

On Computing the Entropy of Braids

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Abstract. We consider in this paper the problem of computing the entropy of a braid. We recall its definition and construct, for each braid, a sequence of real numbers, whose limit is the braid's entropy. We state one conjecture about the convergence speed, and two about the braids that have high entropy, but are written with few letters.

1. Introduction

The formal definition of braid's entropy, as introduced by P. Boyland in [4], is a very natural extension of results presented in [8] about dynamical properties of pseudo-Anosov maps.

In this paper we present a method for computing the entropy of braids without working with train tracks, see [2]. More precisely, we construct for every braid β , an integer sequence c_m whose growth factor is the entropy of β . The major drawback of this method is that it is not an algorithm since we don't get the result in finitely many steps: we have to use an artificial stopping criterion. The major advantage over train-tracks is that it works faster.

The paper is organized as follows. In the first section we quote some results from [8] that justify Boyland's definition and will be used as technical components of proofs. Next section is the central technical part of the paper. Informally, we connect Boyland's definition with Dynnikov and Wiest braid complexity, see [7]. More precisely, we construct for each surface diffeomorphism φ , an integer sequence c_m such that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log c_m = h_{\text{top}}(\varphi),$$

where $h_{\text{top}}(\varphi)$ stands for the topological entropy of φ , see [10] for definitions and properties. The following section gives the formal definition of the braid entropy, introduces integral laminations and their coding as they appear in [6], and finally presents a formula that gives the minimum number of intersections of an integral lamination with the real axis in terms of Dynnikov's coordinates. At this point we describe the method for the computation of

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braid entropies, say a few words about the corresponding computer program and conjecture it might be turned into an algorithm. Last section presents some conjectures about the braids that are written with few letters but have large entropy.

2. Some results on surface diffeomorphisms

One can find in [8] a complete proof of Thurston's results about the classification of surface diffeomorphisms. But the authors also prove many propositions and lemmas that help to understand Thurston's result and that we shall need. Let us quote some of them.

Let M be a compact oriented surface possibly with boundary, then $\mathcal{S}(M)$ will denote the set of homotopy classes of closed and connected simple paths that are not homotopic to zero or to a component of the boundary of M .

If (\mathcal{F}, μ) is a measured foliation of M , see [8] for a definition, and γ a closed and connected simple path then

$$\int_{\gamma} |\mu|$$

will denote the total variation of the differential form μ along γ . If $\alpha \in \mathcal{S}(M)$, let

$$\mathcal{I}(\mathcal{F}, \mu, \alpha) = \inf_{\gamma \in \alpha} \int_{\gamma} |\mu|.$$

Proposition 2.1 – *Let φ be a pseudo-Anosov map with stable and unstable foliations (\mathcal{F}^s, μ^s) and (\mathcal{F}^u, μ^u) . Let $\alpha \in \mathcal{S}(M)$, then*

$$\mathcal{I}(\mathcal{F}^s, \mu^s, \alpha) > 0 \quad \text{and} \quad \mathcal{I}(\mathcal{F}^u, \mu^u, \alpha) > 0.$$

For any two $\alpha, \beta \in \mathcal{S}(M)$, $c(\alpha, \beta)$ will denote the minimum number intersections of a and b , with $a \in \alpha$ and $b \in \beta$.

Let M be a compact oriented surface, A and B two subsets of M . We'll note $\text{Diff}(M)$ the group of diffeomorphisms of M , $\text{Diff}(M, \text{rel } A)$ for the subgroup of $\text{Diff}(M)$ whose elements leave A invariant, and finally $\text{Diff}(M, \text{rel } A, B)$ for the subgroup of $\text{Diff}(M, \text{rel } A)$ whose elements are the identity on B .

If G is a subgroup of $\text{Diff}(M)$ (such as $\text{Diff}(M, \text{rel } A)$), $\varphi, \psi \in G$, we'll write $\varphi \sim_G \psi$ or simply $\varphi \sim \psi$ when φ and ψ are isotopic in G .

