

Reconstruction of lattice sets from infinite X-rays

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May 22, 2002

Abstract

For any *infinite* non-null vectors there is always a subset of \mathbb{Z}^2 whose X-rays along fixed directions are the given vectors. If there are only two directions and the vectors are periodic, then the set can be chosen periodic, it's not true with more than two directions.

1 Introduction

Reconstruction of a *finite* lattice subset (i.e. subsets of \mathbb{Z}^2) from its X-rays is the most basic problem in Discrete Tomography. This problem was positively solved independently by Ryser and Gale in [5, 2] in the two-directions case. Many variants of this problem have been studied since this first result. Discrete Tomography could be applied in several areas like biplane angiography, image processing and image microscopy. For an overview of all these results, see [4].

In usual reconstruction, the X-rays are *non-null* only on a finite segment because only finite lattice sets are considered. In this paper we make the inverse hypothesis: the support of the X-rays is infinite. Simplest such X-rays are the periodic ones. In this case we also can suppose the periodicity to the solutions. In fact, reconstruction of periodic sets from periodic X-rays is motivated by application in high resolution microscopy because it would help to find crystallographic structures (which are supposed to be periodic) from the number of atoms in lines of prescribed directions.

2 Preliminary definitions

A direction is an equivalence class for the relation of parallelism on the straight lines of the plane. It can be given by an equation $p = \lambda x + \mu y = \text{const}$ or by a directing vector $\vec{p} = (-\mu, \lambda)$. If λ and μ are integer then, the direction is a *lattice direction*, and we can suppose that λ and μ are coprime. In this paper we only consider lattice directions.

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A lattice set is a subset of \mathbb{Z}^2 .

A lattice line is a line which contains at least two points of \mathbb{Z}^2 . A lattice line has an equation $p(M) = k$ where p is a lattice direction and k is an integer.

The cardinal of a set E will be denoted $|E|$.

The X-ray of a subset E of \mathbb{Z}^2 along a direction p is the function $X_p E : \mathbb{Z} \rightarrow \mathbb{N} \cup \infty$ given by $X_p E(k) = |\{M \in E : p(M) = k\}|$.

The support of a numerical function f is the set of elements x such that $f(x) \neq 0$.

If $p = ax + by$ and $q = cx + dy$ are two lattice directions (with $a, b, c, d \in \mathbb{Z}^2$), then the determinant of the two directions (denoted $\det(p, q)$) is the quantity $|ad - bc|$, which only depends on the two directions.

3 Reconstruction of Arbitrary Infinite Sets

Basic problem of Discrete Tomography consists in checking if a family of vectors are X-rays of a subset of \mathbb{Z}^2 . In fact, if the vectors are infinite and everywhere non-null then this problem can be trivially solved by the following proposition:

Proposition 1 *For any set \mathcal{D} of directions, and any function $f : \mathcal{D} \times \mathbb{Z} \rightarrow \mathbb{N}^*$, there is a subset E of \mathbb{Z}^2 such that $X_p(E) = f(p, \cdot)$ where $f(p, \cdot) = ((p, x) \mapsto f(p, x))$.*

Proof: Let $i \mapsto (p_i, k_i)$ be a bijection from \mathbb{N} in $\mathcal{D} \times \mathbb{Z}$. We must find a set E which has $f(p_i, k_i)$ points on the line $p_i(M) = k_i$ for any i .

Let

$$\mathcal{S} = \{E \subset \mathbb{Z}^2 : E \text{ is finite and } \forall (p, k) \in \mathcal{D} \times \mathbb{Z} \quad X_p E(k) \leq f(p, k)\}.$$

For any $E \in \mathcal{S}$ we define $N(E)$ by:

$$N(E) = \min\{i : X_{p_i} E(k_i) \neq f(p_i, k_i)\}$$

Let $E \in \mathcal{S}$, $i = N(E)$, $p = p_i$, $k = k_i$. The set R is defined by:

$$R = \{M : p(M) = k \text{ and } \forall q \in \mathcal{D} \setminus \{p\} \quad X_q E(q(M)) < f(q, q(M))\}.$$

If R was finite, then there would be a direction q and an infinity of points M such that $p(M) = k$ and $X_q E(q(M)) \geq f(q, q(M)) > 0$, and so E would be infinite. So R is infinite. We order the points of \mathbb{Z}^2 by $\phi(x) \leq \phi(y)$ where ϕ is any bijection from \mathbb{Z}^2 to \mathbb{N} . Let R' be the set of the first $f(p, k) - X_p E(k)$ points of R and consider the set $F = c(E) = E \cup R'$.

