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On the Cohomology of the Lie Superalgebra of Contact Vector Fields on $S^{1|2}$

B. Agrebaoui † *N. Ben Fraj* † *and S. Omri* ‡

† *Département de Mathématiques, Faculté des Sciences de Sfax, Route de Soukra 3018 Sfax BP 802, Tunisie*
E-mail:bagreba@fss.rnu.tn

† *Institut Supérieur de Sciences Appliquées et Technologie, Sousse, Tunisie*
E-mail: benfraj_nizar@yahoo.fr

‡ *Département de Mathématiques, Faculté des Sciences de Sfax, Route de Soukra, 3018 Sfax BP 802, Tunisie*
E-mail: omri_salem@yahoo.fr

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Abstract

We investigate the first cohomology space associated with the embedding of the Lie superalgebra $\mathcal{K}(2)$ of contact vector fields on the (1,2)-dimensional supercircle $S^{1|2}$ in the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$ of superpseudodifferential operators with smooth coefficients. Following Ovsienko and Roger, we show that this space is ten-dimensional with only even cocycles and we give explicit expressions of the basis cocycles.

1 Introduction

V. Ovsienko and C. Roger [5] calculated the space $H^1(\mathrm{Vect}(S^1),\ \Psi\mathcal{DO}(S^1))$, where $\mathrm{Vect}(S^1)$ is the Lie algebra of smooth vector fields on the circle S^1 and $\Psi\mathcal{DO}(S^1)$ is the space of pseudodifferential operators with smooth coefficients. The action is given by the natural embedding of $\mathrm{Vect}(S^1)$ in $\Psi\mathcal{DO}(S^1)$. They used the results of D. B. Fuchs [3] on the cohomology of $\mathrm{Vect}(S^1)$ with coefficients in weighted densities to determine the cohomology with coefficients in the graded module $Gr(\Psi\mathcal{DO}(S^1))$, namely $H^1(\mathrm{Vect}(S^1),\ Gr^p(\Psi\mathcal{DO}(S^1)))$; here $Gr^p(\Psi\mathcal{DO}(S^1))$ is isomorphic, as $\mathrm{Vect}(S^1)$-module, to the space of weighted densities $\mathcal{F}_p$ of weight $-p$ on S^1. To compute $H^1(\mathrm{Vect}(S^1),\ \Psi\mathcal{DO}(S^1))$, V. Ovsienko and C. Roger applied the theory of spectral sequences to a filtered module over a Lie algebra.

In a recent paper [2], using the same methods as in the paper [5], two of the authors computed $H^1(\mathcal{K}(1),\ \mathcal{S}\Psi\mathcal{DO}(S^{1|1}))$, where $\mathcal{K}(1)$ is the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the supercircle $S^{1|1}$ and $\mathcal{S}\Psi\mathcal{DO}(S^{1|1})$ is the space of superpseudodifferential operators on $S^{1|1}$.

Here, we follow again the same methods by V. Ovsienko and C. Roger [5] to calculate $H^1(\mathcal{K}(2),\ \mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$.

The paper ([5]) contains also the classification of polynomial deformations of the natural embedding of $\mathrm{Vect}(S^1)$ in $\Psi\mathcal{DO}(S^1)$. The multi-parameter deformations of the embedding of $\mathcal{K}(1)$ into $\mathcal{S}\Psi\mathcal{DO}(S^{1|1})$ are classified in ([4]). Our aim is this classification for the case $S^{1|2}$.

2 Definitions and Notations

Let $S^{1|n}$ be the supercircle with local coordinates $(\varphi;\ \theta_1,\dots,\theta_n)$, where $\theta=(\theta_1,\dots,\theta_n)$ are the odd variables. More precisely, let $x=e^{i\varphi}$, in what follows by $S^{1|n}$ we mean the supermanifold $(\mathbb{C}^*)^{1|n}$, whose underlying is $\mathbb{C}\setminus\{0\}$. Any contact structure on $S^{1|n}$ can be given by the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i.$$

Let $\mathcal{K}(n)$ be the Lie superalgebra of vector fields on $S^{1|n}$ whose Lie action on α_n amounts to a multiplication by a function. Any element of $\mathcal{K}(n)$ is of the form (see [1])

$$v_F = F\partial_x + \frac{(-1)^{p(F)+1}}{2}\sum_{i=1}^{n}\eta_i(F)\eta_i,$$

where $F\in C^\infty(S^{1|n}),\ p(F)$ is the parity of F and $\eta_i=\partial_{\theta_i}-\theta_i\partial_x$. The bracket is given by

$$[v_F,\ v_G] = v_{\{F,G\}},$$

where

$$\{F,G\} = FG' - F'G + \frac{(-1)^{p(F)+1}}{2}\sum_{i=1}^{n}\eta_i(F)\eta_i(G).$$

The Lie superalgebra $\mathcal{K}(n)$ is called the Lie superalgebra of contact vector fields.

The superspace of the supercommutative algebra of superpseudodifferential symbols on $S^{1|n}$ with its natural multiplication is spanned by the series

$$\mathcal{SP}(n) = \Big\{ A = \sum_{k=-M}^{\infty}\ \sum_{\epsilon=(\epsilon_1,\dots,\epsilon_n)} a_{k,\,\epsilon}(x,\theta)\xi^{-k}\bar\theta_1^{\epsilon_1}\cdots\bar\theta_n^{\epsilon_n} \,|\, a_{k,\,\epsilon}\in C^\infty(S^{1|n});\ \epsilon_i=0,\,1;\ M\in\mathbb{N}\Big\},$$

where ξ corresponds to ∂_x and $\bar\theta_i$ corresponds to ∂_{θ_i} $(p(\bar\theta_i)=1)$. The space $\mathcal{SP}(n)$ has a structure of the Poisson Lie superalgebra given by the following bracket:

$$\{A,\ B\} = \frac{\partial(A)}{\partial\xi}\frac{\partial(B)}{\partial x} - \frac{\partial(A)}{\partial x}\frac{\partial(B)}{\partial\xi} - (-1)^{p(A)}\sum_{i=1}^{n}\Big(\frac{\partial(A)}{\partial\theta_i}\frac{\partial(B)}{\partial\bar\theta_i} + \frac{\partial(A)}{\partial\bar\theta_i}\frac{\partial(B)}{\partial\theta_i}\Big).$$

The associative superalgebra of superpseudodifferential operators $\mathcal{S}\Psi\mathcal{DO}(S^{1|n})$ on $S^{1|n}$ has the same underlying vector space as $\mathcal{SP}(n)$, but the multiplication is now defined by the following rule:

$$A\circ B=\sum_{\alpha\geq 0,\,\nu_i=0,\,1}\frac{(-1)^{p(A)+1}}{\alpha!}(\partial_\xi^\alpha\partial_{\bar\theta_i}^{\nu_i}A)(\partial_x^\alpha\partial_{\theta_i}^{\nu_i}B).$$

