Stability in Discrete Tomography: Some Positive Results

Sara Brunetti [a] and Alain Daurat [b]

[a] Dipartimento di Scienze Matematiche e Informatiche, Università di Siena, Via del Capitano 15, 53100, Siena, Italy

[b] LSIIT UMR 7005 CNRS-ULP, Pôle API, Boulevard Sébastien Brant, 67400 Illkirch-Graffenstaden, France

Abstract

The problem of reconstructing finite subsets of the integer lattice from X-rays has been studied in discrete mathematics and applied in several fields like image processing, data security, electron microscopy. In this paper we focus on the stability of the reconstruction problem for some special lattice sets. First we prove that if the sets are additive, then a stability result holds for very small errors. Then, we study the stability of reconstructing convex sets from both an experimental and a theoretical point of view. Numerical experiments are conducted by using linear programming that support the conjecture that convex sets are additive with respect to a set of suitable directions, and consequently the reconstruction problem is stable. The theoretical investigation provides a stability result for lattice sets. It is used to prove the following property: if a sequence of lattice convex sets have X-rays in suitable directions which converge to X-rays of a convex body, then it converges to this convex body.

Key words: Discrete Tomography, Stability, Linear Programming, Additivity, Convexity

1 Introduction

A *lattice set* is a non-empty finite subset of the integer lattice $\mathbb{Z}^2$. A *lattice direction* is a direction directed by a vector in $\mathbb{Z}^2 \setminus \{0\}$, and it can also be given by an equation $p(x, y) = ax + by$ with $a, b \in \mathbb{Z}$. Further, the *X-ray* of a lattice

Email addresses: sara.brunetti@unisi.it (Sara Brunetti), daurat@dpt-info.u-strasbg.fr (Alain Daurat).

set E in a lattice direction p is the function X_pE giving the number of points in E on each line parallel to this direction, formally $X_pE(k) = \text{card}(\{M \in E : p(M) = k\})$. Discrete Tomography is the area of mathematics and computer science that deals with the inverse problem of reconstructing lattice sets from a finite set of X-rays. An overview on this subject highlighting the applications, the mathematical foundations, and the algorithms in Discrete Tomography is provided by the book [16].

In this paper we focus on the stability of the reconstruction problem.

Informally, a problem is stable if a small perturbation of the data does not change the corresponding solutions too much. Therefore, the stability problem is of main importance in practical applications where the X-rays are possibly affected by errors. For instance, in electron microscopy, techniques that enable to count the number of atoms lying in a line up to an error of ± 1 are known [14]. So, in case of instability, the reconstructed set can be quite different from the original one even if the error on the data is small. In [1] the authors prove that when $m > 2$, the two sets can be even disjoint, permitting an error of $2(m-1)$ on the X-rays. First, we show in Remark 6 that to obtain a stability result even with a very small error on the data the requirement of uniqueness for the sets is not enough. To this goal, we shall consider the reconstruction of lattice sets with some additional constraints.

In Section 3 we treat the stability of reconstructing additive sets. This class of sets was first introduced by Fishburn et al. in [10]. Here we just recall to the reader that additivity implies uniqueness, whereas the converse is not true. Additionally, the notion of additivity should be regarded as a property of the solutions of the linear program associated to the reconstruction problem. We prove that if the sets are additive, then a stability result holds (Proposition 7).

In Section 4 we study the stability of reconstructing convex sets from both an experimental and a theoretical point of view. In the former, we use linear programming to deal with this problem. Experimental results suggest the conjecture that for the set of directions $\{x, y, 2x + y, -x + 2y\}$, convex sets are additive. This would imply that the results of Section 2 may hold to convex sets so giving a stability result that corresponds to the continuous case where the reconstruction problem for convex bodies is well-posed ([22]). In the latter, the theoretical result (Proposition 18) confirms stability for convex sets by exploiting the result in [22]. The last proposition of this paper allows to use Discrete Tomography to solve Hammer's X-ray problem which is reconstructing a continuous convex body from continuous X-rays. More precisely we prove that a convex body is arbitrarily near lattice convex sets on the condition that the X-rays of the lattice convex sets are near enough the ones of the convex body. In practice it means that we can reconstruct a convex body from discrete X-rays to any precision if the resolution and the error about the X-ray can be as small as wanted. This result is very linked to other tries to approximate the convex body from a sequence of discrete objects reconstructed from the X-rays (see [19]).

2 The Problem

The reconstruction problem is the task of determining any lattice set having the given X-rays. Stability concerns how sensitive is the problem to noisy data. Hence one can ask whether a perturbation of the data correspond solutions that are close. To study the problem we define a measure for the error on the X-rays and one for the distance of two solutions. Let $\mathcal{D}$ be a set of m prescribed lattice directions with $m \geq 2$ and let E, F be lattice sets:

$$DX_{\mathcal{D}}(E,F) = \max_{p \in \mathcal{D}} \sum_{k \in \mathbb{Z}} |X_pE(k) - X_pF(k)|$$

and

$$\mathrm{card}(E \triangle F) = \mathrm{card}((E \setminus F) \cup (F \setminus E)).$$

The formulation of the problem that we consider is the following:

Problem 1 *Let E be known. Determine F maximizing* $\mathrm{card}(E \triangle F)$*, with the constraint that $DX_{\mathcal{D}}(E,F)$ is given.*

Let us introduce some definitions that we need in the following.

Definition 2 *A lattice set E is* additive with respect to $\mathcal{D}$*, or* $\mathcal{D}$-additive*, if there is a function e which gives a value $e_p(k)$ for each line $p = k$ parallel to a direction p of $\mathcal{D}$ such that for all M in $\mathbb{Z}^2$:*

$$M \in E \textit{ if and only if } \sum_{p \in \mathcal{D}} e_p(p(M)) > 0.$$

This definition introduced by Fishburn et al. can be better understood with linear programming: a lattice set E is additive if it is the unique solution of the linear programming problem which looks for a fuzzy set which has the same X-rays than E.

Definition 3 *A lattice set E is* unique with respect to $\mathcal{D}$*, or* $\mathcal{D}$-unique*, if $F \subset \mathbb{Z}^2$ and $X_pE = X_pF$ for any $p \in \mathcal{D}$ imply $E = F$.*

There is an intimate relationship between these two definitions: every $\mathcal{D}$-additive set is $\mathcal{D}$-unique and the converse is true if $m = 2$ (see[10]).

