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REGULARITE ET ASYMPTOTIQUE POUR LES EQUATIONS PRIMITIVES

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mamei...

Regularity Properties and Asymptotics for the Primitive Equations

Abstract

This thesis, containing four chapters, studies the existence, uniqueness and regularity of the solutions for the Primitive Equations of the oceans and the atmosphere in dimensions 2 and 3 (Chapters 1–3), and also the asymptotic behavior of the Primitive Equations when the Rossby number goes to zero (Chapter 4).

In the first chapter we consider the Primitive Equations of the ocean in a two dimensional space with periodic boundary conditions. The equations model a three dimensional motion, in which all the functions depend only on the horizontal west-east and the vertical directions. We prove the existence, globally in time, of a weak solution and the existence and uniqueness of strong solutions. Moreover, we prove the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity.

In the second chapter a model similar to that considered in the first chapter is treated. Working in a two dimensional space with periodical boundary conditions, we prove that, for a forcing term which is analytical in time with values in a Gevrey space, the solutions of the Primitive Equations starting with the initial data in a certain Sobolev space become, for some positive time, elements of a certain Gevrey class. The result also implies that the solutions are real analytic functions.

As a natural continuation of the work from the first two chapters, in the third chapter we consider the Primitive Equations in a 3D domain and we study the Sobolev and Gevrey regularity for the solutions. We obtain, as for the case of the 2D Primitive Equations, the existence of weak solutions and of a unique regular solution, but in this case we have the existence of strong solutions only locally in time. The result obtained on the Gevrey regularity for the three dimensional case is similar to that concerning the 2D case.

The last chapter of the thesis is devoted to the study of the asymptotic behavior when the Rossby number goes to zero, for the Primitive Equations in the form considered in the first chapter. The aim of this work is to average, using the renormalization group method, the oscillations of the exact solution when the Rossby number goes to zero, and to prove that the averaged solution is a good approximation of the exact oscillating solution.

Keywords: Primitive Equations, energy estimates, high-order regularity, Gevrey regularity, analyticity, renormalization group method, error estimate

AMS Classification (2000): 35B65, 35C20, 35Q35, 76D03, 76D50

Régularité et asymptotique pour les Equations Primitives

Résumé

Ce mémoire composé de quatre chapitres, réunit un nombre de résultats sur l'existence, l'unicité et la régularité des solutions pour les Equations Primitives des océans et de l'atmosphère, en dimension deux et trois d'espace (Chapitres 1–3), ainsi qu'une étude sur le comportement asymptotique des Equations Primitives quand le nombre de Rossby tend vers zero (Chapitre 4).

Dans le premier chapitre, on considère les Equations Primitives de l'océan en dimension deux d'espace, avec des conditions aux limites périodiques. Les equations modèlisent un mouvement tri-dimensionel, dans lequel toutes les fonctions dependent seulement de la longitude (direction est-ouest) et de la variable verticale. On montre l'existence globale en temps d'une solution faible pour les Equations Primitives, ainsi que l'existence et l'unicité d'une solution forte. De plus, on prouve l'existence d'une solution plus régulière, jusqu'à la régularité $\mathcal{C}^\infty$.

Dans le deuxième chapitre on considère un modèle semblable à celui du chapitre précédent. On travaille aussi avec des conditions aux limites periodiques et on montre que, pour une force analytique en temps à valeurs dans un espace du type de Gevrey, et une donnée initiale dans un certain espace du type de Sobolev, les solutions des Equations Primitives appartiennent, pour un certain intervalle de temps, à un espace de Gevrey. Le résultat implique aussi que les solutions sont des fonctions réelles analytiques.

Le troisième chapitre est une continuation naturelle des deux premiers chapitres. On considère ici les Equations Primitives en dimension trois d'espace et on étudie la régularité du type de Sobolev et de Gevrey pour les solutions. On obtient, comme pour le cas de la dimension deux d'espace, l'existence d'une solution faible ainsi que l'existence et l'unicité d'une solution forte, mais dans ce cas on a seulement l'existence locale en temps de la solution forte.

Le dernier chapitre de la thèse est dedié à l'étude du comportement asymptotique des Equations Primitives, quand le nombre de Rossby tend vers zero. Les Equations Primitives sont considérées sous la forme introduit au premier chapitre (écart par rapport à une solution simple stratifiée). Le but de ce travail est de "moyenniser" la solution exact très oscillante quand le nombre de Rossby est petit, en utilisant une méthode de renormalisation ; la solution renormalisée est construit est l'on montre que la solution approximative est une bonne approximation de la solution exacte.

Mots clé : Equations Primitives, estimation de l'energie, regularité d'ordre supérieure, régularité du type de Gevrey, méthode de la renormalisation, estimation d'erreur

AMS Classification (2000) : 35B65, 35C20, 35Q35, 76D03, 76D50

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Introduction

This thesis presents a number of results concerning the Primitive Equations (PEs) of the ocean and the atmosphere. The thesis comprises two parts: the first part (Chapters 1–3) contains a qualitative study of the Primitive Equations, that is a study regarding the existence, uniqueness and regularity of solutions and the second part (Chapter 4) is devoted to the asymptotic behavior of the Primitive Equations when a small parameter (the Rossby number) tends to zero.

