{U})}\mathrm{K}_{|\mathrm{U}}$) (truncation with respect to the usual t-structure on U) belongs to ${}^{\mathrm{p}}\mathbf{D}^{\leq 0}(\mathrm{U}, \Lambda)$ (resp. ${}^{\mathrm{p}}\mathbf{D}^{>0}(\mathrm{U}, \Lambda)$) and therefore the usual distinguished triangle

$$\tau_{\leq -\dim(\mathrm{U})}\mathrm{K}_{\mathrm{K}|\mathrm{U}} \to \mathrm{K}_{|\mathrm{U}} \to \tau_{>-\dim(\mathrm{U})}\mathrm{K}_{|\mathrm{U}}$$

defines the required perverse distinguished triangle on U by the formula above. The complement $\mathrm{F} = \mathrm{X} - \mathrm{U}$ has dimension $< \dim(\mathrm{X})$. By induction hypothesis, the conditions above define a t-structure on F and therefore one gets a distinguished triangle

$${}^{p}\tau_{\leq 0}\mathrm{K}_{|\mathrm{F}} \to \mathrm{K}_{|\mathrm{F}} \to {}^{p}\tau_{>0}\mathrm{K}_{|\mathrm{F}}$$

on F . By 2.2 we can glue the trivial t–structure on U and the perverse t–structure on F to a t–structure on $\mathbf{D}^b_c(\mathrm{X},\Lambda)$. It follows that one can glue the distinguished triangles on U and F to a distinguished triangle 3.1.1 which gives the third axiom. □

Remark 3.2. One can also prove the proposition using stratifications as in [5].

The perverse t–structures on $\mathcal{D}^b_c(\mathrm{X},\Lambda_n)$ and $\mathbf{D}^b_c(\mathrm{X},\Lambda)$ extend naturally to the unbounded derived categories $\mathcal{D}_c(\mathrm{X},\Lambda_n)$ and $\mathbf{D}_c(\mathcal{X},\Lambda)$. Let $\mathcal{D}$ denote either of these triangulated categories. For $\mathrm{K}\in\mathcal{D}$ and $a\leq b$ let $\tau_{[a,b]}\mathrm{K}$ denote $\tau_{\geq a}\tau_{\leq b}\mathrm{K}$. The perverse t–structure defines a functor

$$ {}^{\mathrm{p}}\mathcal{H}^0 : \mathcal{D}^b \to \mathcal{D}^b. $$

Lemma 3.3. *There exists integer $a<b$ such that for any $\mathrm{K}\in\mathcal{D}^b$ we have ${}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}) = {}^{\mathrm{p}}\mathcal{H}^0(\tau_{[a,b]}\mathrm{K})$.*

Proof. Consideration of the distinguished triangles

$$ \tau_{\leq b}\mathrm{K} \to \mathrm{K} \to \tau_{>b}\mathrm{K} \to \tau_{\leq b}\mathrm{K}[1] $$

and

$$ \tau_{<a}\mathrm{K} \to \tau_{\leq b}\mathrm{K} \to \tau_{[a,b]}\mathrm{K} \to \tau_{<a}\mathrm{K}[1] $$

implies that it suffices to show that there exists integers $a<b$ such that for K in either $\mathcal{D}^{<a}$ or $\mathcal{D}^{>b}$ we have ${}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K})=0$. By the definition of perverse sheaf we can take a to be any integer smaller than $-\dim(\mathrm{X})$.

To find the integer b, note that since the dualizing sheaf for a scheme of finite type over k has finite quasi–injective dimension [3, I.1.5] and [11, 7.6]. It follows that there exists a constant c such that for any integer b, point $x\in\mathrm{X}$, and $\mathrm{K}\in\mathcal{D}^{>b}$ we have $i_x^!\mathrm{K}\in\mathcal{D}^{>b+c}$. Thus we can take for b any integer greater than $-\dim(\mathrm{X})-c$. □

Choose integers $a<b$ as in the lemma, and define

$$ {}^{\mathrm{p}}\mathcal{H}^0 : \mathcal{D} \to \mathcal{D}^b, \quad \mathrm{K} \mapsto {}^{\mathrm{p}}\mathcal{H}^0(\tau_{[a,b]}\mathrm{K}). $$

One sees immediately that this does not depend on the choice of $a<b$. Define $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) to be the full subcategory of $\mathcal{D}$ of complexes $\mathrm{K}\in\mathcal{D}$ with ${}^{\mathrm{p}}\mathcal{H}^j(\mathrm{K}) := {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}[j]) = 0$ for $j\leq 0$ (resp. $j\geq 0$). The argument in [5, 2.2.1] (which in turn is based in [5, 2.1.4]) shows that this defines a t–structure on $\mathcal{D}$.

