

Continuity of the trace operator with respect to the domain and application to shape optimization

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Abstract

Shape optimization amounts to find the optimal shape of a domain which minimizes a given criterion, often called a cost functional. Here, we are interested in the case where the criterion is computed through the solution of a partial differential equation, the so-called state equation, which makes the optimization problem non-trivial. We use a general parameterization of the unknown boundary in order to preserve the physical general information and we prove the continuity of the cost functional.

1 Introduction

Varying domains constitute an important type of inverse problems which appear in connection with a variety of phenomenon in different fields. Shape Identification, free or moving boundary problems, phase change or elastic contact problems fall into the class of such inverse problems with a priori unknown domains. Some problems that lead to boundary inverse problems are: the dam seepage problem [14]; Riabouchinsky flow past circular disks [21]; Falling droplets [35, 32, 38]; the Alt-Caffarelli or Bernoulli problem [4]. The common structure of these problems is that there are elliptic equations for m unknowns and $m + 1$ mixed Dirichlet-Neumann conditions on the unknown surface. Many more problems also fall into this general category, with applications to aluminium production by electrolytic reduction [13]; groundwater flow [25, 29]; free surface waves [9]; semiconductors [2, 16]; electromagnetic casting [33, 31]; the Dam and Bernoulli problems [10, 37, 28, 22, 17].

There is no previous analytic theory covering this general class of boundary inverse problems, although there has been progress on individual problems.

A variety of techniques exist for numerically solving boundary inverse problems, see [14, 26, 16, 1, 11] and the references therein. Among this, shape optimization technique is a useful method for solving the type of problems mentioned above. This is a variational approach, minimizing an integral functional over the variable domain having an unknown boundary. For an introduction to shape optimization, cf [23, 36] and some examples of its use are given in [5, 8, 15, 18, 34].

A general formulation of such optimal shape approaches can be expressed as follows

$$\begin{cases} \text{Minimize } J(\Omega, u) \\ \text{subject to } \Omega \in \Theta_{ad} \text{ and } u \in \mathcal{U}(\Omega), \end{cases} \quad (1)$$

where $\mathcal{U}(\Omega)$ denotes the set of solutions of a given partial differential equation on the domain Ω (state equation on Ω) and Θ_{ad} is a set of admissible domains.

In order to show the existence of a solution for this type of problems, a widely used method is to show that the space $\mathcal{F} = \{(\Omega, u) \mid \Omega \in \Theta_{ad}, u \in \mathcal{U}(\Omega)\}$ is compact with a suitable topology on \mathcal{F} which is induced from topologies on Θ_{ad} and $\mathcal{U}(\Omega)$, and that the cost functional J is lower semi continuous on \mathcal{F} [23].

In this paper, we are interested in the continuity of cost functionals that can be associated to the determination of the location, size and shape of an unknown portion Γ , of the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^2$, from given, Dirichlet and Neumann, boundary conditions on this unknown part of the boundary. More precisely, we consider the boundary cost functional J defined by $J(\Omega, u) = \int_{\Gamma} (u - g)^2 d\sigma$, which is associated with problems where a condition of the type $u = g$ has to be imposed on Γ . The function g is given and u is the solution of a partial differential equation on Ω . In the case where the unknown boundary Γ is the graph of a function, continuity results for this type of functionals with a suitable topology on \mathcal{F} have been obtained in [6, 10, 24]. However, in many physical problems this assumption on the unknown boundary is too restrictive. The situation where the unknown boundary can not be the graph of a function occurs in many engineering problems such as, for example, the dam problem in non-homogeneous porous media [10, 19, 22], the Stefan problem [20], optimal insulating and electro-chemistry [3, 17] and the semiconductor problem [27, 16]. Here, we use a general parameterization of the unknown boundary in order to preserve the physical general information on this boundary. The topology we use on Θ_{ad} is just that associated to the C^1 convergence of the parameterizations.

The main result of this paper is the continuity of the trace operator from $H^r(\Omega)$ ($1 \geq r > \frac{1}{2}$) to $L^2(\Gamma)$ with a constant independent of Γ . This is proved in section 3. At this stage, it should be noted that, in the case of a family of domains with parallel boundaries, a similar result has been obtained by Lions and Magenes in [30]. The continuity of the cost functional J on the space \mathcal{F} is established in section 4. Actually, we prove the continuity of a more general functional.

2 Notations and definitions

Let D be a fixed C^1 open connected bounded subset of \mathbb{R}^2 . An admissible domain Ω will be an open subset of D . The boundary $\partial\Omega$ of Ω is assumed to consist in two parts $\partial\Omega = \Gamma_0 \cup \Gamma$ such that $\Gamma_0 \cap \Gamma = \emptyset$, $meas(\Gamma_0) > 0$ and $meas(\Gamma) > 0$, where Γ is the free boundary part defined by

$$\Gamma = \Gamma(\varphi) = \{\varphi(t) = (\varphi_1(t), \varphi_2(t)); \quad t \in [0, 1]\},$$

$\varphi : [0, 1] \rightarrow \mathbb{R}^2$ is a parameterisation and Γ is assumed to have its endpoints on the boundary of D (see figure 1). In fact, in this general setting, the curve $\Gamma(\varphi)$ can divide D into two open subsets. This is the case for example when D is simply connected. We have then to choose Ω . We can make for example the following choice : We follow the natural orientation given by the parameterisation and take the open set whose exterior unit normal boundary vector is on the left.

We shall also write $\Omega = \Omega(\varphi)$ to indicate the dependence on the parameterization φ .

