On the Fine Structure of the Projective Line Over GF(2)⊗GF(2)⊗GF(2)

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Abstract
The paper gives a succinct appraisal of the properties of the projective line defined over the direct product ring $R_{\triangle} \equiv$ GF(2)⊗GF(2)⊗GF(2). The ring is remarkable in that except for unity, all the remaining seven elements are zero-divisors, the non-trivial ones forming two distinct sets of three; elementary ('slim') and composite ('fat'). Due to this fact, the line in question is endowed with a very intricate structure. It contains twenty-seven points, every point has eighteen neighbour points, the neighbourhoods of two distant points share twelve points and those of three pairwise distant points have six points in common. Algebraically, the points of the line can be partitioned into three groups: a) the group comprising three distinguished points of the ordinary projective line of order two (the 'nucleus'), b) the group composed of twelve points whose coordinates feature both the unit(y) and a zero-divisor (the 'inner shell') and c) the group of twelve points whose coordinates have both the entries zero-divisors (the 'outer shell'). The points of the last two groups can further be split into two subgroups of six points each; while in the former case there is a perfect symmetry between the two subsets, in the latter case the subgroups have a different footing, reflecting the existence of the two kinds of a zero-divisor. The structure of the two shells, the way how they are interconnected and their link with the nucleus are all fully revealed and illustrated in terms of the neighbour/distant relation. Possible applications of this finite ring geometry are also mentioned.

Keywords: Finite Product Rings – Projective Ring Lines – Neighbour/Distant Relation

1 Introduction

Projective spaces defined over *rings* [1]–[7] represent a very important branch of algebraic geometry, being once endowed with a richer and more intricate structure when compared to that of the corresponding spaces defined over *fields*. This difference is perhaps most pronounced in the case of one-dimensional projective spaces, i. e. lines, where those defined over fields are said to have virtually no intrinsic structure (see, e. g., [6]). The projective line defined over the ring $R_{\triangle} \equiv$ GF(2)⊗GF(2)⊗GF(2), and hereafter denoted as PR$_{\triangle}$(1), serves as an especially nice illustration of this point, visible already at the level of cardinalities and fully revealed when it comes to the neighbour and/or distant relation. As per the former case, although the underlying ring has only eight elements, the line itself possesses as many as twenty-seven points in total, which is to be compared with only nine points of the line defined over the eight-element field, GF(8) (see, e. g., [8]). Concerning the latter issue, whilst the neighbour relation is a mere identity relation for any field line, it acquires, as we shall demonstrate in detail, a highly non-trivial character for PR$_{\triangle}$(1). The paper, when combined with [9] and [10], may serve as an elementary and self-contained introduction into the theory of finite projective ring geometries aimed mainly at physicists and scholars of natural sciences.

2 Rudiments of Ring Theory

In this section we recollect some basic definitions, concepts and properties of rings (see, e. g., [11]–[13]) to be employed in the sequel.

A *ring* is a set R (or, more specifically, $(R, +, *)$) with two binary operations, usually called addition ($+$) and multiplication ($*$), such that R is an abelian group under addition and a semigroup under multiplication, with multiplication being both left and right distributive over addition.[1] A ring in which the multiplication is commutative is a commutative ring. A ring R with a multiplicative identity 1 such that $1r = r1 = r$ for all $r \in R$ is a ring with unity. A ring containing a finite number of elements is a finite ring. In what follows the word ring will always mean a commutative ring with unity.

An element r of the ring R is a *unit* (or an invertible element) if there exists an element r^{-1} such that $rr^{-1} = r^{-1}r = 1$. This element, uniquely determined by r, is called the multiplicative inverse of r. The set of units forms a group under multiplication. A (non-zero) element r of R is said to be a (non-trivial) *zero-divisor* if there exists $s \neq 0$ such that $sr = rs = 0$. An element of a finite ring is either a unit or a zero-divisor. A ring in which every non-zero element is a unit is a *field*; finite (or Galois) fields, often denoted by GF(q), have q elements and exist only for $q = p^n$, where p is a prime number and n a positive integer. The smallest positive integer s such that $s1 = 0$, where $s1$ stands for $1 + 1 + 1 + \ldots + 1$ (s times), is called the *characteristic* of R; if $s1$ is never zero, R is said to be of characteristic zero.