4. The perverse t–structure for stacks of finite type

Let $\mathcal{X}/k$ be an algebraic stack of finite type. Let $\mathcal{D}(\mathcal{X})$ denote either $\mathcal{D}_c(\mathcal{X}, \Lambda_n)$ or $\mathbf{D}_c(\mathcal{X}, \Lambda)$. Fix a smooth surjection $\pi : \mathrm{X} \to \mathcal{X}$ with X of finite type, and define ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathcal{X})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X})$) to be the full subcategory of objects $\mathrm{K} \in \mathcal{D}(\mathcal{X})$ such that $\pi^*\mathrm{K}[d_\pi]$ is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathrm{X})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathrm{X})$), where d_π denotes the relative dimension of X over $\mathcal{X}$ (a locally constant function on X).

Lemma 4.1. *The subcategories ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}$ and ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}$ of $\mathcal{D}$ do not depend on the choice of $\pi : \mathrm{X} \to \mathcal{X}$.*

Proof. It suffices to show that if $f : \mathrm{Y} \to \mathrm{X}$ is a smooth surjective morphism of schemes of relative dimension d_f (a locally constant function on Y), then $\mathrm{K} \in \mathcal{D}(\mathrm{X})$ is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathrm{X})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathrm{X})$) if and only if $f^*\mathrm{K}[d_f]$ is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathrm{Y})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathrm{Y})$). For this note that by [5, 4.2.4] the functor $f^*[d_f]$ is exact for the perverse t–structures. This implies that K is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathrm{X})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathrm{X})$) only if $f^*\mathrm{K}[d_f]$ is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathrm{Y})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathrm{Y})$).
For the other direction, note that if $f^*\mathrm{K}[d_f]$ is in $\mathcal{D}^{\leq 0}(\mathrm{Y})$ (resp. $\mathcal{D}^{\geq 0}(\mathrm{Y})$) then for any integer $i > 0$ (resp. $i < 0$) we have

$$f^*\,{}^{\mathrm{p}}\mathcal{H}^i(\mathrm{K})[d_f] = {}^{\mathrm{p}}\mathcal{H}^i(f^*\mathrm{K}[d_f]) = 0.$$

Since f is surjective it follows that ${}^{\mathrm{p}}\mathcal{H}^i(\mathrm{K}) = 0$ for all $i > 0$ (resp. $i < 0$). □

Theorem 4.2. *The subcategories $({}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathcal{X}), {}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X}))$ define a t–structure on $\mathcal{D}(\mathcal{X})$.*

Proof. Exactly as in the proof of 3.1 using noetherian induction and gluing of t–structures 2.2 one shows that $({}^{\mathrm{p}}\mathcal{D}^{\leq 0}, {}^{\mathrm{p}}\mathcal{D}^{\geq 0})$ define by restriction a t–structure on $\mathcal{D}^b(\mathcal{X})$ (again the only problem is the third axiom for a t–structure since the other two can be verified locally).
The same argument used in the schematic case then extends this t–structure to the unbounded derived category $\mathcal{D}(\mathcal{X})$. □

5. The perverse t–structure for stacks locally of finite type

Assume now that $\mathcal{X}$ is a stack *locally* of finite type over S. We consider either finite coefficients or the adic case and write just $\mathcal{D}(\mathcal{X})$ for the corresponding derived categories $\mathcal{D}_c(\mathcal{X}, \Lambda_n)$ or $\mathbf{D}_c(\mathcal{X}, \Lambda)$.
Define subcategories $({}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathcal{X}), {}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X}))$ of $\mathcal{D}(\mathcal{X})$ by the condition that $\mathrm{K} \in \mathcal{D}(\mathcal{X})$ is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathcal{X})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X})$) if and only if for every open substack $\mathcal{U} \subset \mathcal{X}$ of finite type over k the restriction of K to $\mathcal{U}$ is in ${}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathcal{U})$ (resp. ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{U})$.

Theorem 5.1. *The subcategories* $({}^{\mathrm{p}}\mathcal{D}^{\leq 0}(\mathcal{X}), {}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X}))$ *define a* t*–structure on* $\mathcal{D}(\mathcal{X})$.

Proof. The first two axioms for a t–structure follow immediately from the definition. We now show the third axiom. Write $\mathcal{X}$ as a filtering union of open substacks $\mathcal{X}_i \subset \mathcal{X}$ of finite type. Let $j_i : \mathcal{X}_i \hookrightarrow \mathcal{X}$ be the open immersion. Then for any $\mathrm{M} \in \mathrm{D}_c(\mathcal{X})$, we have for every i a distinguished triangle

$$\mathrm{M}_{i,\leq 0} \to \mathrm{M}|_{\mathcal{X}_i} \to \mathrm{M}_{i,\geq 1} \tag{5.1.1}$$

where $\mathrm{M}_{i,\leq 0} \in {}^{\mathrm{p}}\mathbf{D}^{\leq 0}(\mathcal{X}_i)$ and $\mathrm{M}_{i,\leq 0}$ in ${}^{\mathrm{p}}\mathbf{D}^{\geq 1}(\mathcal{X}_i)$. By the uniqueness statement in [5, 1.3.3] this implies that the formation of this sequence is compatible with restriction to smaller $\mathcal{X}_i$. Since $j_i^* = j_i^!$ for open immersions, we then get a sequence

$$j_{i!}\mathrm{M}_{i,\leq 0} \to j_{i+1,!}\mathrm{M}_{i+1,\leq 0} \to \cdots .$$