Proposition 2.2 – *Let φ be a pseudo-Anosov map with stable and unstable foliations (\mathcal{F}^s, μ^s) , (\mathcal{F}^u, μ^u) and let λ be a real number, $\lambda > 1$, be such that*

$\varphi(\mathcal{F}^s) = 1/\lambda \mathcal{F}^s$ and $\varphi(\mathcal{F}^u) = \lambda \mathcal{F}^u$. Then

$$(1) \quad \lim_{n \rightarrow \infty} \frac{c(\varphi^n \alpha, \beta)}{\lambda^n} = \mathcal{S}(\mathcal{F}^s, \mu^s, \alpha) \mathcal{S}(\mathcal{F}^u, \mu^u, \alpha),$$

$$(2) \quad h_{top}(\varphi) = \log \lambda,$$

$$(3) \quad h_{top}(\varphi) = \inf\{h_{top}(\psi), \psi \in Diff(M), \psi \sim \varphi\}.$$

The following result is a formulation of Thurston's result about the classification of surface diffeomorphisms.

Theorem 2.1 – *Let M be a compact oriented surface possibly with boundary, and φ a diffeomorphism of M . There exists finitely many simple curves of M : C_1, \dots, C_l and a diffeomorphism ψ isotopic to φ such that cutting M along C_1, \dots, C_l produces M_1, \dots, M_k compact, possibly non connected, oriented surfaces satisfying*

- (i) $M = M_1 \cup \dots \cup M_k$;
- (ii) if $j_1 \neq j_2$, $M_{j_1} \cap M_{j_2}$ is the union of several C_i ;
- (iii) φ is isotopic to ψ ;
- (iv) for $i = 1, \dots, l$, $\psi(M_i) = M_i$;
- (v) for $i = 1, \dots, l$, $\psi|_{M_i}$ is periodic or pseudo-Anosov.

Example 2.1 – Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus, and φ be the Dehn twist given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this case the simple closed curve $C_1 = [(0, 0), (1, 0)]$ decomposes M into C_1 and a closed cylinder, M_1 . The map $\varphi|_{M_1}$ is isotopic to the identity.

3. Counting intersections

Proposition 2.2 already suggests a way of computing the topological entropy of a pseudo-Anosov maps φ of a compact oriented surface M : pick α and β in $\mathcal{S}(M)$, and compute

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log c(\varphi^m \alpha, \beta).$$

To do so, one would have to choose some α and β , and devise a method for the computation of $c(\varphi^m \alpha, \beta)$.

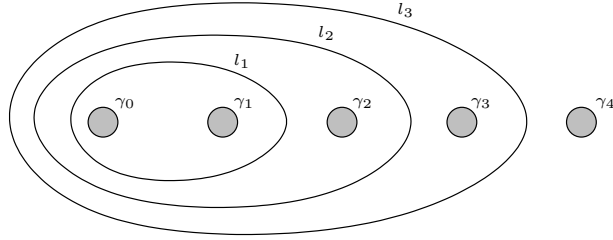


Figure 3.1: A sphere with $n = 6$ open discs removed is represented on the plane using a stereographic projection. The disc γ_5 containing the projection's pole is omitted. In this case the set of curves L is made of 3 disjoint simple closed curves: l_1, l_2, l_3 .

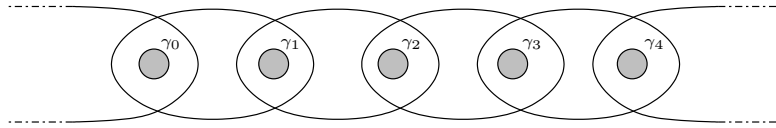


Figure 3.2: The set R corresponding to a sphere with six holes. Here it contains 6 simple closed curves.

To go on with this idea, we first restrict ourselves to surfaces M that are homeomorphic to a sphere with finitely many open discs removed. The boundaries of removed open discs will usually be denoted $\gamma_0, \dots, \gamma_n$, see figure 3.1 or 3.2. We'll usually represent such surfaces on the plane using the stereographic projection with respect to a pole belonging to one of the removed discs. The boundary of the disc containing the pole will usually not be drawn.

We'll consider in M two sets of curves, up to homotopy. The first set L is represented on figure 3.1, and the second set, R , on figure 3.2.

If A and B are two finite subsets of $\mathcal{S}(M)$, $c(A, B)$ will denote

$$c(A, B) = \sum_{a_i \in A, b_j \in B} c(a_i, b_j).$$

Proposition 3.1 – *Let M be a sphere with $n \geq 3$ discs removed, let φ be a diffeomorphism of M and let L and R be the two subsets of $\mathcal{S}(M)$ represented on figure 3.1 and 3.2. Then,*

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log c(\varphi^m L, R) = h_{\text{top}}(\varphi).$$

Proof. Let $\gamma_0, \dots, \gamma_n$ denote the boundaries of the removed discs, and let ψ, M_1, \dots, M_k be the diffeomorphism and the surfaces given by 2.1. Then,

let

$$I_j = \{i \in \{0, \dots, n\}, \gamma_i \subset M_j\}.$$

The decomposition $M = M_1 \cup \dots \cup M_k$ induces a decomposition of $l \in \mathcal{S}(M)$ into finitely many pieces – we consider sufficiently regular elements in l and assume M_j have smooth boundaries with finitely many singularities.