The Primitive Equations govern the motion of the ocean and of the atmosphere and they are derived from the general conservation laws of physics using the Boussinesq and hydrostatic approximations. They comprise: the conservation of horizontal momentum equation, the hydrostatic equation, the continuity equation, the equation for the temperature, the equation for the salinity and the equation of state. These equations read:

(1a) $$\frac{\partial \boldsymbol{v}^*}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)\boldsymbol{v}^* + w^* \frac{\partial \boldsymbol{v}^*}{\partial z^*} + fk \times \boldsymbol{v}^* + \frac{1}{\rho_{\text{ref}}}\nabla p^* = \mu_{\boldsymbol{v}}^* \Delta_{\text{h}}^* \boldsymbol{v}^* + \nu_{\boldsymbol{v}}^* \frac{\partial^2 \boldsymbol{v}^*}{\partial z^{*2}},$$

(1b) $$\frac{\partial p_{\text{full}}^*}{\partial z^*} = -\rho_{\text{full}}^* \, g,$$

(1c) $$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0,$$

(1d) $$\frac{\partial T}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)T + w^* \frac{\partial T}{\partial z^*} = \mu_T \, \Delta_{\text{h}}^* T + \nu_T \frac{\partial^2 T}{\partial z^{*2}},$$

(1e) $$\frac{\partial S}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)S + w^* \frac{\partial S}{\partial z^*} = \mu_S \, \Delta_{\text{h}}^* S + \nu_S \frac{\partial^2 S}{\partial z^{*2}},$$

(1f) $$\rho_{\text{full}}^* = \rho_{\text{ref}} \left[1 - \beta_T(T - T_{\text{ref}}) - \beta_S(S - S_{\text{ref}})\right].$$

Here $\boldsymbol{v}^* = (u^*, v^*)$ is the horizontal velocity, w^* the vertical velocity, p_{full}^* the (full) pressure, ρ_{full}^* the (full) density, T the temperature and S the salinity. Asterisks denote dimensional quantities and ρ_{ref}, T_{ref}, S_{ref} denote reference values respectively for the density, temperature and salinity; g is the gravitational acceleration and f the Coriolis parameter. A simplification of this system is obtained if we assume that $\beta_T \nu_T = \beta_S \nu_S$ and $\beta_T \mu_T = \beta_S \mu_S$ so that (1d)–(1f) can be combined into an equation for ρ:

(2) $$\frac{\partial \rho_{\text{full}}^*}{\partial t^*} + u^* \frac{\partial \rho_{\text{full}}^*}{\partial x^*} + v^* \frac{\partial \rho_{\text{full}}^*}{\partial y^*} + w^* \frac{\partial \rho_{\text{full}}^*}{\partial z^*} = \mu_\rho^* \, \Delta_{\text{h}}^* \rho_{\text{full}}^* + \nu_\rho^* \, \frac{\partial^2 \rho_{\text{full}}^*}{\partial z^{*2}} \, .$$

Details regarding the derivation of these equations can be found in the geophysical literature (see for example, [14], [15]).

The practical interest of these equations, since they are the starting points of dynamical meteorology and climatology, determined many mathematicians to study them from the mathematical and theoretical numerical analysis points of view. We here recall the first rigorous work on existence and uniqueness of solutions by Lions, Temam and Wang (see, for example, [10], [11] and also [18]).

In the first chapter of this thesis we consider the case when the density ρ^*_{full} is of the form

$$\rho^*_{\text{full}}(x,y,z,t) = \rho_{\text{ref}} + \bar{\rho}(z) + \rho^\star(x,y,z,t),$$

where $\bar{\rho} = \bar{\rho}(z)$ is the stratification profile of the density. We introduce the Brunt–Väisälä frequency $N^\star$, which is supposed to be constant:

$$(N^\star)^2 = \frac{g}{\rho_{\text{ref}}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z},$$

meaning that we consider a part of the ocean where the stratification profile is close to be a linear function.

Then the evolution equation (2) for the density becomes:

$$\frac{\partial \rho^*}{\partial t^*} + u^*\frac{\partial \rho^*}{\partial x^*} + v^*\frac{\partial \rho^*}{\partial y^*} + w^*\frac{\partial \rho^*}{\partial z^*} - \frac{\rho_{\text{ref}}}{g}(N^*)^2\, w^* = \mu_\rho\, \Delta^*_{\text{h}}\rho^* + \nu_\rho\,\frac{\partial^2 \rho^*}{\partial z^{*2}}\,. \tag{3}$$

We non-dimensionalize these equations and we consider that all the functions are independent on the y variable. The model obtained then reads:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}}\,\Delta u + S_u, \tag{4a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u = \nu_{\boldsymbol{v}}\,\Delta v + S_v, \tag{4b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{4c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{4d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{\varepsilon}w = \nu_\rho\,\Delta\rho + S_\rho, \tag{4e}$$

where (S_u, S_v, S_ρ) is the forcing term, and the dimensionless parameters are the Rossby number ϵ, the Burger number N and the inverse Reynolds numbers $\nu_{\boldsymbol{v}}$ and ν_ρ. The details regarding the derivation of this system are recalled in the Appendix of the first chapter.

In the first chapter, we consider system (4) with periodic boundary conditions and in order to insure the well-posedness of the problem, we impose that u, v and p are even in z and w and ρ are odd in z, a case relevant for studies in turbulence (see, e.g., [3]). We then prove the existence, globally in time, of a weak solution corresponding to an initial data in L^2 and the existence and uniqueness of a strong solution, when the same initial data is in H^1. Moreover, we find the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity.

In the second chapter we consider a model related to (4), without stratification; that is equation (4e) is replaced by:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = \nu_\rho \,\Delta\rho + S_\rho. \tag{5}$$

In (5), the density ρ is the deviation of the full density from a reference value: $\rho = \rho_{\text{full}} - \rho_{\text{ref}}$. The model obtained in this way is supplemented with periodic boundary conditions and we study the Gevrey regularity of the solution.

We prove that, considering a forcing term which is an analytical function in time with values in some Gevrey space, the solutions of the Primitive Equations starting with initial data in the Sobolev space H^1 immediately become and remain for some positive time, elements of a certain Gevrey class. We prove that the unique solution of the PEs is the restriction to the real time axis $t \geq 0$ of a complex function analytic in the variable t in some neighborhood of the real axis.

This work was inspired by the article of Foias and Temam [9], where the authors proved similar results regarding the Navier–Stokes equations. As in [9], the idea is to write the PEs in their evolution form, to extend this equation to the complex time and then to obtain appropriate a priori estimates. Since the form of the PEs is more complicated than that of the Navier–Stokes equations, some additional technical difficulties appear. The difficulties can be overcomed by splitting the initial evolution problem into a linear problem and a remaining nonlinear problem and derive a priori estimates separately, first for the linear problem and then for the nonlinear one.