This composition rule induces the supercommutator defined by:

$$[A,\ B]=A\circ B-(-1)^{p(A)p(B)}B\circ A.$$

3 The space of weighted densities on $S^{1|2}$

Recall the definition of the $\mathrm{Vect}(S^1)$-module of weighted densities on S^1. Consider the 1-parameter action of $\mathrm{Vect}(S^1)$ on $C^\infty(S^1)$ given by

$$L^\lambda_{X(x)\partial}(f(x))=X(x)f'(x)+\lambda X'(x)f(x),$$

where $f\in C^\infty(S^1)$ and $\lambda\in\mathbb{R}$. Denote $\mathcal{F}_\lambda$ the $\mathrm{Vect}(S^1)$-module structure on $C^\infty(S^1)$ defined by this action. Note that the adjoint $\mathrm{Vect}(S^1)$-module is isomorphic to $\mathcal{F}_{-1}$. Geometrically, $\mathcal{F}_\lambda$ is the space of weighted densities of weight λ on S^1, i.e., the set of all expressions: $f(x)(dx)^\lambda$, where $f\in C^\infty(S^1)$. We have analogous definition of weighted densities in the supercase (see [2]) with dx replaced by α_n.

Consider the 1-parameter action of $\mathcal{K}(n)$ on $C^\infty(S^{1|n})$ given by the rule:

$$\mathfrak{L}^\lambda_{v_F}(G)=v_F(G)+\lambda F'\cdot G, \tag{3.1}$$

where $F,\,G\in C^\infty(S^{1|n}), F'\equiv\partial_xF$. We denote this $\mathcal{K}(1)$-module by $\Im_\lambda$ and the $\mathcal{K}(2)$-module by $\mathfrak{F}_\lambda$. Geometrically, the space $\mathfrak{F}_\lambda$ is the space of all weighted densities on $S^{1|2}$ of weight λ:

$$\phi=f(x,\theta)\alpha_2^\lambda,\ f(x,\theta)\in C^\infty(S^{1|2}). \tag{3.2}$$

Remarks 3.1. *1) The adjoint $\mathcal{K}(2)$-module is isomorphic to $\mathfrak{F}_{-1}$. This isomorphism induces a contact bracket on $C^\infty(S^{1|2})$ given by:*

$$\{F,G\}=\mathfrak{L}^{-1}_{v_F}(G)=FG'-F'G+\frac{(-1)^{p(F)+1}}{2}\sum_{i=1}^2(\eta_iF)(\eta_iG). \tag{3.3}$$

2) As a $\mathrm{Vect}(S^1)$-module, the space of weighted densities $\mathfrak{F}_\lambda$ is isomorphic to

$$\mathcal{F}_\lambda\oplus\Pi(\mathcal{F}_{\lambda+\frac12}\oplus\mathcal{F}_{\lambda+\frac12})\oplus\mathcal{F}_{\lambda+1}.$$

4 The structure of $\mathcal{SP}(2)$ as a $\mathcal{K}(2)$-module

The natural embedding of $\mathcal{K}(2)$ into $\mathcal{SP}(2)$ defined by

$$\pi(v_F) = F\xi + \frac{(-1)^{p(F)+1}}{2}\sum_{i=1}^{2}\eta_i(F)\zeta_i, \quad \text{where, } \zeta_i = \bar\theta_i - \theta_i\xi, \tag{4.1}$$

induces a $\mathcal{K}(2)$-module structure on $\mathcal{SP}(2)$.

Setting $\deg x = \deg\theta_i = 0,\ \deg\xi = \deg\bar\theta_i = 1$ for all i, we endow the Poisson superalgebra $\mathcal{SP}(2)$ with a $\mathbb{Z}$-grading:

$$\mathcal{SP}(2) = \widetilde{\bigoplus}_{n\in\mathbb{Z}}\mathcal{SP}_n, \tag{4.2}$$

where $\widetilde{\bigoplus}_{n\in\mathbb{Z}} = (\bigoplus_{n<0})\bigoplus\prod_{n\geq 0}$ and

$$\mathcal{SP}_n = \left\{F\xi^{-n} + G\xi^{-n-1}\bar\theta_1 + H\xi^{-n-1}\bar\theta_2 + T\xi^{-n-2}\bar\theta_1\bar\theta_2 \mid F,\, G,\ H,\ T \in C^\infty(S^{1|2})\right\}$$

is the homogeneous subspace of degree $-n$. Each element of $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$ can be expressed as

$$A = \sum_{k\in\mathbb{Z}}(F_k + G_k\xi^{-1}\bar\theta_1 + H_k\xi^{-1}\bar\theta_2 + T_k\xi^{-2}\bar\theta_1\bar\theta_2)\xi^{-n},$$

where $F_k,\, G_k,\ H_k,\ T_k \in C^\infty(S^{1|2})$. We define the *order* of A to be

$$\mathrm{ord}(A) = \sup\{k \mid F_k \neq 0 \text{ or } G_k \neq 0 \text{ or } H_k \neq 0 \text{ or } T_k \neq 0\}.$$

This definition of order equips $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$ with a decreasing filtration as follows: set

$$\mathbf{F}_n = \{A \in \mathcal{S}\Psi\mathcal{DO}(S^{1|2}),\, \mathrm{ord}(A) \leq -n\},$$

where $n \in \mathbb{Z}$. So one has

$$\ldots \subset \mathbf{F}_{n+1} \subset \mathbf{F}_n \subset \ldots \tag{4.3}$$

This filtration is compatible with the multiplication and the Poisson bracket, that is, for $A \in \mathbf{F}_n$ and $B \in \mathbf{F}_m$, one has $A \circ B \in \mathbf{F}_{n+m}$ and $\{A, B\} \in \mathbf{F}_{n+m-1}$. This filtration makes $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$ an associative filtered superalgebra. Consider the associated graded space

$$Gr(\mathcal{S}\Psi\mathcal{DO}(S^{1|2})) = \widetilde{\bigoplus}_{n\in\mathbb{Z}}\mathbf{F}_n/\mathbf{F}_{n+1}.$$

