

# A MULTIPLICITY RESULT FOR A NONLINEAR DEGENERATE PROBLEM ARISING IN THE THEORY OF ELECTORRHEOLOGICAL FLUIDS\*

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ABSTRACT. We study the boundary value problem  $-\operatorname{div}(a(x, \nabla u)) = \lambda(u^{\gamma-1} - u^{\beta-1})$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$  and  $\operatorname{div}(a(x, \nabla u))$  is a  $p(x)$ -Laplace type operator, with  $1 < \beta < \gamma < \inf_{x \in \Omega} p(x)$ . We prove that if  $\lambda$  is large enough then there exist at least two nonnegative weak solutions. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with adequate variational methods and a variant of Mountain Pass Lemma.

**2000 Mathematics Subject Classification:** 35D05, 35J60, 35J70, 58E05, 68T40, 76A02.

**Key words:**  $p(x)$ -Laplace operator, generalized Lebesgue-Sobolev space, critical point, weak solution, electrorheological fluids.

## 1 Introduction and preliminary results

Most materials can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces,  $L^p$  and  $W^{1,p}$ , where  $p$  is a fixed constant. For some materials with inhomogeneities, for instance electrorheological fluids (sometimes referred to as “smart fluids”), this is not adequate, but rather the exponent  $p$  should be able to vary. This leads us to the study of variable exponent Lebesgue and Sobolev spaces,  $L^{p(x)}$  and  $W^{1,p(x)}$ , where  $p$  is a real-valued function.

This paper is motivated by phenomena which are described by nonlinear boundary value problems of the type

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = f(x, u), & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is a bounded domain with smooth boundary,  $1 < p(x)$  and  $p(x) \in C(\overline{\Omega})$ . The interest in studying such problems consists in the presence of the  $p(x)$ -Laplace type operator  $\operatorname{div}(a(x, \nabla u))$ . We remember that the  $p(x)$ -Laplace operator is defined by  $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ .

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The study of differential equations and variational problems involving  $p(x)$ -growth conditions is a consequence of their applications. Materials requiring such more advanced theory have been studied experimentally since the middle of last century. The first major discovery on electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. He noticed that such fluids (for instance lithium polymetachrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For a general account of the underlying physics confer [13] and for some technical applications [18]. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids we refer to [1, 4, 5, 10, 13, 20].

Variable exponent Lebesgue spaces appeared in the literature for the first time already in a 1931 article by W. Orlicz [17], who proved various results (including Hölder's inequality) in a discrete framework. Orlicz also considered the variable exponent function space  $L^{p(x)}$  on the real line, and proved the Hölder inequality in this setting, too. Next, Orlicz abandoned the study of variable exponent spaces, to concentrate on the theory of the function spaces that now bear his name. The first systematic study of spaces with variable exponent (called *modular spaces*) is due to Nakano [16]. In the appendix of this book, Nakano mentions explicitly variable exponent Lebesgue spaces as an example of the more general spaces he considers [16, p. 284]. Despite their broad interest, these spaces have not reached the same main-stream position as Orlicz spaces. Somewhat later, a more explicit version of such spaces, namely modular function spaces, were investigated by Polish mathematicians. We refer to the book of Musielak [15] for a nice presentation of the modular function spaces. This book, although not dealing specifically with the spaces that interest us, is still specific enough to contain several interesting results regarding variable exponent spaces. Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers, notably Sharapudinov. These investigations originated in a paper by Tsenov [23]. The question raised by Tsenov and solved by Sharapudinov [21] is the minimization of  $\int_a^b |u(x) - v(x)|^{p(x)} dx$ , where  $u$  is a fixed function and  $v$  varies over a finite dimensional subspace of  $L^{p(x)}([a, b])$ . Sharapudinov also introduces the Luxemburg norm for the Lebesgue space and shows that this space is reflexive if the exponent satisfies  $1 < p^- \leq p^+ < \infty$ . In the 80's Zhikov started a new line of investigation, that was to become intimately related to the study of variable exponent spaces, namely he considered variational integrals with non-standard growth conditions.

We recall in what follows some definitions and basic properties of the generalized Lebesgue–Sobolev spaces  $L^{p(x)}(\Omega)$  and  $W_0^{1,p(x)}(\Omega)$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ .

Throughout this paper we assume that  $p(x) > 1$ ,  $p(x) \in C^{0,\alpha}(\overline{\Omega})$  with  $\alpha \in (0, 1)$ .

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any  $h \in C_+(\overline{\Omega})$  we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any  $p(x) \in C_+(\overline{\Omega})$ , we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u; u \text{ is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [14, Theorem 2.5], the Hölder inequality holds [14, Theorem 2.1], they are reflexive if and only if  $1 < p^- \leq p^+ < \infty$  [14, Corollary 2.7] and continuous functions are dense if  $p^+ < \infty$  [14, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [14, Theorem 2.8]: if  $0 < |\Omega| < \infty$  and  $p, q$  are variable exponent so that  $p_1(x) \leq p_2(x)$  almost everywhere in  $\Omega$  then there exists the continuous embedding  $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ , whose norm does not exceed  $|\Omega| + 1$ .