An *ideal* $\mathcal{I}$ of R is a subgroup of $(R, +)$ such that $a\mathcal{I} = \mathcal{I}a \subseteq \mathcal{I}$ for all $a \in R$. An ideal of the ring R which is not contained in any other ideal but R itself is called a *maximal* ideal. If an ideal is of the form Ra for some element a of R it is called a *principal* ideal, usually denoted by $\langle a \rangle$. A ring with a unique maximal ideal is a *local* ring. Let R be a ring and $\mathcal{I}$ one of its ideals. Then $\overline{R} \equiv R/\mathcal{I} = \{a + \mathcal{I} \mid a \in R\}$ together with addition $(a + \mathcal{I}) + (b + \mathcal{I}) = a + b + \mathcal{I}$ and multiplication $(a + \mathcal{I})(b + \mathcal{I}) = ab + \mathcal{I}$ is a ring, called the quotient, or factor, ring of R with respect to $\mathcal{I}$; if $\mathcal{I}$ is maximal, then $\overline{R}$ is a field. A very important ideal of a ring is that represented by the intersection of all maximal ideals; this ideal is called the *Jacobson radical*.

A mapping π: $R \mapsto S$ between two rings $(R, +, *)$ and $(S, \oplus, \otimes)$ is a ring *homomorphism* if it meets the following constraints: $\pi(a + b) = \pi(a) \oplus \pi(b)$, $\pi(a * b) = \pi(a) \otimes \pi(b)$ and $\pi(1) = 1$ for any two elements a and b of R. From this definition it is readily discerned that $\pi(0) = 0$, $\pi(-a) = -\pi(a)$, a unit of R is sent into a unit of S and the set of elements $\{a \in R \mid \pi(a) = 0\}$, called the *kernel* of π, is an ideal of R. A *canonical*, or *natural*, map $\overline{\pi}$: $R \rightarrow \overline{R} \equiv R/\mathcal{I}$ defined by $\overline{\pi}(r) = r + \mathcal{I}$ is clearly a ring homomorphism with kernel $\mathcal{I}$. A bijective ring homomorphism is called a ring *iso*morphism; two rings R and S are called isomorphic, denoted by $R \cong S$, if there exists a ring isomorphism between them.

Finally, we mention a couple of relevant examples of rings: a polynomial ring, $R[x]$, viz. the set of all polynomials in one variable x and with coefficients in a ring R, and the ring $R_\otimes$ that is a (finite) direct product of rings, $R_\otimes \equiv R_1 \otimes R_2 \otimes \ldots \otimes R_n$, where both addition and multiplication are carried out componentwise and where the individual rings need not be the same.

3 The Ring $R_\triangle$, its Fundamental Quotient Rings and Canonical Homomorphisms

The ring $R_\triangle \equiv$ GF(2)$\otimes$GF(2)$\otimes$GF(2) is, like GF(2) itself, of characteristic two and consists of the following eight elements

$$\begin{aligned} R_\triangle \;\; = \;\; & \{[0,0,0] \equiv a, [1,0,0] \equiv b, [0,1,0] \equiv c, [0,0,1] \equiv d, \\ & [1,1,0] \equiv e, [1,0,1] \equiv f, [0,1,1] \equiv g, [1,1,1] \equiv h\} \end{aligned} \tag{1}$$

which comprise just one unit,

$$R_\triangle^* = \{h = b + c + d\}, \tag{2}$$

[1] It is customary to denote multiplication in a ring simply by juxtaposition, using ab in place of $a * b$.