The filtration (4.3) is also compatible with the natural action of $\mathcal{K}(2)$ on $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$. Indeed, if $v_F \in \mathcal{K}(2)$ and $A \in \mathbf{F}_n$, then

$$v_F(A) = [v_F, A] \in \mathbf{F}_n.$$

The induced $\mathcal{K}(2)$-module on the quotient $\mathbf{F}_n/\mathbf{F}_{n+1}$ is isomorphic to the $\mathcal{K}(2)$-module $\mathcal{SP}_n$. Therefore, the $\mathcal{K}(2)$-module $Gr(\mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$, is isomorphic to the graded $\mathcal{K}(2)$-module $\mathcal{SP}(2)$, that is

$$\mathcal{SP}(2) \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathbf{F}_n/\mathbf{F}_{n+1}.$$

Recall that a C^∞ function on $S^{1|2}$ has the form $F = f_0 + f_1\theta + f_2\theta + f_{12}\theta_1\theta_2$ with $f_0, f_1, f_2, f_{12} \in C^\infty(S^1)$ and a C^∞ function on $S_i^{1|1}(i = 1, 2)$, where $S_i^{1|1}$ is the supercircle with local coordinates (φ, θ_i), has the form $F = f_0 + f_i\theta_i$ $(f_{12} = f_{3-i} = 0)$ with $f_0, f_i \in C^\infty(S^1)$. Then the Lie superalgebra $\mathcal{K}(2)$ has two subsuperalgebras $\mathcal{K}(1)_i$ for $i = 1,\ 2$ isomorphic to $\mathcal{K}(1)$ defined by

$$\mathcal{K}(1)_i = \Big\{ v_F = F\partial_x + \frac{(-1)^{p(F)+1}}{2} \sum_{i=1}^{2} \eta_i(F)\eta_i \mid F \in C^\infty(S_i^{1|1}) \Big\}.$$

Therefore, $\mathcal{SP}(2)$ and $\mathfrak{F}_\lambda$ are $\mathcal{K}(1)_i$-modules.
For $i = 1, 2$, let $\Im_\lambda^i$ be the $\mathcal{K}(1)_i$-module of weighted densities of weight λ on $S_i^{1|1}$.

Proposition 4.1. *1) As a $\mathcal{K}(1)_i$-module, $i = 1,\ 2$, we have*

$$\mathcal{SP}_n \simeq \mathfrak{F}_n \oplus \Pi(\mathfrak{F}_{n+\frac{1}{2}} \oplus \mathfrak{F}_{n+\frac{1}{2}}) \oplus \mathfrak{F}_{n+1} \ \textit{for} \ \ n = 0, -1.$$

2) For $n \neq 0, -1$:

a) The following subspace of $\mathcal{SP}_n$:

$$\mathcal{SP}_{n,\ i} = \left\{ \begin{array}{ll} B_F^{(n,i)} = & F\theta_{3-i}\bar{\theta}_{3-i}\xi^{-n-1} + \theta_{3-i}(\eta_{3-i} - \frac{1}{2}\eta_i)(F)\zeta_i\zeta_{3-i}\xi^{-n-2} \mid \\ & F \in C^\infty(S^{1|2}) \end{array} \right\} \quad (4.4)$$

is a $\mathcal{K}(1)_i$- module, $i = 1, 2$, isomorphic to $\mathfrak{F}_{n+1}$.
b) As a $\mathcal{K}(1)_i$-module we have

$$\mathcal{SP}_n/\mathcal{SP}_{n,\ i} \simeq \mathfrak{F}_n \oplus \Pi(\mathfrak{F}_{n+\frac{1}{2}} \oplus \mathfrak{F}_{n+\frac{1}{2}}), \quad i = 1, 2.$$

Proof. First, note that for $n = 0,\ -1$, the $\mathcal{K}(1)_i$-module $\mathcal{SP}_n$ with the grading (4.2) is the direct sum of four $\mathcal{K}(1)_i$-modules, $i = 1, 2$.

For $n=0$, the four $\mathcal{K}(1)_i$-modules are

$$\mathcal{SP}_{(0,\ 0)} = \left\{A_F^{(0,\ 0)} = F \mid \ F \in C^\infty(S^{1|2})\right\},$$

$$\mathcal{SP}_{(0,\ \frac{1}{2},\ i)} = \left\{\begin{array}{lcl} A_F^{(0,\ \frac{1}{2},\ i)} & = & \theta_i F - (1-2\theta_{3-i}\partial_{\theta_{3-i}})(F)\bar\theta_i\xi^{-1} - \\ & & \theta_{3-i}\partial_{\theta_i}(F)\bar\theta_{3-i}\xi^{-1} + F'\theta_{3-i}\bar\theta_i\bar\theta_{3-i}\xi^{-2} \mid \\ & & F\in C^\infty(S^{1|2})\end{array}\right\},$$

$$\widetilde{\mathcal{SP}}_{(0,\ \frac{1}{2},\ i)} = \left\{\begin{array}{lcl} \widetilde{A}_F^{(0,\ \frac{1}{2},\ i)} & = & \theta_i(\partial_{\theta_{3-i}} - 2\partial_{\theta_i} + 2\theta_{3-i}\partial_{\theta_{3-i}}\partial_{\theta_i})(F)\bar\theta_{3-i}\xi^{-1} + \\ & & \frac{1}{2}(3F - (-1)^{p(F)}F)\bar\theta_{3-i}\xi^{-1} + \\ & & (-1)^{p(F)}(\partial_{\theta_{3-i}} - \partial_{\theta_i} + \theta_i\partial_x)(F)\bar\theta_i\bar\theta_{3-i}\xi^{-2} \mid \\ & & F\in C^\infty(S^{1|2})\end{array}\right\},$$

$$\mathcal{SP}_{(0,\ 1,\ i)} = \left\{\begin{array}{lcl} A_F^{(0,\ 1,\ i)} & = & F\theta_{3-i}\bar\theta_{3-i}\xi^{-1} + \theta_{3-i}(\eta_{3-i} - \frac{1}{2}\eta_i)(F)\zeta_i\zeta_{3-i}\xi^{-2} \mid \\ & & F\in C^\infty(S^{1|2})\end{array}\right\}.$$