For each component of l that belongs to M_j , we form a new simple closed curve l' with that component and connect its ends following the boundary of M_j . There are two choices for this. If there are more than three γ_i inside M_j we choose l' such that $l' \in \mathcal{S}(M_j)$. We shall denote l_{I_j} the union of all these curves, see figure 3.3.

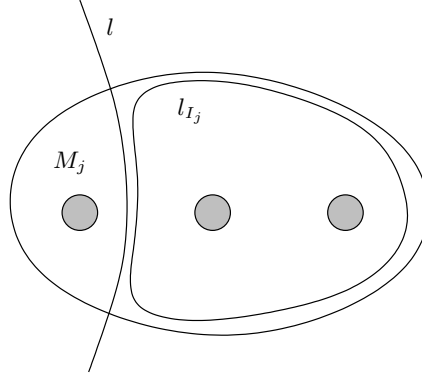


Figure 3.3: Construction of l_{I_j} from l . If there are more than three holes in M_j , we construct l_{I_j} in such a way that it belongs to $\mathcal{S}(M_j)$.

We'll also form two sets of curves R_{I_j} and R'_{I_j} from R and M_j . First, for $i, j \in 0, \dots, n$, let $r_{i,j}$ be the simple closed curve formed with the boundaries of γ_i and γ_j . Figure 3.2 represents for instance, from left to right, $r_{5,0}$, $r_{0,1}$, $r_{1,2}$, $r_{2,3}$, $r_{3,4}$ and $r_{4,5}$. Let

$$I_j = \{i_1, i_2, \dots, i_p\}, \quad \text{with } i_1 < i_2 < \dots < i_p,$$

then,

$$R_{I_j} = r_{i_1, i_2} \cup r_{i_2, i_3} \dots \cup r_{i_{p-1}, i_p} \cup r_{i_p, i_1}$$

and

$$R'_{I_j} = \bigcup_{\substack{\gamma_i \text{ or } \gamma_{i+1} \text{ is inside} \\ M_j \text{ but not both.}}} r_{i, i+1}.$$

This construction is represented on figure 3.4. For any $l \in \mathcal{S}(M)$, $c(l, R)$

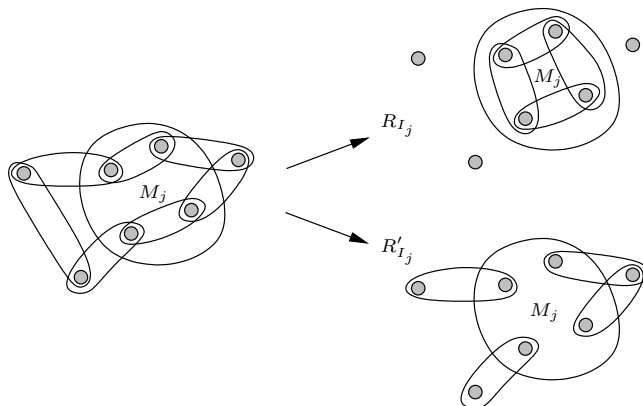


Figure 3.4: Construction of R_{I_j} and R'_{I_j} from R and M_j . Each γ_i is represented.

may be written as follows

$$c(l, R) = \sum_{\substack{l_{I_j} \in \mathcal{S}(M_j) \\ \#I_j \geq 2}} c(l_{I_j}, R_{I_j}) + \sum_j c(l, R'_{I_j}),$$

where $\#I_j$ is the cardinal of I_j . For any $l \in \mathcal{S}(M)$, the sequence $c(\psi^m l, R'_{I_j})$

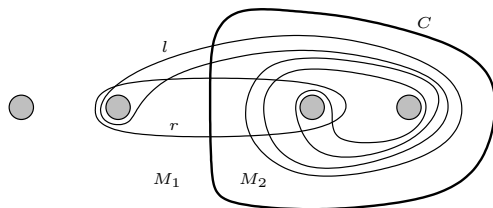


Figure 3.5: A Dehn twist δ in M_2 along C produces a twist of l , and $c(\delta^m l, r)$ grows linearly with m .

cannot grow faster than linearly with m because only Dehn twists along M_j 's boundaries can make it grow, as illustrated on figure 3.5, hence