As a last remark we recall that if p and q are two directions, then a p-line does not always intersect a q-line. Indeed $\mathbb{Z}^2$ can be split in $\det(p,q)$ pq-lattice such that in each pq-lattice a p-line intersects with any q-line. Precisely a pq-lattice has the form:

$$L_i^{pq} = \{M \ : \ p(M) = i \pmod{\det(p,q)} \text{ and } q(M) = \kappa i \pmod{\det(p,q)}\}$$

where κ only depends on the directions p and q (see for example [7]). Moreover we denote by $\langle i, j\rangle_{pq}$ the point M such that $p(M) = i$ and $q(M) = j$. Notice that this point is in $\mathbb{Z}^2$ only if $p = i$ and $q = j$ are in the same pq-lattice.

3 Stability for Additive Sets

In this section we study the stability of reconstructing $\mathcal{D}$-additive sets. We begin to study Problem 1 with E and F verifying the constraint $DX_{\mathcal{D}}(E,F) \leq 1$.

In the first two lemmas additivity is not required.

The condition $DX_{\mathcal{D}}(E,F) \leq 1$ permits the X-rays of the two sets to differ by one in at most a line for each direction. Then, $p \in \mathcal{D}$ and an integer k_p exists such that $|X_pE(k_p) - X_pF(k_p)| = 1$ and $X_pE(k) = X_pF(k)$ for $k \neq k_p$.

Lemma 4 *If $p \in \mathcal{D}$ and an integer k_p exist such that $|X_pE(k_p) - X_pF(k_p)| = 1$, then for every $q \in \mathcal{D}$ there is an integer k_q such that $|X_qF(k_q) - X_qE(k_q)| = 1$ and $\langle k_p, k_q\rangle_{pq} \in \mathbb{Z}^2$.*

PROOF. Let L_i^{pq} be the pq-lattice containing the line $p = k_p$, or equivalently $k_p \in p(L_i^{pq})$. Suppose that $X_pF(k_p) - X_pE(k_p) = +1$. Thus, we have that

$$\sum_{k \in p(L_i^{pq})} X_pF(k) = 1 + \sum_{k \in p(L_i^{pq})} X_pE(k).$$

Using the consistency of the X-rays for F and E, the previous identity leads to the following

$$\sum_{k \in q(L_i^{pq})} X_qF(k) = 1 + \sum_{k \in q(L_i^{pq})} X_qE(k),$$

for all q in $\mathcal{D}$. From this, the thesis easily follows. $\square$

In the next lemma we show that all the lines with error 1 have a common point and this point is in $\mathbb{Z}^2$. In the following, we assume that $\text{card}(F) > \text{card}(E)$ and for any $p \in \mathcal{D}$ the integer k_p is as in the previous lemma.

Lemma 5 *If $DX_{\mathcal{D}}(E,F)=1$, then a point $W \in \mathbb{Z}^2$ exists such that*

$$X_pF(k) = X_pE(k)+1, \text{ if } k = p(W)$$
$$X_pF(k) = X_pE(k), \qquad \text{otherwise}$$

for all the directions p in $\mathcal{D}$.

PROOF. Let p, q and r be directions in $\mathcal{D}$ and suppose that $A = \langle k_p, k_q \rangle_{pq}$, $B = \langle k_p, k_r \rangle_{pr}$, $C = \langle k_q, k_r \rangle_{qr}$ are three distinct points. Let a, b be such that $r = ap + bq$. Thus, summing up we can write:

$$\sum_{M \in F} r(M) = a \sum_{M \in F} p(M) + b \sum_{M \in F} q(M)$$

and by grouping line by line we obtain:

$$\sum_k kX_rF(k) = a \sum_k kX_pF(k) + b \sum_k kX_qF(k).$$

We can exhibit the corresponding identity for the set E. As a result of the difference of these two identities we obtain that $k_r = ak_p + bk_q$ and so $r(A) = r(B) = r(C)$. Thus, the three points A, B and C coincide and the claim is proved. □

Remark 6 *Given any three lattice directions we can construct two sets E, F in such a way that they are (non-additive) sets of uniqueness. (We do not give the proof for reasons of space limit and we refer the reader to [10]). Figure 1 illustrates two such sets verifying the constraint $DX_{\mathcal{D}}(E,F)=1$. Since they are disjoint, Proposition 7 does not hold for $\mathcal{D}$-unique sets.*

Since uniqueness is not sufficient to have stable solutions for the reconstruction problem, we suppose that E and F are $\mathcal{D}$-additive, that is $E = \{M : e(M) > 0\}$ and $F = \{M : f(M) > 0\}$.

Proposition 7 *Let E and F be $\mathcal{D}$-additive lattice sets. If $DX_{\mathcal{D}}(E,F)=1$, then* $\mathrm{card}(E \triangle F) = 1$.

PROOF. Let W be as in Lemma 5. At first suppose that $W \notin E$ and let $E' = E \cup \{W\}$. For each direction p in $\mathcal{D}$ we have that $X_pE' = X_pF$. Finally, since additivity of F implies uniqueness of F, we conclude that $F = E \cup \{W\}$. On the contrary, if $W \in E$ we study the following:

$$\Phi_E = \sum_{M \in \mathbb{Z}^2} \sum_{p \in \mathcal{D}} e_p(p(M))(1_E(M) - 1_F(M)).$$

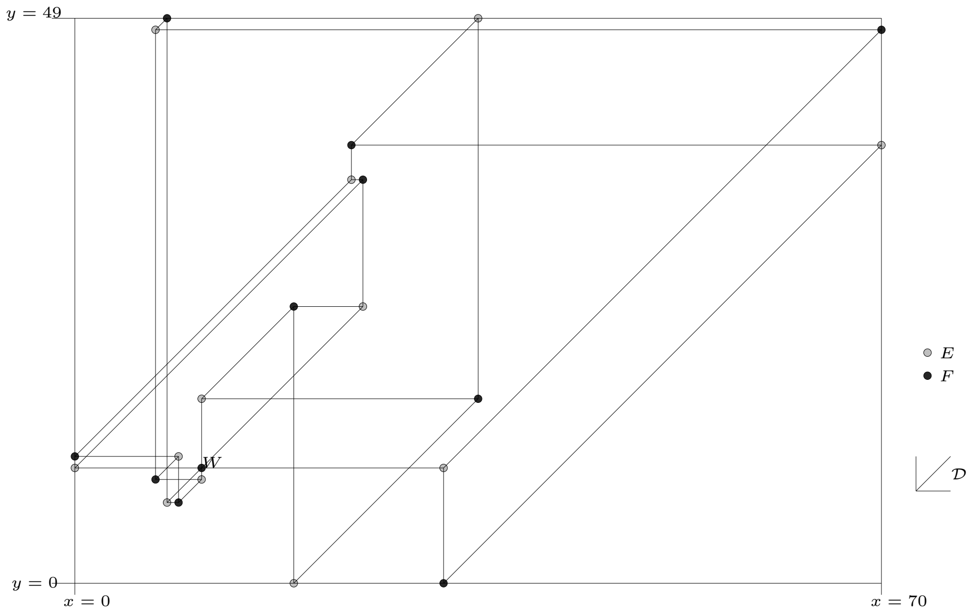

Fig. 1. E and F are non-additive sets of uniqueness such that $DX_{\mathcal{D}}(E,F)=1$ and $E\cap F=\emptyset$.