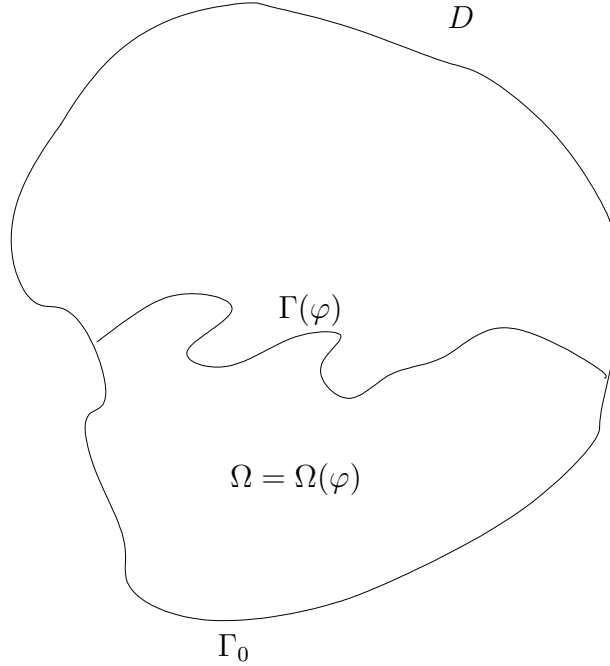


Figure 1: An example of the considered domain $\Omega = \Omega(\varphi)$.

We shall denote by $\nu(x)$ the exterior unit normal vector to ∂D at $x \in \partial D$.

Define V_{ad} to be the set of vector functions $\varphi \in C^1([0, 1], \mathbb{R}^2)$ satisfying the following conditions :

$$|\varphi(t)| \leq C_0, \quad \forall t \in [0, 1],$$

$$C_1|t - t'| \leq |\varphi(t) - \varphi(t')| \leq C_2|t - t'|, \quad \forall t, t' \in [0, 1],$$

$$\varphi([0, 1]) \subset \overline{D} \quad \text{and, for } t = 0 \text{ or } 1, \varphi(t) \in \partial D,$$

$$\text{for } t = 0 \text{ or } 1, |\nu(\varphi(t)) \cdot \varphi'(t)| \geq \epsilon_0, \tag{2}$$

$$\text{and, for all } t \in [0, 1], d(\varphi(t), \partial D) \geq \epsilon_1 t(1 - t), \tag{3}$$

where $C_0, C_1, C_2, \epsilon_0$ and ϵ_1 are positive fixed constants, $d(\varphi(t), \partial D)$ is the distance of the point $\varphi(t)$ from the boundary ∂D , and we are using the standard notations $|(a, b)| = \sqrt{a^2 + b^2}$, $(a, b) \cdot (a', b') = aa' + bb'$. Note that the assumptions (2) and (3) are made to insure that the curve $\Gamma(\varphi)$ touches the boundary ∂D only two times (with the endpoints) and in a transverse manner. The consequence of this is that the domain $\Omega(\varphi)$ is Lipschitz regular.

Clearly, V_{ad} is a closed and bounded subset of $C^1([0, 1], \mathbb{R}^2)$.

Now, the set U_{ad} of admissible functions will be any compact subset of V_{ad} . In other words, U_{ad} is a subset of V_{ad} whose elements and their derivatives are equicontinuous as it follows from Ascoli-Arzelà theorem. An example of such a set U_{ad} is that of a closed subset of V_{ad} which is bounded in $C^{1,\delta}([0, 1], \mathbb{R}^2)$ for some δ such that $0 < \delta \leq 1$.

The space of admissible domains is then defined by :

$$\Theta_{ad} = \{\Omega = \Omega(\varphi) \subset D; \varphi \in U_{ad}\}.$$

Note that the elements of Θ_{ad} are uniformly Lipschitz open sets of \mathbb{R}^2 and so they satisfy the uniform cone property [34].

Assume that $u = u(\Omega) \in H^1(\Omega)$ is the solution of a well posed problem such as

$$\begin{cases} L_{\Omega} u = f & \text{in } \Omega \\ L_{\partial\Omega} u = h & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where $\Omega \in \Theta_{ad}$ and L_{Ω} and $L_{\partial\Omega}$ are partial differential operators on Ω and $\partial\Omega$ respectively.

The cost functional J we are interested in is then defined on the set

$$\mathcal{F} = \{(\Omega(\varphi), u(\Omega)); \Omega(\varphi) \in \Theta_{ad} \text{ and } u(\Omega) \text{ solves (4) on } \Omega(\varphi)\}$$

by

$$J(\Omega(\varphi), u(\Omega)) = \int_{\Gamma(\varphi)} (u(\Omega) - g)^2 d\sigma,$$

where g is a given function on D having essentially the same regularity as $u(\Omega)$. For simplicity and without any restriction on our study, we shall assume that $g = 0$.

Now, the customary problem of shape optimization or optimal shape design is

$$\text{to minimize } J(\Omega, u) \text{ on } \mathcal{F}. \quad (5)$$

As we have already said, this minimization problem is usually solved by endowing the set \mathcal{F} with a topology for which \mathcal{F} is compact and J is lower semi-continuous. Let us therefore define the topology we shall work with. First, we define the convergence of a sequence $(\varphi_n)_n \subset U_{ad}$ by

$$\varphi_n \longrightarrow \varphi \iff \begin{cases} \varphi_n \longrightarrow \varphi \text{ uniformly on } [0, 1] \\ \varphi'_n \longrightarrow \varphi' \text{ uniformly on } [0, 1] \end{cases} \quad (6)$$

that is, iff $\varphi_n \rightarrow \varphi$ in the C^1 topology. Then, the convergence of a sequence $(\Omega_n)_n \subset \Theta_{ad}$, such that $\Omega_n = \Omega(\varphi_n)$, to $\Omega = \Omega(\varphi) \in \Theta_{ad}$ is simply defined by

$$\Omega_n \longrightarrow \Omega \iff \varphi_n \longrightarrow \varphi. \quad (7)$$