We denote by  $L^{q(x)}(\Omega)$  the conjugate space of  $L^{p(x)}(\Omega)$ , where  $1/p(x) + 1/q(x) = 1$ . For any  $u \in L^{p(x)}(\Omega)$  and  $v \in L^{q(x)}(\Omega)$  the Hölder type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \quad (2)$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the  $L^{p(x)}(\Omega)$  space, which is the mapping  $\rho_{p(x)} : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

If  $(u_n), u \in L^{p(x)}(\Omega)$  and  $p^+ < \infty$  then the following relations holds true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} \quad (3)$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-} \quad (4)$$

$$|u_n - u|_{p(x)} \rightarrow 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \rightarrow 0. \quad (5)$$

Spaces with  $p^+ = \infty$  have been studied by Edmunds, Lang and Nekvinda [6].

Next, we define  $W_0^{1,p(x)}(\Omega)$  as the closure of  $C_0^\infty(\Omega)$  under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

The space  $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$  is a separable and reflexive Banach space. We note that if  $q \in C_+(\overline{\Omega})$  and  $q(x) < p^*(x)$  for all  $x \in \overline{\Omega}$  then the embedding  $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$  is compact and continuous, where  $p^*(x) = \frac{Np(x)}{N-p(x)}$  if  $p(x) < N$  or  $p^*(x) = +\infty$  if  $p(x) \geq N$ . We refer to [7, 8, 11, 14] for further properties of variable exponent Lebesgue-Sobolev spaces.

## 2 The main result

Assume that  $a(x, \xi) : \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is the continuous derivative with respect to  $\xi$  of the mapping  $A : \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}$ ,  $A = A(x, \xi)$ , that is,  $a(x, \xi) = \frac{d}{d\xi} A(x, \xi)$ . Suppose that  $a$  and  $A$  satisfy the following hypotheses:

(A1) The following equality holds

$$A(x, 0) = 0,$$

for all  $x \in \overline{\Omega}$ .

(A2) There exists a positive constant  $c_1$  such that

$$|a(x, \xi)| \leq c_1(1 + |\xi|^{p(x)-1}),$$

for all  $x \in \overline{\Omega}$  and  $\xi \in \mathbb{R}^N$ .

(A3) The following inequality holds

$$0 \leq (a(x, \xi) - a(x, \psi)) \cdot (\xi - \psi),$$

for all  $x \in \overline{\Omega}$  and  $\xi, \psi \in \mathbb{R}^N$ , with equality if and only if  $\xi = \psi$ .

(A4) There exists  $k > 0$  such that

$$A\left(x, \frac{\xi + \psi}{2}\right) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \psi) - k|\xi - \psi|^{p(x)}$$

for all  $x \in \overline{\Omega}$  and  $\xi, \psi \in \mathbb{R}^N$ .

(A5) The following inequalities hold true

$$|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x) A(x, \xi),$$

for all  $x \in \overline{\Omega}$  and  $\xi \in \mathbb{R}^N$ .

### Examples.

1. Set  $A(x, \xi) = \frac{1}{p(x)}|\xi|^{p(x)}$ ,  $a(x, \xi) = |\xi|^{p(x)-2}\xi$ , where  $p(x) \geq 2$ . Then we get the  $p(x)$ -Laplace operator

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

2. Set  $A(x, \xi) = \frac{1}{p(x)}[(1 + |\xi|^2)^{p(x)/2} - 1]$ ,  $a(x, \xi) = (1 + |\xi|^2)^{(p(x)-2)/2}\xi$ , where  $p(x) \geq 2$ . Then we obtain the generalized mean curvature operator

$$\operatorname{div}((1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u).$$

In this paper we study problem (1) in the particular case

$$f(x, t) = \lambda(t^{\gamma-1} - t^{\beta-1})$$

with  $1 < \beta < \gamma < \inf_{x \in \overline{\Omega}} p(x)$  and  $t \geq 0$ . More precisely, we consider the degenerate boundary value problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \lambda(u^{\gamma-1} - u^{\beta-1}), & \text{for } x \in \Omega \\ u = 0, & \text{for } x \in \partial\Omega \\ u \geq 0, & \text{for } x \in \Omega. \end{cases} \quad (6)$$

We say that  $u \in W_0^{1,p(x)}(\Omega)$  is a *weak solution* of problem (6) if  $u \geq 0$  a. e. in  $\Omega$  and

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u^{\gamma-1} \varphi \, dx + \lambda \int_{\Omega} u^{\beta-1} \varphi \, dx = 0$$

for all  $\varphi \in W_0^{1,p(x)}(\Omega)$ .

Our main result asserts that problem (6) has at least two nontrivial weak solutions provided that  $\lambda > 0$  is large enough and operators  $A$  and  $a$  satisfy conditions (A1)-(A5). More precisely, we prove

**Theorem 1.** *Assume hypotheses (A1)-(A5) are fulfilled. Then there exists  $\lambda^* > 0$  such that for all  $\lambda > \lambda^*$  problem (6) has at least two distinct non-negative, nontrivial weak solutions, provided that  $p^+ < \min\{N, Np^-(N - p^-)\}$ .*

**Remark.** By Theorem 4.3 in [9] problem (6) has at least a weak solution in the particular case  $a(x, \xi) = |\xi|^{p(x)-1}\xi$ . However, the proof in [9] does not state the fact that the solution is non-negative and not even nontrivial in the case when  $f(x, 0) = 0$ .

We point out that our result is inspired by [19, Theorem 1.2], where a related property is proved in the case of the  $p$ -Laplace operators. We point out that the extension from  $p$ -Laplace operator to  $p(x)$ -Laplace operator is not trivial, since the  $p(x)$ -Laplacian has a more complicated structure than the  $p$ -Laplace operator, for example it is inhomogeneous.