and seven zero-divisors,

$$R_\triangle \backslash R_\triangle^* = \{a, b, c, d, e = b + c, f = b + d, g = c + d\}. \quad (3)$$

The addition and multiplication read, respectively, as follows

$\oplus$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	a	e	f	c	d	h	g
c	c	e	a	g	b	h	d	f
d	d	f	g	a	h	b	c	e
e	e	c	b	h	a	g	f	d
f	f	d	h	b	g	a	e	c
g	g	h	d	c	f	e	a	b
h	h	g	f	e	d	c	b	a

$\otimes$	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	a	a	b	b	a	b
c	a	a	c	a	c	a	c	c
d	a	a	a	d	a	d	d	d
e	a	b	c	a	e	b	c	e
f	a	b	a	d	b	f	d	f
g	a	a	c	d	c	d	g	g
h	a	b	c	d	e	f	g	h

from where we readily discern that $a \equiv 0$ and $h \equiv 1$ and find out that the ring has three maximal — and principal as well — ideals

$$\mathcal{I}_e \equiv \langle e \rangle = \{a, b, c, e\}, \quad (4)$$

$$\mathcal{I}_f \equiv \langle f \rangle = \{a, b, d, f\}, \quad (5)$$

and

$$\mathcal{I}_g \equiv \langle g \rangle = \{a, c, d, g\}, \quad (6)$$

and three other principal ideals

$$\mathcal{I}_1 \equiv \langle b \rangle = \{a, b\} = \mathcal{I}_e \cap \mathcal{I}_f, \quad (7)$$

$$\mathcal{I}_2 \equiv \langle c \rangle = \{a, c\} = \mathcal{I}_e \cap \mathcal{I}_g, \quad (8)$$

and

$$\mathcal{I}_3 \equiv \langle d \rangle = \{a, d\} = \mathcal{I}_f \cap \mathcal{I}_g. \quad (9)$$

These ideals give rise to the fundamental quotient rings, all of characteristic two, namely

$$R_\triangle/\mathcal{I}_e \cong R_\triangle/\mathcal{I}_f \cong R_\triangle/\mathcal{I}_g \cong \mathrm{GF}(2) \quad (10)$$

and

$$R_\triangle/\mathcal{I}_1 \cong R_\triangle/\mathcal{I}_2 \cong R_\triangle/\mathcal{I}_3 \cong \mathrm{GF}(2) \otimes \mathrm{GF}(2), \quad (11)$$

which yield two canonical homomorphisms

$$R_\triangle \to \mathrm{GF}(2) \quad (12)$$

and

$$R_\triangle \to \mathrm{GF}(2) \otimes \mathrm{GF}(2), \quad (13)$$

respectively. To conclude this section, it is worth mentioning that there exist two kinds of a subring of $R_\triangle$ which are isomorphic to GF(2)$\otimes$GF(2), differing from each other in whether or not is their unity inherited from $R_\triangle$. As per the former case, an example is furnished by the subset $R_\circ \equiv \{a, b, g, h\}$ with the addition and multiplication inherited from the parent ring, viz.

$\oplus$	a	b	g	h
a	a	b	g	h
b	b	a	h	g
g	g	h	a	b
h	h	g	b	a

$\otimes$	a	b	g	h
a	a	a	a	a
b	a	b	a	b
g	a	a	g	g
h	a	b	g	h

.

That $R_\circ \cong \mathrm{GF}(2)\otimes\mathrm{GF}(2)$ stems from the following correspondence $a = [0, 0]$, $b = [1, 0]$, $g = [0, 1]$ and $h = [1, 1]$, and we see that h is indeed the unity in both $R_\triangle$ and $R_\circ$. An example of the latter case is provided by the subset $R_\bullet \equiv \{a, b, c, e\}$, with the following addition and multiplication tables

$\oplus$	a	b	c	e
a	a	b	c	e
b	b	a	e	c
c	c	e	a	b
e	e	c	b	a

$\otimes$	a	b	c	e
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
e	a	b	c	e

.