Denote by \tilde{u} the uniform extension of u from Ω to an open smooth bounded domain B (a disc, for example), such that $\bar{D} \subset B$ (see [12]). We define the convergence of a sequence $(u_n)_n$ of solutions of (4) on $\Omega(\varphi_n)$ to u the solution of (4) on $\Omega(\varphi)$ by

$$u_n \longrightarrow u \iff \tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } H^1(B). \quad (8)$$

Finally, the topology we put on \mathcal{F} is the one induced by the convergence defined by

$$(\Omega_n, u_n) \longrightarrow (\Omega, u) \iff \begin{cases} \Omega_n \longrightarrow \Omega \\ u_n \longrightarrow u. \end{cases} \quad (9)$$

Thus, the compactness of \mathcal{F} with respect to this topology depends on the type of the state problem (4) and on the compactness of Θ_{ad} with respect to the convergence (7). At this stage, we have to say that it is not our objective here to prove the compactness of \mathcal{F} , the setting being too general. But since U_{ad} is compact, one can obtain the compactness of \mathcal{F} from the continuity of the state problem (4), which is based on the following (reasonable) condition : There exists a constant $C > 0$ such that $\|u(\Omega)\|_{1,\Omega} \leq C, \forall \Omega \in \Theta_{ad}$. Anyhow, the chosen topology is rather natural, is used in many applied problems and, as we shall see, it allows to prove the continuity of the cost functional J and even the continuity of more general boundary functionals (see section 4).

3 Continuity of the trace operator

In this section, we give a proof of the trace theorem which allows us to estimate the norm of the trace operator by a constant independent of the free boundary Γ .

Theorem 1 *Let u be in $H^r(B)$, $1 \geq r > \frac{1}{2}$, and $\Omega(\varphi) \in \Theta_{ad}$. Then, there exists a constant K independent of φ such that*

$$\|u\|_{0,\Gamma(\varphi)} \leq K \|u\|_{r,B}$$

where $\|\cdot\|_{0,\Gamma(\varphi)}$ is the $L^2(\Gamma(\varphi))$ -norm and $\|\cdot\|_{r,B}$ is the $H^r(B)$ -norm.

The proof of this result is based on the following constructions and techniques.

Let $\varphi \in U_{ad}$. By using standard techniques, we can construct a continuous extension operator from $C^1([0, 1], \mathbb{R}^2)$ to $\tilde{C}^1([-1, 2], \mathbb{R}^2) = \{\varphi \in C^1(\mathbb{R}, \mathbb{R}^2); \text{supp}(\varphi) \subset [-1, 2]\}$, such that the extension $\tilde{\varphi}$ of φ to \mathbb{R} satisfies

$$\|\tilde{\varphi}'\|_{L^\infty(\mathbb{R})} = \|\varphi'\|_{L^\infty([0,1])}, \quad (10)$$

where $\|\cdot\|_{L^\infty}$ is the L^∞ -norm.

Now, let $\chi \in \mathcal{D}(\mathbb{R})$ such that $\chi \geq 0$, $\int_{\mathbb{R}} \chi dx = 1$ and $\text{supp}(\chi) \subset [-1, 1]$. We define the function $\psi_k = (\psi_{1,k}, \psi_{2,k})$, $k \in \mathbb{N}^*$, on \mathbb{R} as a regularized function of $\tilde{\varphi}$, that is,

$$\psi_k(t) = \int_{\mathbb{R}} \tilde{\varphi}(t - \tau) \chi(k\tau) k d\tau.$$

Note that

$$\psi'_k(t) = \int_{\mathbb{R}} \tilde{\varphi}'(t - \tau) \chi(k\tau) k d\tau.$$

Since $\tilde{\varphi}'$ is continuous, it is well known that ψ'_k converges uniformly to $\tilde{\varphi}'$ on any compact subset of \mathbb{R} as $k \longrightarrow \infty$. In particular,

$$\lim_{k \longrightarrow \infty} \|\psi'_k - \tilde{\varphi}'\|_{L^\infty([0,1])} = 0.$$

In fact, we can show that for any $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}^*$ independent of φ such that, for $k \geq k_\varepsilon$,

$$\|\psi'_k - \tilde{\varphi}'\|_{L^\infty([0,1])} < \varepsilon.$$

Indeed, we have

$$\begin{aligned} |\psi'_k(t) - \tilde{\varphi}'(t)| &= \left| \int_{\mathbb{R}} (\tilde{\varphi}'(t - \tau) - \tilde{\varphi}'(t)) \chi(k\tau) k d\tau \right| \\ &\leq \int_{\mathbb{R}} |\tilde{\varphi}'(t - \frac{\tau}{k}) - \tilde{\varphi}'(t)| \chi(\tau) d\tau. \end{aligned}$$

Since the extension operator is continuous from $C^1([0, 1], \mathbb{R}^2)$ to $\tilde{C}^1([-1, 2], \mathbb{R}^2)$, the functions $\tilde{\varphi}'$, are equicontinuous and uniformly continuous when φ describes U_{ad} . So, for $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ independent of φ such that

$$\forall t, t' \in \mathbb{R}, |t - t'| \leq \gamma_\varepsilon \quad \text{implies that} \quad |\tilde{\varphi}'(t) - \tilde{\varphi}'(t')| \leq \varepsilon, \quad \forall \varphi \in U_{ad}. \quad (11)$$

Hence, for k such that $\frac{|\tau|}{k} \leq \frac{1}{k} \leq \gamma_\varepsilon$ and all $t \in \mathbb{R}$,

$$|\psi'_k(t) - \tilde{\varphi}'(t)| \leq \int_{\mathbb{R}} \varepsilon \chi(\tau) d\tau = \varepsilon.$$

So, we can take $k_\varepsilon = \lceil \frac{1}{\gamma_\varepsilon} \rceil + 1$, where $\lceil \cdot \rceil$ denotes the integral part.