Let us compute the X-rays of $F = c(E)$. The X-rays of F along the lines which do not contain any point of R' are the same than the ones of E . Consider a direction $q \neq p$ and a point $M \in R'$ we have:

$$X_q F(q(M)) = X_q E(q(M)) + 1 \leq (f(q, q(M)) - 1) + 1 = f(q, q(M)).$$

moreover $X_p F(k) = X_p E(k) + (f(p, k) - X_p E(k)) = f(p, k)$ so $F \in \mathcal{S}$ and $N(F) > N(E)$.

So we have proved that for any set $E \in \mathcal{S}$, there exists a set $c(E) \in \mathcal{S}$ such that $N(c(E)) > N(E)$.

We define inductively the sequence of sets (E_n) by $E_0 = \emptyset$ and $E_{n+1} = c(E_n)$. We have $N(E_n) \geq n$. Let us define E_∞ by

$$E_\infty = \bigcup_{n \in \mathbb{N}} E_n.$$

For any $i \in \mathbb{N}$ and $j \geq i$ we have $X_{p_i} E_j(k_i) = f(p_i, k_i)$, so $X_{p_i} E_\infty(k_i) = f(p_i, k_i)$. So E_∞ is the searched set. \square

Remark 1 *In the proof, the set R' has not been defined as any subset of $f(p, k) - X_p E(k)$ elements of R , because with this vague definition, Axiom of Choice would be needed for the existence of the function c .*

Remark 2 *If $\mathcal{D} = \{x, y\}$, the proposition remains true for any function $f : \mathcal{D} \times \mathbb{Z} \rightarrow \mathbb{N}$ such that the supports of the two functions $f(x, \cdot)$ and $f(y, \cdot)$ are infinite.*

Remark 3 *If $\mathcal{D} = \{p, q\}$ then the discrete plane \mathbb{Z}^2 is the union of $\det(p, q)$ lattices \mathcal{L}_i such that there exist module-isomorphisms $\phi_i : \mathcal{L}_i \rightarrow \mathbb{Z}^2$ which transform the p -lines into the horizontal lines and the q -lines into the vertical ones. (see for example [1])*

So if $|\mathcal{D}| = 2$ the proposition remains true for any function $f : \mathcal{D} \times \mathbb{Z} \rightarrow \mathbb{N}$ if the set $S_{p,i} = \{x : f(p, x) > 0 \text{ and } (x \bmod \det(p, q) = i)\}$ is infinite for any $p \in \mathcal{D}$ and $i \in \{0, \dots, \det(p, q) - 1\}$.

4 Reconstruction of Periodic Sets

A function $f : \mathbb{Z} \rightarrow \mathbb{N}$ is periodic of period p if $f(x + p) = f(x)$ for any x . A lattice set E is periodic of period $\vec{u} \in \mathbb{Z}^2$ if E is invariant by the translation of vector \vec{u} .

Now we are interested by the following algorithmic problem (\mathcal{D} is a finite set of directions):

RECPER(\mathcal{D})

Data: a function $f : \mathcal{D} \times \mathbb{Z} \rightarrow \mathbb{N}^*$ such that for any $p \in \mathcal{D}$ the function $f(p, \cdot)$ is periodic.

Question: does there exist a *periodic* lattice set E such that $X_p(E) = f(p, \cdot)$ for any p ?

This problem is well-posed because a function f , data of this problem, can be finitely represented, by the periodicity of the partial functions $f(p, \cdot)$.

In fact with two directions, this problem is trivial because we have a proposition which is the periodic version of proposition 1.

Proposition 2 *For any pair \mathcal{D} of directions and any $f : \mathcal{D} \times \mathbb{Z} \rightarrow \mathbb{N}$, if for any $p \in \mathcal{D}$ the function $f(p, \cdot)$ is periodic, then there exists a periodic subset E of \mathbb{Z}^2 such that $X_p(E) = f(p, \cdot)$ for any p .*

We recall Ryser's characterization of X-rays of finites sets along two directions ([5, 2]):

Theorem 3 (Gale-Ryser) *Let $(h_i)_{0 \leq i < m}$ and $(v_j)_{0 \leq j < n}$ two finite sequences of non-negative integers. There exists a finite set $E \subset \{0, \dots, n-1\} \times \{0, \dots, m-1\}$ such that $X_y E = h$ and $X_x E = v$ if and only if:*

$$\sum_{i=0}^{m-1} h_i = \sum_{j=0}^{n-1} v_j \quad (1)$$

$$\sum_{k=0}^j \bar{v}_k \geq \sum_{k=0}^j \tilde{v}_k \text{ for any } 0 \leq j < n \quad (2)$$

where \tilde{v} is the sequence v reordered decreasingly, and $\bar{v}_j = |\{i : h_i > j\}|$.