For $n=-1$, the four $\mathcal{K}(1)_i$-modules are

$$\mathcal{SP}_{(-1,\ 0)} = \left\{A_F^{(-1,\ 0)} = F\xi + \tfrac{(-1)^{p(F)+1}}{2}\Big(\eta_1(F)\zeta_1 + \eta_2(F)\zeta_2\Big) \mid \ F\in C^\infty(S^{1|2})\right\},$$

$$\mathcal{SP}_{(-1,\ \frac{1}{2},\ i)} = \left\{\begin{array}{lcl} A_F^{(-1,\ \frac{1}{2},\ i)} & = & F\zeta_i - (\theta_{3-i}\eta_i + \theta_i\partial_{\theta_{3-i}})(F)\bar\theta_{3-i} - \\ & & (-1)^{p(F)}\partial_{\theta_{3-i}}(F)\bar\theta_i\bar\theta_{3-i}\xi^{-1} \mid \ F\in C^\infty(S^{1|2})\end{array}\right\},$$

$$\widetilde{\mathcal{SP}}_{(-1,\ \frac{1}{2},\ i)} = \left\{\widetilde{A}_F^{(-1,\ \frac{1}{2},\ i)} = F\zeta_i + (1-\theta_{3-i}\eta_i)(F)\bar\theta_{3-i} \mid \ F\in C^\infty(S^{1|2})\right\},$$

$$\mathcal{SP}_{(-1,\ 1,\ i)} = \left\{\begin{array}{lcl} A_F^{(-1,\ 1,\ i)} & = & F\theta_{3-i}\bar\theta_{3-i} + \theta_{3-i}(\eta_{3-i} - \frac{1}{2}\eta_i)(F)\zeta_i\zeta_{3-i}\xi^{-1} \mid \\ & & F\in C^\infty(S^{1|2})\end{array}\right\}.$$

The action of $\mathcal{K}(1)_i$ on $\mathcal{SP}_{(n,\ 0)}$ and on $\mathcal{SP}_{(n,\ 1,\ i)}$ for $n=0,\ -1$ is induced by the embedding (4.1) as follows

$$v_G\cdot A_F^{(n,\ 0)} = \left\{\pi(v_G),\ A_F^{(n,\ 0)}\right\} = A^{(n,\ 0)}_{\mathfrak{L}^n_{v_G}(F)} \quad \text{and} \quad v_G\cdot A_F^{(n,\ 1,\ i)} = \left\{\pi(v_G),\ A_F^{(n,\ 1,\ i)}\right\} = A^{(n,\ 1,\ i)}_{\mathfrak{L}^{n+1}_{v_G}(F)},$$

where $F\in C^\infty(S^{1|2})$ and $G\in C^\infty(S_i^{1|1})$. Therefore, the natural maps

$$\begin{array}{rccc} \psi^i_{n,\ 0}: & \mathfrak{F}_n & \longrightarrow & \mathcal{SP}_{(n,\ 0)} \\ & F\alpha_2^n & \longmapsto & A_F^{(n,\ 0)} \end{array} \quad \text{and} \quad \begin{array}{rccc} \psi^i_{n,\ 1}: & \mathfrak{F}_{n+1} & \longrightarrow & \mathcal{SP}_{(n,\ 1,\ i)} \\ & F\alpha_2^{n+1} & \longmapsto & A_F^{(n,\ 1,\ i)} \end{array} \tag{4.5}$$

provide us with isomorphisms of $\mathcal{K}(1)_i$-modules, $i=1,\ 2$.

The action of $\mathcal{K}(1)_i$ on $\mathcal{SP}_{(n,\ \frac{1}{2},\ i)}$ and on $\widetilde{\mathcal{SP}}_{(n,\ \frac{1}{2},\ i)}$ for $n = 0,\ -1$ is given by

$$v_G \cdot A_F^{(n,\ \frac{1}{2},\ i)} = \left\{\pi(v_G),\ A_F^{(n,\ \frac{1}{2},\ i)}\right\} = A^{(n,\ \frac{1}{2},\ i)}_{\mathfrak{L}_{v_G}^{n+\frac{1}{2}}(F)} \quad \text{and} \quad v_G \cdot \widetilde{A}_F^{(n,\ \frac{1}{2},\ i)} = \left\{\pi(v_G),\ \widetilde{A}_F^{(n,\ \frac{1}{2},\ i)}\right\} = \widetilde{A}^{(n,\ \frac{1}{2},\ i)}_{\mathfrak{L}_{v_G}^{n+1}(F)},$$

where $F \in C^\infty(S^{1|2})$ and $G \in C^\infty(S_i^{1|1})$. Therefore, the natural maps

$$\begin{array}{rcl} \psi^i_{n,\ \frac{1}{2}} : \Pi(\mathfrak{F}_{n+\frac{1}{2}}) & \longrightarrow & \mathcal{SP}_{(n,\ \frac{1}{2},\ i)} \\ \Pi(F\alpha_2^{n+\frac{1}{2}}) & \longmapsto & A_F^{(n,\ \frac{1}{2},\ i)} \end{array} \quad \text{and} \quad \begin{array}{rcl} \widetilde{\psi}^i_{n,\ \frac{1}{2}} : \Pi(\mathfrak{F}_{n+\frac{1}{2}}) & \longrightarrow & \widetilde{\mathcal{SP}}_{(n,\ \frac{1}{2},\ i)} \\ \Pi(F\alpha_2^{n+\frac{1}{2}}) & \longmapsto & \widetilde{A}_F^{(n,\ \frac{1}{2},\ i)} \end{array} \tag{4.6}$$

provide us with isomorphisms of $\mathcal{K}(1)_i$-modules.

Second, for $n \neq 0,\ -1$, the action of $\mathcal{K}(1)_i$ on $\mathcal{SP}_{n,\ i}$ is given by

$$v_G \cdot B_F^{(n,\ i)} = \left\{\pi(v_G),\ B_F^{(n,\ i)}\right\} = B^{(n,\ i)}_{\mathfrak{L}_{v_G}^{n+1}(F)},$$

where $F \in C^\infty(S^{1|2})$ and $G \in C^\infty(S_i^{1|1})$. Therefore, $\mathcal{SP}_{n,\ i} \simeq \mathfrak{F}_{n+1}$ as a $\mathcal{K}(1)_i$-module. The induced $\mathcal{K}(1)_i$-module on the quotient $\mathcal{SP}_n/\mathcal{SP}_{n,\ i}$ has the direct sum decomposition of the three $\mathcal{K}(1)_i$- modules, $\mathcal{SP}_{(n,\ 0,\ i)}$, $\mathcal{SP}_{(n,\ \frac{1}{2},\ i)}$ and $\widetilde{\mathcal{SP}}_{(n,\ \frac{1}{2},\ i)}$, defined by

$$\mathcal{SP}_{(n,\ 0\ i)} = \left\{ \begin{array}{rcl} A_F^{(n,\ 0\ i)} & = & F\xi^{-n} + \frac{(-1)^{p(F)}}{2}(\frac{1}{2n+1}\theta_{3-i}\eta_{3-i}\eta_i - \eta_i)(F)\zeta_i\xi^{-n-1} + \\ & & (\partial_{\theta_{3-i}} + \frac{3n+1}{2n+1}\theta_i\partial_{\theta_{3-i}}\partial_{\theta_i})(F)\bar{\theta}_{3-i}\xi^{-n-1} + \\ & & \frac{n+1}{2n+1}(\theta_{3-i}\eta_i^3 + \eta_i\eta_{3-i})(F)\bar{\theta}_{3-i}\bar{\theta}_i\xi^{-n-2} \mid \\ & & F \in C^\infty(S^{1|2}) \end{array} \right\},$$

