ON A NONHOMOGENEOUS QUASILINEAR EIGENVALUE PROBLEM IN SOBOLEV SPACES WITH VARIABLE EXPONENT*

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ABSTRACT. We consider the nonlinear eigenvalue problem $-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda |u|^{q(x)-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary and p, q are continuous functions on $\overline{\Omega}$ such that $1 < \inf_\Omega q < \inf_\Omega p < \sup_\Omega q$, $\sup_\Omega p < N$, and $q(x) < Np(x)/\left(N - p(x)\right)$ for all $x \in \overline{\Omega}$. The main result of this paper establishes that any $\lambda > 0$ sufficiently small is an eigenvalue of the above nonhomogeneous quasilinear problem. The proof relies on simple variational arguments based on Ekeland's variational principle.

2000 Mathematics Subject Classification: 35D05, 35J60, 35J70, 58E05, 68T40, 76A02.

Key words: $p(x)$-Laplace operator, nonlinear eigenvalue problem, Sobolev space with variable exponent, Ekeland's variational principle.

1 Introduction and preliminary results

A basic result in the elementary theory of linear partial differential equations asserts that the spectrum of the Laplace operator in $H_0^1(\Omega)$ is discrete, where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary. More precisely, the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

has an unbounded sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$. This celebrated result goes back to the Riesz-Fredholm theory of self-adjoint and compact operators on Hilbert spaces. The anisotropic case

$$\begin{cases} -\Delta u = \lambda a(x) u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

was considered by Bocher [3], Hess and Kato [12], Minakshisundaram and Pleijel [15, 17]. For instance, Minakshisundaram and Pleijel proved that the above eigenvalue problem has an unbounded sequence

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of positive eigenvalues if $a \in L^\infty(\Omega)$, $a \geq 0$ in Ω, and $a > 0$ in $\Omega_0 \subset \Omega$, where $|\Omega_0| > 0$. Eigenvalue problems for homogeneous quasilinear problems have been intensively studied in the last decades (see, e.g., Anane [2]).

This paper is motivated by recent advances in elastic mechanics and electrorheological fluids (sometimes referred to as "smart fluids"), where some processes are modeled by nonhomogeneous quasilinear operators (see Diening [4], Halsey [11], Ruzicka [18], Zhikov [21], and the references therein). We refer mainly to the $p(x)$–Laplace operator $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \text{div}\,(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous.

In this paper we are concerned with the nonhomogeneous eigenvalue problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u, & \text{for} \quad x \in \Omega \\ u = 0, & \text{for} \quad x \in \partial\Omega\,, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, and p, q are continuous on $\overline{\Omega}$.

The case $p(x) = q(x)$ was considered by Fan, Zhang and Zhao in [10] who, using the Ljusternik-Schnirelmann critical point theory, established the existence of a sequence of eigenvalues. Denoting by Λ the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that $\sup \Lambda = +\infty$ and they pointed out that only under additional assumptions we have $\inf \Lambda > 0$. We remark that for the p-Laplace operator (corresponding to $p(x) \equiv p$) we always have $\inf \Lambda > 0$.

In this paper we study problem (1) under the basic assumption

$$1 < \min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x) < \max_{x \in \overline{\Omega}} q(x). \tag{2}$$

Our main result establishes the existence of a continuous family of eigenvalues for problem (1) in a neighborhood of the origin. More precisely, we show that there exists $\lambda^\star > 0$ such that *any* $\lambda \in (0, \lambda^\star)$ is an eigenvalue for problem (1).

We start with some preliminary basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details we refer to the book by Musielak [16] and the papers by Edmunds et al. [5, 6, 7], Kovacik and Rákosník [13], Mihăilescu and Rădulescu [14], and Samko and Vakulov [19].

Assume that $p \in C(\overline{\Omega})$ and $p(x) > 1$, for all $x \in \overline{\Omega}$.

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x \in \Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponent so that $p_1(x) \le p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{p^{'}(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p^{'}(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^{'}(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^{'-}}\right) |u|_{p(x)} |v|_{p^{'}(x)} \tag{3}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{4}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{5}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \tag{6}$$

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. We note that if $s(x) \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \ge N$.

2 The main result

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1) if there exists $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{q(x)-2} uv \, dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. We point out that if λ is an eigenvalue of the problem (1) then the corresponding $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ is a weak solution of (1).