Now, to the given $\varphi \in U_{ad}$, we associate the function $\Phi \in C^1([0, 1] \times \mathbb{R}, \mathbb{R}^2)$ defined by

$$\Phi(t, s) = \varphi(t) + s \psi'_{k_\varepsilon}(t)^\perp, \quad s \in \mathbb{R}, \quad (12)$$

where we are using the notation $(a, b)^\perp = (-b, a)$. In what follows, we shall omit the index k_ε and write ψ instead of ψ_{k_ε} .

The following lemma is basic for the proof of Theorem 1.

Lemma 1 *There exists a small enough $s_0 > 0$ such that s_0 is independent of $\varphi \in U_{ad}$ and the following three assertions hold.*

(i) *The Jacobian, $J\Phi$, of Φ is such that*

$$|J\Phi| \geq \frac{1}{2} C_1^2 \quad \text{on} \quad [0, 1] \times [-s_0, s_0]. \quad (13)$$

(ii) *There exists $C_3 > 0$ independent of φ such that*

$$|\Phi(t, s) - \Phi(t', s')| \leq C_3 |(t - t', s - s')|, \quad \forall (t, s), (t', s') \in [0, 1] \times [-s_0, s_0]. \quad (14)$$

(iii) *Φ is injective in $[0, 1] \times [-s_0, s_0]$, and, more precisely,*

$$|\Phi(t, s) - \Phi(t', s')| \geq \frac{\sqrt{2}}{4} C_1 |(t - t', s - s')|, \quad \forall (t, s), (t', s') \in [0, 1] \times [-s_0, s_0]. \quad (15)$$

where C_1 is the same constant as that in the definition of U_{ad} .

Proof :

(i) We have that

$$\Phi' = \begin{pmatrix} \varphi'_1 - s \psi''_2 & -\psi'_2 \\ \varphi'_2 + s \psi''_1 & \psi'_1 \end{pmatrix}.$$

So,

$$\begin{aligned} J\Phi = \det(\Phi') &= \varphi'_1 \psi'_1 + \varphi'_2 \psi'_2 + s(\psi''_1 \psi'_2 - \psi'_1 \psi''_2) \\ &= |\varphi'|^2 + \varphi' \cdot (\psi - \varphi)' - s \psi'' \cdot \psi'^{\perp}. \end{aligned}$$

It follows from the definition of $\psi = \psi_{k_\varepsilon}$ that

$$\|\psi'^{\perp}\|_{L^\infty([0,1])} \leq \|\psi'\|_{L^\infty(\mathbb{R})} \leq \|\tilde{\varphi}'\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \chi(k_\varepsilon \tau) k_\varepsilon d\tau \|\tilde{\varphi}'\|_{L^\infty([0,1])} \leq C_2 \quad (16)$$

and

$$\|\psi''\|_{L^\infty([0,1])} \leq \|\psi''\|_{L^\infty(\mathbb{R})} \leq \|\tilde{\varphi}'\|_{L^\infty(\mathbb{R})} k_\varepsilon^2 \int_{\mathbb{R}} |\chi'(k_\varepsilon \tau)| d\tau \leq C_2 k_\varepsilon \int_{\mathbb{R}} |\chi'(\tau)| d\tau \equiv C'_2 k_\varepsilon. \quad (17)$$

Using equations (16), (17) and the inequality $C_1 \leq \|\varphi'\|_{L^\infty([0,1])} \leq C_2$, which follows from the definition of U_{ad} , we obtain

$$|J\Phi| \geq C_1^2 - C_2 \varepsilon - |s| C_2 C'_2 k_\varepsilon \geq C_1^2 - C_2 (\varepsilon + s_0 C'_2 k_\varepsilon), \quad \forall s, |s| \leq s_0.$$

Finally, choosing ε and s_0 so small that

$$C_2 (\varepsilon + s_0 C'_2 k_\varepsilon) \leq \frac{1}{2} C_1^2,$$

we obtain

$$|J\Phi| \geq \frac{1}{2} C_1^2 \quad \text{on} \quad [0, 1] \times [-s_0, s_0].$$

(ii) We have, for all $(t, s) \in [0, 1] \times [-s_0, s_0]$,

$$\begin{aligned} |\varphi'(t) + s \psi''(t)^{\perp}| &\leq \|\varphi'\|_{L^\infty([0,1])} + s \|\psi''\|_{L^\infty([0,1])} \\ &\leq C_2 + s_0 C'_2 k_\varepsilon, \end{aligned}$$

and

$$|\psi'(t)^{\perp}| \leq \|\psi'\|_{L^\infty([0,1])} \leq C_2.$$

Hence, if $C_3 C_2 (1 + s_0 k_\varepsilon \|\chi'\|_{L^1})$, we obtain, by Taylor formula,

$$|\Phi(t, s) - \Phi(t', s')| \leq C_3 |(t - t', s - s')|,$$

for all $(t, s), (t', s') \in [0, 1] \times [-s_0, s_0]$.