### 3 Proof of Theorem 1

Let  $E$  denote the generalized Sobolev space  $W_0^{1,p(x)}(\Omega)$ .

Define the energy functional  $I : E \rightarrow \mathbb{R}$  by

$$I(u) = \int_{\Omega} A(x, \nabla u) \, dx - \frac{\lambda}{\gamma} \int_{\Omega} u_+^{\gamma} \, dx + \frac{\lambda}{\beta} \int_{\Omega} u_+^{\beta} \, dx,$$

where  $u_+(x) = \max\{u(x), 0\}$ .

We first establish some basic properties of  $I$ .

**Proposition 1.** *The functional  $I$  is well-defined on  $E$  and  $I \in C^1(E, \mathbb{R})$  with the derivative given by*

$$\langle I'(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u_+^{\gamma-1} \varphi \, dx + \lambda \int_{\Omega} u_+^{\beta-1} \varphi \, dx,$$

for all  $u, \varphi \in E$ .

With that end in view we define the functional  $\Lambda : E \rightarrow \mathbb{R}$  by

$$\Lambda(u) = \int_{\Omega} A(x, \nabla u) \, dx, \quad \forall u \in E.$$

**Lemma 1.** (i) *The functional  $\Lambda$  is well-defined on  $E$ .*

(ii) *The functional  $\Lambda$  is of class  $C^1(E, \mathbb{R})$  and*

$$\langle \Lambda'(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi \, dx,$$

for all  $u, \varphi \in E$ .

*Proof.* (i) For any  $x \in \Omega$  and  $\xi \in \mathbb{R}^N$  we have

$$A(x, \xi) = \int_0^1 \frac{d}{dt} A(x, t\xi) \, dt = \int_0^1 a(x, t\xi) \cdot \xi \, dt.$$

Using hypotheses (A2) we get

$$\begin{aligned} A(x, \xi) &\leq c_1 \int_0^1 (1 + |\xi|^{p(x)-1} t^{p(x)-1}) |\xi| \, dt \\ &\leq c_1 |\xi| + \frac{c_1}{p(x)} |\xi|^{p(x)} \\ &\leq c_1 |\xi| + \frac{c_1}{p^-} |\xi|^{p(x)}, \quad \forall x \in \overline{\Omega}, \xi \in \mathbb{R}^N. \end{aligned} \tag{7}$$

The above inequality and (A5) imply

$$0 \leq \int_{\Omega} A(x, \nabla u) \, dx \leq c_1 \int_{\Omega} |\nabla u| \, dx + \frac{c_1}{p^-} \int_{\Omega} |\nabla u|^{p(x)} \, dx, \quad \forall u \in E.$$

Using inequality (2) and relations (3) and (4) we deduce that  $\Lambda$  is well defined on  $E$ .

(ii) **Existence of the Gâteaux derivative.** Let  $u, \varphi \in E$ . Fix  $x \in \Omega$  and  $0 < |r| < 1$ . Then, by the mean value theorem, there exists  $\nu \in [0, 1]$  such that

$$|A(x, \nabla u(x) + r \nabla \varphi(x)) - A(x, \nabla u)| / |r| = |a(x, \nabla u(x) + \nu r \nabla \varphi(x))| |\nabla \varphi(x)|.$$

Using condition (A2) we obtain

$$\begin{aligned} |A(x, \nabla u(x) + r \nabla \varphi(x)) - A(x, \nabla u)| / |r| &\leq [c_1 + c_1 (|\nabla u(x)| + |\nabla \varphi(x)|)^{p(x)-1}] |\nabla \varphi(x)| \\ &\leq [c_1 + c_1 2^{p^+} (|\nabla u(x)|^{p(x)-1} + |\nabla \varphi(x)|^{p(x)-1})] |\nabla \varphi(x)|. \end{aligned}$$

Next, by inequality (2), we have

$$\int_{\Omega} c_1 |\nabla \varphi| \, dx \leq |c_1|_{\frac{p(x)}{p(x)-1}} \cdot |\nabla \varphi|_{p(x)}$$

and

$$\int_{\Omega} |\nabla u|^{p(x)-1} |\nabla \varphi| \, dx \leq \| |\nabla u|^{p(x)-1} \|_{\frac{p(x)}{p(x)-1}} \cdot |\nabla \varphi|_{p(x)}.$$

The above inequalities imply

$$c_1 [1 + 2^{p^+} (|\nabla u(x)|^{p(x)-1} + |\nabla \varphi(x)|^{p(x)-1})] |\nabla \varphi(x)| \in L^1(\Omega).$$

It follows from the Lebesgue theorem that

$$\langle \Lambda'(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi \, dx.$$

**Continuity of the Gâteaux derivative.** Assume  $u_n \rightarrow u$  in  $E$ . Let us define  $\theta(x, u) = a(x, \nabla u)$ . Using hypotheses (A2) and Proposition 2.2 in [9] we deduce that  $\theta(x, u_n) \rightarrow \theta(x, u)$  in  $(L^{q(x)}(\Omega))^N$ , where  $q(x) = \frac{p(x)}{p(x)-1}$ . By inequality (2) we obtain

$$|\langle \Lambda'(u_n) - \Lambda'(u), \varphi \rangle| \leq |\theta(x, u_n) - \theta(x, u)|_{q(x)} |\nabla \varphi|_{p(x)}$$

and so

$$\|\Lambda'(u_n) - \Lambda'(u)\| \leq |\theta(x, u_n) - \theta(x, u)|_{q(x)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The proof of Lemma 1 is complete. □

**Lemma 2.** *If  $u \in E$  then  $u_+, u_- \in E$  and*

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u \leq 0] \\ \nabla u, & \text{if } [u > 0], \end{cases} \quad \nabla u_- = \begin{cases} 0, & \text{if } [u \geq 0] \\ \nabla u, & \text{if } [u < 0] \end{cases}$$

where  $u_{\pm} = \max\{\pm u(x), 0\}$  for all  $x \in \Omega$ .