We can easily verify that $R_\bullet \cong \mathrm{GF}(2)\otimes\mathrm{GF}(2)$ by making the following identifications $a = [0, 0]$, $b = [1, 0]$, $c = [0, 1]$ and $e = [1, 1]$, and notice that the current unity, e, is a *zero-divisor* in $R_\triangle$.

4 The Projective Line over $R_\triangle$ and its Fine Structure

Given a ring R and $\mathrm{GL}_2(R)$, the general linear group of invertible two-by-two matrices with entries in R, a pair $(\alpha, \beta) \in R^2$ is called *admissible* over R if there exist $\gamma, \delta \in R$ such that [14]

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(R). \tag{14}$$

The projective line over R, PR(1), is defined as the set of classes of ordered pairs $(\varrho\alpha, \varrho\beta)$, where ϱ is a unit and (α, β) admissible [6],[7],[14],[15]. Such a line carries two non-trivial, mutually complementary relations of neighbour and distant. In particular, its two distinct points X: $(\varrho\alpha, \varrho\beta)$ and Y: $(\varrho\gamma, \varrho\delta)$ are called *neighbour* (or, also *parallel*) if

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \notin \mathrm{GL}_2(R) \tag{15}$$

and *distant* otherwise, i. e., if condition (14) is met. The neighbour relation is reflexive (every point is obviously neighbour to itself) and symmetric (i. e., if X is neighbour to Y then also Y is neighbour to X), but, in general, not transitive (i. e., X being neighbour to Y and Y being neighbour to Z does not necessarily mean that X is neighbour to Z — see, e. g., [2],[7]). Given a point of PR(1), the set of all neighbour points to it will be called its *neighbourhood*.[2] Obviously, if R is a field then 'neighbour' simply reduces to 'identical' and 'distant' to 'different'.

Our next task is to apply this general definitions and concepts to our ring $R_\triangle$. To this end in view, one first notice that Eqs. (14) and (15) acquire, respectively, the following simple forms

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1 \tag{16}$$

and

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R_\triangle \backslash R^*_\triangle. \tag{17}$$

Employing the former of the two equations, we find that $\mathrm{PR}_\triangle(1)$ consists of the following twenty-seven points:

1) the three distinguished points (the 'nucleus')

$$U: (1, 0),\quad V: (0, 1),\quad W: (1, 1), \tag{18}$$

which represent the ordinary projective line of order two (PG(1,2)) embedded in $\mathrm{PR}_\triangle(1)$;

2) the twelve points of the 'inner shell' whose coordinates feature both the unity and a zero-divisor,

$$\begin{gathered} I^S_1: (1, b),\ \ I^S_2: (1, c),\ \ I^S_3: (1, d),\ \ I^F_1: (1, e),\ \ I^F_2: (1, f),\ \ I^F_3: (1, g), \\ J^S_1: (b, 1),\ \ J^S_2: (c, 1),\ \ J^S_3: (d, 1),\ \ J^F_1: (e, 1),\ \ J^F_2: (f, 1),\ \ J^F_3: (g, 1), \end{gathered} \tag{19}$$

[2]To avoid any confusion, the reader must be warned here that some authors (e. g. [7],[15]) use this term for the set of *distant* points instead.