Rewriting it as

$$\sum_{M\in E}\sum_{p\in\mathcal{D}} e_p(p(M))(1_E(M)-1_F(M)) + \sum_{M\notin E}\sum_{p\in\mathcal{D}} e_p(p(M))(1_E(M)-1_F(M)),$$

we notice that $\Phi_E > 0$, because the additivity of E implies that if M is in E, then $e(M) > 0$ and $1_E(M) = 1$ holds, and otherwise $e(M) \leq 0$ and $1_E(M) = 0$. We can also explicit the terms X_pE and X_pF in Φ_E so obtaining that

$$\Phi_E = \sum_{k\notin p(W)}\sum_{p\in\mathcal{D}} e_p(k)(X_pE(k)-X_pF(k)) + \sum_{p\in\mathcal{D}} e_p(p(W))(X_pE(p(W))-X_pF(p(W)))$$

that is strictly less than zero. □

Remark 8 *Let us notice that in the proof, additivity for F and just uniqueness for E are needed.*

If we consider the case where the error is larger than 1, we have instability even when the error is just equal to 2, if the number of lattice directions is larger than 2. More in detail, the instability follows from the result of [1, Theorem 1] because the sets constructed in the proof of [1] are actually $\mathcal{D}$-additive. Therefore we can restate it as follows:

Proposition 9 (see [1]) *For any n and a set $\mathcal{D}$ of $m \geq 3$ directions there exist E and F $\mathcal{D}$-additive such that* $\mathrm{card}(E) = \mathrm{card}(F) \geq n$, $DX_{\mathcal{D}}(E,F) = 2$ *and $E\cap F=\emptyset$.*

4 Stability for Convex Sets

In this section we study Problem 1 for convex sets from both an experimental and a theoretical point of view.

Any *convex set* is the intersection of a convex polygon and the digital plane $\mathbb{Z}^2$. The definition of convex set pass through that of convex polygon, and this can be used to determine results for the discrete case from the continuous case. In this way, convex sets are uniquely determined by their X-rays taken in a suitable set of directions [11], and this set of directions distinguishes convex bodies [13]. So in the "continuous" plane an analogous result holds, and additionally the reconstruction problem is stable [22]. Moreover we notice that there is a connection between additive sets and convex sets, since an euclidean ball is additive with respect to two orthogonal directions ([9]).

Experiments support the conjecture that convex sets are additive for a suitable set of directions, and indeed they accord with Proposition 7. The experimental results are better for a larger error thanks to the property of convexity.

In the second part, we conduct a theoretical study that confirms stability for convex sets.

4.1 Experimental results

In this section we experimentally study the stability of the reconstruction of convex sets via linear programming. Our experiments support the suspect that the results in the continuous have a correspondence in the "digital" plane.

Actually we consider in this section a class of lattice sets which is more general than the convex sets [5].

For each point $M = (x_M, y_M) \in \mathbb{Z}^2$ the four quadrants around M are defined by the following formulas:

$$
\begin{aligned}
R_0(M) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \le x_M \text{ and } y \le y_M\},\\
R_1(M) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \ge x_M \text{ and } y \le y_M\},\\
R_2(M) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \ge x_M \text{ and } y \ge y_M\},\\
R_3(M) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \le x_M \text{ and } y \ge y_M\}.
\end{aligned}
$$

Definition 10 *A lattice set E is* Q-convex *if and only if for each $M \notin E$ there exists $i \in \{0,1,2,3\}$ such that $R_i(M) \cap E = \emptyset$.*

An example of Q-convex set is given on the left part of Figure 2.

We have generated 184 Q-convex sets of semi-perimeter from 4 to 370 using an uniform generator ([4], inspired from [17]). Then their X-rays in the set of directions $\mathcal{D} = \{x, y, 2x + y, -x + 2y\}$ have been computed. (These directions have been chosen because the X-rays along them uniquely determine the convex sets ([11]) and they contain the horizontal and vertical directions). We then used these X-rays and any error $e \in \{0, 1, 2, 3\}$ as input data in the following linear-program:

Maximizing

$$\sum_{(i,j)\in E} (1 - v_{i,j}) + \sum_{(i,j)\notin E} v_{i,j} \tag{1}$$

such that

$$\sum_{p(i,j)=k} v_{i,j} = X_p E(k) + er^+_{p,k} - er^-_{p,k} \tag{2}$$

$$\sum_k er^+_{p,k} + er^-_{p,k} \leq e \tag{3}$$

$$0 \leq v_{i,j} \leq 1,\ er^+_{p,k} \geq 0,\ er^-_{p,k} \geq 0 \tag{4}$$

We solved the linear program with the software soplex which implements the simplex algorithm ([23]). Notice that solving this problem with $v_{i,j} \in \mathbb{Z}$ permits to exactly find the maximum of card$(E \triangle F)$ where F describes *all* the lattice sets such that $DX_{\mathcal{D}}(E, F) \leq e$. Unfortunately, integer-linear-program is an $\mathbb{NP}$-hard problem, and hence we solved the *relaxed* problem where the unknown variables can be fractional: this computation provides an upper bound to card$(E \triangle F)$. Figure 2 illustrates (on the right side-hand) a solution of the linear programming for card$(E) = 200$ and $e = 3$. The different grey-scale colors of the squares correspond to different values of $v_{i,j}$.

The complete results are summarized in Figures 3 and 4. In Figure 3 the upper bound to card$(E \triangle F)$ is divided by card(E), so that each value gives an upper bound to the relative distance from a given set. Moreover the black squares show the values of the maximum of the quantity (1) when the constraints (2),(3) are replaced by $X_p E(k) - 1 \leq \sum_{p(i,j)=k} v_{i,j} \leq X_p E(k) + 1$: these values give an upper bound to card$(E \triangle F)$ when $DX'_{\mathcal{D}}(E, F) = \max_{p\in\mathcal{D}} \max_{k\in\mathbb{Z}} |X_p E(k) - X_p F(k)| = 1$.