*Proof.* Let  $u \in E$  be fixed. Then there exists a sequence  $(\varphi_n) \in C_0^\infty(\Omega)$  such that

$$|\nabla(\varphi_n - u)|_{p(x)} \rightarrow 0.$$

Since  $1 < p^- \leq p(x)$  for all  $x \in \Omega$ , it follows that  $L^{p(x)}$  is continuously embedded in  $L^{p^-}(\Omega)$  and thus

$$|\nabla(\varphi_n - u)|_{p^-} \rightarrow 0.$$

Hence  $u \in W_0^{1,p^-}(\Omega)$ . We obtain

$$u_+, u_- \in W_0^{1,p^-}(\Omega) \subset W_0^{1,1}(\Omega). \tag{8}$$

On the other hand, Theorem 7.6 in [12] implies

$$\nabla u_+ = \begin{cases} 0, & \text{if } [u \leq 0] \\ \nabla u, & \text{if } [u > 0], \end{cases} \quad \nabla u_- = \begin{cases} 0, & \text{if } [u \geq 0] \\ \nabla u, & \text{if } [u < 0]. \end{cases}$$

By the above equalities we deduce that

$$|u_+(x)|^{p(x)} \leq |u(x)|^{p(x)}, \quad |\nabla u_+(x)|^{p(x)} \leq |\nabla u|^{p(x)}, \quad \text{a.e. } x \in \Omega \quad (9)$$

and

$$|u_-(x)|^{p(x)} \leq |u(x)|^{p(x)}, \quad |\nabla u_-(x)|^{p(x)} \leq |\nabla u|^{p(x)}, \quad \text{a.e. } x \in \Omega. \quad (10)$$

Since  $u \in E$  we have

$$|u(x)|^{p(x)}, |\nabla u(x)|^{p(x)} \in L^1(\Omega). \quad (11)$$

By (9), (10) and (11) and Lebesgue theorem we obtain that  $u_+, u_- \in L^{p(x)}(\Omega)$  and  $\rho_{p(x)}(|\nabla u_+|) < \infty$ ,  $\rho_{p(x)}(|\nabla u_-|) < \infty$ . It follows that

$$u_+, u_- \in W^{1,p(x)}(\Omega) \quad (12)$$

where  $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\}$  (see [11] for more details).

By (8) and (12) we conclude that

$$u_+, u_- \in W^{1,p(x)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Since  $p \in C^{0,\alpha}(\overline{\Omega})$ , Theorem 2.6 and Remark 2.9 in [11] show that  $E = W^{1,p(x)}(\Omega) \cap W_0^{1,1}(\Omega)$ . Thus  $u_+, u_- \in E$  and the proof of Lemma 2 is complete.  $\square$

By Lemmas 1 and 2 it is clear that Proposition 1 holds true. Thus, the weak solutions of (6) are exactly the critical points of  $I$ . The above remark shows that we can prove Theorem 1 using the critical points theory. More exactly, we first show that for  $\lambda > 0$  large enough, the functional  $I$  has a global minimizer  $u_1 \geq 0$  such that  $I(u_1) < 0$ . Next, by means of the Mountain Pass Theorem, a second critical point  $u_2$  with  $I(u_2) > 0$  is obtained.

**Remark.** If  $u$  is a critical point of  $I$  then using Lemma 2 and condition (A5) we have

$$\begin{aligned} 0 &= \langle I'(u), u_- \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla u_- \, dx - \lambda \int_{\Omega} (u_+)^{\gamma-1} u_- \, dx + \lambda \int_{\Omega} (u_+)^{\beta-1} u_- \, dx \\ &= \int_{\Omega} a(x, \nabla u) \cdot \nabla u_- \, dx = \int_{\Omega} a(x, \nabla u_-) \cdot \nabla u_- \, dx \geq \int_{\Omega} |\nabla u_-|^{p(x)} \, dx. \end{aligned}$$

Thus we deduce that  $u \geq 0$ . It follows that the nontrivial critical points of  $I$  are non-negative solutions of (6).

**Lemma 3.** *The functional  $\Lambda$  is weakly lower semi-continuous.*

*Proof.* By Corollary III.8 in [3], it is enough to show that  $\Lambda$  is inferior semi-continuous. For this purpose, we fix  $u \in E$  and  $\epsilon > 0$ . Since  $\Lambda$  is convex (by condition (A4)), we deduce that for any  $v \in E$  the following inequality holds

$$\int_{\Omega} A(x, \nabla v) \, dx \geq \int_{\Omega} A(x, \nabla u) \, dx + \int_{\Omega} a(x, \nabla u) \cdot (\nabla v - \nabla u) \, dx.$$