$$\mathcal{SP}_{(n,\ \frac{1}{2},\ i)} = \left\{ \begin{array}{rcl} A_F^{(n,\ \frac{1}{2},\ i)} & = & (\theta_{3-i}\partial_{\theta_{3-i}} - 1)(F)\zeta_i\xi^{-n-1} + \\ & & \frac{1}{2n+1}(n\theta_i\theta_{3-i}\partial_x - \theta_{3-i}\partial_{\theta_i})(F)\bar{\theta}_{3-i}\xi^{-n-1} + \\ & & \frac{n+1}{2n+1}F'\theta_{3-i}\bar{\theta}_i\bar{\theta}_{3-i}\xi^{-n-2} \mid \quad F \in C^\infty(S^{1|2}) \end{array} \right\},$$

$$\widetilde{\mathcal{SP}}_{(n,\ \frac{1}{2},\ i)} = \left\{ \begin{array}{rcl} \widetilde{A}_F^{(n,\ \frac{1}{2},\ i)} & = & (-1)^{p(F)}\theta_{3-i}(1 + \theta_i\partial_{\theta_{3-i}} - \frac{n}{2n+1}\theta_i\partial_{\theta_i})(F)\xi^{-n} + \\ & & (\theta_{3-i}\partial_{\theta_{3-i}} - \frac{n}{2n+1}\theta_{3-i}\eta_i)(F)\bar{\theta}_i\xi^{-n-1} + \\ & & (-1)^{p(F)}(\theta_{3-i}\partial_x + \eta_{3-i})(F)\zeta_i\bar{\theta}_{3-i}\xi^{-n-2} \mid \\ & & F \in C^\infty(S^{1|2}) \end{array} \right\}.$$

The action of $\mathcal{K}(1)_i$ on $\mathcal{SP}_{(n,\ j,\ i)}$ and on $\widetilde{\mathcal{SP}}_{(n,\ \frac{1}{2},\ i)}$ is induced by the the action of $\mathcal{K}(1)_i$ on $\mathcal{SP}_n/\mathcal{SP}_{n,\ i}$ and a direct computation shows that one has:

$$v_G \cdot A_F^{(n,\ j,\ i)} = A^{(n,\ j,\ i)}_{\mathfrak{L}_{v_G}^{n+j}(F)} \quad \text{for} \quad j = 0,\ \frac{1}{2} \quad \text{and} \quad v_G \cdot \widetilde{A}_F^{(n,\ \frac{1}{2},\ i)} = \widetilde{A}^{(n,\ \frac{1}{2},\ i)}_{\mathfrak{L}_{v_G}^{n+\frac{1}{2}}(F)},$$

where $F \in C^\infty(S^{1|2})$ and $G \in C^\infty(S_i^{1|1}), i = 1, 2$. Therefore, the natural maps

$$\begin{array}{rcl} \psi^i_{n,\,0}: \mathfrak{F}_n & \longrightarrow & \mathcal{SP}_{(n,\,0,\,i)} \\ F\alpha_2^n & \longmapsto & A_F^{(n,\,0,\,i)} \end{array}, \qquad \begin{array}{rcl} \psi^i_{n,\,\frac{1}{2}}: \Pi(\mathfrak{F}_{n+\frac{1}{2}}) & \longrightarrow & \mathcal{SP}_{(n,\,\frac{1}{2},\,i)} \\ \Pi(F\alpha_2^{n+\frac{1}{2}}) & \longmapsto & A_F^{(n,\,\frac{1}{2},\,i)} \end{array}$$

$$\text{and} \quad \begin{array}{rcl} \widetilde{\psi}^i_{n,\,\frac{1}{2}}: \Pi(\mathfrak{F}_{n+\frac{1}{2}}) & \longrightarrow & \widetilde{\mathcal{SP}}_{(n,\,\frac{1}{2},\,i)} \\ \Pi(F\alpha_2^{n+\frac{1}{2}}) & \longmapsto & \widetilde{A}_F^{(n,\,\frac{1}{2},\,i)} \end{array} \tag{4.7}$$

provide us with isomorphisms of $\mathcal{K}(1)_i$-modules. This completes the proof.

5 The first cohomology space $H^1(\mathcal{K}(2), \mathcal{SP}(2))$

Let us first recall some fundamental concepts from cohomology theory ([3]). Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a super vector space $V = V_0 \oplus V_1$. The space $\mathrm{Hom}(\mathfrak{g}, V)$ is $\mathbb{Z}_2$-graded via

$$\mathrm{Hom}(\mathfrak{g}, V)_b = \oplus_{a \in \mathbb{Z}_2} \mathrm{Hom}(\mathfrak{g}_a, V_{a+b});\ b \in \mathbb{Z}_2. \tag{5.1}$$

According to the $\mathbb{Z}_2$-grading (5.1), each $c \in Z^1(\mathfrak{g}, V)$, is broken to $(c', c'') \in \mathrm{Hom}(\mathfrak{g}_0, V) \oplus \mathrm{Hom}(\mathfrak{g}_1, V)$ subject to the following three equations:

$$\begin{array}{llcll} (E_1) & c'([g_1, g_2]) - g_1.c'(g_2) + g_2.c'(g_1) & = 0 & \text{for any} & g_1, g_2 \in \mathfrak{g}_0, \\ (E_2) & c''([g, h]) - g.c''(h) + h.c'(g) & = 0 & \text{for any} & g \in \mathfrak{g}_0, h \in \mathfrak{g}_1, \\ (E_3) & c'([h_1, h_2]) - h_1 c''(h_2) - h_2 c''(h_1) & = 0 & \text{for any} & h_1, h_2 \in \mathfrak{g}_1. \end{array} \tag{5.2}$$

Proposition 5.1. *1)*

$$H^1(\mathcal{K}(1)_i, \mathfrak{F}_\lambda)_0 \simeq \begin{cases} \mathbb{R}^3 & \text{if } \lambda = 0, \\ \mathbb{R} & \text{if } \lambda = 1, \\ 0 & \text{otherwise}. \end{cases}$$