(iii) Let us now show that Φ is injective in $[0, 1] \times [-s_0, s_0]$, for s_0 small enough and independent of φ . To this end, we shall show that there exists a constant M such that

$$|\Phi(t, s) - \Phi(t', s')| \geq M |(t - t', s - s')|, \quad \forall (t, s), (t', s') \in [0, 1] \times [-s_0, s_0].$$

Let $(t, s), (t', s') \in [0, 1] \times [-s_0, s_0]$. We have

$$\begin{aligned} \Phi(t, s) - \Phi(t', s') &= \varphi(t) - \varphi(t') + s \psi'(t)^{\perp} - s' \psi'(t')^{\perp} \\ &= \varphi(t) - \varphi(t') + (s - s') \psi'(t)^{\perp} + s' (\psi'(t)^{\perp} - \psi'(t')^{\perp}). \end{aligned} \quad (18)$$

Now, we discuss two cases :

First case, if $|s - s'| \leq \eta |t - t'|$, η small enough, we have

$$\begin{aligned}
|\Phi(t, s) - \Phi(t', s')| &\geq C_1 |t - t'| - \eta |t - t'| \|\psi'\|_{L^\infty([0,1])} - s_0 \|\psi''\|_{L^\infty([0,1])} |t - t'| \\
&\geq (C_1 - \eta \|\psi'\|_{L^\infty([0,1])} - s_0 \|\psi''\|_{L^\infty([0,1])}) |t - t'| \\
&\geq (C_1 - \eta C_2 - s_0 C'_2 k_\varepsilon) |t - t'| \\
&\geq (C_1 - \eta C_2 - s_0 C'_2 k_\varepsilon) \left(\frac{1}{2} |t - t'|^2 + \frac{1}{2} \frac{1}{\eta^2} |s - s'|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Assuming that $\eta \leq 1$, we obtain

$$\|\Phi(t, s) - \Phi(t', s')\| \geq \frac{1}{\sqrt{2}} (C_1 - \eta C_2 - s_0 C'_2 k_\varepsilon) |(t - t', s - s')|.$$

Second case, if $|s - s'| \geq \eta |t - t'|$, we can rewrite the terms $\varphi(t) - \varphi(t')$ and $(s - s') \psi'(t)^\perp$ in (18) as

$$\varphi(t') - \varphi(t) = (t' - t) \varphi'(t) + (t - t') \int_0^1 (\varphi'(t + \tau(t - t')) - \varphi'(t)) d\tau$$

and

$$(s - s') \psi'^\perp(t) = (s - s') (\varphi'(t))^\perp + (s - s') (\psi'(t) - \varphi'(t))^\perp.$$

We know that

$$\|\psi' - \varphi'\|_{L^\infty([0,1])} \leq \varepsilon$$

and, from (11), that there exists $\gamma_\varepsilon > 0$ independent of φ such that

$$|t - t'| \leq \gamma_\varepsilon \quad \text{implies that} \quad |\varphi'(t + \tau(t - t')) - \varphi'(t)| \leq \varepsilon, \quad \forall \tau \in [0, 1]. \quad (19)$$

Hence,

$$|s - s'| \leq \eta \gamma_\varepsilon \quad \text{implies that} \quad \left| \int_0^1 (\varphi'(t + \tau(t - t')) - \varphi'(t)) d\tau \right| \leq \varepsilon. \quad (20)$$

Thus, for $s_0 \leq \frac{1}{2} \eta \gamma_\varepsilon$, we have

$$\begin{aligned}
&|\Phi(t, s) - \Phi(t', s')| \\
&\geq |-(t - t') \varphi'(t) + (s - s') \varphi'(t)^\perp| - \varepsilon |t - t'| - \varepsilon |s - s'| - s_0 \|\psi''\|_{L^\infty([0,1])} |t - t'| \\
&\geq \left(|t - t'|^2 |\varphi'(t)|^2 + |s - s'|^2 |\varphi'(t)^\perp|^2 \right)^{\frac{1}{2}} - (2\varepsilon + s_0 C'_2 k_\varepsilon) |(t - t', s - s')| \\
&\geq (C_1 - 2\varepsilon - s_0 C'_2 k_\varepsilon) |(t - t', s - s')|.
\end{aligned}$$

The constants ε , η and s_0 can be chosen such that, for example,

$$\eta C_2 + s_0 C'_2 k_\varepsilon \leq \frac{C_1}{2}, \quad 2\varepsilon + s_0 C'_2 k_\varepsilon \leq \frac{C_1}{2} \quad \text{and} \quad s_0 \leq \frac{1}{2} \eta \gamma_\varepsilon.$$

Hence, it suffices to take

$$\eta = \frac{C_1}{4C_2}, \quad \varepsilon = \frac{C_1}{8} \quad \text{and} \quad s_0 = \min \left\{ \frac{C_1 \gamma_\varepsilon}{8C_2}, \frac{C_2}{8C_2 k_\varepsilon \|\chi\|_{L^1}} \right\}.$$

This shows that there exists s_0 independent of φ , such that Φ is injective in $[0, 1] \times [-s_0, s_0]$ and

$$|\Phi(t, s) - \Phi(t', s')| \geq \frac{\sqrt{2}}{4} C_1 |(t - t', s - s')|, \quad \forall (t, s), (t', s') \in [0, 1] \times [-s_0, s_0].$$

■

Proof of theorem 1 :

Let us denote by $\mathcal{I} =]0, 1[$ and $\mathcal{J} =]-s_0, s_0[$ and let us consider $u \in H^r(B)$ ($\frac{1}{2} < r \leq 1$). We have that

$$u(\varphi(t)) = u|_{\Gamma} \circ \varphi(t), \quad \forall t \in [0, 1],$$

and, on the other hand,

$$u(\varphi(t)) = u(\Phi(t, s))|_{s=0} \equiv v(t, s)|_{s=0}, \quad \forall t \in [0, 1],$$

where Φ is the function defined in (12) and $v = u \circ \Phi$. From the above lemma, we have that Φ is a C^1 diffeomorphism from $\mathcal{I} \times \mathcal{J}$ onto an open set of \mathbb{R}^2 which is some tubular neighborhood of Γ , and thus $v \in H^r(\mathcal{I} \times \mathcal{J})$. Now,