Proof of Proposition 2: By the same argument than in remark 2, we can suppose that $\mathcal{D} = \{p, q\}$ with $p = x$, $q = y$. Let m be the period of $f(q, \cdot)$, n be the period of $f(p, \cdot)$, $h_i = f(q, i)$, $v_j = f(p, j)$.

Let $V = \sum_{j=0}^{n-1} v_j$, $H = \sum_{i=0}^{m-1} h_i$. Then the sequences $(h_i)_{0 \leq i < mV}$ and $(v_j)_{0 \leq j < nH}$ verify the condition (1) of theorem 3.

Now we fix an integer l and we define $h^l = (h_i)_{0 \leq i < mVl}$ and $v^l = (v_j)_{0 \leq j < nHl}$.

Let $(\tilde{v}_j)_{0 \leq j < n}$ (resp. $(\tilde{v}_j^l)_{0 \leq j < nHl}$) be the sequence $(v_j)_{0 \leq j < n}$ (resp. v^l) reordered decreasingly, $\bar{v}_j = |\{0 \leq i < m : h_i > j\}|$, $\bar{v}_j^l = |\{0 \leq i < mVl : h_i > j\}|$ and finally $m' = \max h_i$. We have

$$\begin{aligned} \bar{v}^l &= (Vl\bar{v}_0, Vl\bar{v}_1, \dots, Vl\bar{v}_{m'-1}, \underbrace{0, 0, \dots, 0}_{Hl-m' \text{ times}}) \\ \tilde{v}^l &= (\underbrace{\tilde{v}_0, \tilde{v}_0, \dots, \tilde{v}_0}_{Hl \text{ times}}, \underbrace{\tilde{v}_1, \dots, \tilde{v}_1}_{Hl \text{ times}}, \tilde{v}_2, \dots, \dots, \underbrace{\tilde{v}_{m-1}, \dots, \tilde{v}_{m-1}}_{Hl \text{ times}}) \end{aligned}$$

So if we take l such that

$$Hl \geq m'$$

$$Vl \sum_{k=0}^j \bar{v}_k \geq (j+1)\tilde{v}_0 \text{ for any } 0 \leq j < m'$$

then the sequences v^l and h^l verify the conditions (1) and (2). So there exists a set E whose X-rays are h^l and v^l . Then $E + (nHl, mVl)\mathbb{Z}$ is the searched set. \square

This proposition is not true with more than two directions:

Proposition 4 *There is no periodic lattice set such that the X-rays along x , y , $x + y$ are the constant function 1.*

It is in fact a simple corollary of the following property:

Lemma 5 *If E is periodic of period $\vec{u} \in \mathbb{Z}^2$ and has X-rays $X_p E, X_q E : \mathbb{Z} \rightarrow \mathbb{N}$ along two directions p and q then we have:*

$$\sum_{k=0}^{|\vec{p}(\vec{u})|-1} X_p E(k) = \sum_{k=0}^{|\vec{q}(\vec{u})|-1} X_q E(k). \quad (3)$$

Proof: We define the two following finite sets:

$$E_1 = \{M \in E : 0 \leq p(M) < |\vec{p}(\vec{u})|\}, \quad E_2 = \{M \in E : 0 \leq q(M) < |\vec{q}(\vec{u})|\}$$

We have $E = E_1 + \vec{u}\mathbb{Z} = E_2 + \vec{u}\mathbb{Z}$ and

$$|E_1| = \sum_{k=0}^{|\vec{p}(\vec{u})|-1} X_p E(k), \quad |E_2| = \sum_{k=0}^{|\vec{q}(\vec{u})|-1} X_q E(k)$$

So we only have to prove $|E_1| = |E_2|$.