The respective nontrivial 1*-cocycles are*

$$\begin{array}{ll} C_0(v_F) = \frac{1}{4}(3F + (-1)^{p(F)}F),\ C_1(v_F) = F',\ C_2(v_F) = \bar{\eta}_i(F')\theta_{3-i} & \text{if } \lambda = 0, \\ C_3(v_F) = \bar{\eta}_i(F'')\theta_{3-i} & \text{if } \lambda = 1, \end{array} \tag{5.3}$$

where $\bar{\eta}_i = \partial_{\theta_i} + \theta_i \partial_x$, $v_F \in \mathcal{K}(1)_i$ *and* $F = f_0 + f_i\theta_i$.
2)

$$H^1(\mathcal{K}(1)_i, \mathfrak{F}_\lambda)_1 \simeq \begin{cases} \mathbb{R} & \text{if } \lambda = \frac{1}{2}, \frac{3}{2}, \\ \mathbb{R}^2 & \text{if } \lambda = -\frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

It is spanned by the following 1*-cocycles:*

$$\begin{cases} C_4(v_F) = \frac{1}{4}(3F + (-1)^{p(F)}F)\theta_{3-i}, \quad C_5(v_F) = F'\theta_{3-i} & \text{if } \lambda = -\frac{1}{2}, \\ C_6(v_F) = \bar{\eta}_i(F') & \text{if } \lambda = \frac{1}{2}, \\ C_7(v_F) = \bar{\eta}_i(F'') & \text{if } \lambda = \frac{3}{2}. \end{cases} \tag{5.4}$$

To prove Proposition 5.1, we need the following result (see [2]).

Proposition 5.2. *[2]*
1) The space $H^1(\mathcal{K}(1)_i, \Im^i_\lambda)_0, i = 1, 2$, has the following structure:

$$H^1(\mathcal{K}(1)_i, \Im^i_\lambda)_0 \simeq \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The space $H^1(\mathcal{K}(1)_i, \Im^i_0)_0$ is generated by the cohomology classes of the 1*-cocycles*

$$c_0(v_F) = \frac{1}{4}(3F + (-1)^{p(F)}F) \quad \text{and} \quad c_1(v_F) = F'. \tag{5.5}$$

2)

$$H^1(\mathcal{K}(1)_i, \Im^i_\lambda)_1 \simeq \begin{cases} \mathbb{R} & \text{if } \lambda = \frac{1}{2}, \frac{3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

It is spanned by the nontrivial 1*-cocycles*

$$\begin{cases} c_2(v_F) = \bar{\eta}_i(F') & \text{if } \lambda = \frac{1}{2}, \\ c_3(v_F) = \bar{\eta}_i(F'') & \text{if } \lambda = \frac{3}{2}. \end{cases} \tag{5.6}$$

Proof of Proposition 5.1: Let $F\alpha_2^\lambda = (f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2)\alpha_2^\lambda \in \mathfrak{F}_\lambda$. The map

$$\begin{array}{rccl} \Phi: & \mathfrak{F}_\lambda & \longrightarrow & \Im^i_\lambda \oplus \Im^i_{\lambda+\frac{1}{2}} \\ & F\alpha_2^\lambda & \longmapsto & ((1 - \theta_{3-i}\partial_{\theta_{3-i}})(F)\alpha_{1,i}^\lambda,\ (-1)^{p(F)+1}\partial_{\theta_{3-i}}(F)\alpha_{1,i}^{\lambda+\frac{1}{2}}), \end{array}$$

where $\alpha_{1,i} = dx + \theta_i d\theta_i, i = 1, 2$, provides us with an isomorphism of $\mathcal{K}(1)_i$-modules. This map induces the following isomorphism between cohomology spaces:

$$H^1(\mathcal{K}(1)_i,\ \mathfrak{F}_\lambda) \simeq H^1(\mathcal{K}(1)_i,\ \Im^i_\lambda) \oplus H^1(\mathcal{K}(1)_i,\ \Im^i_{\lambda+\frac{1}{2}}).$$

We deduce from this isomorphism and Proposition 5.2, the 1-cocycles (5.3–5.4). □

The space $H^1(\mathcal{K}(2),\ \mathcal{SP}(2))$ inherits the grading (4.2) of $\mathcal{SP}(2)$, so it suffices to compute it in each degree. The main result of this section is the following.

Theorem 5.3. *The space $H^1(\mathcal{K}(2),\ \mathcal{SP}_n)$ is purely even. It has the following structure:*

$$H^1(\mathcal{K}(2), \mathcal{SP}_n) \simeq \begin{cases} \mathbb{R}^3 & \text{if } n = -1 \\ \mathbb{R}^6 & \text{if } n = 0 \\ \mathbb{R} & \text{if } n = 1 \\ 0 & \text{otherwise}. \end{cases}$$

For $n=-1$, the nontrivial 1-cocycles are:

$$\begin{aligned}
\Upsilon_1(v_F) &= \eta_1\eta_2(F)\xi^{-1}\zeta_1\zeta_2,\\
\Upsilon_2(v_F) &= F'\xi^{-1}\zeta_1\zeta_2,\\
\Upsilon_3(v_F) &= \Big(\frac{1}{4}(F+(-1)^{p(F)+1}F)+\eta_2\eta_1(F\theta_1\theta_2)\Big)\xi^{-1}\zeta_1\zeta_2,
\end{aligned}$$

For $n=0$, the nontrivial 1-cocycles are:

$$\begin{aligned}
\Upsilon_4(v_F) &= \frac{1}{4}(F+(-1)^{p(F)+1}F)+\eta_2\eta_1(F\theta_1\theta_2),\\
\Upsilon_5(v_F) &= F',\\
\Upsilon_6(v_F) &= \eta_1\eta_2(F),\\
\Upsilon_7(v_F) &= (-1)^{p(F)}\Big(\eta_1(F')\zeta_1+\eta_2(F')\zeta_2\Big)\xi^{-1},\\
\Upsilon_8(v_F) &= F''\xi^{-2}\zeta_1\zeta_2+(-1)^{p(F)}\Big(\eta_2(F')\zeta_1-\eta_1(F')\zeta_2\Big)\xi^{-1},\\
\Upsilon_9(v_F) &= \eta_1\eta_2(F')\xi^{-2}\zeta_1\zeta_2,
\end{aligned}$$

For $n=1$, the nontrivial 1-cocycle is:

$$\Upsilon_{10}(v_F)=\frac{2}{3}F'''\xi^{-3}\zeta_1\zeta_2+(-1)^{p(F)}\Big(\eta_2(F'')\zeta_1-\eta_1(F'')\zeta_2\Big)\xi^{-2}+2\eta_1\eta_2(F')\xi^{-1}.$$

To prove Theorem 5.3, we need first to proof the following lemma:

Lemma 5.4. *Let C be a even (resp. odd) 1-cocycle from $\mathcal{K}(2)$ to $\mathcal{SP}_n$, $n\in\mathbb{Z}$. If its restriction to $\mathcal{K}(1)_1$ and to $\mathcal{K}(1)_2$ is a coboundary, then C is a coboundary.*