The third chapter extends to dimension 3 the work of the first two chapters. We consider the model from the second chapter, extended to a three dimensional domain and we study the Sobolev and Gevrey regularity of the solutions. The technical tools used are the same as in the previous chapters. Considering the classical differences between the 2D and 3D Navier–Stokes equations, the results obtained are, as expected, weaker than the results for the 2D case. We thus obtain the existence of a unique very regular solution, but only locally in time.

In the last chapter of this thesis we study the small–Rossby number asymptotics for the Primitive Equations. When the Rossby number goes to zero, the exact solutions present fast oscillations which one would like to avoid dealing with by averaging. We are working with the Primitive Equations in a two dimensional domain, as given by model (4).

In order to average the exact solution, we use the so-called renormalization group method, which was introduced by Schochet in [16] and reformulated in a form close to the one we use here by Ziane [19]. Independently, the method was introduced in a physical context by Chen, Goldenfeld and Oono [5] and used in the mathematical context, for rotating fluids and geophysical flows by Chemin [4], Embid-Majda [6] and Grenier [8]. A lot of mathematical literature regarding the applications of the renormalized group method is available and we mention here the work of Gallagher, where asymptotic expansions are deduced [7], and also of Babin-Mahalov-Nicolaenko [1], [2], and of Moise, Temam, Ziane [12], [13]. For ODEs Temam and Wirosoetisno showed in [17] that the method can be applied to higher orders.

Here we apply the method to obtain a first order approximation. The idea from [19] is to write system (4) as an abstract evolution problem of the form:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{1}{\varepsilon} LU = \mathcal{F}(U), \\ &U(0) = U_0, \end{aligned} \tag{6}$$

where $\varepsilon > 0$ is a small parameter and L is an antisymmetric operator (which explains why the solutions display large oscillations for ε small); here L corresponds to the Coriolis force.

We then write (6) using the fast variable $s = t/\varepsilon$ and consider $F(s, \cdot) = e^{Ls}\mathcal{F}(e^{-Ls}\cdot)$. We split F into a time independent (resonant) part F_r and the remaining (nonresonant) part F_n, which depends on s. We also introduce the operator:

$$F_{np}(s, U) = \int_0^s F_n(s', U)\, \mathrm{d}s'. \tag{7}$$

We then search for an approximate solution of the form:

$$\tilde{U}^1(s) = e^{-Ls}\{\bar{U}(s) + \varepsilon F_{np}(s, \bar{U}(s))\}, \tag{8}$$

where $\bar{U}$ is the solution of the renormalized group equation:

$$\begin{cases} \dfrac{\mathrm{d}\bar{U}}{\mathrm{d}s} = \varepsilon F_r(\bar{U}), \\ \\ \bar{U}(0) = U_0. \end{cases} \tag{9}$$

We first show that the renormalized group system keeps the properties of the original system, namely energy conservation in the inviscid case (orthogonality for the nonlinear term) and dissipation rate (coercivity) in the dissipative case. Using these properties we prove the existence of weak or of very regular solutions for the system (9), depending on the regularity of the initial data. We then prove the main result of the chapter, consisting in showing that the renormalized solution is a good approximation of the exact oscillating solution:

$$|\tilde{U}^1(s) - U(s)|_{L^2} \sim \mathcal{O}(\varepsilon). \tag{10}$$

This chapter is ended by some appendices containing technical results as the derivation of the renormalized group system and two different ways of approximating small denominators.

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Introduction

Cette thèse réunit un certain nombre de résultats relatifs aux Equations Primitives de l'océan et de l'atmosphère. Elle est constituée de deux parties : la première partie (Chapitres 1–3) présente une étude qualitative des Equations Primitives qui concerne l'existence, l'unicité ainsi que la régularité des solutions. La deuxième partie (Chapitre 4) est dediée à l'étude du comportement asymptotique des Equations Primitives lorqu'un petit paramètre (le nombre de Rossby) tend vers zero.

Les Equations Primitives sont les équations qui modélisent le mouvement de l'océan et de l'atmosphère, et elles sont obtenues à partir des lois fondamentales de la physique, en utilisant les approximations de Boussinesq et hydrostatique. Ces lois comprennent : l'équation de conservation du moment horizontal, l'équation hydrostatique, l'équation de conservation de la masse, l'équation de la température (conservation de l'énergie), l'équation de la salinité et l'équation d'état. Les équations s'expriment comme suit :

$$\text{(1a)}\quad \frac{\partial \boldsymbol{v}^*}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)\boldsymbol{v}^* + w^* \frac{\partial \boldsymbol{v}^*}{\partial z^*} + fk \times \boldsymbol{v}^* + \frac{1}{\rho_{\text{ref}}} \nabla p^* = \mu^*_{\boldsymbol{v}}\, \Delta^*_{\text{h}} \boldsymbol{v}^* + \nu^*_{\boldsymbol{v}} \frac{\partial^2 \boldsymbol{v}^*}{\partial z^{*2}},$$

$$\text{(1b)}\quad \frac{\partial p^*_{\text{full}}}{\partial z^*} = -\rho^*_{\text{full}}\, g,$$

$$\text{(1c)}\quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0,$$

$$\text{(1d)}\quad \frac{\partial T}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)T + w^* \frac{\partial T}{\partial z^*} = \mu_T\, \Delta^*_{\text{h}} T + \nu_T \frac{\partial^2 T}{\partial z^{*2}},$$

$$\text{(1e)}\quad \frac{\partial S}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)S + w^* \frac{\partial S}{\partial z^*} = \mu_S\, \Delta^*_{\text{h}} S + \nu_S \frac{\partial^2 S}{\partial z^{*2}},$$

$$\text{(1f)}\quad \rho^*_{\text{full}} = \rho_{\text{ref}} \big[1 - \beta_T(T - T_{\text{ref}}) - \beta_S(S - S_{\text{ref}})\big].$$