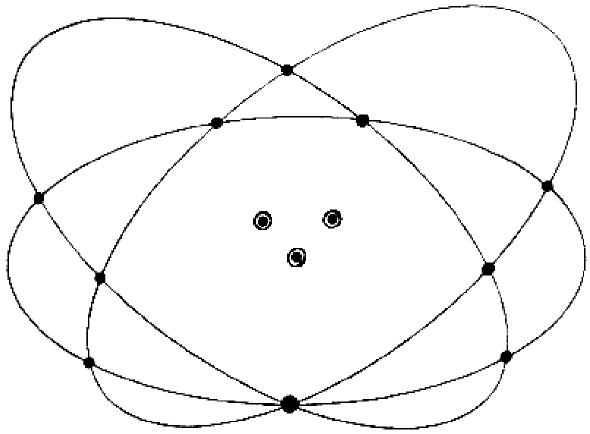

Figure 1: A schematic sketch of the structure of the projective line $PR_{\triangle}(1)$. Choosing any three pairwise distant points (represented by the three double circles), the remaining points of the line are all located on the neighbourhoods of the three points (three sets of points located on three different ellipses centered at the points in question). Every small bullet represents *two* distinct points of the line, while the big bullet at the bottom stands for as many as *six* different points.

and which form two symmetric sets of six points each, the sets themselves subject to further splitting according as the zero-divisor is elementary (slim, 'S') or composite (fat, 'F') — see Eq. (3); and

3) the twelve points of the 'outer shell' whose coordinates have zero-divisors in both the entries,

$$\begin{aligned} &S_1^+ : (e,d),\quad S_2^+ : (f,c),\quad S_3^+ : (g,b),\quad S_1^- : (d,e),\quad S_2^- : (c,f),\quad S_3^- : (b,g), \\ &F_1^+ : (e,f),\quad F_2^+ : (e,g),\quad F_3^+ : (f,g),\quad F_1^- : (f,e),\quad F_2^- : (g,e),\quad F_3^- : (g,f), \end{aligned} \tag{20}$$

and which comprise two asymmetric sets of cardinality six according as both the entries are composite zero-divisors ('F') or not ('S'); here the sets are further subdivided in terms of the parity ('+' or '−') of the coordinates' entries. With the help of Eq. (17) we can discover how these three sets, and elements within them, are related to each other.

To get the idea about the general structure of the line, we first consider the nucleus and find out that these three mutually distant points have the following neighbourhoods ($i = 1, 2, 3$):

$$U: \ \{I_i^S,\ I_i^F,\ S_i^+,\ S_i^-,\ F_i^+,\ F_i^-\}, \tag{21}$$

$$V: \ \{J_i^S,\ J_i^F,\ S_i^+,\ S_i^-,\ F_i^+,\ F_i^-\}, \tag{22}$$

$$W: \ \{I_i^S,\ I_i^F,\ J_i^S,\ J_i^F,\ F_i^+,\ F_i^-\}. \tag{23}$$

Now, as the coordinate system on this line can *always* be chosen in such a way that the coordinates of *any* three mutually distant points are made identical to those of U, V and W, from the last three expressions we discern that the neighbourhood of any point of the line features eighteen distinct points, the neighbourhoods of any two distant points share twelve points and the neighbourhoods of any three mutually distant points have six points in common — as depicted in Fig. 1. As in the case of the lines defined over $\mathrm{GF}(2)[x]/\langle x^3 - x\rangle$ and $\mathrm{GF}(2) \otimes \mathrm{GF}(2)$ [9], the neighbour relation is not transitive; however, a novel feature, not encountered in the previous cases, is here a non-zero overlapping between the neighbourhoods of *three* pairwise distant points, which can be attributed to the existence of three maximal ideals of $R_{\triangle}$.

Table 1: The neighbour/distant relation between the points of the inner shell of the projective line $PR_{\triangle}(1)$.