Define $\mathrm{M}_{\leq 0}$ to be the homotopy colimit of this sequence. There is a natural map $\mathrm{M}_{\leq 0} \to \mathrm{M}$ and take $\mathrm{M}_{\geq 1}$ to be the cone. The following lemma implies that the third axiom holds and hence proves 5.1. □

Lemma 5.2. *For any* i*, the restriction of the distinguished triangle*

$$\mathrm{M}_{\leq 0} \to \mathrm{M} \to \mathrm{M}_{\geq 1} \tag{5.2.1}$$

$\mathcal{X}_i$ *is isomorphic to 5.1.1. In particular,* $\mathrm{M}_{\leq 0} \in {}^{\mathrm{p}}\mathcal{D}^{\leq 0}$ *and* $\mathrm{M}_{\geq 1} \in {}^{\mathrm{p}}\mathcal{D}^{\geq 1}$.

Proof. Let i_0 be any nonnegative integer. By [14, 1.7.1], one has a distinguished triangle

$$\bigoplus_{i \geq i_0} j_{i!}\mathrm{M}_{i,\leq 0} \to \bigoplus_{i \geq i_0} j_{i!}\mathrm{M}_{i,\leq 0} \to \mathrm{M}_{\leq 0}.$$

Because j_i^* is exact and commutes with direct sums, one gets by restriction a distinguished triangle

$$\bigoplus_{i \geq i_0} \mathrm{M}_{i_0,\leq 0} \to \bigoplus_{i \geq i_0} \mathrm{M}_{i_0,\leq 0} \to \mathrm{M}_{\leq 0|\mathcal{X}_{i_0}}.$$

where the inductive system is given by the identity morphism of $\mathrm{M}_{i_0,\leq 0}$. By[14, 1.6.6], one gets $\mathrm{M}_{\leq 0|\mathcal{X}_{i_0}} = \mathrm{M}_{i_0,\leq 0}$. The lemma follows. □

We define the *perverse t–structure* on $\mathcal{D}$ to be the t–structure given by 5.1. By the very definition, it coincides with the usual one if $\mathcal{X}$ is a scheme.. A complex in the heart of the perverse t-structure is by definition a perverse sheaf.

Remark 5.3. By [5, 1.3.6], the category of perverse sheaves a stack $\mathcal{X}$ is an abelian category.

Remark 5.4. If we work with Λ_0–coefficients, then it follows from the case of schemes that Verdier duality interchanges the categories ${}^{\mathrm{p}}\mathcal{D}^{\leq}(\mathcal{X}, \Lambda_0)$ and ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X}, \Lambda_0)$. For other coefficients this does not hold due to the presence of torsion.

Remark 5.5. If the normalized complex P is perverse on $\mathcal{X}$ and $\mathrm{U} \to \mathcal{X}$ is in Lisse-ét$(\mathcal{X})$, then $\mathrm{P}_{\mathrm{U},n} \in \mathbf{D}^b(\mathrm{U}_{\text{ét}}, \Lambda_n)$ is perverse on $\mathrm{U}_{\text{ét}}$. In particular, one has $\mathcal{E}xt^i(\mathrm{P}_{\mathrm{U},n}, \mathrm{P}_{\mathrm{U},n}) = 0$ if $i < 0$. By the gluing lemma, perversity is a local condition for the lisse-étale topology. For instance, it follows that the category perverse sheaf on $\mathcal{X} = [\mathrm{X}/\mathrm{G}]$ (X is a scheme of finite type acting on by an algebraic group G) is equivalent to the category of G-equivariant perverse sheaves on X[2].

In the case of finite coefficients, one can also define ${}^{\mathrm{p}}\mathcal{H}^0$ by gluing. Let us consider a diagram

(5.5.1)
$$\begin{array}{ccc} \mathrm{V} & \xrightarrow{\ \sigma\ } & \mathrm{U} \\ & {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle u} & \\ & \mathcal{X} & \end{array}$$

with a 2-commutative triangle and $u, v \in$ Lisse-ét$(\mathcal{X})$ of relative dimension d_u, d_v. Let R be a Gorenstein ring of dimension 0.

Lemma 5.6. *Let* $\mathrm{K} \in \mathcal{D}_c^b(\mathcal{X}, \mathrm{R})$. *There exists a unique* ${}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}) \in \mathcal{D}^b(\mathcal{X}, \mathrm{R})$ *such that*

$$[{}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K})]_{\mathrm{U}}[d_u] = {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}_{\mathrm{U}}[d_u]) \in \mathcal{D}_c^b(\mathrm{U}_{\text{ét}}, \mathrm{R})$$

(functorially).

Proof. Because ${}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}_{\mathrm{U}})$ is perverse, one has by [5, 2.1.21]

$$\mathcal{E}xt^i({}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}_{\mathrm{U}}[d_u]), {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K}_{\mathrm{U}}[d_u])) = 0 \text{ for } i < 0.$$

Let $\mathrm{W} = \mathrm{U} \times_{\mathcal{X}} \mathrm{V}$ which is an algebraic space.