Our main result is given by the following theorem.

Theorem 1. *Assume that condition (2) is fulfilled,* $\max_{x \in \overline{\Omega}} p(x) < N$ *and* $q(x) < p^\star(x)$ *for all* $x \in \overline{\Omega}$. *Then there exists* $\lambda^\star > 0$ *such that any* $\lambda \in (0, \lambda^\star)$ *is an eigenvalue for problem (1).*

The above result implies

$$\inf_{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)} \, dx}{\displaystyle\int_\Omega |u|^{q(x)} \, dx} = 0.$$

Thus, for any positive constant C, there exists $u_0 \in W_0^{1,p(x)}(\Omega)$ such that

$$C \int_\Omega |u_0|^{q(x)} \, dx \geq \int_\Omega |\nabla u_0|^{p(x)} \, dx.$$

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$.

For any $\lambda > 0$ the energy functional corresponding to problem (1) is defined as $J_\lambda : E \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx.$$

Standard arguments imply that $J_\lambda \in C^1(E, \mathbb{R})$ and

$$\langle J_\lambda^{'}(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_\Omega |u|^{q(x)-2} uv \, dx,$$

for all u, $v \in E$. Thus the weak solutions of (1) coincide with the critical points of J_λ. If such an weak solution exists and is nontrivial then the corresponding λ is an eigenvalue of problem (1).

Lemma 1. *There exists* $\lambda^\star > 0$ *such that for any* $\lambda \in (0, \lambda^\star)$ *there exist* ρ, $a > 0$ *such that* $J_\lambda(u) \geq a > 0$ *for any* $u \in E$ *with* $\|u\| = \rho$.

Proof. Since $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ it follows that E is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant c_1 such that

$$|u|_{q(x)} \leq c_1 \|u\|, \quad \forall\, u \in E. \tag{7}$$

We fix $\rho \in (0,1)$ such that $\rho < 1/c_1$. Then relation (7) implies

$$|u|_{q(x)} < 1, \quad \forall\, u \in E, \text{ with } \|u\| = \rho.$$

Furthermore, relation (5) yields

$$\int_\Omega |u|^{q(x)}\, dx \le |u|_{q(x)}^{q^-}, \quad \forall\, u \in E, \text{ with } \|u\| = \rho. \tag{8}$$

Relations (7) and (8) imply

$$\int_\Omega |u|^{q(x)}\, dx \le c_1^{q^-} \|u\|^{q^-}, \quad \forall\, u \in E, \text{ with } \|u\| = \rho. \tag{9}$$

Taking into account relations (5) and (9) we deduce that for any $u \in E$ with $\|u\| = \rho$ the following inequalities hold true

$$\begin{aligned} J_\lambda(u) &\ge \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\, dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\, dx \\ &\ge \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-} c_1^{q^-} \|u\|^{q^-} \\ &= \frac{1}{p^+}\rho^{p^+} - \frac{\lambda}{q^-} c_1^{q^-} \rho^{q^-} \\ &= \rho^{q^-}\left(\frac{1}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-} c_1^{q^-}\right). \end{aligned}$$

By the above inequality we remark that if we define

$$\lambda^\star = \frac{\rho^{p^+-q^-}}{2p^+} \cdot \frac{q^-}{c_1^{q^-}} \tag{10}$$

then for any $\lambda \in (0, \lambda^\star)$ and any $u \in E$ with $\|u\| = \rho$ there exists $a = \frac{\rho^{p^+}}{2p^+} > 0$ such that

$$J_\lambda(u) \ge a > 0.$$

The proof of Lemma 1 is complete. □

Lemma 2. *There exists $\varphi \in E$ such that $\varphi \ge 0$, $\varphi \ne 0$ and $J_\lambda(t\varphi) < 0$, for $t > 0$ small enough.*

Proof. Assumption (2) implies that $q^- < p^-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < p^-$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \le q^- + \epsilon_0 < p^-$ for all $x \in \Omega_0$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\text{supp}(\varphi) \supset \overline{\Omega}_0$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le \varphi \le 1$ in Ω. Then using the above information for any $t \in (0,1)$ we have

$$\begin{aligned} J_\lambda(t\varphi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)}\,dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\varphi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\varphi|^{p(x)}\,dx - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\varphi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\varphi|^{p(x)}\,dx - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|\varphi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\varphi|^{p(x)}\,dx - \frac{\lambda\cdot t^{q^-+\epsilon_0}}{q^+}\int_{\Omega_0} |\varphi|^{q(x)}\,dx. \end{aligned}$$