Using condition (A2) and inequality (2) we have

$$\begin{aligned} \int_{\Omega} A(x, \nabla v) \, dx &\geq \int_{\Omega} A(x, \nabla u) \, dx - \int_{\Omega} |a(x, \nabla u)| |\nabla v - \nabla u| \, dx \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - c_1 \int_{\Omega} |\nabla(v - u)| \, dx - c_1 \int_{\Omega} |\nabla u|^{p(x)-1} |\nabla(v - u)| \, dx \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - c_2 \|1\|_{q(x)} \|\nabla(v - u)\|_{p(x)} - c_3 \|\nabla u\|_{q(x)}^{p(x)-1} \|\nabla(v - u)\|_{p(x)} \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - c_4 \|v - u\| \\ &\geq \int_{\Omega} A(x, \nabla u) \, dx - \epsilon \end{aligned}$$

for all  $v \in E$  with  $\|v - u\| < \delta = \epsilon/c_4$ , where  $c_2, c_3, c_4$  are positive constants, and  $q(x) = \frac{p(x)}{p(x)-1}$ . We conclude that  $\Lambda$  is weakly lower semi-continuous. The proof of Lemma 3 is complete.  $\square$

**Lemma 4.** *There exists  $\lambda_1 > 0$  such that*

$$\lambda_1 = \inf_{u \in E, \|u\| > 1} \frac{\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx}{\int_{\Omega} |u|^{p^-} \, dx}.$$

*Proof.* We know that  $E$  is continuously embedded in  $L^{p^-}(\Omega)$ . It follows that there exists  $C > 0$  such that

$$\|u\| \geq C \|u\|_{p^-}, \quad \forall u \in E.$$

On the other hand, by (3) we have

$$\int_{\Omega} |\nabla u|^{p(x)} \, dx \geq \|u\|^{p^-}, \quad \forall u \in E \text{ with } \|u\| > 1.$$

Combining the above inequalities we obtain

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx \geq \frac{C^{p^-}}{p^+} \int_{\Omega} |u|^{p^-} \, dx, \quad \forall u \in E \text{ with } \|u\| > 1.$$

The proof of Lemma 4 is complete.  $\square$

**Proposition 2.** (i) *The functional  $I$  is bounded from below and coercive.*  
(ii) *The functional  $I$  is weakly lower semi-continuous.*

*Proof.* (i) Since  $1 < \beta < \gamma < p^-$  we have

$$\lim_{t \rightarrow \infty} \frac{\frac{1}{\gamma}t^\gamma - \frac{1}{\beta}t^\beta}{t^{p^-}} = 0.$$

Then for any  $\lambda > 0$  there exists  $C_\lambda > 0$  such that

$$\lambda \left( \frac{1}{\gamma}t^\gamma - \frac{1}{\beta}t^\beta \right) \leq \frac{\lambda_1}{2}t^{p^-} + C_\lambda, \quad \forall t \geq 0,$$

where  $\lambda_1$  is defined in Lemma 4.

Condition (A5) and the above inequality show that for any  $u \in E$  with  $\|u\| > 1$  we have

$$\begin{aligned} I(u) &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{\lambda_1}{2} \int_{\Omega} |u|^{p^-} dx - C_\lambda \mu(\Omega) \\ &\geq \frac{1}{2} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - C_\lambda \mu(\Omega) \\ &\geq \frac{1}{2p^+} \|u\|^{p^-} - C_\lambda \mu(\Omega). \end{aligned}$$

This shows that  $I$  is bounded from below and coercive.

(ii) Using Lemma 3 we deduce that  $\Lambda$  is weakly lower semi-continuous. We show that  $I$  is weakly lower semi-continuous. Let  $(u_n) \subset E$  be a sequence which converges weakly to  $u$  in  $E$ . Since  $\Lambda$  is weakly lower semi-continuous we have

$$\Lambda(u) \leq \liminf_{n \rightarrow \infty} \Lambda(u_n). \quad (13)$$

On the other hand, since  $E$  is compactly embedded in  $L^\gamma(\Omega)$  and  $L^\beta(\Omega)$  it follows that  $(u_{n+})$  converges strongly to  $u_+$  both in  $L^\gamma(\Omega)$  and in  $L^\beta(\Omega)$ . This fact together with relation (13) imply

$$I(u) \leq \liminf_{n \rightarrow \infty} I(u_n).$$

Therefore,  $I$  is weakly lower semi-continuous. The proof of Proposition 2 is complete.  $\square$

By Proposition 2 and Theorem 1.2 in [22] we deduce that there exists  $u_1 \in E$  a global minimizer of  $I$ . The following result implies that  $u_1 \neq 0$ , provided that  $\lambda$  is sufficiently large.

**Proposition 3.** *There exists  $\lambda^* > 0$  such that  $\inf_E I < 0$ .*

*Proof.* Let  $\Omega_1 \subset \Omega$  be a compact subset, large enough and  $u_0 \in E$  be such that  $u_0(x) = t_0$  in  $\Omega_1$  and  $0 \leq u_0(x) \leq t_0$  in  $\Omega \setminus \Omega_1$ , where  $t_0 > 1$  is chosen such that

$$\frac{1}{\gamma}t_0^\gamma - \frac{1}{\beta}t_0^\beta > 0.$$

We have

$$\begin{aligned} \frac{1}{\gamma} \int_{\Omega} u_0^\gamma dx - \frac{1}{\beta} \int_{\Omega} u_0^\beta dx &\geq \frac{1}{\gamma} \int_{\Omega_1} u_0^\gamma dx - \frac{1}{\beta} \int_{\Omega_1} u_0^\beta dx - \frac{1}{\beta} \int_{\Omega \setminus \Omega_1} u_0^\beta dx \\ &\geq \frac{1}{\gamma} \int_{\Omega_1} u_0^\gamma dx - \frac{1}{\beta} \int_{\Omega_1} u_0^\beta dx - \frac{1}{\beta} t_0^\beta \mu(\Omega \setminus \Omega_1) > 0 \end{aligned}$$

and thus  $I(u_0) < 0$  for  $\lambda > 0$  large enough. The proof of Proposition 3 is complete.  $\square$

Since Proposition 3 holds true it follows that  $u_1 \in E$  is a nontrivial weak solution of problem (6).