[2]See [12, III.15]

Assume for simplicity that W is even a scheme (certainly of finite type over S). One has a commutative diagram with cartesian square

$$\begin{array}{ccc} \mathrm{W} & \xrightarrow{\tilde v} & \mathrm{U} \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle \tilde u} & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle u} \\ \mathrm{V} & \xrightarrow[v]{} & \mathcal{X} \end{array}$$

with $\tilde u, \tilde v$ smooth of relative dimension d_u, d_v and s is a graph section. In particular, $\tilde u^*[d_u]$ and $\tilde v^*[d_v]$ are t-exact (for the perverse t-structure) by [5, 4.2.4].
Therefore, we get

$$\begin{aligned} \tilde v^*\, {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_U}[d_u])[-d_u] &= \tilde v^*[d_v]\, {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_U}[d_u])[-d_u-d_v] \\ &= {}^{\mathrm{p}}\mathcal{H}^0(\tilde v^*\mathrm{K_U}[d_u+d_v])[-d_u-d_v] \\ &= {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_W}[d_u+d_v])[-d_u-d_v] \\ &= \tilde u^*\, {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_V}[d_v])[-d_v] \end{aligned}$$

Pulling back by s, we get

$${}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_V}[d_\mathrm{V}])[-d_v] = s^*\tilde u^*\, {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_V}[d_\mathrm{V}])[-d_v] = s^*\tilde v^*\, {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_U}[d_u])[-d_u] = \sigma^*\, {}^{\mathrm{p}}\mathcal{H}^0(\mathrm{K_U}[d_u])[-d_u].$$

The lemma follows from [10, 2.3.3]. □

Remark 5.7. It follows from the construction of the perverse t–structure on $\mathcal{D}_c(\mathcal{X}, \mathrm{R})$ that the above defined functor ${}^{\mathrm{p}}\mathcal{H}^0$ agrees with the one defined by the perverse t–structure.

6. Intermediate extension

Let $\mathcal{X}$ be an algebraic k–stack of finite type, and let $i : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a closed substack with complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$. For a perverse sheaf P on $\mathcal{U}$ we define the *intermediate extension*, denoted $j_{!*}\mathrm{P}$, to be the image in the abelian category of perverse sheaves on $\mathcal{X}$ of the morphism

$${}^{\mathrm{p}}\mathcal{H}^0(j_!\mathrm{P}) \to {}^{\mathrm{p}}\mathcal{H}^0(j_*\mathrm{P}).$$

Lemma 6.1. *The perverse sheaf $j_{!*}\mathrm{P}$ is the unique perverse sheaf with $j^*(j_{!*}\mathrm{P}) = \mathrm{P}$ and ${}^{\mathrm{p}}\mathcal{H}^0(i^*(j_{!*}\mathrm{P})) = 0$.*

Proof. Let us first verify that $j_{!*}\mathrm{P}$ has the indicated properties. Since j is an open immersion, the functor j^* is t–exact and hence the first property $j^*j_{!*}\mathrm{P} = \mathrm{P}$ is immediate. The equality ${}^{\mathrm{p}}\mathcal{H}^0(i^*(j_{!*}\mathrm{P})) = 0$ follows from [5, 1.4.23].

Let F be a second perverse sheaf with these properties. Then $j^*\mathrm{F} = \mathrm{P}$ defines a morphism $j_!\mathrm{P} \to \mathrm{F}$ which since $j_!$ is right exact for the perverse t–structure (this follows immediately from [5, 2.2.5] and a reduction to the case of schemes) factors through a morphism ${}^{\mathrm{p}}\mathcal{H}^0(j_!\mathrm{P}) \to \mathrm{F}$. Adjunction also defines a morphism $\mathrm{F} \to j_*\mathrm{P}$ which since j_* is left exact for the perverse t–structure (again by loc. cit.) defines a morphism $\mathrm{F} \to {}^{\mathrm{p}}\mathcal{H}^0(j_*\mathrm{P})$. It follows that the morphism ${}^{\mathrm{p}}\mathcal{H}^0(j_!\mathrm{P}) \to {}^{\mathrm{p}}\mathcal{H}^0(j_*\mathrm{P})$ factors through F whence we get a morphism $\rho : j_{!*}\mathrm{P} \to \mathrm{F}$ of perverse sheaves. The kernel and cokernel of this morphism is a perverse sheaf supported on $\mathcal{Y}$. The assumption ${}^{\mathrm{p}}\mathcal{H}^0(i^*\mathrm{F}) = {}^{\mathrm{p}}\mathcal{H}^0(i^*j_{!*}\mathrm{P}) = 0$ then implies that the kernel and cokernel are zero. □

Lemma 6.2. *Let $f : \mathrm{X} \to \mathcal{X}$ be a smooth morphism of relative dimension d with* X *a scheme. Let*

$$\mathrm{Y} \xrightarrow{\;i'\;} \mathrm{X} \xleftarrow{\;j'\;} \mathrm{U}$$

be the pullbacks of $\mathcal{Y}$ and $\mathcal{U}$. Then $f^*[d]j_{!*} = j'_{!*}f^*[d]$.