Proof. Let C be a even (resp. odd) 1-cocycle of $\mathcal{K}(2)$ with coefficients in $\mathcal{SP}_n$ such that its restriction to $\mathcal{K}(1)_1$ and to $\mathcal{K}(1)_2$ is a coboundary. Using the condition of a 1-cocycle, we prove that there exists $G\in\mathcal{SP}_n$ such that

$$C(v_{f_0+f_i\theta_i})=\{v_{f_0+f_i\theta_i}\ ,\ G\}\ \text{for any}\ f_0,\ f_i\in C^\infty(S^1)\ \text{and}\ i=1,\ 2$$

and

$$C(v_{f_{12}\theta_1\theta_2})=\{v_{f_{12}\theta_1\theta_2}\ ,\ G\}\ \text{for any}\ f_{12}\in C^\infty(S^1).$$

We deduce that $C(v_F)=\{v_F\ ,\ G\}$, for any $F\in C^\infty(S^{1|2})$, and therefore C is a coboundary of $\mathcal{K}(2)$. □

Proof of Theorem 5.3 : According to Lemma 5.4, the restriction of any nontrivial 1-cocycle of $\mathcal{K}(2)$ with coefficients in $\mathcal{SP}_n$ to $\mathcal{K}(1)_1$ or to $\mathcal{K}(1)_2$ is a nontrivial 1-cocycle.

Using Proposition 4.1 and Proposition 5.1, we obtain:

$$H^1(\mathcal{K}(1)_i,\mathcal{SP}_n)\simeq\left\{\begin{array}{ll}\mathbb{R}^7 & \text{if } n=-1\\ \mathbb{R}^6 & \text{if } n=0.\end{array}\right.$$

In the case $n=-1$, the space $H^1(\mathcal{K}(1)_i,\mathcal{SP}_{-1})$ is spanned by the following 1-cocyles:

$$\begin{aligned}
\beta_l^i(v_F) &= \psi^i_{-1,\ 1}(C_l(v_F)), \quad l=0,\ 1,\ 2,\\
\beta_4^i(v_F) &= \psi^i_{-1,\ \frac{1}{2}}(\Pi(C_4(v_F))),\\
\widetilde{\beta}_4^i(v_F) &= \widetilde{\psi}^i_{-1,\ \frac{1}{2}}(\Pi(C_4(v_F))),\\
\beta_5^i(v_F) &= \psi^i_{-1,\ \frac{1}{2}}(\Pi(C_5(v_F))),\\
\widetilde{\beta}_5^i(v_F) &= \widetilde{\psi}^i_{-1,\ \frac{1}{2}}(\Pi(C_5(v_F))).
\end{aligned}$$

In the case $n=0$, the space $H^1(\mathcal{K}(1)_i,\mathcal{SP}_0)$ is spanned by the following 1-cocyle:

$$\begin{aligned}
\beta_{l+6}^i(v_F) &= \psi^i_{0,\ 0}(C_l(v_F)), \quad l=0,\ 1,\ 2,\\
\beta_9^i(v_F) &= \psi^i_{0,\ 1}(C_3(v_F)),\\
\beta_{10}^i(v_F) &= \psi^i_{0,\ \frac{1}{2}}(\Pi(C_6(v_F))),\\
\widetilde{\beta}_{10}^i(v_F) &= \widetilde{\psi}^i_{0,\ \frac{1}{2}}(\Pi(C_6(v_F))),
\end{aligned}$$

where the cocycles $C_0,\cdots,C_6$ are defined by the formulae (5.3)–(5.4) and $\psi^i_{n,\ j}$, $\widetilde{\psi}^i_{n,\ j}$ are as in (4.5)–(4.6).

According to the same propositions, we obtain $H^1(\mathcal{K}(1)_i,\mathcal{SP}_n/\mathcal{SP}_{n,\ i})$ and $H^1(\mathcal{K}(1)_i,\mathcal{SP}_{n,\ i})$ for $n\neq 0,-1$ and $i=1,\ 2$. By direct computations, one can now deduce $H^1(\mathcal{K}(1)_i,\mathcal{SP}_n)$.

Second, note that any nontrivial 1-cocycle of $\mathcal{K}(2)$ with coefficients in $\mathcal{SP}_n$ should retain the following general form: $\Upsilon=\Upsilon^0+\Upsilon^1+\Upsilon^2+\Upsilon^3$ where $\Upsilon^0:\mathrm{Vect}(S^1)\longrightarrow\mathcal{SP}_n$, $\Upsilon^1,\Upsilon^2:\mathcal{F}_{-\frac{1}{2}}\longrightarrow\mathcal{SP}_n$ and $\Upsilon^3:\mathcal{F}_0\longrightarrow\mathcal{SP}_n$ are linear maps. The space $H^1(\mathcal{K}(1)_i,\mathcal{SP}_n), i=1,\ 2$, determines the linear maps Υ^0, Υ^1 and Υ^2. The 1-cocycle conditions determines Υ^3. More precisely, we get:

For $n=-1$, the space $H^1(\mathcal{K}(2),\mathcal{SP}_{-1})$ is generated by the nontrivial cocycles Υ_1, Υ_2 and Υ_3 corresponding to the cocycles β_2^i, β_5^i and β_4^i, respectively, via their restrictions to $\mathcal{K}(1)_i$.

For $n=0$, the space $H^1(\mathcal{K}(2),\mathcal{SP}_0)$ is spanned by the nontrivial cocycles $\Upsilon_4,\Upsilon_5,\Upsilon_6,\widetilde{\Upsilon}_7$, $\widetilde{\Upsilon}_8$ and Υ_9 corresponding to the cocycles β_6^i, β_7^i, β_8^i, β_{10}^i, $\widetilde{\beta}_{10}^i$ and β_9^i, respectively, via their restrictions to $\mathcal{K}(1)_i$, where $\widetilde{\Upsilon}_7=\Upsilon_7+\Upsilon_9$ and $\widetilde{\Upsilon}_8=\Upsilon_8+\Upsilon_6$.

Finally, for $n=1$, the space $H^1(\mathcal{K}(2),\mathcal{SP}_1)$ is generated by the nontrivial cocycle Υ_{10} corresponding to the cocycle $\psi^i_{1,\ 0}(C_3(v_F))$ with $\psi^i_{1,\ 0}$ as in (4.7) via its restriction to $\mathcal{K}(1)_i$.

Theorem 5.3 is proved. □

6 The space $H^1(\mathcal{K}(2),\mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$

6.1 The spectral sequence for a filtered module over a Lie (super)algebra

The reader should refer to [6], for the details of the homological algebra used to construct spectral sequences. We will merely quote the results for a filtered module M with decreasing filtration $\{M_n\}_{n\in\mathbb{Z}}$ over a Lie (super)algebra $\mathfrak{g}$ so that $M_{n+1}\subset M_n$, $\cup_{n\in\mathbb{Z}}M_n=M$ and $\mathfrak{g}M_n\subset M_n$.