Fix  $\lambda \geq \lambda^*$ . Set

$$g(x, t) = \begin{cases} 0, & \text{for } t < 0 \\ t^{\gamma-1} - t^{\beta-1}, & \text{for } 0 \leq t \leq u_1(x) \\ u_1(x)^{\gamma-1} - u_1(x)^{\beta-1}, & \text{for } t > u_1(x) \end{cases}$$

and

$$G(x, t) = \int_0^t g(x, s) ds.$$

Define the functional  $J : E \rightarrow \mathbb{R}$  by

$$J(u) = \int_{\Omega} A(x, \nabla u) dx - \lambda \int_{\Omega} G(x, u) dx.$$

The same arguments as those used for functional  $I$  imply that  $J \in C^1(E, \mathbb{R})$  and

$$\langle J'(u), \varphi \rangle = \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx - \lambda \int_{\Omega} g(x, u) \varphi dx,$$

for all  $u, \varphi \in E$ .

On the other hand, we point out that if  $u \in E$  is a critical point of  $J$  then  $u \geq 0$ . The proof can be carried out as in the case of functional  $I$ .

Next, we prove

**Lemma 5.** *If  $u$  is a critical point of  $J$  then  $u \leq u_1$ .*

*Proof.* We have

$$\begin{aligned} 0 &= \langle J'(u) - J'(u_1), (u - u_1)_+ \rangle \\ &= \int_{\Omega} (a(x, \nabla u) - a(x, \nabla u_1)) \cdot \nabla (u - u_1)_+ dx - \lambda \int_{\Omega} [g(x, u) - (u_1^{\gamma-1} - u_1^{\beta-1})] (u - u_1)_+ dx \\ &= \int_{[u > u_1]} (a(x, \nabla u) - a(x, \nabla u_1)) \cdot \nabla (u - u_1) dx. \end{aligned}$$

By condition (A3) we deduce that the above equality holds if and only if  $\nabla u = \nabla u_1$ . It follows that  $\nabla u(x) = \nabla u_1(x)$  for all  $x \in \omega := \{y \in \Omega; u(y) > u_1(y)\}$ . Hence

$$\int_{\omega} |\nabla(u - u_1)|^{p(x)} dx = 0$$

and thus

$$\int_{\Omega} |\nabla(u - u_1)_+|^{p(x)} dx = 0.$$

By relation (4) we obtain

$$\|(u - u_1)_+\| = 0.$$

Since  $u - u_1 \in E$  by Lemma 2 we have that  $(u - u_1)_+ \in E$ . Thus we obtain that  $(u - u_1)_+ = 0$  in  $\Omega$ , that is,  $u \leq u_1$  in  $\Omega$ . The proof of Lemma 5 is complete.  $\square$

In the following we determine a critical point  $u_2 \in E$  of  $J$  such that  $J(u_2) > 0$  via the Mountain Pass Theorem. By the above lemma we will deduce that  $0 \leq u_2 \leq u_1$  in  $\Omega$ . Therefore

$$g(x, u_2) = u_2^{\gamma-1} - u_2^{\beta-1} \quad \text{and} \quad G(x, u_2) = \frac{1}{\gamma}u_2^\gamma - \frac{1}{\beta}u_2^\beta$$

and thus

$$J(u_2) = I(u_2) \quad \text{and} \quad J'(u_2) = I'(u_2).$$

More exactly we find

$$I(u_2) > 0 = I(0) > I(u_1) \quad \text{and} \quad I'(u_2) = 0.$$

This shows that  $u_2$  is a weak solution of problem (6) such that  $0 \leq u_2 \leq u_1$ ,  $u_2 \neq 0$  and  $u_2 \neq u_1$ .

In order to find  $u_2$  described above we prove

**Lemma 6.** *There exists  $\rho \in (0, \|u_1\|)$  and  $a > 0$  such that  $J(u) \geq a$ , for all  $u \in E$  with  $\|u\| = \rho$ .*

*Proof.* Let  $u \in E$  be fixed, such that  $\|u\| < 1$ . It is clear that there exists  $\delta > 1$  such that

$$\frac{1}{\gamma}t^\gamma - \frac{1}{\beta}t^\beta \leq 0, \quad \forall t \in [0, \delta].$$

For  $\delta$  given above we define

$$\Omega_u = \{x \in \Omega; u(x) > \delta\}.$$

If  $x \in \Omega \setminus \Omega_u$  with  $u(x) < u_1(x)$  we have

$$G(x, u) = \frac{1}{\gamma}u_+^\gamma - \frac{1}{\beta}u_+^\beta \leq 0.$$

If  $x \in \Omega \setminus \Omega_u$  with  $u(x) > u_1(x)$  then  $u_1(x) \leq \delta$  and we have

$$G(x, u) = \frac{1}{\gamma}u_1^\gamma - \frac{1}{\beta}u_1^\beta \leq 0.$$

Thus we deduce that

$$G(x, u) \leq 0, \quad \text{on } \Omega \setminus \Omega_u.$$

Provided that  $\|u\| < 1$  by condition (A5) and relation (4) we get

$$\begin{aligned} J(u) &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_{\Omega_u} G(x, u) dx \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \lambda \int_{\Omega_u} G(x, u) dx \end{aligned} \tag{14}$$