Proof. Let P be a perverse sheaf on $\mathcal{U}$ and let $\bar{\mathrm{P}}$ denote $j_{!*}\mathrm{P}$. The functor $f^*[d]$ is t-exact, and hence preserves perversity. It follows that $\bar{\mathrm{P}}' = f^*[d]\bar{\mathrm{P}}$ is perverse and is an extension of the perverse sheaf $\mathrm{P}' = f^*[d]\mathrm{P}$ (in particular the statement of the lemma makes sense!). By the uniqueness in 6.1 it suffices to show that ${}^{\mathrm{p}}\mathcal{H}^0(i'^*\bar{\mathrm{P}}') = 0$. But, keeping in mind that $f^*[d]$ commutes with ${}^{\mathrm{p}}\mathcal{H}^0$, the first point is for instance a consequence of smooth base change. □

Remark 6.3. In the case of finite coefficients, one can also define the intermediate extension using 6.2 and gluing.

7. Gluing perverse sheaves

In this section we work either with finite coefficients or with adic coefficients.

Let $\mathcal{X}$ be a stack locally of finite type over k, and define a fibered category $\mathcal{P}$ (*not* in groupoids) on Lisse-ét$(\mathcal{X})$ by

$$\mathrm{U} \mapsto (\text{category of perverse sheaves on U}).$$

Proposition 7.1. *The fibered category $\mathcal{P}$ is a stack and the natural functor*

$$(\text{perverse sheaves on } \mathcal{X}) \to \mathcal{P}(\mathcal{X})$$

is an equivalence of categories.

Proof. For a smooth surjective morphism of stacks $f : \mathcal{Y} \to \mathcal{X}$ let $\mathrm{Des}_{\mathcal{Y}/\mathcal{X}}$ denote the category of pairs (P, σ), where P is a perverse sheaf on $\mathcal{Y}$ and $\sigma : \mathrm{pr}_1^*\mathrm{P} \to \mathrm{pr}_2^*\mathrm{P}$ is an isomorphism over $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ satisfying the usual cocycle condition on $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$. To prove the proposition it suffices to show that the natural functor

$$\text{(perverse sheaves on } \mathcal{X}) \to \mathrm{Des}_{\mathcal{Y}/\mathcal{X}} \tag{7.1.1}$$

is an equivalence of categories.

Now if P and P′ are perverse sheaves on a stack, then $\mathcal{E}xt^i(\mathrm{P}, \mathrm{P}') = 0$ for all $i < 0$. Indeed this can be verified locally where it follows from the first axiom of a t–structure. That 7.1.1 is an equivalence in the finite coefficients case then follows from the gluing lemma [10, 2.3.3 and 2.3.4].

For the adic case, note that by the discussion in [11, §5] if P and P′ are two perverse sheaves on a stack $\mathcal{X}$ with normalized complexes $\hat{\mathrm{P}}$ and $\hat{\mathrm{P}}'$ then

$$\mathrm{Ext}^i_{\mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)}(\hat{\mathrm{P}}, \hat{\mathrm{P}}') = \mathrm{Ext}^i_{\mathbf{D}_c(\mathcal{X}, \Lambda)}(\mathrm{P}, \mathrm{P}'),$$

where $\mathcal{X}^{\mathbb{N}}$ denotes the topos of projective systems of sheaves on Lisse-ét$(\mathcal{X})$ and $\Lambda_\bullet$ denotes $\varprojlim \Lambda_n$. It follows that for any object $(\mathrm{P}, \sigma) \in \mathrm{Des}_{\mathcal{Y}/\mathcal{X}}$ we have $\mathcal{E}xt^i_{\mathcal{D}_c(\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet)}(\hat{\mathrm{P}}, \hat{\mathrm{P}}) = 0$ for $i < 0$. By the gluing lemma [10, 2.3.3] the pair (P, σ) is therefore induced by a unique complex on $\mathcal{X}$ which is a perverse sheaf since this can be verified after pulling back to $\mathcal{Y}$. Similarly if P and P′ are two perverse sheaves on $\mathcal{X}$ with normalized complexes $\hat{\mathrm{P}}$ and $\hat{\mathrm{P}}'$, then $\mathcal{E}xt^i_{\mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet)}(\hat{\mathrm{P}}, \hat{\mathrm{P}}') = 0$ for $i < 0$ and therefore by [10, 2.3.4] the functor of morphisms $\hat{\mathrm{P}} \to \hat{\mathrm{P}}'$ is a sheaf. □

Remark 7.2. Using the above argument one can define the category of perverse sheaves on a stack without defining the t–structure.