These experimental results bring the following comments:

- If $DX_{\mathcal{D}}(E, F) = 0$, then we always found a null relative distance. In other words, according to our experiments every Q-convex set is $\mathcal{D}$-additive. In fact this property was first conjectured by L. Thorens ([21]) (with additivity replaced by uniqueness), and can be seen as a variant of Conjecture 4.6 of [2] and Theorem 5.7 of [11]. We can set out the conjecture as follows:

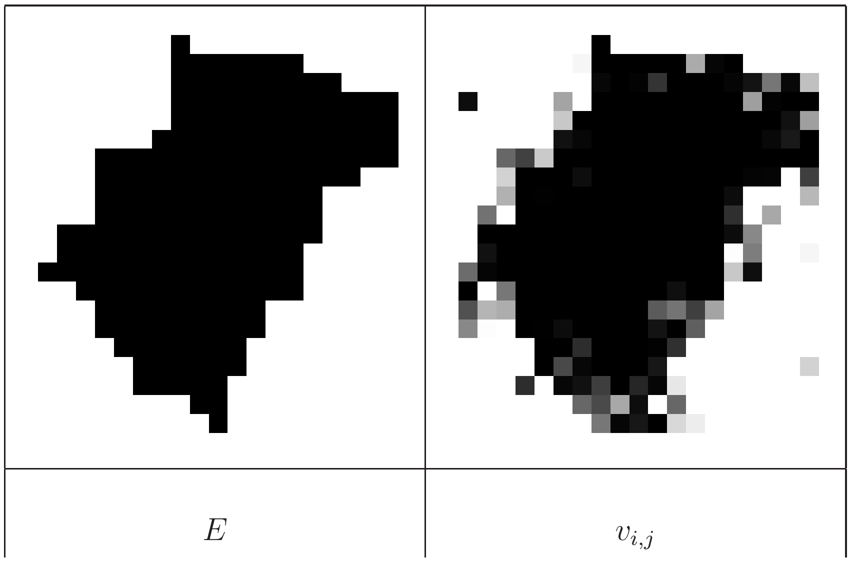

Fig. 2. A Q-convex set E and the corresponding extremal values of $v_{i,j}$ for $e = 3$. In this case we have $\mathrm{card}(E) = 200$ and $\sum_{(i,j)\in E}(1 - v_{i,j}) + \sum_{(i,j)\in E^c} v_{i,j} = 33.7$.

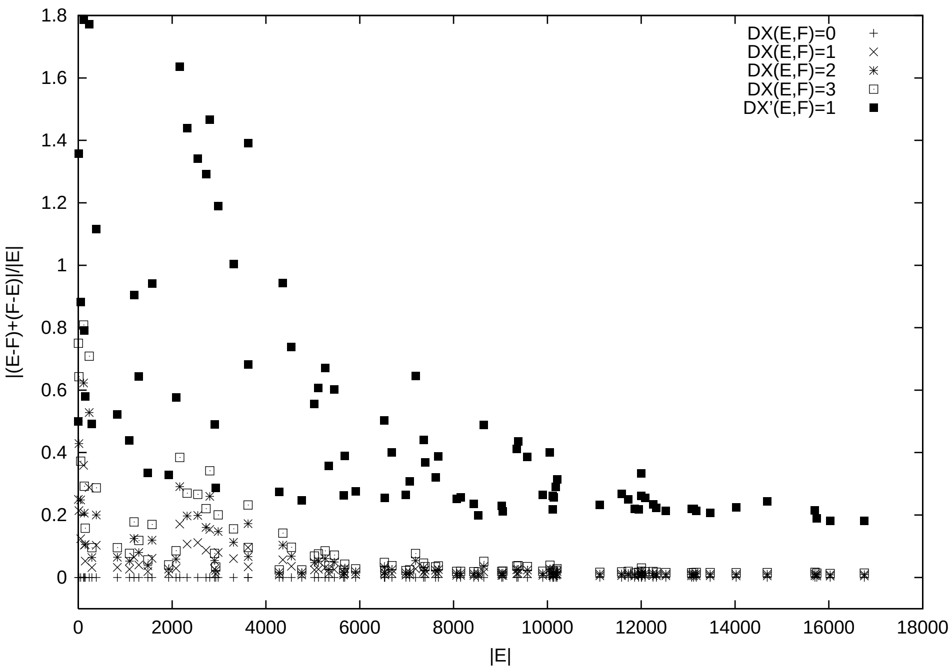

Fig. 3. An upper bound to $\frac{\mathrm{card}(E\triangle F)}{\mathrm{card}(E)}$ for the Q-convex generated sets. (Only 40 % of the 184 generated sets have been represented for readability)

Conjecture 11 *If $\mathcal{D}$ is a set of directions which contains $\{x, y\}$, such that all the directions are not in the same quadrant and they uniquely determine the convex sets, then every Q-convex set is $\mathcal{D}$-additive.*
Notice that the property about the quadrants is necessary because there is a counter-example with $\mathcal{D} = \{x, y, x + y, x + 5y\}$.

- For a fixed error e, the relative distance looks to converge to zero as $\mathrm{card}(E)$

grows. If we divide by $\sqrt{\text{card}(E)}$ instead of $\text{card}(E)$, this ratio seems to be bounded so that in *average* $\text{card}(E \triangle F) = O(\sqrt{\text{card}(E)})$ according to our experiments (see Figure 4). It must be noticed that in the case $e = 1$, the maximum error for lattice sets is always 1 for the generated cases according to the result of Proposition 7. Since the theoretical result holds for additive sets, the experiments could be interpreted as a further evidence of the conjecture.

- If $DX'_{\mathcal{D}}(E, F) = 1$, then the relative distance does not seem to converge to zero, but the computed values are only upper bounds, that is, we do not know if the fractional values mirror instability or they are just an artifact introduced by relaxing the integral constraints of the problem. In the former case, the reconstruction of convex sets would not be applied easily in the continuous world (as in medical imaging), because a rounding error of the measurements can always be of ± 1.