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log c(\psi^m l, R) = \max_{l_{I_j} \in \mathcal{S}(M_j), \#I_j \geq 2} \lim_{m \rightarrow \infty} \frac{1}{m} \log c(\varphi^m l_{I_j}, R_{I_j}).$$

Remark that if $\#I_j \leq 2$, then

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log c(\varphi^m l_{I_j}, R_{I_j}) = 0.$$

Let h_j be the topological entropy of $\psi|_{M_j}$, using proposition 2.2 we get

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log c(\psi^m l, R) = \max_{l_{I_j} \in \mathcal{S}(M_j), \#I_j \geq 2} h_j.$$

If $\#I_j \geq 3$, there exists $l \in L$ such that $l_{I_j} \in \mathcal{S}(M_j)$, hence

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{1}{m} \log c(\psi^m L, R) &= \max_{\#I_j \geq 3} h_j \\ &= \max_{j \in \{1, \dots, k\}} h_j \end{aligned}$$

This last maximum is precisely the topological entropy of ψ , and consequently, the topological entropy of φ . \square

4. The entropy of braids

4.1 Definitions

The braid group can be defined in many ways, see [3] for instance. Here, we shall use a definition that shows the connection with surface diffeomorphism.

Definition 4.1 – *Let n be an integer, $n \geq 2$, and M the surface obtained from the unit disc D^2 with n disjoint open discs removed; we'll write Γ_n the union of their boundaries. The braid group with n strands is defined as*

$$Br(n) = \text{Diff}(D^2, \text{rel } \Gamma_n, \partial D^2) / \sim$$

Definition 4.2 – *Let $\beta \in Br(n)$. The entropy of β is defined as*

$$h(\beta) = \inf_{\varphi \in \beta} h_{\text{top}}(\varphi).$$

Proposition 2.2 should make this definition rather clear.

4.2 Integral laminations

We already came across integral laminations in proposition 3.1. Here is a more formal definition.

Definition 4.3 – *Let M be a compact and oriented surface. An integral lamination of M is a set L of disjoint non homotopic simple closed curves of M . Integral laminations are considered up to homotopy. The set of integral laminations of M will be denoted $\mathcal{L}(M)$.*

I. Dynnikov introduced in [6] a coding for integral laminations when M_n is a sphere with $n + 3$ open discs removed. We represented such a surface

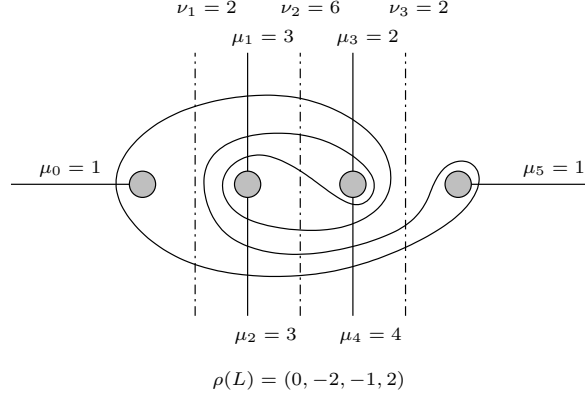


Figure 4.1: Dynnikov's coding of an integral lamination L of M_n , with $n = 2$. There are 8 intersections between L and the real axis.

on figure 4.1. Following our convention one component of M 's boundary is not represented. We consider $2n + 2$ half lines (continuous lines) and $n + 1$ vertical lines (dotted lines) as represented on figure 4.1. We consider ν_1, \dots, ν_n the minimum number of intersection of L with each dotted line, and $\mu_0, \mu_1, \mu_2, \dots, \mu_{2n-1}, \mu_{2n}$, the minimum number of intersection of L with the half lines. Now for $i = 1, \dots, n$, let

$$a_i = \frac{\mu_{2i} - \mu_{2i-1}}{2}$$

$$b_i = \frac{\nu_i - \nu_{i+1}}{2}.$$

and

$$\rho(L) = (a_1, b_1, \dots, a_n, b_n).$$

The following result appears in [6].

Proposition 4.1 – *Let n be an integer, $n \geq 2$. The map ρ defines a bijection between $\mathcal{L}(M_n)$ and \mathbb{Z}^{2n} .*

Example 4.1 – For the integral lamination L represented on figure 3.1, $\rho(L) = (0, 1, 0, 1, 0, 1)$. We'll denote L_0^n the integral lamination whose coding is $(0, 1, 0, 1, \dots, 0, 1)$.