8. Simple objects

Let $\mathcal{X}$ be an algebraic stack of finite type over k. Let $\mathcal{D}_c^b(\mathcal{X}, \mathbb{Q}_l)$ denote the category $\mathcal{D}_c^b(\mathcal{X}, \mathbb{Z}_l) \otimes \mathbb{Q}$ (see for example [11, 3.21]). The perverse t–structure on $\mathcal{D}_c^b(\mathcal{X}, \mathbb{Z}_l)$ defines a t–structure on $\mathcal{D}_c^b(\mathcal{X}, \mathbb{Q}_l)$ which we also call the perverse t–structure. An object in the heart of this t–structure is called a perverse $\mathbb{Q}_l$–sheaf. One check easily that the category of perverse $\mathbb{Q}_l$–sheaves is canonically equivalent to the category $\mathrm{Perv}_{\mathbb{Z}_l} \otimes \mathbb{Q}$, where $\mathrm{Perv}_{\mathbb{Z}_l}$ denotes the category

of perverse sheaves of $\mathbb{Z}_l$–modules. In particular, as in 7.1,the corresponding fibred category is a stack ($\mathbb{Q}_l$-perverse sheaves can be glued).
In what follows we consider only $\mathbb{Q}_l$–coefficients for some l invertible in k.

Remark 8.1. Verdier duality interchanges ${}^{\mathrm{p}}\mathcal{D}^{\leq}(\mathcal{X}, \mathbb{Q}_l)$ and ${}^{\mathrm{p}}\mathcal{D}^{\geq 0}(\mathcal{X}, \mathbb{Q}_l)$. Indeed this can be verified on a smooth covering of $\mathcal{X}$ and hence follows from the case of schemes.

Theorem 8.2 (stack version of [5, 4.3.1])**.** (i) *In the category of perverse sheaves on $\mathcal{X}$, every object is of finite length. The category of perverse sheaves is artinian and noetherian.*
(ii) *Let $j : \mathcal{V} \hookrightarrow \mathcal{X}$ be the inclusion of an irreducible substack such that $(\mathcal{V} \otimes_k \bar{k})_{\mathrm{red}}$ is smooth. Let* L *be a smooth $\mathbb{Q}_l$–sheaf on $\mathcal{V}$ which is irreducible in the category of smooth $\mathbb{Q}_l$–sheaves on $\mathcal{V}$. Then $j_{!*}(\mathrm{L}[\dim(\mathcal{V})])$ is a simple perverse sheaf on $\mathcal{X}$ and every simple perverse sheaf is obtained in this way.*

Proof. Statement (i) can be verified on a quasi–compact smooth covering of $\mathcal{X}$ and hence follows from the case of schemes [5, 4.3.1 (i)].
For (ii) note first that if $\mathcal{X}$ is irreducible and smooth, L is a smooth $\mathbb{Q}_l$–sheaf on $\mathcal{X}$, $j : \mathcal{U} \hookrightarrow \mathcal{X}$ is a noempty open substack, then the perverse sheaf $\mathrm{F} := \mathrm{L}[\dim(\mathcal{X})]$ satisfies $\mathrm{F} = j_{!*}j^*\mathrm{F}$. Indeed it suffices to verify this locally in the smooth topology on $\mathcal{X}$ where it follows from the case of schemes [5, 4.3.2].
Let $\mathrm{Mod}_{\mathcal{X}}(\mathbb{Z}_l)$ denote the category of smooth adic sheaves of $\mathbb{Z}_l$–modules on $\mathcal{X}$ so that the category of smooth $\mathbb{Q}_l$–sheaves is equal to $\mathrm{Mod}_{\mathcal{X}}(\mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

Lemma 8.3. *Let $\mathcal{X}$ be a normal algebraic stack of finite type over k, and let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ be a dense open substack. Then the natural functor*

$$\mathrm{Mod}_{\mathcal{X}}(\mathbb{Z}_l) \to \mathrm{Mod}_{\mathcal{U}}(\mathbb{Z}_l)$$

is fully faithful and its essential image is closed under subobjects.

Proof. Note first that the result is standard in the case when $\mathcal{X}$ is a scheme (in this case when $\mathcal{X}$ is connected the result follows from the surjectivity of the map $\pi_1(\mathcal{U}) \to \pi_1(\mathcal{X})$). Let $\mathrm{V} \to \mathcal{X}$ be a smooth surjection with V a scheme, and let $\mathrm{U} \subset \mathrm{V}$ denote the inverse image of $\mathcal{U}$. Also define V' to be $\mathrm{V} \times_{\mathcal{X}} \mathrm{V}$ and let $\mathrm{U}' \subset \mathrm{V}'$ be the inverse image of $\mathcal{U}$. Assume first that V' is a scheme (in general V' will only be an algebraic space). For any two $\mathrm{F}_1, \mathrm{F}_2 \in \mathrm{Mod}_{\mathcal{X}}(\mathbb{Z}_l)$ we have exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{X}}(\mathrm{F}_1, \mathrm{F}_2) \to \mathrm{Hom}_{\mathrm{V}}(\mathrm{F}_1|_{\mathrm{V}}, \mathrm{F}_2|_{\mathrm{V}}) \rightrightarrows \mathrm{Hom}_{\mathrm{V}'}(\mathrm{F}_1|_{\mathrm{V}'}, \mathrm{F}_2|_{\mathrm{V}'})$$

and

$$0 \to \mathrm{Hom}_{\mathcal{U}}(\mathrm{F}_1, \mathrm{F}_2) \to \mathrm{Hom}_{\mathrm{U}}(\mathrm{F}_1|_{\mathrm{V}}, \mathrm{F}_2|_{\mathrm{V}}) \rightrightarrows \mathrm{Hom}_{\mathrm{U}'}(\mathrm{F}_1|_{\mathrm{U}'}, \mathrm{F}_2|_{\mathrm{U}'}).$$