Suppose for example $|E_1| > |E_2|$. For any $x \in E_1$, there are $y_x \in E_2$ and $n_x \in \mathbb{Z}$ such that $x = y_x + n_x \vec{u}$. We have $|E_1| > |E_2|$ so there are two distinct points $x_1, x_2 \in E_1$ such that $y_{x_1} = y_{x_2} = y$. We have

$$\begin{aligned} x_1 &= y + n_{x_1} \vec{u} \\ x_2 &= y + n_{x_2} \vec{u} \end{aligned}$$

So $x_1 = x_2 + (n_{x_1} - n_{x_2}) \vec{u}$ and then $p(x_1) = p(x_2) + (n_{x_1} - n_{x_2})p(\vec{u})$ with $0 \leq p(x_1), p(x_2) < |\vec{p}(\vec{u})|$. So $n_{x_1} - n_{x_2} = 0$ and $x_1 = x_2$ which leads to a contradiction. \square

Proof of Proposition 4: Suppose there is a set E periodic of period $\vec{u} = (u_x, u_y)$ which has the prescribed X-rays.

By using Lemma 5 with the directions x, y we have:

$$\sum_{k=0}^{|u_x|-1} 1 = \sum_{k=0}^{|u_y|-1} 1$$

so $|u_x| = |u_y|$.

But again by Lemma 5 with the pair of directions $(x, x+y)$, we also have $|u_x| = |u_x + u_y|$ so $\vec{u} = 0$ contradiction. \square

We can think that if there exists a vector \vec{u} which verifies the equation (3) for any pair of direction in \mathcal{D} then there is solution with period a multiple of \vec{u} . For example, does there exist a periodic set whose X-rays along $x, y, x+y$ are respectively the periodic functions (1), (1), (4, 1, 1, 1, 1) (second set of figure 1) ? In fact I even do not know if $RECPER(\{x, y, x+y\})$ is decidable.

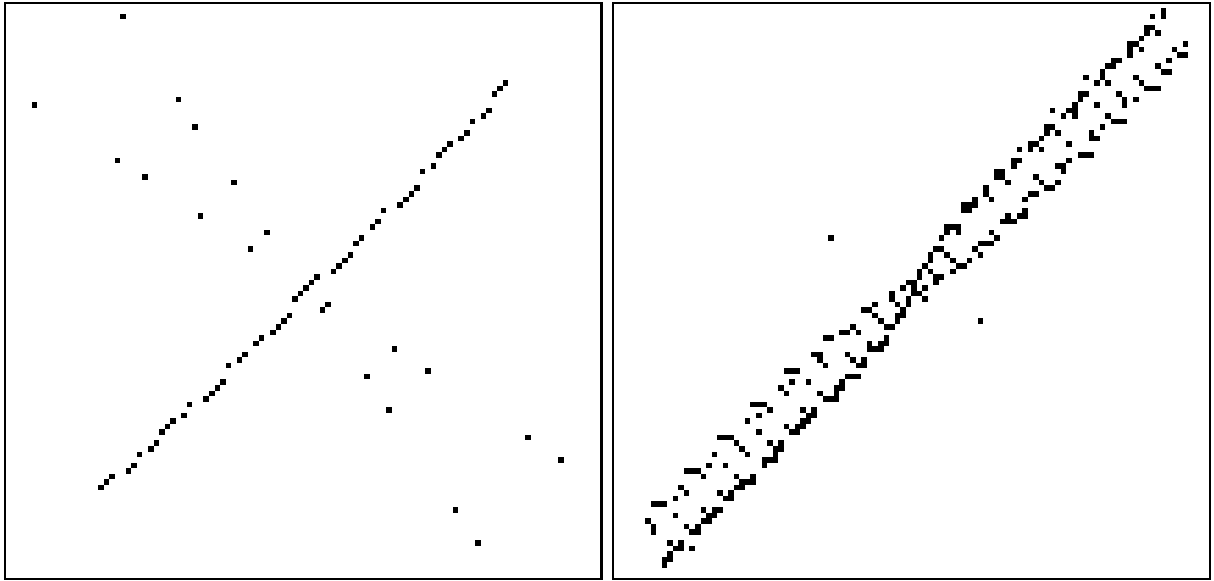


Figure 1: The X-rays along $x, y, x + y$ are the periodic functions $(1), (1), (1)$ for the first one, and $(3), (3), (4, 1, 1, 1, 1)$ for the second one. By proposition 3 the first set is not periodic, for the second one it is more complex.

References

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