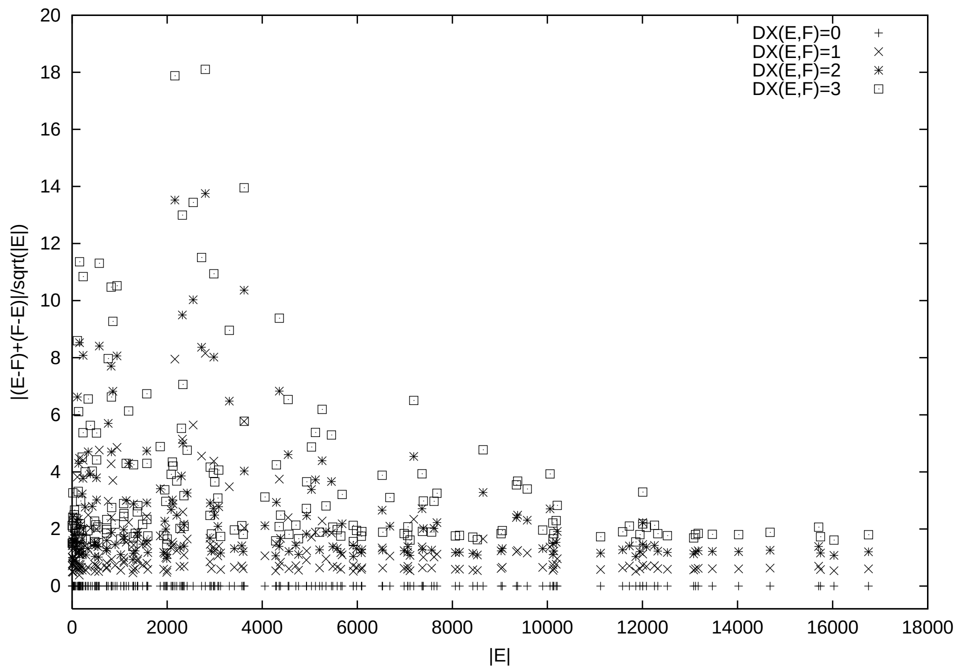

Fig. 4. An upper bound to $\frac{\text{card}(E \triangle F)}{\sqrt{\text{card}(E)}}$ for the 184 generated Q-convex sets

4.2 Theoretical results

In this section we first exploit a stability result for convex bodies [22] to deal with the corresponding problem for convex lattice sets and then we use this result to choose that it is possible to reconstruct convex bodies from X-rays, by the intermediate of lattice convex sets and Discrete Tomography.

4.2.1 Preliminaries

A *convex body* is a compact convex subset of $\mathbb{R}^2$ with non-empty interior. We denote the set of all the convex bodies by $\mathcal{K}_*$. The X-ray X_pU of the convex body U in direction p is the function giving the length of each chord of U parallel to p. More precisely $X_pU(\alpha)$ is the length of the intersection of U with the line $p = \alpha$. The *Steiner symmetral* $S_p(U)$ of U in direction p is the closure of the union of all open segments on lines parallel to p of the same length as X_pU centered about a fixed line orthogonal to p. So the Steiner symmetral $S_p(U)$ and the X-ray X_pU contains exactly the same information.

Definition 12 *A set of direction $\mathcal{D}$ is a* Gardner-McMullen set of directions *if any convex body is characterized by all its X-rays in the directions of $\mathcal{D}$.*

We recall a result of [12, Proposition 6.1,Theorem 4.5]:

Theorem 13 (Gardner-Gritzmann) *A set $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ of four* lattice *directions is a Gardner-McMullen set of directions if and only if the cross-ratio of the directions arranged in order of increasing angle with the positive x-axis is not in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$.*

Example 14 *This theorem implies that the set $\mathcal{D} = \{x, y, 2x + y, -x + 2y\}$ is a Gardner-Gritzmann set of directions.*

In the following we suppose that $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ is a Gardner-McMullen set of four directions. So the mapping $\mu : \mathcal{K}_* \mapsto (S_{p_1}U, S_{p_2}U, S_{p_3}U, S_{p_4}U)$ is injective.

Let $\mathcal{K}_*$ be endowed with Nikodym's distance:

$$d_N(U, V) = m(U \triangle V),$$

where $m(U)$ denotes the Lebesgue measure on $\mathbb{R}^2$. Now are we in place to state the stability result for convex bodies (see Theorem of [22, section 3.1]):

if $\mathcal{K}_*$ is endowed with the topology induced by the Nikodym's distance, μ is continuous and continuously invertible from $\mu(\mathcal{K}_*)$.

We shall reformulate this theorem. Consider the map $\sigma_{\mathcal{D}} : U \mapsto (X_pU)_{p\in\mathcal{D}}$; if $\mathcal{D}$ is a Gardner-McMullen set of direction, then $\sigma_{\mathcal{D}}(U)$ is injective. Let $\mathcal{X}_{\mathcal{D}}$ be the range of $\sigma_{\mathcal{D}}$. We endow $\mathcal{X}_{\mathcal{D}}$ with the following distance:

$$d_{\mathcal{X}}((f_p)_{p\in\mathcal{D}}, (g_p)_{p\in\mathcal{D}}) = \max_{p\in\mathcal{D}} \int_{-\infty}^{+\infty} \frac{|f_p(\alpha) - g_p(\alpha)|}{\sqrt{a_p^2 + b_p^2}} \mathrm{d}\alpha,$$

where $(f_p)_{p\in\mathcal{D}}, (g_p)_{p\in\mathcal{D}}$ are in $\mathcal{X}_{\mathcal{D}}$, a_p and b_p are defined by $p(x, y) = a_px + b_py$. (Notice that each integral in the definition of this distance corresponds exactly to the Nikodym distance if X-rays are considered as Steiner symmetral.) We also use the notation $d_{\mathcal{X}_\mathcal{D}}(U,V) = d_\mathcal{X}(\sigma_\mathcal{D}(U), \sigma_\mathcal{D}(V))$.

The Theorem of [22, section 3.1] can be rewritten as follows:

Theorem 15 (Volčič) *Let $\mathcal{D}$ be a Gardner-McMullen set of four lattice directions the inverse $\sigma_\mathcal{D}^{-1}$ of the function $\sigma_\mathcal{D}$ is a continuous function from $\mathcal{X}_\mathcal{D}$ to $\mathcal{K}_*$.*

For any bounded set $E \subset \mathbb{R}^2$ we define $R_{\max}(E) = \max_{M\in E} \|M\|$ where $\|\cdot\|$ is the euclidean norm. The set $\mathcal{K}^1_\varepsilon = \{U \in \mathcal{K}_* \ : \ R_{\max}(U) \leq 1 \text{ and } m(U) \geq \varepsilon\}$ is a compact subset of $\mathcal{K}_*$; it follows that $\sigma_\mathcal{D}(\mathcal{K}^1_\varepsilon)$ is a compact subset of $\mathcal{X}_\mathcal{D}$ and so the function $\sigma_\mathcal{D}^{-1}$ restricted to $\sigma_\mathcal{D}(\mathcal{K}^1_\varepsilon)$ is uniformly continuous. So we can give a more precise formulation of the previous theorem:

Corollary 16 *Let $\mathcal{D}$ be a Gardner-McMullen set of four lattice directions. For any $\varepsilon > 0$ there exists $\eta > 0$ such that any $U, V \in \mathcal{K}^1_\varepsilon$ satisfy:*

$$d_{\mathcal{X}_\mathcal{D}}(U,V) < \eta \Longrightarrow d_N(U,V) < \varepsilon.$$

4.2.2 A stability result for lattice convex sets

In this section $\mathcal{D}$ is a Gardner-McMullen set of four lattice directions. Since a Gardner-McMullen set of lattice direction uniquely determines convex lattice sets [11], we use the result enunciated in Corollary 16 to get a stability result for convex lattice sets.