$$\|u\|_{0, \Gamma(\varphi)} \leq C_2 \|u \circ \varphi\|_{0, \mathcal{I}} = C_2 \|v(t, s)|_{s=0}\|_{0, \mathcal{I}},$$

and, according to the standard result on the continuity of the trace operator from $H^r(\mathcal{I} \times \mathcal{J})$ to $L^2(\mathcal{I} \times \{0\})$ (for $r > \frac{1}{2}$), there exists a constant β , independent of v , such that $\|v|_{s=0}\|_{0, \mathcal{I}} \leq \beta \|v\|_{r, \mathcal{I} \times \mathcal{J}}$ for all $v \in H^r(\mathcal{I} \times \mathcal{J})$. Hence,

$$\|u\|_{0, \Gamma(\varphi)} \leq C_2 \beta \|v\|_{r, \mathcal{I} \times \mathcal{J}}.$$

Next, there exists a constant C_4 independent of φ , such that

$$\|v\|_{r, \mathcal{I} \times \mathcal{J}} \leq C_4 \|u\|_{r, B}.$$

Indeed, we have

$$\int \int_{\mathcal{I} \times \mathcal{J}} |u(\Phi(t, s))|^2 dt ds = \int \int_{\tilde{\Omega}} |u(x, y)|^2 |\det((\Phi^{-1})'(x, y))| dx dy$$

where $\tilde{\Omega} = \Phi(\mathcal{I} \times \mathcal{J})$ and $(x, y) = \Phi(t, s)$, and it follows from Lemma 1 that

$$\begin{aligned} \int \int_{\mathcal{I} \times \mathcal{J}} |u(\Phi(t, s))|^2 dt ds &= \int \int_{\tilde{\Omega}} |u(x, y)|^2 \frac{dx dy}{|\det \Phi'(\Phi^{-1}(x, y))|} \\ &\leq \frac{2}{C_1^2} \int \int_{\tilde{\Omega}} |u(x, y)|^2 dx dy \\ &\leq \frac{2}{C_1^2} \|u\|_{0, \tilde{\Omega}}^2. \end{aligned}$$

Thus,

$$\|u \circ \Phi\|_{0, \mathcal{I} \times \mathcal{J}}^2 \leq \frac{2}{C_1^2} \|u\|_{0, \tilde{\Omega}}^2. \quad (21)$$

On the other hand, setting $\mathcal{U} = \mathcal{I} \times \mathcal{J}$, we have from Lemma 1

$$\begin{aligned} &\int \int_{\mathcal{U} \times \mathcal{U}} \frac{|u \circ \Phi(t, s) - u \circ \Phi(t', s')|^2}{|(t, s) - (t', s')|^{2r+2}} dt dt' ds ds' \\ &= \int \int_{\tilde{\Omega} \times \tilde{\Omega}} \frac{|u(x, y) - u(x', y')|^2}{|\Phi^{-1}(x, y) - \Phi^{-1}(x', y')|^{2r+2}} \times |\det((\Phi^{-1})'(x, y))| |\det((\Phi^{-1})'(x', y'))| dx dx' dy dy' \\ &\leq \frac{4}{C_1^4} \int \int_{\tilde{\Omega} \times \tilde{\Omega}} \frac{|u(x, y) - u(x', y')|^2}{|\Phi^{-1}(x, y) - \Phi^{-1}(x', y')|^{2r+2}} dx dx' dy dy', \end{aligned}$$

Since Φ is Lipschitz of constant C_3 , we have that

$$|\Phi^{-1}(x, y) - \Phi^{-1}(x', y')| \geq \frac{1}{C_3} |(x - x', y - y')| \quad \forall (x, y), (x', y') \in \tilde{\Omega} \times \tilde{\Omega}. \quad (22)$$

Therefore,

$$\begin{aligned} & \int \int_{\mathcal{U} \times \mathcal{U}} \frac{|u \circ \Phi(t, s) - u \circ \Phi(t', s')|^2}{|(t, s) - (t', s')|^{2r+2}} dt dt' ds ds' \\ & \leq \frac{4}{C_1^4} C_3^{2+2r} \int \int_{\tilde{\Omega} \times \tilde{\Omega}} \frac{|u(x, y) - u(x', y')|^2}{|(x, y) - (x', y')|^{2r+2}} dx dx' dy dy'. \\ & \leq \frac{4 C_3^{2+2r}}{C_1^4} \|u\|_{r, B}^2 \end{aligned} \quad (23)$$

(21) and (23) imply the theorem when $r < 1$.

When $r = 1$, we estimate the partial derivatives of in the same manner (one can also estimate $\|u\|_{r, B}$ by $\|u\|_{1, B}$). This achieves the proof of the theorem. \blacksquare

Now, as a consequence of Theorem 1, we state and prove the following convergence result which essentially says that $u|_{\Gamma_n} \rightarrow u|_{\Gamma}$. It will also be needed in next section.