From this and the case of schemes the full faithfulness follows.

For the second statement, let $\mathrm{M} \in \mathrm{Mod}_{\mathcal{X}}(\mathbb{Z}_l)$ be a sheaf and $\mathrm{L}_{\mathcal{U}} \subset \mathrm{M}|_{\mathcal{U}}$ a subobject. By the case of schemes the pullback $\mathrm{L}_{\mathrm{U}} \subset \mathrm{M}_{\mathrm{V}}|_{\mathrm{U}}$ to U extends uniquely to a subobject $\mathrm{L}_{\mathrm{V}} \subset \mathrm{M}_{\mathrm{V}}$. Moreover, the pullback of L_{V} to V' via either projection is the unique extension of $\mathrm{L}_{\mathrm{U}'}$ to a subobject of $\mathrm{M}_{\mathrm{V}'}$. It follows that the descent data on M_{V} induces descent data on L_{V} restriction to the tautological descent data on L_{U}. The descended subobject $\mathrm{L} \subset \mathrm{M}$ is then the desired extension of $\mathrm{L}_{\mathcal{U}}$.

In all of the above we assumed that V' is a scheme. This proves in particular the result when $\mathcal{X}$ is an algebraic space. Repeating the above argument allowing V' to be an algebraic space we then obtain the result for a general stack. □

Tensoring with $\mathbb{Q}_l$ we see that the restriction map

$$\mathrm{Mod}_{\mathcal{X}}(\mathbb{Q}_l) \to \mathrm{Mod}_{\mathcal{U}}(\mathbb{Q}_l)$$

is also fully faithful with essential image closed under subobjects.

Lemma 8.4. *Let $\mathcal{X}$ be a normal algebraic stack and $j : \mathcal{U} \hookrightarrow \mathcal{X}$ a dense open substack. If* L *is a smooth irreducible $\mathbb{Q}_l$–sheaf on $\mathcal{X}$ then the restriction of* L *to $\mathcal{U}$ is also irreducible.*

Proof. Immediate from the preceding lemma. □

Lemma 8.5. *Let $\mathcal{X}$ be a smooth algebraic stack of finite type and* L *a smooth $\mathbb{Q}_l$–sheaf on $\mathcal{X}$ which is irreducible. Then the perverse sheaf* $\mathrm{F} := \mathrm{L}[\dim(\mathcal{X})]$ *is simple.*

Proof. This follows from the same argument proving [5, 4.3.3] (note that the reference at the end of the proof should be 1.4.25). □

We can now prove 8.2. That the perverse sheaf $j_{!*}\mathrm{F}$ is simple follows from [5, 1.4.25] applied to $\mathcal{U} \hookrightarrow \overline{\mathcal{U}}$ (where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ in $\mathcal{X}$) and [5, 1.4.26] applied to $\overline{\mathcal{U}} \hookrightarrow \mathcal{X}$.

To see that every simple perverse sheaf is of this form, let F be a simple perverse sheaf on $\mathcal{X}$. Then there exists a dense open substack $j : \mathcal{U} \hookrightarrow \mathcal{X}$ such that $\mathrm{F}_{\mathrm{U}} = \mathrm{L}[\dim(\mathcal{U})]$ and such that

$(\mathcal{U} \otimes_k \bar{k})_{\rm red}$ is smooth over $\bar{k}$. By [5, 1.4.25] the map $j_{!*}\mathrm{F}_{\mathrm{U}} \to \mathrm{F}$ is a monomorphism whence an isomorphism since F is simple. This completes the proof of 8.2. □

9. Weights

In this section we work over a finite field $k = \mathbb{F}_q$. Fix an algebraic closure $\bar{k}$ of k, and for any integer $n \geq 1$ let $\mathbb{F}_{q^n}$ denote the unique subfield of $\bar{k}$ with q^n elements. Following [5] we write objects (e.g. stacks, schemes, sheaves etc.) over k with a subscript 0 and their base change to $\bar{k}$ without a subscript. So for example, $\mathcal{X}_0$ denotes a stack over k and $\mathcal{X}$ denotes $\mathcal{X}_0 \otimes_k \bar{k}$. In what follows we work with $\overline{\mathbb{Q}}_l$–coefficients for some prime l invertible in k.