At first we need a lemma which is a direct consequence of Pick's theorem. We recall that a lattice polygon is a polygon whose vertexes are in $\mathbb{Z}^2$, and a simple polygon is a polygon whose edges have a non-empty intersection only if they are consecutive.

Lemma 17 *If $P \subset \mathbb{R}^2$ is simple lattice polygon, then* $\operatorname{card}(P \cap \mathbb{Z}^2) \leq 2m(P) + 2$.

PROOF. By Pick's theorem [20,15] we have $m(P) = i + \frac{b}{2} - 1$ where i is the number of lattice points which are in the interior of P and b is the number of lattice points which are in the border of P.

So $\operatorname{card}(P \cap \mathbb{Z}^2) = i + b \leq 2i + b = 2(m(P) + 1)$. □

In the sequel each direction p of $\mathcal{D}$ has the form $p(x,y) = a_p x + b_p y$ with a_p, b_p integer.

Proposition 18 *For any* $\varepsilon > 0$ *and* $K > 1$, *there exists* $\eta > 0, M > 0$ *such that any lattice convex non-segment sets* E *and* F *such that* $\frac{\mathrm{card}(E)}{(R_{\max}(E))^2}, \frac{\mathrm{card}(F)}{(R_{\max}(F))^2} \geq \varepsilon$, $R_{\max}(E), R_{\max}(F) \geq M$, $\frac{1}{K} \leq \frac{R_{\max}(E)}{R_{\max}(F)} \leq K$ *satisfy:*

$$\frac{DX(E,F)}{(\max(R_{\max}(E), R_{\max}(F)))^2} < \eta \Longrightarrow \frac{\mathrm{card}(E \triangle F)}{(\max(R_{\max}(E), R_{\max}(F)))^2} < \varepsilon + \frac{17}{\max(R_{\max}(E), R_{\max}(F))}$$

PROOF. We define $\varepsilon_c = \min(\frac{\varepsilon}{4K^2}, \frac{\varepsilon}{2})$. Let η_c given by Corollary 16 applied to ε_c. We take M such that $\frac{6}{M} \leq \frac{\eta_c}{2}$, $\frac{1}{(KM)^2} \leq \varepsilon_c$ and $M \geq 8$. So we suppose that E and F are sets which satisfy the conditions of the propositions.

Let us consider sets $E_c = \frac{1}{N}\mathrm{conv}(E)$, $F_c = \frac{1}{N}\mathrm{conv}(F)$ and the number $N = \max(R_{\max}(E), R_{\max}(F))$.

The sets E_c and F_c are convex polygons of $\mathbb{R}^2$, and since they are not segments, E_c and F_c are convex bodies. Additionally, they are simple *lattice* polygons being their vertices in $(\mathbb{Z}/N)^2$. By Lemma 17 applied to $P = NE_c$ we have $m(E_c) \geq \frac{1}{N^2}(\frac{\mathrm{card}(E)}{2} - 1)$. So:

$$\begin{aligned} m(E_c) &\geq \frac{1}{N^2}(\frac{\mathrm{card}(E)}{2} - 1) \\ &\geq \frac{\mathrm{card}(E)}{2(KR_{\max}(E))^2} - \frac{1}{(KR_{\max}(E))^2} \\ &\geq \frac{\varepsilon}{2K^2} - \frac{1}{(KM)^2} \\ &\geq 2\varepsilon_c - \varepsilon_c = \varepsilon_c \end{aligned}$$

Similarly $m(F_c) \geq \varepsilon_c$. So $E_c, F_c \in \mathcal{K}^1_{\varepsilon}$.

Now we suppose that $\frac{DX(E,F)}{N^2} < \eta$ with $\eta = \frac{\eta_c}{4}$ and we estimate $d_{\mathcal{X}_{\mathcal{D}}}(E_c, F_c)$. We have that:

$$\frac{X_pE(n) - 1}{N} \leq X_pE_c(\frac{n}{N}) \leq \frac{X_pE(n) + 1}{N}$$
$$\frac{X_pF(n) - 1}{N} \leq X_pF_c(\frac{n}{N}) \leq \frac{X_pF(n) + 1}{N}$$

so that

$$|X_pE_c(\frac{n}{N}) - X_pF_c(\frac{n}{N})| \leq \frac{|X_pE(n) - X_pF(n)| + 2}{N}.$$

Since

$$|X_pE_c(\alpha)-X_pF_c(\alpha)| \leq \max(|X_pE_c(\frac{\lfloor \alpha N \rfloor}{N})-X_pF_c(\frac{\lfloor \alpha N \rfloor}{N})|, |X_pE(\frac{\lceil \alpha N \rceil}{N})-X_pF(\frac{\lceil \alpha N \rceil}{N})|)$$

we get

$$\begin{aligned}\int_{-\infty}^{+\infty} |X_pE_c(\alpha) - X_pF_c(\alpha)|\mathrm{d}\alpha &\leq \frac{2}{N} \sum_{n=\lceil -N\sqrt{a_p^2+b_p^2}\rceil}^{\lfloor N\sqrt{a_p^2+b_p^2}\rfloor} \frac{|X_pE(n) - X_pF(n)| + 2}{N} \\ &= \frac{2}{N^2} \sum_{n=-\infty}^{+\infty} |X_pE(n) - X_pF(n)| + \frac{2(2N\sqrt{a_p^2+b_p^2}+1)}{N^2}.\end{aligned}$$

Finally, a_p and b_p are integer and, so $\sqrt{a_p^2+b_p^2} \geq 1$, and we conclude that

$$\begin{aligned} d_{\mathcal{X}_\mathcal{D}}(E_c, F_c) &\leq \frac{2}{N^2} DX(E,F) + \frac{6}{N} \\ &\leq 2\eta + \frac{6}{M} \\ &\leq \frac{\eta_c}{2} + \frac{\eta_c}{2} = \eta_c. \end{aligned}$$

By Corollary 16 we have that $d_N(E_c, F_c) \leq \varepsilon_c \leq \frac{\varepsilon}{2}$.

The symmetric difference $E_c \triangle F_c$ is the union of components of $E_c \setminus F_c$ and of $F_c \setminus E_c$. Let C_j denotes the closure of the jth component and $E_c \triangle F_c = \cup_{j=1}^k C_j$. Each component C_j is a simple polygon $(S_0A_0A_1A_2 \dots A_lS_1B_0B_1B_2B_m)$ where $A_0, \dots A_l$ are consecutive vertexes of E_c, $B_0, \dots B_m$ are consecutive vertexes of F_c, and S_0, S_1 are intersection of an edge of E_c and an edge of F_c.