Ici $\boldsymbol{v}^* = (u^*, v^*)$ représente la vitesse horizontale, w^* est la vitesse verticale, p^*_{full} est la pression totale, T la température et S la salinité. Les quantité avec astérisques respectivement correspondent à des valeurs dimensionnées et ρ_{ref}, T_{ref}, S_{ref} représentent les valeurs de référence (valeurs moyennes) pour la densité, la température et la salinité ; g est la constante universelle de gravitation et f est le paramètre de Coriolis. Une simplification de ce système est obtenue en supposant que $\beta_T\nu_T = \beta_S\nu_S$ et $\beta_T\mu_T = \beta_S\mu_S$, et combinant alors (1d)–(1f) on obtient l'équation suivante pour la densité :

$$\frac{\partial \rho^*_{\mathrm{full}}}{\partial t^*} + u^* \frac{\partial \rho^*_{\mathrm{full}}}{\partial x^*} + v^* \frac{\partial \rho^*_{\mathrm{full}}}{\partial y^*} + w^* \frac{\partial \rho^*_{\mathrm{full}}}{\partial z^*} = \mu^*_\rho \, \Delta^*_{\mathrm{h}} \rho^*_{\mathrm{full}} + \nu^*_\rho \, \frac{\partial^2 \rho^*_{\mathrm{full}}}{\partial z^{*2}} \, . \tag{2}$$

Les détails concernant la façon d'obtenir ces équations se trouvent dans la littérature géophysique (voir e.g., [14], [15]).

Les Equations Primitives sont le point de départ pour les sciences de l'atmosphère et des océans et leur intérêt pratique a amené nombre de mathématiciens a les étudier du point de vue mathématique et du point de vue de l'analyse numérique. On rappelle ici le travail fondateur de Lions, Temam et Wang sur l'existence et l'unicité de solution pour les Equations Primitives (voir, par exemple, [10], [11] et l'article de synthèse [18]).

Dans le premier chapitre de la thèse on considère le cas où la densité ρ^*_{full} est de la forme

$$\rho^*_{\mathrm{full}}(x, y, z, t) = \rho_{\mathrm{ref}} + \bar{\rho}(z) + \rho^\star(x, y, z, t),$$

où $\bar{\rho} = \bar{\rho}(z)$ est le profil de la stratification pour la densité. On introduit la fréquence de Brunt–Väisälä $N^\star$, qui est supposé constante :

$$(N^\star)^2 = \frac{g}{\rho_{\mathrm{ref}}} \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z};$$

en fait cela signifie qu'on considère une partie de l'océan où le profil de la stratification est presque une fonction linéaire. Alors, l'équation de l'évolution (2) pour la densité devient :

$$\frac{\partial \rho^*}{\partial t^*} + u^* \frac{\partial \rho^*}{\partial x^*} + v^* \frac{\partial \rho^*}{\partial y^*} + w^* \frac{\partial \rho^*}{\partial z^*} - \frac{\rho_{\mathrm{ref}}}{g} (N^*)^2 \, w^* = \mu_\rho \, \Delta^*_{\mathrm{h}} \rho^* + \nu_\rho \, \frac{\partial^2 \rho^*}{\partial z^{*2}} \, . \tag{3}$$

On écrit les équations sous leur forme adimensionnée et on considère toutes les fonctions comme indépendentes de la variable y. Le modèle ainsi obtenu est le suivant :

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} - \frac{1}{\varepsilon} v + \frac{1}{\varepsilon} \frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}} \, \Delta u + S_u, \tag{4a}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} + \frac{1}{\varepsilon} u = \nu_{\boldsymbol{v}} \, \Delta v + S_v, \tag{4b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{4c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{4d}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} - \frac{N^2}{\varepsilon} w = \nu_\rho \, \Delta \rho + S_\rho, \tag{4e}$$

où (S_u, S_v, S_ρ) est la force extérieure et les paramètres adimensionés sont le nombre de Rossby ε, ainsi que l'inverse des nombres de Reynolds $\nu_{\boldsymbol{v}}$ et ν_ρ. Les détails concernant l'obtention de ce système sont rappelés dans l'Appendice du premier chapitre.

Dans le premier chapitre, on considère le système (4) avec des conditions aux limites périodiques et pour garantir le caractère bien posé du problème, on exige que u, v et p

soient paires, et w et ρ soient impaires en z. On prouve l'existence globale en temps d'une solution faible correspondant à une donnée initiale dans L^2, ainsi que l'existence (pour tout temps) et l'unicité de la solution forte quand la donnée initiale est dans H^1. De plus, on prouve l'existence de solutions très réguliéres, jusqu'à la régularité $\mathcal{C}^\infty$.

Dans le deuxième chapitre on considère un modèle semblable au (4) mais sans stratification ; faisant $N^* = 0$ dans (4e) on obtient :

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = \nu_\rho \Delta\rho + S_\rho. \tag{5}$$

Dans (5), la densité ρ est la différence entre la densité totale et sa valeur de référence : $\rho = \rho_{\text{full}} - \rho_{\text{ref}}$. Le modèle ainsi obtenu est completé par des conditions aux limites periodiques et on étudie la régularité du type de Gevrey pour la solution du système.

On montre que, en considérant comme force extérieure une fonction analytique en temps avec des valeurs dans un espace de type Gevrey, les solutions des Equations Primitives correspondant à une donnée initiale dans l'espace de Sobolev H^1 sont, dans un certain intervalle de temps, éléments d'un espace de type Gevrey. On montre que la solution des Equations Primitives est la restriction à l'axe réel $t \geq 0$ d'une fonction complexe de la variable t, analytique dans un voisinage de l'axe réel.

Ce travail a été inspiré par un article de Foias et Temam [9], où les auteurs ont prouvé des résultats similaires sur les Equations de Navier–Stokes. Comme dans [9], l'idée consiste à écrire les Equations Primitives comme une équation d'évolution, de prolonger cette équation dans le temps complexe et d'obtenir des extimations a priori. Comme la forme des Equations Primitives est plus compliquée que celle des Equations de Navier–Stokes, quelques difficultés additionelles apparaissent. Les difficultés peuvent être surmontées si on décompose l'équation d'évolution initiale en une partie linéaire et la partie nonlinéaire restante, et on obtient des estimations a priori séparément, premièrement pour le problème linéaire, puis pour le probléme nonlinéaire.