Corollary 1 *Let $(\varphi_n)_n \subset U_{ad}$ be a sequence such that $\varphi_n \rightarrow \varphi$ in the sense of (6), that is for the C^1 convergence, and let $u \in H^1(B)$. Then*

$$\lim_{n \rightarrow \infty} u \circ \varphi_n u \circ \varphi \quad \text{in } L^2([0, 1]).$$

Proof :

First, it follows from the density of $\mathcal{D}(\bar{B})$ in $H^1(B)$ (see [?]) that, for a given $\varepsilon > 0$, there exists $v_\varepsilon \in \mathcal{D}(\bar{B})$ such that

$$\|v_\varepsilon - u\|_{1, B} \leq \frac{\varepsilon}{3\sqrt{C_1} K},$$

where K is the constant of Theorem 1. Next, we can write

$$\begin{aligned} \|u \circ \varphi_n - u \circ \varphi\|_{0, [0, 1]} &= \|u \circ \varphi_n - v_\varepsilon \circ \varphi_n + v_\varepsilon \circ \varphi_n - v_\varepsilon \circ \varphi + v_\varepsilon \circ \varphi - u \circ \varphi\|_{0, [0, 1]} \\ &\leq \|u \circ \varphi_n - v_\varepsilon \circ \varphi_n\|_{0, [0, 1]} + \|v_\varepsilon \circ \varphi_n - v_\varepsilon \circ \varphi\|_{0, [0, 1]} \\ &\quad + \|v_\varepsilon \circ \varphi - u \circ \varphi\|_{0, [0, 1]}. \end{aligned} \quad (24)$$

Now, by Theorem 1,

$$\|v_\varepsilon \circ \varphi_n - u \circ \varphi_n\|_{0, [0, 1]} = \left(\int_0^1 |v_\varepsilon \circ \varphi_n(t) - u \circ \varphi_n(t)|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{C_1} K \|v_\varepsilon - u\|_{1, B},$$

and according to the Lebesgue convergence theorem, we have

$$\lim_{n \rightarrow \infty} \|v_\varepsilon \circ \varphi_n - v_\varepsilon \circ \varphi\|_{0, [0, 1]} = \lim_{n \rightarrow \infty} \left(\int_0^1 |v_\varepsilon \circ \varphi_n(t) - v_\varepsilon \circ \varphi(t)|^2 dt \right)^{\frac{1}{2}} = 0$$

Thus,

$$\|v_\varepsilon \circ \varphi_n - u \circ \varphi_n\|_{0, [0, 1]} = \left(\int_0^1 |v_\varepsilon \circ \varphi_n(t) - u \circ \varphi_n(t)|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{C_1} K \|v_\varepsilon - u\|_{1, B} \leq \frac{\varepsilon}{3}, \quad (25)$$

$$\|v_\varepsilon \circ \varphi - u \circ \varphi\|_{0,[0,1]} = \left(\int_0^1 |v_\varepsilon \circ \varphi(t) - u \circ \varphi(t)|^2 dt \right)^{\frac{1}{2}} \leq \sqrt{C_1} K \|v_\varepsilon - u\|_{1,B} \leq \frac{\varepsilon}{3}, \quad (26)$$

and there exists $N \in \mathbb{N}^*$ such that $n \geq N$ implies

$$\|v_\varepsilon \circ \varphi_n - v_\varepsilon \circ \varphi\|_{0,[0,1]} = \left(\int_0^1 |v_\varepsilon \circ \varphi_n(t) - v_\varepsilon \circ \varphi(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{3}. \quad (27)$$

Hence, for $n \geq N$,

$$\|u \circ \varphi_n - u \circ \varphi\|_{0,[0,1]} = \left(\int_0^1 |u \circ \varphi_n(t) - u \circ \varphi(t)|^2 dt \right)^{\frac{1}{2}} \leq \varepsilon, \quad (28)$$

which ends the proof of the corollary. \blacksquare

4 Applications

We now apply Theorem 1 to study the continuity of the boundary cost functional

$$J(\Omega, u) = \int_\Gamma |u|^2 d\sigma$$

which appears in many problems of optimal shape design, often as a shape optimization formulation of the Dirichlet type condition of a free boundary problem. Of course, Theorem 2 below can also be considered as a continuity result for the trace operator with respect to the couple (u, Γ) .

Theorem 2 *The functional J is continuous on \mathcal{F} with the topology induced by the convergence defined in (9).*

Proof :

Let $\{(\Omega_n, u_n)\}_n$ be a sequence of \mathcal{F} , such that $\Omega_n = \Omega(\varphi_n)$, $\Omega = \Omega(\varphi)$ and

$$(\Omega_n, u_n) \longrightarrow (\Omega, u) \quad \text{as } n \longrightarrow \infty.$$

In what follows, the functions under consideration are of course the extensions \tilde{u} , $\tilde{u}_n \in H^1(B)$, but for simplicity we shall drop the “ \sim ”. To show that $J(\Omega_n, u_n) \longrightarrow J(\Omega, u)$, let us prove that $\sqrt{J(\Omega_n, u_n)} \longrightarrow \sqrt{J(\Omega, u)}$. We have:

$$\begin{aligned} & \left| \sqrt{J(\Omega_n, u_n)} - \sqrt{J(\Omega, u)} \right| \\ &= \left| \|u_n \circ \varphi_n \cdot |\varphi'_n|^{\frac{1}{2}}\|_{0,[0,1]} - \|u \circ \varphi \cdot |\varphi'|^{\frac{1}{2}}\|_{0,[0,1]} \right| \\ &\leq \left\| \|u_n \circ \varphi_n \cdot |\varphi'_n|^{\frac{1}{2}} - u \circ \varphi \cdot |\varphi'|^{\frac{1}{2}}\|_{0,[0,1]} \right\| \\ &\leq \left\| \|u_n \circ \varphi_n - u \circ \varphi_n\|_{0,[0,1]} \cdot \|\varphi'_n\|^{\frac{1}{2}} + \|(u \circ \varphi_n - u \circ \varphi) \cdot |\varphi'_n|^{\frac{1}{2}}\|_{0,[0,1]} \right. \\ &\quad \left. + \|u \circ \varphi \cdot (|\varphi'_n|^{\frac{1}{2}} - |\varphi'|^{\frac{1}{2}})\|_{0,[0,1]} \right\| \\ &\leq \|u_n - u\|_{0,\Gamma_n} + \sqrt{C_2} \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]} + \frac{1}{\sqrt{C_1}} \|u\|_{0,\Gamma} \sup_{[0,1]} \left| |\varphi'_n|^{\frac{1}{2}} - |\varphi'|^{\frac{1}{2}} \right| \\ &\leq \|u_n - u\|_{r,B} + \sqrt{C_2} \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]} + \frac{1}{2C_1} \|u\|_{0,\Gamma} \sup_{[0,1]} |\varphi'_n - \varphi'|, \end{aligned} \quad (29)$$