We need to adapt slightly the definition we gave earlier for the braid group.

Proposition 4.2 – *Let n be an integer, $n \geq 2$, M_n the sphere with $n + 3$ open discs removed. Let $\gamma_0, \dots, \gamma_{n+2}$ denote their boundaries, then,*

$$Br(n) = \text{Diff}(M, \text{rel } \gamma_1 \cup \dots \cup \gamma_n, \gamma_0 \cup \gamma_{n+1} \cup \gamma_{n+2}) / \sim .$$

Using this definition, we choose the $n - 1$ generators of $\text{Br}(n)$ to be the diffeomorphisms whose action is depicted on 4.2. One checks easily that these generators satisfy the usual relations, see [8] for instance. The action

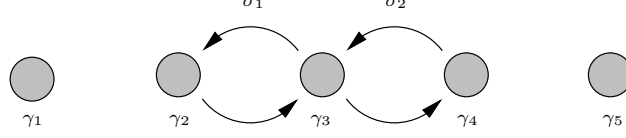


Figure 4.2: Here $n = 3$. The border of one disc is not represented. The braid group with 3 strands acts on M_n . We illustrated the actions of the generators σ_1 and σ_2 .

of $\text{Br}(n)$ on $\mathcal{L}(M_n)$ is coded using I. Dynnikov's formulae. For any real number a , we'll write

$$a^+ = \max(a, 0) \quad a^- = \min(a, 0).$$

Proposition 4.3 – *Let n be an integer $n \geq 2$, and $L \in \mathcal{L}(M_n)$ with $\rho(M) = (a_1, b_1, \dots, a_n, b_n)$. For each integer i , $1 \leq i \leq n - 1$, let*

$$\begin{aligned} \rho(\sigma_i L) &= (a'_1, b'_1, \dots, a'_n, b'_n) \\ \rho(\sigma_i^{-1} L) &= (a''_1, b''_1, \dots, a''_n, b''_n) \\ c &= a_i - b_i^- - a_{i+1} + b_{i+1}^+ \\ d &= a_i + b_i^- - a_{i+1} - b_{i+1}^+. \end{aligned}$$

Then,

$$\begin{aligned} a'_j &= a_j, \quad b'_j = b_j \quad \text{for } j \neq i, i+1 \\ \left\{ \begin{array}{l} a'_i = a_i + b_i^+ + (b_{i+1}^+ - c)^+ \\ b'_i = b_{i+1} - c^+ \\ a'_{i+1} = a_{i+1} + b_{i+1}^- + (b_i^- + c)^- \\ b'_{i+1} = b_i + c^+ \end{array} \right. \end{aligned}$$

and

$$\begin{aligned} a''_j &= a_j, \quad b''_j = b_j \quad \text{for } j \neq i, i+1 \\ \left\{ \begin{array}{l} a''_i = a_i - b_i^+ - (b_{i+1}^+ + d)^+ \\ b''_i = b_{i+1} + d^- \\ a''_{i+1} = a_{i+1} - b_{i+1}^- - (b_i^- - d)^- \\ b''_{i+1} = b_i - d^-. \end{array} \right. \end{aligned}$$

We may use these formulas to count the minimum number of intersection of an integral lamination with the real axis.

Proposition 4.4 – Let n be an integer, $n \geq 2$, $L \in \mathcal{L}(M_n)$ and $c(L)$ denote the minimum number of intersections between L and the real axis. If $\rho(L) = (a_1, b_1, \dots, a_n, b_n)$, then

$$c(L) = \sum_{i=1}^n |b_i| + \sum_{i=1}^{n-1} |a_{i+1} - a_i| + |a_1| + |a_n| + \nu_1/2 + \nu_n/2.$$

Proof. Four types of intersections may occur as represented on figure 4.1. Counting them results in the formula for $c(L)$. \square

If R is the set of curves we introduced earlier, see 3.2, and L the integral lamination represented on figure 3.1, then

$$c(L, R) = 2c(L).$$

This leads to the description of a method for the estimation of braids entropy. Let n be an integer, $n \geq 1$, $\beta \in \text{Br}(n)$, recall that L_0^n is the lamination whose coordinate is $(0, 1, 0, 1, \dots, 0, 1)$, and choose $\varepsilon > 0$.