Le troisième chapitre est une extension, en dimension trois, du travail fait dans les deux premiers chapitres. On considère le modèle du deuxième chapitre mais cette fois-ci en dimension trois d'espace et on étudie la régularité de type Sobolev et Gevrey pour les solutions. Les méthodes utilisées sont semblables à celles des chapitres précédents mais des difficultés techniques nouvelles apparaissent. En comparant aux différences classiques entre les Equations de Navier–Stokes en dimension deux d'espace et en dimension trois, les résultats obtenus pour les Equations Primitives sont, comme prévu, plus faibles que leurs équivalents en dimension deux. Ainsi, on obtient l'existence d'une solution très réguliére, mais seulement localement en temps.

Dans le dernier chapitre de la thèse, on étudie l'asymptotique au petit nombre de Rossby pour les Equations Primitives. Quand le nombre de Rossby tend vers zero, la solution exacte présente des oscillations qui peuvent être "moyennées" par une méthode convenable provenant de la théorie de la renormalisation. On travaille avec les Equations Primitives en dimension deux d'espace, sous la forme (4) étudiée dans le premier chapitre.

La méthode de renormalisation utilisée est celle introduite par Schochet [16] et reformulée dans un contexte mathématique par Ziane [19]. Indépendamment, la méthode a e'té introduit, dans un contexte physique, par Chen, Goldenfeld et Oono, et utilisée dans un contexte mathématique pour les fluides en rotation et pour les fluides géophysique, par Chemin [4], Embid-Majda [6] et Grenier [8]. Les applications de la méthode de renormalisation a fourni beaucoup de litterature mathématique et on mentionne ici le travail de Gallagher [7] qui deduit des expansions asymptotiques, et aussi le travail de Babin-Mahalov-Nicolaenko [1], [2], et de Moise, Temam, Ziane [12], [13]. Pour les equations differentielles ordinaire, Temam et Wirosoetisno ont montré dans [17] que la méthode peut être appliquée aux ordres supérieurs.

On applique ici cette méthode de renormalisation pour obtenir une approximation d'ordre un. L'idée de [19] est d'écrire le système (4) comme un problème d'évolution abstrait de la forme :

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{1}{\varepsilon} LU = \mathcal{F}(U), \\ &U(0) = U_0, \end{aligned} \tag{6}$$

où $\varepsilon > 0$ est un paramètre petit et L est un opérateur antisymétrique (qui provoque les oscillations des solutions) ; L correspond à la force de Coriolis. On ecrit (6) dans la variable rapide $s = t/\varepsilon$ et on introduit $F(s, \cdot) = e^{Ls}\mathcal{F}(e^{-Ls}\cdot)$. On partage F en sa partie F_r indépendante du temps (la partie resonante) et la partie restante F_n, dépendante du temps s. Puis on considère l'opérateur :

$$F_{np}(s, U) = \int_0^s F_n(s', U)\ \mathrm{d}s'. \tag{7}$$

La solution approximative cherchée est de la forme :

$$\tilde{U}^1(s) = e^{-Ls}\{\bar{U}(s) + \varepsilon F_{np}(s, \bar{U}(s))\}, \tag{8}$$

où $\bar{U}$ est la solution de l'équation renormalisée :

$$\begin{cases} \dfrac{\mathrm{d}\bar{U}}{\mathrm{d}s} = \varepsilon F_r(\bar{U}), \\ \\ \bar{U}(0) = U_0. \end{cases} \tag{9}$$

On montre d'abord que le système renormalisé garde les propriétés du système initial, c'est à dire la conservation de l'énergie pour le cas sans viscosité (propriété d'orthogonalité du terme nonlinéaire) et de dissipation (coercivité) en présence de viscosité. En utilisant ces propriétés, on montre l'existence et l'unicité pour tout temps d'une solution faible ou d'une solution très réguliére pour le système (9) ; la régularité de la solution dépendant bien sur de la régularité de la donnée initiale. Outre cette étude du système renormalisé (**??**), un autre résultat principal de ce chapitre est de montrer que la solution renormalisée est une bonne approximation pour la solution exacte qui est oscillante :

$$|\tilde{U}^1(s) - U(s)|_{L^2} \sim \mathcal{O}(\varepsilon). \tag{10}$$

Ce chapitre se termine par des appendices contenant les résultats techniques utilisés, tels que la dérivation du système renormalisé et deux manières différentes de traiter les petits dénominateurs qui apparaissent dans l'étude.

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Chapitre 1

Existence and Regularity Results for the Primitive Equations in Two Space Dimensions

Resultats d'existence et régularité pour les Equations Primitives en deux dimensions

Ce chapitre est constitué de l'article **Existence and Regularity Results for the Primitive Equations in Two Space Dimensions**, ecrit en collaboration avec R. Temam et D. Wirosoetisno, article paru en 2004 dans *Communication on pure and applied analysis*, volume 3, numero 1, pages 115-131. Dans cet article on montre l'existence globalement en temps d'une solution faible pour les Equations Primitives en dimension deux d'espace, avec conditions aux limites periodiques, ainsi que l'existence et l'unicité globalement en temps d'une solution très régulière, jusqu'à la régularité $\mathcal{C}^\infty$, si les données sont suffisament régulières.

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Existence and Regularity Results for the Primitive Equations in Two Space Dimensions

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Abstract: Our aim in this article is to present some existence, uniqueness and regularity results for the Primitive Equations of the ocean in space dimension two with periodic boundary conditions. We prove the existence of weak solutions for the PEs, the existence and uniqueness of strong solutions and the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity.

1.1 Introduction

The objective of this article is to derive various results of existence and regularity of solutions for the Primitive Equations of the ocean (PEs) in two space dimensions. These results, besides their intrinsic interest, are needed in [9] which is another motivation of this work.