Since  $p^+ < \min\{N, \frac{Np^-}{N-p^-}\}$  it follows that  $p^+ < p^*(x)$  for all  $x \in \bar{\Omega}$ . Then there exists  $q \in (p^+, \frac{Np^-}{N-p^-})$  such that  $E$  is continuously embedded in  $L^q(\Omega)$ . Thus there exists a positive constant  $C > 0$  such that

$$\|u\|_q \leq C\|u\|, \quad \forall u \in E.$$

Using the definition of  $G$ , Hölder's inequality and the above estimate, we obtain

$$\begin{aligned} \lambda \int_{\Omega_u} G(x, u) dx &= \lambda \int_{\Omega_u \cap [u < u_1]} \left( \frac{1}{\gamma} u_+^\gamma - \frac{1}{\beta} u_+^\beta \right) dx + \lambda \int_{\Omega_u \cap [u > u_1]} \left( \frac{1}{\gamma} u_1^\gamma - \frac{1}{\beta} u_1^\beta \right) dx \\ &\leq \frac{2\lambda}{\gamma} \int_{\Omega_u} u_+^\gamma dx \\ &\leq \frac{2\lambda}{\gamma} \int_{\Omega_u} u_+^{p^+} dx \\ &\leq \frac{2\lambda}{\gamma} \left( \int_{\Omega_u} u_+^q dx \right)^{p^+/q} [\mu(\Omega_u)]^{1-p^+/q} \\ &\leq C \frac{2\lambda}{\gamma} [\mu(\Omega_u)]^{1-p^+/q} \|u\|^{p^+}. \end{aligned} \tag{15}$$

By (14) and (15) we infer that it is enough to show that  $\mu(\Omega_u) \rightarrow 0$  as  $\|u\| \rightarrow 0$  in order to prove Lemma 6.

Let  $\epsilon > 0$ . We choose  $\Omega_\epsilon \subset \Omega$  a compact subset, such that  $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ . We denote by  $\Omega_{u,\epsilon} := \Omega_u \cap \Omega_\epsilon$ . Then it is clear that

$$C[\mu(\Omega)]^{1-p^+/q} \|u\|^{p^+} \geq \left( \int_{\Omega} |u|^q dx \right)^{p^+/q} \geq \left( \int_{\Omega_{u,\epsilon}} |u|^q dx \right)^{p^+/q} \geq \delta^{p^+} [\mu(\Omega_{u,\epsilon})]^{p^+/q}.$$

The above inequality implies that  $\mu(\Omega_{u,\epsilon}) \rightarrow 0$  as  $\|u\| \rightarrow 0$ .

Since  $\Omega_u \subset \Omega_{u,\epsilon} \cup (\Omega \setminus \Omega_\epsilon)$  we have

$$\mu(\Omega_u) \leq \mu(\Omega_{u,\epsilon}) + \epsilon$$

and  $\epsilon > 0$  is arbitrary. We find that  $\mu(\Omega_u) \rightarrow 0$  as  $\|u\| \rightarrow 0$ . This concludes the proof of Lemma 6.  $\square$

**Lemma 7.** *The functional  $J$  is coercive.*

*Proof.* For each  $u \in E$  with  $\|u\| > 1$  by condition (A5), relation (3) and inequality (2) we have

$$\begin{aligned} J(u) &\geq \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_{[u > u_1]} G(x, u) dx - \lambda \int_{[u < u_1]} G(x, u) dx \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{\gamma} \int_{[u > u_1]} u_1^\gamma dx + \frac{\lambda}{\beta} \int_{[u > u_1]} u_1^\beta dx - \frac{\lambda}{\gamma} \int_{[u < u_1]} u_+^\gamma dx + \frac{\lambda}{\beta} \int_{[u < u_1]} u_+^\beta dx \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{\gamma} \int_{\Omega} u_1^\gamma dx - \frac{\lambda}{\gamma} \int_{\Omega} u_+^\gamma dx \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{\gamma} [\mu(\Omega)]^{1-\gamma/p^-} C_1 \|u\|^\gamma - C_2 \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - C_2 3 \|u\|^\gamma - C_2, \end{aligned}$$

where  $C_1, C_2$  and  $C_3$  are positive constants. Since  $\gamma < p^-$  the above inequality implies that  $J(u) \rightarrow \infty$  as  $\|u\| \rightarrow \infty$ , that is,  $J$  is coercive. The proof of Lemma 7 is complete.  $\square$

The following result yields a sufficient condition which ensures that a weakly convergent sequence in  $E$  converges strongly, too.