- ① Write β as a word using standard generators $\sigma_1, \dots, \sigma_{n-1}$.

$$\beta = \prod \sigma_{i_k}^{a_{i_k}}, \quad a_{i_k} \in \mathbb{Z}.$$

- ② For $m = 1, 2, \dots$, compute $\rho(\beta^m L_0^n)$ and $c_m = 1/m \log c(\beta^m L_0^n)$ using Dynnikov's formulae, proposition 4.4 and forgetting about $\nu_1/2$ and $\nu_2/2$ since they don't change.

- ③ Stop when $|c_{m+1} - c_m| < \varepsilon$.

As we said this method is not an algorithm, nevertheless the corresponding computer program behaves well. We recover the well-known topological entropy of $\beta = \sigma_1 \sigma_2^{-1} \in \text{Br}(3)$

$$h(\beta) = \log \frac{3 + \sqrt{5}}{2} \simeq 0.962$$

Moreover the sequences c_m seem to decrease like $\log m/m$. We have the following conjecture.

Conjecture 4.1 – Let n be an integer, $n \geq 2$. There exists a positive constant $C_n \in \mathbb{R}$ such that for any braid $\beta \in \text{Br}(n)$ and its corresponding sequence $(c_m)_{m \geq 0}$,

$$|c_m - h(\beta)| \leq C_n \frac{\log m}{m}.$$

This last conjecture, if it was correct, would turn our method into an algorithm for the computation of $h(\beta)$ with precision ε .

Finally, notice that we consider integer sequences $c(\beta^m L_0^n)$ that grow rapidly, like geometric ones. We had to use a special library for handling large integers. We used NTL, see [12]. Our program is freely available at <http://www.ceremade.dauphine.fr/~msfr>.

5. Braids with maximum entropy

We were primarily interested in braids that have large entropy, but are written with few letters.

Definition 5.1 – Let n be an integer $n \geq 2$, and $Br(n)$ the braid group with n strands and standard generators $\sigma_1, \dots, \sigma_{n-1}$. If $\beta \in Br(n)$, the length of β , $l(\beta)$ is the minimum number of σ_i needed to write β .

Definition 5.2 – We say that a braid β has maximal entropy if

$$h(\beta) = \max\{h(\beta'), l(\beta') \leq l(\beta)\}.$$

Notice here that we do not refer to the number of strands. Using the program we got the two following conjectures.

Conjecture 5.1 – Braids of maximal entropy belong to $Br(3)$ or $Br(4)$.

Definition 5.3 – Let n be an integer $n \geq 2$, and $\beta \in Br(n)$. If

$$\beta = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_k}^{\epsilon_k}$$

with

$$i_{k+1} = i_k \pm 1, |\epsilon_k| = 1, \epsilon_{k+1} = -\epsilon_k,$$

then β is said to be alternated (with respect with the standard generators).

Conjecture 5.2 – Braids with maximal entropy are alternated.

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References

- [1] V.I. Arnol'd and A. Avez. *Problèmes ergodiques de la mécanique classique*. Gauthier-Villars, 1967.
- [2] M. Bestvina and M. Handel. Train-tracks for surface homeomorphisms. *Topology*, 34(1):109–140, 1995.
- [3] J. Birman. *Braids, links, and mapping class groups*. Annals of Mathematics Studies. Princeton University Press and University of Tokyo Press., 1975.

- [4] P. Boyland. Topological methods in surface dynamics. *Topology Appl.*, 58(3):223–298, 1994.
- [5] P. Boyland, H. Aref, and M. Stremler. Topological fluid mechanics of stirring. *J. Fluid Mech.*, 403:277–304, 2000.
- [6] I.A. Dynnikov. On a Yang-Baxter mapping and the Dehornoy ordering. *Russ. Math. Surv.*, 57(3):592–594, 2002.
- [7] I.A. Dynnikov and B. Wiest. On the complexity of braids. Paper available on line at <http://hal.ccsd.cnrs.fr/>, 2004.
- [8] A. Fathi, F. Laudenbach, and V. Poenaru. *Travaux de Thurston sur les surfaces. Séminaire Orsay. 2nd ed.*, volume 66-67 of *Astérisque*. Centre National de la Recherche Scientifique, 1991.
- [9] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math., Inst. Hautes Étud. Sci.*, 1980.
- [10] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*. Cambridge University Press, 1995.
- [11] B. Kolev. Entropie topologique et représentation de Burau. *C. R. Acad. Sci., Paris, Sér. I*, 309(13), 1989.
- [12] V. Shoup. Ntl: A library for doing number theory. The library is available at <http://www.shoup.net>, 2001.
- [13] W. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Am. Math. Soc., New Ser.*, 19(2):417–431, 1988.