Let $\mathcal{X}_0/k$ be a stack of finite type, and let $\mathrm{Fr}_q : \mathcal{X} \to \mathcal{X}$ be the Frobenius morphism. Recall that if T is a $\bar{k}$–scheme then

$$\mathrm{Fr}_q : \mathcal{X}(\mathrm{T}) = \mathcal{X}_0(\mathrm{T}) \to \mathcal{X}_0(\mathrm{T}) = \mathcal{X}(\mathrm{T})$$

is the pullback functor along the Frobenius morphism of T (which is a k–morphism). We let Fr_{q^n} denote the n–th iterate of Fr_q. If $x : \mathrm{Spec}(\mathbb{F}_{q^n}) \to \mathcal{X}_0$ is a morphism, we then obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\bar{k}) & \xrightarrow{\mathrm{Fr}_{q^n}} & \mathrm{Spec}(\bar{k}) \\ \bar{x}\downarrow & & \downarrow\bar{x} \\ \mathcal{X} & \xrightarrow{\mathrm{Fr}_{q^n}} & \mathcal{X}. \end{array}$$

If F_0 is a sheaf on $\mathcal{X}_0$, then the commutativity of this diagram over $\mathcal{X}_0$ defines an automorphism $\mathrm{F}^*_{q^n} : \mathrm{F}_{\bar{x}} \to \mathrm{F}_{\bar{x}}$.

Definition 9.1. (i) A sheaf F_0 on $\mathcal{X}_0$ is *punctually pure* of weight w ($w \in \mathbb{Z}$) if for every $n \geq 1$ and every $x \in \mathcal{X}_0(\mathbb{F}_{q^n})$ the eigenvalues of the automorphism $\mathrm{F}^*_{q^n} : \mathrm{F}_{\bar{x}} \to \mathrm{F}_{\bar{x}}$ are algebraic numbers all of whose complex conjugates have absolute value $q^{nw/2}$.

(ii) A sheaf F_0 on $\mathcal{X}_0$ is *mixed* if it admits a finite filtration whose successive quotients are punctually pure. The weights of the graded pieces are called the weights of F_0.

(iii) A complex $\mathrm{K} \in \mathcal{D}^b_c(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ is *mixed* if for all i the sheaf $\mathcal{H}^i(\mathrm{K})$ is mixed.

(iv) A complex $\mathrm{K} \in \mathcal{D}^b_c(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ is of *weight* $\leq w$ if for every i the mixed sheaf $\mathcal{H}^i(\mathrm{K})$ has weights $\leq w + i$.

(v) A complex $\mathrm{K} \in \mathcal{D}^b_c(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ is of *weight* $\geq w$ if the Verdier dual of K is of weight $\leq -w$.

(vi) A complex $\mathrm{K} \in \mathcal{D}^b_c(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ is *pure of weight* w if it is of weight $\leq w$ and $\geq w$.

In particular we can talk about a mixed (or pure etc) perverse sheaf.

Theorem 9.2 (Stack version of [5, 5.3.5]). *A mixed perverse sheaf* F_0 *on* $\mathcal{X}_0$ *admits a unique filtration* W *such that the graded pieces* $\mathrm{gr}_i^{\mathrm{W}}\mathrm{F}_0$ *are pure of weight* i*. Every morphism of mixed perverse sheaves is strictly compatible with the filtrations.*

Proof. By descent theory (and the uniqueness) it suffices to construct the filtration locally in the smooth topology. Hence the result follows from the case of schemes. □

The filtration W in the theorem is called the *weight filtration*.

Corollary 9.3. *Any subquotient of a mixed perverse sheaf* F_0 *is mixed. If* F_0 *is mixed of weight* $\leq w$ *(resp.* $\geq w$*) then any subquotient is also of weight* $\leq w$ *(resp.* $\geq w$*).*

Proof. The weight filtration on F_0 induces a filtration on any subquotient whose successive quotients are pointwise pure. This implies the first statement. The second statement can be verified on a smooth cover of $\mathcal{X}_0$ and hence follows from [5, 5.3.1]. □

One verifies immediately that the subcategory of the category of constructible sheaves on $\mathcal{X}_0$ consisting of mixed sheaves is closed under the formation of subquotients and extensions. In particular we can define $\mathcal{D}^b_m(\mathcal{X}_0, \overline{\mathbb{Q}}_l) \subset \mathcal{D}^b_c(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ to be the full subcategory consisting of complexes whose cohomology sheaves are mixed. The category $\mathcal{D}^b_m(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ is a triangulated subcategory.

Proposition 9.4. *The perverse* t*–structure induces a* t*–structure on* $\mathcal{D}^b_m(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$*.*

Proof. It suffices to show that the subcategory $\mathcal{D}^b_m(\mathcal{X}_0, \overline{\mathbb{Q}}_l) \subset \mathcal{D}^b_c(\mathcal{X}_0, \overline{\mathbb{Q}}_l)$ is stable under the perverse truncations $\tau_{\leq 0}$ and $\tau_{\geq 0}$. This can be verified locally on $\mathcal{X}_0$ and hence follows from the case of schemes. □

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