We consider the polygon C'_i union of the three following polygons:

- $\mathrm{conv}((S_0A_0B_0) \cap (\mathbb{Z}/N)^2)$
- the polygon $(A_0A_1 \dots A_lB_0B_1 \dots B_l)$
- $\mathrm{conv}((S_1A_lB_m) \cap (\mathbb{Z}/N)^2)$

This polygon is included in C_i and is a simple polygon whose vertexes are all in $(\mathbb{Z}/N)^2$, so by Lemma 17 we have $\mathrm{card}(C_i \cap (\mathbb{Z}/N)^2) = \mathrm{card}(C'_i \cap (\mathbb{Z}/N)^2) \leq 2m(C'_i) + 2 \leq 2m(C_i) + 2$.

So finally

$$\mathrm{card}(E \triangle F) = \sum_{j=1}^{k} \mathrm{card}(C_i \cap (\mathbb{Z}/N)^2) \leq \sum_{j=1}^{k} (2m(C_j) + 2) = 2d_N(E_c, F_c) + 2k.$$

The vertexes of E_c and F_c are in $\mathbb{Z}/N^2 \cap [-1,1]^2$ so the polygons E_c and F_c have less than $2(2N+1)$ vertexes. Each component C_i contains at least one vertex of E_c or one vertex of F_c so $k \leq 4(2N+1)$. Moreover $d_N(E_c, F_c) \leq \frac{\varepsilon}{2}$ so $\mathrm{card}(E \triangle F) \leq \varepsilon N^2 + 8(2N+1) = \varepsilon N^2 + 16N + 8$. We have supposed that $N \geq M \geq 8$ so finally $\mathrm{card}(E \triangle F) \leq \varepsilon N^2 + 17N$. $\square$

This upper bound overestimates the symmetric difference because we actually count also points of the border of $E_c \cap F_c$ and the number of components is less than the number of vertexes of E_c and F_c, that in turn is less than $4N$.

4.2.3 Reconstruction of a convex body from noisy discrete X-rays

In this section we always suppose that $\mathcal{D}$ is a Gardner-McMullen set of directions. If F is a convex body then we know that it is completely determined by its continuous X-rays in $\mathcal{D}$. Anyway it does not give an algorithm to reconstruct F from its X-rays. The aim of this section is to use previous Proposition to show that it is possible, in theory, to reconstruct F by using Discrete Tomography. For this we fix an integer n, and we suppose that we have a lattice convex set E_n which could be seen as an approximation of F to the resolution $\frac{1}{n}$. But as we do not know F, the only assertions about E_n is the nearness of the discrete X-rays of E_n and the continuous X-rays of F. Proposition 20 will show that assertions which only consider the X-rays of E_n exist such that the set E_n converges, in a certain sense, to F when n tends to infinity.

If $p = ax + by$ is a direction, $\|p\|$ designs $\sqrt{a^2+b^2}$. We start with an easy lemma which will be useful in the following:

Lemma 19 *If E is any bounded subset of $\mathbb{R}^2$ and p, p' are two directions then*

$$\max\left(\frac{|\alpha_1|}{\|p\|}, \frac{|\alpha_2|}{\|p\|}, \frac{|\alpha_1'|}{\|p'\|}, \frac{|\alpha_2'|}{\|p'\|}\right) \leq R_{\max}(E)$$
$$\leq \max(\|\langle \alpha_1, \alpha_1' \rangle_{pp'}\|, \|\langle \alpha_1, \alpha_2' \rangle_{pp'}\|, \|\langle \alpha_2, \alpha_1' \rangle_{pp'}\|, \|\langle \alpha_2, \alpha_2' \rangle_{pp'}\|)$$

where $\alpha_1 = \inf_{z\in E} p(z),\ \alpha_2 = \sup_{z\in E} p(z),\ \alpha_1' = \inf_{z\in E} p'(z),\ \alpha_2' = \sup_{z\in E} p'(z)$.

Proposition 20 *Let F be a convex body, and $(E_n)_{n\in\mathbb{N}}$ a sequence of non-segment convex lattice sets such that for any $p \in \mathcal{D}$ there hold:*

$$\frac{1}{n}\max\{k \in \mathbb{Z} \ :\ X_pE_n(k) \neq 0\} \underset{n\to\infty}{\longrightarrow} \sup\{\alpha \in \mathbb{R} \ :\ X_pF(\alpha) \neq 0\} \qquad (5)$$

$$\frac{1}{n}\min\{k \in \mathbb{Z} \ :\ X_pE_n(k) \neq 0\} \underset{n\to\infty}{\longrightarrow} \inf\{\alpha \in \mathbb{R} \ :\ X_pF(\alpha) \neq 0\} \qquad (6)$$

$$\frac{1}{n}\max_{p\in\mathcal{D}} \sum_{k\in\mathbb{Z}} \left|\frac{X_pE_n(k)}{n} - X_pF\left(\frac{k}{n}\right)\right| \underset{n\to\infty}{\longrightarrow} 0 \qquad (7)$$

then
$$\frac{1}{n^2}\text{card}(E_n\Delta(nF\cap\mathbb{Z}^2)) \underset{n\to\infty}{\longrightarrow} 0.$$

PROOF. Let $(F_n)_{n\in\mathbb{N}}$ be the sequence of convex lattice sets, defined by $F_n = nF\cap\mathbb{Z}^2$. To prove this proposition we are going to show that an integer N exists such that for $n > N$ the sets E_n and F_n verify the conditions of Proposition 18. The thesis follows by applying the proposition.

At first we derive some conditions we need to our goal.

Since $\frac{\text{card}(F_n)}{n^2} \underset{n\to\infty}{\longrightarrow} m(F)$ and $\frac{R_{\max}(F_n)}{n} \underset{n\to\infty}{\longrightarrow} R_{\max}(F)$, it follows that

$$\frac{\text{card}(F_n)}{(R_{\max}(F_n))^2} \underset{n\to\infty}{\longrightarrow} \frac{m(F)}{(R_{\max}(F))^2} > 0. \tag{8}$$

We have
$$\frac{X_pF_n(k)-1}{n} \leq X_pF(\frac{k}{n}) \leq \frac{X_pF_n(k)+1}{n},$$
and, hence by condition (7)

$$\frac{1}{n^2}DX(E_n,F_n) \underset{n\to\infty}{\longrightarrow} 0. \tag{9}$$

As a consequence of this and $|\,\text{card}(E_n)-\text{card}(F_n)\,| \leq DX(E_n,F_n)$, we obtain that $\frac{\text{card}(E_n)-\text{card}(F_n)}{n^2} \underset{n\to\infty}{\longrightarrow} 0$.