**Lemma 8.** *Assume that the sequence  $(u_n)$  converges weakly to  $u$  in  $E$  and*

$$\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \leq 0.$$

*Then  $(u_n)$  converges strongly to  $u$  in  $E$ .*

*Proof.* Using relation (7) we have that there exists a positive constant  $c_5$  such that

$$A(x, \xi) \leq c_5(|\xi| + |\xi|^{p(x)}), \quad \forall x \in \bar{\Omega}, \xi \in \mathbb{R}^N.$$

The above inequality implies

$$A(x, \nabla u_n) \leq c_5(|\nabla u_n| + |\nabla u_n|^{p(x)}), \quad \forall x \in \bar{\Omega}, n. \quad (16)$$

The fact that  $u_n$  converges weakly to  $u$  in  $E$  implies that there exists  $R > 0$  such that  $\|u_n\| \leq R$  for all  $n$ . By relation (16), inequalities (2), (3) and (4) we deduce that  $\{\int_{\Omega} A(x, \nabla u_n) \, dx\}$  is bounded. Then, up to to a subsequence, we deduce that  $\int_{\Omega} A(x, \nabla u_n) \, dx \rightarrow c$ . By Lemma 3 we obtain

$$\int_{\Omega} A(x, \nabla u) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} A(x, \nabla u_n) \, dx = c.$$

On the other hand, since  $\Lambda$  is convex, we have

$$\int_{\Omega} A(x, \nabla u) \, dx \geq \int_{\Omega} A(x, \nabla u_n) \, dx + \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u - \nabla u_n) \, dx.$$

Next, by the hypothesis  $\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx \leq 0$ , we conclude that  $\int_{\Omega} A(x, \nabla u) \, dx = c$ .

Taking into account that  $(u_n + u)/2$  converges weakly to  $u$  in  $E$  and using Lemma 3 we have

$$c = \int_{\Omega} A(x, \nabla u) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} A\left(x, \nabla \frac{u_n + u}{2}\right) \, dx. \quad (17)$$

We assume by contradiction that  $u_n$  does not converge to  $u$  in  $E$ . Then by (5) it follows that there exist  $\epsilon > 0$  and a subsequence  $(u_{n_m})$  of  $(u_n)$  such that

$$\int_{\Omega} |\nabla(u_{n_m} - u)|^{p(x)} \, dx \geq \epsilon, \quad \forall m. \quad (18)$$

By condition (A4) we have

$$\frac{1}{2}A(x, \nabla u) + \frac{1}{2}A(x, \nabla u_{n_m}) - A\left(x, \nabla \frac{u + u_{n_m}}{2}\right) \geq k|\nabla(u_{n_m} - u)|^{p(x)}. \quad (19)$$

Relations (18) and (19) yield

$$\frac{1}{2} \int_{\Omega} A(x, \nabla u) \, dx + \frac{1}{2} \int_{\Omega} A(x, \nabla u_{n_m}) \, dx - \int_{\Omega} A \left( x, \nabla \frac{u + u_{n_m}}{2} \right) \geq k \int_{\Omega} |\nabla(u_{n_m} - u)|^{p(x)} \, dx \geq k\epsilon.$$

Letting  $m \rightarrow \infty$  in the above inequality we obtain

$$c - k\epsilon \geq \limsup_{m \rightarrow \infty} \int_{\Omega} A \left( x, \nabla \frac{u + u_{n_m}}{2} \right) \, dx$$

and that is a contradiction with (17). It follows that  $u_n$  converges strongly to  $u$  in  $E$  and Lemma 8 is proved.  $\square$

PROOF OF THEOREM 1 COMPLETED. Using Lemma 6 and the Mountain Pass Theorem (see [2] with the variant given by Theorem 1.15 in [24]) we deduce that there exists a sequence  $(u_n) \subset E$  such that

$$J(u_n) \rightarrow c > 0 \quad \text{and} \quad J'(u_n) \rightarrow 0 \tag{20}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E); \gamma(0) = 0, \gamma(1) = u_1\}.$$

By relation (20) and Lemma 7 we obtain that  $(u_n)$  is bounded and thus passing eventually to a subsequence, still denoted by  $(u_n)$ , we may assume that there exists  $u_2 \in E$  such that  $u_n$  converges weakly to  $u_2$ . Since  $E$  is compactly embedded in  $L^i(\Omega)$  for any  $i \in [1, p^-]$ , it follows that  $u_n$  converges strongly to  $u_2$  in  $L^i(\Omega)$  for all  $i \in [1, p^-]$ . Hence

$$\langle \Lambda'(u_n) - \Lambda'(u_2), u_n - u_2 \rangle = \langle J'(u_n) - J'(u_2), u_n - u_2 \rangle + \lambda \int_{\Omega} [g(x, u_n) - g(x, u_2)](u_n - u_2) \, dx = o(1),$$

as  $n \rightarrow \infty$ . By Lemma 8 we deduce that  $u_n$  converges strongly to  $u_2$  in  $E$  and using relation (20) we find

$$J(u_2) = c > 0 \quad \text{and} \quad J'(u_2) = 0.$$

Therefore,  $J(u_2) = c > 0$  and  $J'(u_2) = 0$ . By Lemma 5 we deduce that  $0 \leq u_2 \leq u_1$  in  $\Omega$ . Therefore

$$g(x, u_2) = u_2^{\gamma-1} - u_2^{\beta-1} \quad \text{and} \quad G(x, u_2) = \frac{1}{\gamma} u_2^{\gamma} - \frac{1}{\beta} u_2^{\beta}$$

and thus

$$J(u_2) = I(u_2) \quad \text{and} \quad J'(u_2) = I'(u_2).$$

We conclude that  $u_2$  is a critical point of  $I$  and thus a solution of (6). Furthermore,  $I(u_2) = c > 0$  and  $I(u_2) > 0 > I(u_1)$ . Thus  $u_2$  is not trivial and  $u_2 \neq u_1$ . The proof of Theorem 1 is now complete.  $\square$

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