We consider the PEs in their nondimensional form (see Section 1.5) :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}}\,\Delta u + S_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u = \nu_{\boldsymbol{v}}\,\Delta v + S_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{1.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{\varepsilon}w = \nu_{\rho}\,\Delta \rho + S_{\rho}. \tag{1.1e}$$

All the independent variables (t, x, z) and the dependent variables (u, v, w, ρ, p) are dimensionless, as are the forcing and source terms (S_u, S_v, S_ρ). Here $(u,\ v,\ w)$ are the three components of the velocity vector and, as usual, we denote by p and ρ the pressure and density deviations, respectively, from prescribed background states. The (dimensionless) parameters are the Rossby number ε, the Burger number N, and the inverse (eddy) Reynolds numbers $\nu_{\boldsymbol{v}}$ and ν_ρ.

Some motivations on the physical background and the derivation of these equations are given in the Appendix (Section 1.5). The two spatial directions are $0x$ and $0z$, corresponding to the west–east and vertical directions in the so-called f-plane approximation for geophysical flows (for details, see the Appendix); $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$.

The article is organized as follows: We start in Section 4.3.2 by recalling the variational formulation of problem (2.1) under suitable assumptions and we prove the existence of weak solutions for the PEs. We continue in Section 1.3 by proving the existence and uniqueness of strong solutions. Finally in Section 1.4 we prove the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity. We thought that it is useful to end the article with an Appendix (Section 1.5) containing some physical explanations regarding the PEs and the derivation of (2.1).

We mention here the similar works of Bresch, Kazhikhov and Lemoine [2] and of Ziane [13], who consider different boundary conditions and do not consider the higher regularity results needed in [9]; see also [11]. For the non-dimensional form of the PEs, we refer here for example to [4], [8], and [12] but a substantial amount of literature is available on this subject.

1.2 Existence of the Weak Solutions for the PEs

We work in a limited domain

$$\mathcal{M} = (0,\ L_1) \times (-L_3/2,\ L_3/2), \tag{2.1}$$

and, since this is needed in [9], we assume space periodicity with period $\mathcal{M}$, that is, all functions are taken to satisfy $f(x + L_1, z, t) = f(x, z, t) = f(x, z + L_3, t)$ when extended to $\mathbb{R}^2$. Moreover, we assume that the following symmetries hold:

$$
\begin{aligned}
u(x,z,t) &= u(x,-z,t), & S_u(x,z,t) &= S_u(x,-z,t),\\
v(x,z,t) &= v(x,-z,t), & S_v(x,z,t) &= S_v(x,-z,t),\\
\rho(x,z,t) &= -\rho(x,-z,t), & S_\rho(x,z,t) &= -S_\rho(x,-z,t),\\
w(x,z,t) &= -w(x,-z,t), & p(x,z,t) &= p(x,-z,t).
\end{aligned}
$$

(Here u, v and p are said to be even in z, and w and ρ odd in z.)

We note that these conditions are often used in numerical studies of rotating stratified turbulence (see e.g., [1]).

Our aim is to solve the problem (2.1) with initial data

$$u = u_0,\ v = v_0,\ \rho = \rho_0 \quad \text{at } t = 0. \tag{2.2}$$

Hence the natural function spaces for this problem are as follows:

$$
\begin{aligned}
V = \{&(u,\, v,\, \rho) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3,\\
&u,\, v \text{ even in } z,\, \rho \text{ odd in } z,\ \textstyle\int_{-L_3/2}^{L_3/2} u(x,z')\,\mathrm{d}z' = 0\},
\end{aligned} \tag{2.3}
$$

$$H = \text{closure of } V \text{ in } (\dot{L}^2(\mathcal{M}))^3. \tag{2.4}$$

Here the dot above $\dot{H}^1_{\text{per}}$ or $\dot{L}^2$ denotes the functions with average in $\mathcal{M}$ equal to zero. These spaces are endowed with Hilbert scalar products; in H the scalar product is

$$(U,\, \tilde{U})_H = (u,\, \tilde{u})_{L^2} + (v,\, \tilde{v})_{L^2} + \kappa(\rho,\, \tilde{\rho})_{L^2}, \tag{2.5}$$

and in $\dot{H}^1_{\text{per}}$ and V the scalar product is (using the same notation when there is no ambiguity):

$$((U,\, \tilde{U})) = ((u,\, \tilde{u})) + ((v,\, \tilde{v})) + \kappa((\rho,\, \tilde{\rho})); \tag{2.6}$$

where we have written $\mathrm{d}\mathcal{M}$ for $\mathrm{d}x\,\mathrm{d}z$, and

$$((\phi,\, \tilde{\phi})) = \int_{\mathcal{M}} \Big(\frac{\partial\phi}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial\tilde{\phi}}{\partial z}\Big)\,\mathrm{d}\mathcal{M}. \tag{2.7}$$

The positive constant κ is defined below. We have

$$|U|_H \leq c_0\|U\|,\ \forall\, U \in V. \tag{2.8}$$

where $c_0 > 0$ is a positive constant related to κ and the Poincaré constant in $\dot{H}^1_{\text{per}}(\mathcal{M})$. More generally, the c_i, c_i', c_i'' will denote various positive constants. Inequality (1.9) implies that $\|U\| = ((U,\, U))^{1/2}$ is indeed a norm on V.

We first show how we can express the diagnostic variables w and p in terms of the prognostic variables u, v and ρ. For each $U = (u,\, v,\, \rho) \in V$ we can determine uniquely $w = w(U)$ from (2.1d),

$$w(U) = w(x,z,t) = -\int_0^z u_x(x,z',t)\,\mathrm{d}z', \tag{2.9}$$

since $w(x,0)=0$, w being odd in z. Furthermore, writing that $w(x,-L_3/2,t) = w(x,L_3/2,t)$, we also have

$$\int_{-L_3/2}^{L_3/2} u_x(x,z',t)\,\mathrm{d}z' = 0. \tag{2.10}$$

As for the pressure, we obtain from (2.1c),

$$p(x,z,t) = p_{\mathrm{s}}(x,t) - \int_0^z \rho(x,z',t)\,\mathrm{d}z', \tag{2.11}$$

where $p_{\mathrm{s}} = p(x,0,t)$ is the surface pressure. Thus, we can uniquely determine the pressure p in terms of ρ up to p_{s}.