	I_1^S	I_2^S	I_3^S	I_1^F	I_2^F	I_3^F	J_1^S	J_2^S	J_3^S	J_1^F	J_2^F	J_3^F
I_1^S	−	−	−	−	−	+	−	+	+	−	−	+
I_2^S	−	−	−	−	+	−	+	−	+	−	+	−
I_3^S	−	−	−	+	−	−	+	+	−	+	−	−
I_1^F	−	−	+	−	−	−	−	−	+	−	−	−
I_2^F	−	+	−	−	−	−	−	+	−	−	−	−
I_3^F	+	−	−	−	−	−	+	−	−	−	−	−
J_1^S	−	+	+	−	−	+	−	−	−	−	−	+
J_2^S	+	−	+	−	+	−	−	−	−	−	+	−
J_3^S	+	+	−	+	−	−	−	−	−	+	−	−
J_1^F	−	−	+	−	−	−	−	−	+	−	−	−
J_2^F	−	+	−	−	−	−	−	+	−	−	−	−
J_3^F	+	−	−	−	−	−	+	−	−	−	−	−

Table 2: The neighbour/distant relation between the points of the outer shell of the projective line $PR_{\triangle}(1)$.

	F_1^+	F_2^+	F_3^+	F_1^-	F_2^-	F_3^-	S_1^+	S_2^+	S_3^+	S_1^-	S_2^-	S_3^-
F_1^+	−	−	+	−	+	−	−	+	−	+	−	−
F_2^+	−	−	−	+	−	+	−	−	+	+	−	−
F_3^+	+	−	−	−	+	−	−	−	+	−	+	−
F_1^-	−	+	−	−	−	+	+	−	−	−	+	−
F_2^-	+	−	+	−	−	−	+	−	−	−	−	+
F_3^-	−	+	−	+	−	−	−	+	−	−	−	+
S_1^+	−	−	−	+	+	−	−	−	−	+	−	−
S_2^+	+	−	−	−	−	+	−	−	−	−	+	−
S_3^+	−	+	+	−	−	−	−	−	−	−	−	+
S_1^-	+	+	−	−	−	−	+	−	−	−	−	−
S_2^-	−	−	+	+	−	−	−	+	−	−	−	−
S_3^-	−	−	−	−	+	+	−	−	+	−	−	−

The intricacies of the geometry are fully revealed if one examines the neighbour/distant relation between the points of the shells as summarized in the following tables (Tables 1–3), where the sign '$+/-$' means, respectively, distant/neighbour; thus, for example, from the first table we read off that the points J_1^S and I_3^F are distant, whilst from the third table we find out that F_2^+ and J_2^S are neighbour. We readily see that the inner shell exhibits a more pronounced asymmetry in the coupling between the individual elements than the outer one, as it is also obvious from the comparison of the two parts of Fig. 2. Further, as per the interconnection between the two shells, one finds out that whereas the 'fat' points of the outer shell are symmetrically (two-and-two) coupled with the points of the two sets of the inner shell, the 'slim' points show a strong (three-and-one) asymmetry in this respect. Noteworthy is also a very intricate character of the coupling between the 'fat' points of the outer shell, which sharply contrasts the triviality of the corresponding coupling in the inner shell. To complete the picture, we also give the table (Table 4) showing the connection between the nucleus and the shells. There are obviously a number of other interesting sub-configurations of the line, like a seven-point configuration comprising a point of the nucleus and one of the sets of either shell and a fifteen-point configuration featuring the nucleus and a shell.

Table 3: The neighbour/distant relation between the points of the two shells of the projective line $PR_{\triangle}(1)$.

	I_1^S	I_2^S	I_3^S	I_1^F	I_2^F	I_3^F	J_1^S	J_2^S	J_3^S	J_1^F	J_2^F	J_3^F
F_1^+	−	+	−	−	−	+	−	−	+	−	−	+
F_2^+	+	−	−	−	+	−	−	−	+	−	+	−
F_3^+	+	−	−	+	−	−	−	+	−	+	−	−
F_1^-	−	−	+	−	−	+	−	+	−	−	−	+
F_2^-	−	−	+	−	+	−	+	−	−	−	+	−
F_3^-	−	+	−	+	−	−	+	−	−	+	−	−
S_1^+	−	−	−	+	−	−	−	−	+	−	+	+
S_2^+	−	−	−	−	+	−	−	+	−	+	−	+
S_3^+	−	−	−	−	−	+	+	−	−	+	+	−
S_1^-	−	−	+	−	+	+	−	−	−	+	−	−
S_2^-	−	+	−	+	−	+	−	−	−	−	+	−
S_3^-	+	−	−	+	+	−	−	−	−	−	−	+

Table 4: The neighbour/distant relation between the points of the nucleus and those of the inner/outer shell of the projective line $PR_{\triangle}(1)$.