Consider the natural filtration induced on the space of cochains by setting:

$$F^n(C^*(\mathfrak{g},\ M)) = C^*(\mathfrak{g},\ M_n),$$

then we have:

$$dF^n(C^*(\mathfrak{g},\ M)) \subset F^n(C^*(\mathfrak{g},\ M)) \ \text{ (i.e., the filtration is preserved by } d);$$
$$F^{n+1}(C^*(\mathfrak{g},\ M)) \subset F^n(C^*(\mathfrak{g},\ M)) \text{ (i.e. the filtration is decreasing).}$$

Then there is a spectral sequence $(E_r^{*,*}, d_r)$ for $r \in \mathbb{N}$ with d_r of degree $(r, 1-r)$ and

$$E_0^{p,q} = F^p(C^{p+q}(\mathfrak{g},\ M))/F^{p+1}(C^{p+q}(\mathfrak{g},\ M)) \quad \text{and} \quad E_1^{p,q} = H^{p+q}(\mathfrak{g},\ \mathrm{Grad}^p(M)).$$

To simplify the notations, we have to replace $F^n(C^*(\mathfrak{g},\ M))$ by F^nC^*. We define

$$Z_r^{p,q} = F^pC^{p+q}\bigcap d^{-1}(F^{p+r}C^{p+q+1}),$$
$$B_r^{p,q} = F^pC^{p+q}\bigcap d(F^{p-r}C^{p+q-1}),$$
$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}).$$

The differential d maps $Z_r^{p,q}$ into $Z_r^{p+r,q-r+1}$, and hence includes a homomorphism

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

The spectral sequence converges to $H^*(C, d)$, that is

$$E_\infty^{p,q} \simeq F^pH^{p+q}(C, d)/F^{p+1}H^{p+q}(C, d),$$

where $F^pH^*(C, d)$ is the image of the map $H^*(F^pC, d) \to H^*(C, d)$ induced by the inclusion $F^pC \to C$.

6.2 Computing $H^1(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$

Now we can check the behavior of the cocycles $\Upsilon_1, \dots, \Upsilon_{10}$ under the successive differentials of the spectral sequence. Cocycles Υ_1, Υ_2 and Υ_3 belong to $E_1^{-1,2}$, cocycles $\Upsilon_4, \dots, \Upsilon_9$ belong to $E_1^{0,1}$ and cocycle Υ_{10} belongs to $E_1^{1,0}$. Consider a cocycle in $\mathcal{SP}(2)$, but compute its differential as if it were with values in $\mathcal{S}\Psi\mathcal{DO}(S^{1|2})$ and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image under d_1. The higher order differentials d_r can be calculated by iteration of this procedure, the space $E_r^{p+r,q-r+1}$ contains the subspace coming from $H^{p+q+1}(\mathcal{K}(2);\ \mathrm{Grad}^{p+1}(\mathcal{S}\Psi\mathcal{DO}(S^{1|2})))$.

It is now easy to see that the cocycles $\Upsilon_1, \dots, \Upsilon_6$ will survive in the same form. Computing supplementary higher order terms for the other cocycles, we obtain

Theorem 6.1. *The space $H^1(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(S^{1|2}))$ is purely even. It is spanned by the classes of the following nontrivial 1-cocycles*

$$
\begin{aligned}
\Theta_1(v_F) = & \quad \eta_1\eta_2(F)\xi^{-1}\zeta_1\zeta_2, \\
\Theta_2(v_F) = & \quad F'\xi^{-1}\zeta_1\zeta_2, \\
\Theta_3(v_F) = & \quad \Big(\frac{1}{4}(F + (-1)^{p(F)+1}F) + \eta_2\eta_1(F\theta_1\theta_2)\Big)\xi^{-1}\zeta_1\zeta_2, \\
\Theta_4(v_F) = & \quad \frac{1}{4}(F + (-1)^{p(F)+1}F) + \eta_2\eta_1(F\theta_1\theta_2), \\
\Theta_5(v_F) = & \quad F', \\
\Theta_6(v_F) = & \quad \eta_1\eta_2(F), \\
\Theta_7(v_F) = & \quad \sum_{n=0}^{\infty}\frac{(-1)^{p(F)+n}}{n+1}\Big(\eta_1(F^{(n+1)})\zeta_1 + \eta_2(F^{(n+1)})\zeta_2\Big)\xi^{-n-1} + \\
& \quad \textstyle\sum_{n=0}^{\infty}\frac{2(-1)^n}{n+2}F^{(n+2)}\xi^{-n-1}, \\
\Theta_8(v_F) = & \quad \sum_{n=0}^{\infty}(-1)^{p(F)+n}\Big(\eta_2(F^{(n+1)})\zeta_1 - \eta_1(F^{(n+1)})\zeta_2\Big)\xi^{-n-1} + \\
& \quad \textstyle\sum_{n=0}^{\infty}(-1)^n F^{(n+2)}\xi^{-n-2}\zeta_1\zeta_2 + \\
& \quad \textstyle\sum_{n=1}^{\infty}(-1)^n\eta_1\eta_2(F^{(n)})\xi^{-n}, \\
\Theta_9(v_F) = & \quad \sum_{n=0}^{\infty}(-1)^n\eta_1\eta_2(F^{(n+1)})\xi^{-n-2}\zeta_1\zeta_2 + \\
& \quad \textstyle\sum_{n=1}^{\infty}(-1)^{p(F)+n}\frac{n}{n+1}\Big(\eta_1(F^{(n+1)})\zeta_1 + \eta_2(F^{(n+1)})\zeta_2\Big)\xi^{-n-1} + \\
& \quad \textstyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}F^{(n+2)}\xi^{-n-1}, \\
\Theta_{10}(v_F) = & \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{n+2}F^{(n+2)}\xi^{-n-2}\zeta_1\zeta_2 + \\
& \quad \textstyle\sum_{n=1}^{\infty}(-1)^{p(F)+n}\frac{2n}{n+1}\eta_1(F^{(n+1)})\xi^{-n-1}\zeta_2 + \\
& \quad \textstyle\sum_{n=1}^{\infty}(-1)^{p(F)+n+1}\frac{2n}{n+1}\eta_2(F^{(n+1)})\xi^{-n-1}\zeta_1 + \\
& \quad \textstyle\sum_{n=1}^{\infty}2(-1)^{n+1}\eta_1\eta_2(F^{(n)})\xi^{-n}.
\end{aligned}
$$

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