Therefore

$$J_\lambda(t\varphi) < 0$$

for $t < \delta^{1/(p^- - q^- - \epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\frac{\lambda\cdot p^-}{q^+}\int_{\Omega_0}|\varphi|^{q(x)}\,dx}{\int_\Omega |\nabla\varphi|^{p(x)}\,dx}\right\}.$$

Finally, we point out that $\int_\Omega |\nabla\varphi|^{p(x)}\,dx > 0$. Indeed, it is clear that

$$\int_{\Omega_0} |\varphi|^{q(x)}\,dx \leq \int_\Omega |\varphi|^{q(x)}\,dx \leq \int_{\Omega_0} |\varphi|^{q^-}\,dx.$$

On the other hand, $W_0^{1,p(x)}(\Omega)$ is continuously embedded in $L^{q^-}(\Omega)$ and thus, there exists a positive constant c_2 such that

$$|\varphi|_{q^-} \leq c_2\|\varphi\|.$$

The last two inequalities imply that

$$\|\varphi\| > 0$$

and combining that fact with relations (4) or (5) we deduce that

$$\int_\Omega |\nabla\varphi|^{p(x)}\,dx > 0.$$

The proof of Lemma 2 is complete. □

PROOF OF THEOREM 1. Let $\lambda^\star > 0$ be defined as in (10) and $\lambda \in (0,\lambda^\star)$. By Lemma 1 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{11}$$

On the other hand, by Lemma 2, there exists $\varphi \in E$ such that $J_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, relations (9) and (5) imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\|u\|^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} J_\lambda < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} J_\lambda(u_\epsilon) &< \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \\ J_\lambda(u_\epsilon) &< J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda\,,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + t \cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda^{'}(u_\epsilon), v\rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|J_\lambda^{'}(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda^{'}(w_n) \to 0. \tag{12}$$

It is clear that $\{w_n\}$ is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. Since $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ we deduce that E is compactly embedded in $L^{q(x)}(\Omega)$, hence $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$. So, by relations (6) and (3),

$$\lim_{n\to\infty} \int_\Omega |w_n|^{q(x)-2} w_n(w_n - w)\, dx = 0.$$

On the other hand, relation (12) yields

$$\lim_{n\to\infty} \langle J_\lambda^{'}(w_n), w_n - w\rangle = 0.$$

Using the above information we find

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^{p(x)-2} \nabla w_n \nabla(w_n - w)\, dx = 0. \tag{13}$$

Relation (13) and the fact that $\{w_n\}$ converges weakly to w in E enable us to apply Theorem 3.1 in Fan and Zhang [9] in order to obtain that $\{w_n\}$ converges strongly to w in E. So, by (12),

$$J_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad J_\lambda^{'}(w) = 0. \tag{14}$$

We conclude that w is a nontrivial weak solution for problem (1) and thus any $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem (1).

The proof of Theorem 1 is complete. $\square$

Let us now assume that the hypotheses of Theorem 1 are fulfilled and, furthermore,

$$\max_{\overline{\Omega}} p(x) < \max_{\overline{\Omega}} q(x).$$

Then, using similar arguments as in the proof of Lemma 2, we find some $\psi \in E$ such that

$$\lim_{t \to \infty} J_\lambda(t\psi) = -\infty.$$

That fact combined with Lemma 1 and the mountain pass theorem (see [1]) implies that there exists a sequence $\{u_n\}$ in E such that

$$J_\lambda(u_n) \to \overline{c} > 0 \quad \text{and} \quad J_\lambda^{'}(u_n) \to 0 \text{ in } E^\star. \tag{15}$$

However, relation (15) is not useful because we can not show that the sequence $\{u_n\}$ is bounded in E since the functional J_λ does not satisfy a relation of the Ambrosetti-Rabinowitz type. This enable us to affirm that we can not obtain a critical point for J_λ by using this method.

On the other hand, we point out that we will fail in trying to show that the functional J_λ is coercive since by relation (2) we have $q^+ > p^-$. Thus, we can not apply (as in the homogeneous case) a result as Theorem 1.2 in Struwe [20] in order to obtain a critical point of the functional J_λ.

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