	I_1^S	I_2^S	I_3^S	I_1^F	I_2^F	I_3^F	J_1^S	J_2^S	J_3^S	J_1^F	J_2^F	J_3^F
U	−	−	−	−	−	−	+	+	+	+	+	+
V	+	+	+	+	+	+	−	−	−	−	−	−
W	−	−	−	−	−	−	−	−	−	−	−	−

	F_1^+	F_2^+	F_3^+	F_1^-	F_2^-	F_3^-	S_1^+	S_2^+	S_3^+	S_1^-	S_2^-	S_3^-
U	−	−	−	−	−	−	−	−	−	−	−	−
V	−	−	−	−	−	−	−	−	−	−	−	−
W	−	−	−	−	−	−	+	+	+	+	+	+

5 Conclusion

We have carried out an in-depth examination of the structure of the projective line $PR_{\triangle}(1)$. The line was shown to be endowed, from the algebraic point of view, with three distinct 'layers'; the nucleus comprising the ordinary projective (sub)line of order two, the inner shell made of points whose coordinates feature both the unity and a zero-divisor and the outer shell consisting of the points whose coordinates have zero-divisors in both the entries. The intricacies of the structure of the line were fully revealed by employing the concepts of neighbour/distant, as summarized in Tables 1 to 4 and partially illustrated in Fig. 2. We believe that this remarkable finite ring geometry — and its multifaceted sub-configurations — will soon find interesting applications in quantum physics (to address, for example, the properties of finite-dimensional quantum systems following and properly generalizing the strategy pursued in [16]), chemistry (when dealing with highly-complex systems of molecules), particle physics, crystallography, biology and other natural sciences as well.

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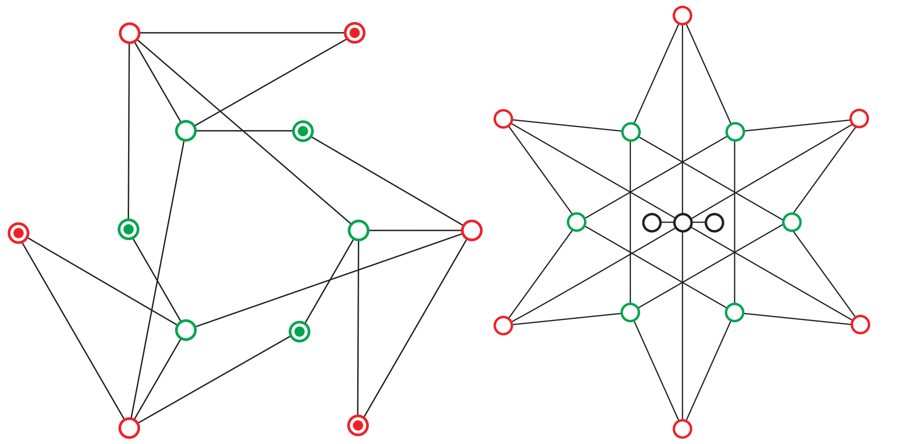

Figure 2: A schematic illustration of the fine structure of the inner (*left*) and outer (*right*) shells of the projective line $PR_{\triangle}(1)$, where any two distant points are joined by a line segment. In both the cases, the two sets of points are distinguished by different colours; in the former case, in addition, the filled circles denote the 'fat' points and in the latter case also the nucleus (three points in the center, the middle one being W) and its relation to the shell in question are shown.

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