We choose arbitrarily two directions p,p' of $\mathcal{D}$. Let $\alpha_1 = \min_{z\in F} p(z)$, $\alpha_2 = \max_{z\in F} p(z)$, $\alpha'_1 = \min_{z\in F} p'(z)$, $\alpha'_2 = \max_{z\in F} p'(z)$. By Lemma 19 applied to E_n, the conditions (5),(6) and the continuity of the function $(\alpha,\alpha') \to \langle\alpha,\alpha'\rangle_{pp'}$, there exists an integer N_1 such that for $n > N_1$ we have: $M_1 \leq \frac{R_{\max}(E_n)}{n} \leq M_2$ with $M_1 = \frac{1}{2}\max(\frac{|\alpha_1|}{\|p\|}, \frac{|\alpha_2|}{\|p\|}, \frac{|\alpha'_1|}{\|p'\|}, \frac{|\alpha'_2|}{\|p'\|})$ and
$M_2 = 2\max(\|\langle\alpha_1,\alpha'_1\rangle_{pp'}\|, \|\langle\alpha_1,\alpha'_2\rangle_{pp'}\|, \|\langle\alpha_2,\alpha'_1\rangle_{pp'}\|, \|\langle\alpha_2,\alpha'_2\rangle_{pp'}\|)$.Thus, by this and the previous deduction, we get

$$\frac{\text{card}(E_n)}{(R_{\max}(E_n))^2} \geq \frac{\text{card}(E_n)}{(nM_2)^2} \underset{n\to\infty}{\longrightarrow} \frac{m(F)}{(M_2)^2} > 0. \tag{10}$$

Moreover, an integer $N_2 > N_1$ exists such that for $n > N_2$ there holds: $\frac{M_1}{2R_{\max}(F)} \leq \frac{R_{\max}(E_n)}{R_{\max}(F_n)} \leq \frac{2M_2}{R_{\max}(F)}$.

Now we are going to use these properties to show that we can apply Proposition 18 to E_n and F_n. To prove the thesis, we have to find, for any $\varepsilon > 0$, an N such that $\frac{1}{n^2}\text{card}(E_n\Delta(nF\cap\mathbb{Z}^2)) \leq \varepsilon$ for $n > N$. Let $K = \max(\frac{2M_2}{R_{\max}(F)}, \frac{2R_{\max}(F)}{M_1})$

and $\varepsilon' = \frac{\varepsilon}{2(2KR_{\max}(F))^2}$. Without loss of generality let us suppose that $0 < \varepsilon' < \frac{m(F)}{2(R_{\max}(F))^2}, \frac{m(F)}{2(M_2)^2}$,

- We have that $\frac{1}{K} \le \frac{R_{\max}(E_n)}{R_{\max}(F_n)} \le K$ for $n > N_2$.
- By the conditions (8) and (10) there exists an integer N_3 such that $\frac{\mathrm{card}(E_n)}{(R_{\max}(E_n))^2}, \frac{\mathrm{card}(F_n)}{(R_{\max}(F_n))^2} \ge \min(\frac{m(F)}{2(R_{\max}(F))^2}, \frac{m(F)}{2(M_2)^2}) \ge \varepsilon'$. Hence $\frac{\mathrm{card}(E_n)}{(R_{\max}(E_n))^2}, \frac{\mathrm{card}(F_n)}{(R_{\max}(F_n))^2} \ge \varepsilon'$.
- For any fixed $M > 0$ there exists an integer N_3 such that for any $n > N_3$ we have $R_{\max}(E_n) \ge M$ and $R_{\max}(F_n) \ge M$.
- For any fixed $\eta > 0$, by property (9), an integer N_4 exists such that $\frac{1}{n^2} DX(E_n, F_n) < \eta(\frac{R_{\max}(F)}{2K})^2$ for any $n > N_4$.
- There exists an integer N_5 such that $\frac{R_{\max}(F_n)}{n} \ge \frac{R_{\max}(F)}{2}$ for $n > N_5$.

So, for any $n > N = \max(N_2, N_3, N_4, N_5)$, the sets E_n and F_n satisfy the conditions of Proposition 18 (with ε' instead of ε and η and M chosen as in the proposition). Therefore we have:

$$\frac{DX(E_n, F_n)}{(\max(R_{\max}(E_n), R_{\max}(F_n)))^2} < \eta \Longrightarrow \frac{\mathrm{card}(E_n \triangle F_n)}{(\max(R_{\max}(E_n), R_{\max}(F_n)))^2} < \varepsilon' + \frac{17}{\max(R_{\max}(E_n), R_{\max}(F_n))}.$$

We have $\max(R_{\max}(E_n), R_{\max}(F_n))) \ge \frac{R_{\max}(F_n)}{K} \ge \frac{nR_{\max}(F)}{2K}$ and $\max(R_{\max}(E_n), R_{\max}(F_n)) \le KR_{\max}(F_n) \le 2KnR_{\max}(F)$.

So $\frac{DX(E_n,F_n)}{(\max(R_{\max}(E_n),R_{\max}(F_n)))^2} \le \frac{n^2\eta(\frac{R_{\max}(F)}{2K})^2}{(\frac{nR_{\max}(F)}{2K})^2} = \eta$, then

$$\begin{aligned}
\mathrm{card}(E_n \Delta F_n) &\le \varepsilon'(R_{\max}(E_n), R_{\max}(F_n)))^2 + 17\max(R_{\max}(E_n), R_{\max}(F_n)) \\
&\le \varepsilon'(2KnR_{\max}(F))^2 + 17(2KnR_{\max}(F)) \\
&\le n^2\frac{\varepsilon}{2} + 34KR_{\max}(F)n \\
&\le n^2\frac{\varepsilon}{2} + n^2\frac{\varepsilon}{2} \qquad \text{for } n > N_6 = \frac{68KR_{\max}(F)}{\varepsilon} \\
&= \varepsilon n^2
\end{aligned}$$

□

Remark 21 *This last proposition gives a positive result for the reconstruction of a convex body from X-rays. Nevertheless it does not really give a concrete algorithm to reconstruct this convex body for at least two reasons:*

- *the result is not quantitative: the difference* $\mathrm{card}(E_n \Delta \mathrm{conv}(nF \cap \mathbb{Z}^2))$ *is not bounded by a precise function on the X-rays errors.*

- *when the resolution is fixed, we have to reconstruct a lattice convex set from approximative X-rays. Algorithms are known in the exact case ([5]) or for more general classes than lattice convex sets ([3,6]) but not exactly in this case.*

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