$\frac{1}{2} < r < 1$, by Theorem 1. The theorem follows from Corollary 1 and the compactness of the injection of $H^1(B)$ into $H^r(B)$. \blacksquare

Last, we establish the following more general continuity result. It concerns a general boundary cost functional of the form

$$J(\Omega, u) = \int_{\Gamma} f(x, u(x)) d\sigma \quad (30)$$

where f is a real continuous function defined in $\mathbb{R}^2 \times \mathbb{R}$ (or $\mathbb{R}^2 \times \mathbb{C}$ if u is allowed to take complex values) such that

$$|f(x, u)| \leq C (1 + |u|^2) \quad (31)$$

for all x and u , C being a positive constant. Note that it follows from (31) that J is well defined on \mathcal{F} .

Theorem 3 *Under the above assumptions, J is continuous on \mathcal{F} with the topology induced by the convergence defined in (9).*

Proof :

Taking the same sequence (Ω_n, u_n) as in the beginning of the proof of Theorem 2, we can write

$$J(\Omega_n, u_n) = I_1(n) + I_2(n) + J(\Omega, u)$$

where

$$I_1(n) = \int_0^1 [f(\varphi_n(t), u_n(\varphi_n(t))) - f(\varphi(t), u(\varphi(t)))] |\varphi_n'(t)| dt,$$

$$I_2(n) = \int_0^1 f(\varphi(t), u(\varphi(t))) (|\varphi_n'(t)| - |\varphi'(t)|) dt.$$

$I_2(n)$ is easy to estimate. Indeed, obviously,

$$|I_2(n)| \leq C \int_0^1 (1 + |u(\varphi(t))|^2) dt \sup_{[0,1]} |\varphi_n' - \varphi'|,$$

so,

$$\lim_{n \rightarrow \infty} I_2(n) = 0.$$

To treat $I_1(n)$, we need the following lemma which improves Corollary 1.

Lemma 2 *Let $(\varphi_n) \subset U_{ad}$ be a sequence such that $\varphi_n \rightarrow \varphi \in U_{ad}$ in the sense of (6), and let $u, u_n \in H^1(B)$ such that u_n converges weakly to u in $H^1(B)$. Then, $u_n \circ \varphi_n \rightarrow u \circ \varphi$ in $L^2([0, 1])$.*

Proof :

We have

$$\begin{aligned} \|u_n \circ \varphi_n - u \circ \varphi\|_{0,[0,1]} &\leq \|u_n \circ \varphi_n - u \circ \varphi_n\|_{0,[0,1]} + \|u \circ \varphi_n - u \circ \varphi\|_{0,[0,1]} \\ &\leq \frac{1}{\sqrt{C_1}} \|u_n - u\|_{0,\Gamma_n} + \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]} \\ &\leq \frac{K}{\sqrt{C_1}} \|u_n - u\|_{r,B} + \|(u \circ \varphi_n - u \circ \varphi)\|_{0,[0,1]}, \end{aligned}$$

$\Gamma_n = \Gamma(\varphi_n)$, $\frac{1}{2} < r < 1$, by Theorem 1. Thus, the lemma follows from Corollary 1 and the compactness of the canonical injection of $H^1(B)$ into $H^r(B)$. ■

End of proof of Theorem 3: Since $u_n \circ \varphi_n \rightarrow u \circ \varphi$ in $L^2([0, 1])$, there exists a subsequence $u_{n_k} \circ \varphi_{n_k}$ and $v \in L^2([0, 1])$ such that (see[7])

$$u_{n_k} \circ \varphi_{n_k} \rightarrow u \circ \varphi, \text{ a.e. and } |u_{n_k} \circ \varphi_{n_k}| \leq v, \text{ a.e.} \quad (32)$$

Consider now $I_1(n_k)$. We have :

- $[f(\varphi_{n_k}(t), u_{n_k}(\varphi_{n_k}(t))) - f(\varphi(t), u(\varphi(t)))] |\varphi'_{n_k}(t)| \rightarrow 0, \text{ a.e.}$
- $|f(\varphi_{n_k}(t), u_{n_k}(\varphi_{n_k}(t))) - f(\varphi(t), u(\varphi(t)))| |\varphi'_{n_k}(t)| \leq C_2 C(2 + v(t)^2 + |u(\varphi(t))|^2), \text{ a.e.}$

So, it follows from the Lebesgue convergence theorem that $I_1(n_k) \rightarrow 0$. However, this does not allow us to conclude. Therefore, consider $\underline{\lim} J(\Omega_n, u_n)$. Since $(J(\Omega_n, u_n))$ is a bounded real sequence, this limit exists and is equal to $\lim J(\Omega_{n_k}, u_{n_k})$ for some subsequence. Now,

$$\underline{\lim} J(\Omega_n, u_n) = \lim J(\Omega_{n_k}, u_{n_k}) = \lim J(\Omega_{n_{k_l}}, u_{n_{k_l}})$$

where (n_{k_l}) is such that the subsequence $u_{n_{k_l}} \circ \varphi_{n_{k_l}}$ satisfies (32). Hence, by the argument given above,

$$\underline{\lim} J(\Omega_n, u_n) = \lim J(\Omega_{n_{k_l}}, u_{n_{k_l}}) = J(\Omega, u).$$

Of course, the same argument works for $\overline{\lim} J(\Omega_n, u_n)$, and this proves the theorem. ■

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