It is appropriate to use Fourier series and we write, e.g., for u,

$$u(x,z,t) = \sum_{(k_1,k_3)\in\mathbb{Z}} u_{k_1,k_3}(t)\mathrm{e}^{i(k_1'x+k_3'z)}, \tag{2.12}$$

where for notational conciseness we set $k_1' = 2\pi k_1/L_1$ and $k_3' = 2\pi k_3/L_3$. Since u is real and even in z, we have $u_{-k_1,-k_3} = \bar{u}_{k_1,k_3} = \bar{u}_{k_1,-k_3}$, where $\bar{u}$ denotes the complex conjugate of u. Regarding the pressure, we obtain from (2.1c):

$$\begin{aligned}
p(x,z,t) &= p(x,0,t) - \int_0^z \sum_{(k_1,\,k_3)} \rho_{k_1,\,k_3} e^{i(k_1'x+k_3'z')}\mathrm{d}z' \\
&= \sum_{k_1} p_{\mathrm{s}k_1} e^{ik_1'x} - \sum_{(k_1,\,k_3),\,k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'} e^{ik_1'x}(e^{ik_3'z}-1) \\
&\qquad \text{[using the fact that } \rho_{k_1,\,0}=0,\ \rho \text{ being odd in } z] \\
&= \sum_{k_1} \big(p_{\mathrm{s}k_1} + \sum_{k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'}\big) e^{ik_1'x} - \sum_{(k_1,\,k_3),\,k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'} e^{i(k_1'x+k_3'z)} \\
&= \sum_{k_1} p_{\star k_1} e^{ik_1'x} - \sum_{(k_1,\,k_3),\,k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'} e^{i(k_1'x+k_3'z)},
\end{aligned}$$

where we denoted by p_{s} the surface pressure and $p_\star = \sum_{k_1\in\mathbb{Z}} \; p_{\star\, k_1} e^{ik_1'x}$, which is the average of p in the vertical direction, is defined by

$$p_{\star,k_1} = p_{\mathrm{s}k_1} + \sum_{k_3\neq 0} \frac{\rho_{k_1,\,k_3}}{ik_3'}.$$

Note that p is fully determined by ρ, up to one of the terms p_{s} or $p_\star$ which are connected by the relation above.

We now obtain the variational formulation of problem (2.1). For that purpose we consider a test function $\tilde{U} = (\tilde{u},\,\tilde{v},\,\tilde{\rho}) \in V$ and we multiply (2.1a), (2.1b) and (2.1e), respectively by $\tilde{u}$, $\tilde{v}$ and $\kappa\,\tilde{\rho}$, where the constant κ (which was already introduced in (1.6) and (2.4)) will be chosen later. We add the resulting equations and integrate over $\mathcal{M}$. We find:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_H + b(U,U,\tilde{U}) + a(U,\tilde{U}) + \frac{1}{\varepsilon}e(U,\tilde{U}) = (S,\tilde{U})_H, \qquad \forall\, \tilde{U}\in V. \tag{2.13}$$

Here we set

$$\begin{aligned}
a(U,\tilde{U}) &= \nu_{\boldsymbol{v}}((u,\tilde{u})) + \nu_{\boldsymbol{v}}((v,\tilde{v})) + \kappa\nu_\rho((\rho,\tilde{\rho})),\\
e(U,\tilde{U}) &= \int_{\mathcal{M}} (u\tilde{v} - v\tilde{u})\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (\rho\tilde{w} - \kappa N^2\, w\tilde{\rho})\,\mathrm{d}\mathcal{M},\\
b(U,U^\sharp,\tilde{U}) &= \int_{\mathcal{M}} \Big(u\frac{\partial u^\sharp}{\partial x} + w(U)\frac{\partial u^\sharp}{\partial z}\Big)\tilde{u}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} \Big(u\frac{\partial v^\sharp}{\partial x} + w(U)\frac{\partial v^\sharp}{\partial z}\Big)\tilde{v}\,\mathrm{d}\mathcal{M}\\
&\qquad + \kappa\int_{\mathcal{M}} \Big(u\frac{\partial \rho^\sharp}{\partial x} + w(U)\frac{\partial \rho^\sharp}{\partial z}\Big)\tilde{\rho}\,\mathrm{d}\mathcal{M}.
\end{aligned}$$

We now choose $\kappa = 1/N^2$ and this way we find $e(U,U) = 0$. Also it can be easily seen that:

$$\begin{aligned}
&a : V\times V \to \mathbb{R} \text{ is bilinear, continuous, coercive, } a(U,U) \geq c_1\|U\|^2,\\
&e : V\times V \to \mathbb{R} \text{ is bilinear, continuous, } e(U,U) = 0,\\
&b \text{ is trilinear, continuous from } V\times V_2\times V \text{ into } \mathbb{R},\\
&\qquad\qquad \text{and from } V\times V\times V_2 \text{ into } \mathbb{R},
\end{aligned} \tag{2.14}$$

where V_2 is the closure of $V \cap (H^2_{\text{per}}(\mathcal{M}))^3$ in $(H^2_{\text{per}}(\mathcal{M}))^3$. Furthermore,

$$\begin{aligned}
&b(U,\tilde{U},U^\sharp) = -b(U,U^\sharp,\tilde{U}),\\
&b(U,\tilde{U},\tilde{U}) = 0,
\end{aligned} \tag{2.15}$$

when U, $\tilde{U}$, $U^\sharp \in V$ with $\tilde{U}$ or $U^\sharp$ in V_2. We also have the following:

Lemma 1.2.1. *There exists a constant $c_2 > 0$ such that, for all $U \in V$, $\tilde{U} \in V_2$ and $U^\sharp \in V$:*

$$\begin{aligned}
|b(U,U^\sharp,\tilde{U})| &\leq c_2|U|^{1/2}_{L^2}\|U\|^{1/2}\|U^\sharp\||\tilde{U}|^{1/2}_{L^2}\|\tilde{U}\|^{1/2}\\
&\quad + c_2\|U\|\|U^\sharp\|^{1/2}|U^\sharp|^{1/2}_{V_2}|\tilde{U}|^{1/2}_{L^2}\|\tilde{U}\|^{1/2}.
\end{aligned} \tag{2.16}$$

Proof: We only estimate two typical terms; the other terms are estimated exactly in the same way. Using the Hölder, Sobolev and interpolation inequalities, we write:

$$\begin{aligned}
\Big|\int_{\mathcal{M}} u\frac{\partial u^\sharp}{\partial x}\tilde{u}\,\mathrm{d}\mathcal{M}\Big| &\leq |u|_{L^4}\Big|\frac{\partial u^\sharp}{\partial x}\Big|_{L^2}|\tilde{u}|_{L^4}\\
&\leq c_1'|u|^{1/2}_{L^2}\|u\|^{1/2}\Big|\frac{\partial u^\sharp}{\partial x}\Big|_{L^2}|\tilde{u}|^{1/2}_{L^2}\|\tilde{u}\|^{1/2},
\end{aligned}$$

$$\begin{aligned}
\Big|\int_{\mathcal{M}} w(U)\frac{\partial u^\sharp}{\partial z}\tilde{u}\,\mathrm{d}\mathcal{M}\Big| &\leq |w(U)|_{L^2}\Big|\frac{\partial u^\sharp}{\partial z}\Big|_{L^4}|\tilde{u}|_{L^4}\\
&\leq c_2'\|u\|\Big|\frac{\partial u^\sharp}{\partial z}\Big|^{1/2}_{L^2}\Big\|\frac{\partial u^\sharp}{\partial z}\Big\|^{1/2}|\tilde{u}|^{1/2}\|\tilde{u}\|^{1/2};
\end{aligned}$$

(2.16) follows from these estimates and the analogous estimates for the other terms.

We now recall the result regarding the existence of weak solutions for the PEs of the ocean; see [7]. In [7] the existence of the weak solutions is established in three space dimensions with different boundary conditions, but the proof applies as well to two dimensions with our boundary conditions.

Theorem 1.2.1. *Given $U_0 \in H$ and $S \in L^\infty(\mathbb{R}_+;\, H)$, there exists at least one solution U of (2.13), $U \in L^\infty(\mathbb{R}_+;\, H) \cap L^2(0, t_\star;\, V)$, $\forall\, t_\star > 0$, with $U(0) = U_0$.*

The proof of this theorem is based on the a priori estimates given below, which gives, as in [7], that $U \in L^\infty(0,\, t_\star;\, H)$, $\forall\, t_\star > 0$; however, as shown below, we have in fact,

$$U \in L^\infty(\mathbb{R}_+;\, H).$$

Taking $\tilde{U} = U$ in equation (2.13), after some simple computations and using (2.14), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_1\|U\|^2 \le c_1'|S|_\infty^2\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_0c_1|U|_H^2 \le c_1'|S|_\infty^2\,, \tag{2.17}$$

where $|S|_\infty$ is the norm of S in $L^\infty(\mathbb{R}_+;\, H)$. Using the Gronwall inequality, we infer from (2.17) that:

$$|U(t)|_H^2 \le |U(0)|_H^2\, \mathrm{e}^{-c_1c_0t} + \frac{c_1'}{c_1c_0}(1 - \mathrm{e}^{-c_1c_0t})\, |S|_\infty^2\,, \quad \forall\, t > 0. \tag{2.18}$$

Hence

$$\limsup_{t\to\infty} |U(t)|_H^2 \le \frac{c_1'}{c_1c_0}|S|_\infty^2 =: r_0^2\,,$$

and any ball $B(0,\, r_0')$ in H with $r_0' > r_0$ is an absorbing ball; that is, for all U_0, there exists $t_0 = t_0(|U_0|_H)$ depending increasingly on $|U_0|_H$ (and depending also on r_0', $|S|_\infty$ and other data), such that $|U(t)|_H \le r_0'$, $\forall\, t \ge t_0(|U_0|_H)$. Furthermore, integrating equation (2.17) from t to $t + r$, with $r > 0$ arbitrarily chosen, we find:

$$\int_t^{t+r} \|U(t')\|^2\, \mathrm{d}t' \le K_1, \qquad \text{for all } t \ge t_0(|U_0|_H), \tag{2.19}$$

where K_1 denotes a constant depending on the data but not on U_0. As mentioned before, (3.10) implies also that

$$U \in L^\infty(\mathbb{R}_+;\, H), \qquad |U(t)|_H \le \max(|U_0|_H,\, r_0).$$

Remark 1.2.1. We notice that, in the inviscid case ($\nu_{\boldsymbol{v}} = \nu_\rho = 0$ with $S = 0$), taking $\tilde{U} = U$ in (2.13), we find, at least formally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|_{L^2}^2 + |v|_{L^2}^2 + \frac{1}{N^2}|\rho|_{L^2}^2\right) = 0. \tag{2.20}$$

The physical meaning of (2.20) is that the sum of the kinetic energy (given by $\frac{1}{2}(|u|_{L^2}^2 + |v|_{L^2}^2)$) and the available potential energy (given by $\frac{1}{2N^2}|\rho|_{L^2}^2$) is conserved in time. This is the physical justification of the introduction of the constant $\kappa = N^{-2}$ in (1.6).

Remark 1.2.2. We note that the result deduced in (2.14) regarding the trilinear form b is weaker than the equivalent one for the Navier-Stokes equations since here the vertical velocity is a diagnostic variable which is deduced as an integral in the z variable of u_x. Consequently, for the 2D Primitive Equations we can not prove the uniqueness of the weak solution (result available for the 2D Navier-Stokes equations).

1.3 Existence and Uniqueness of Strong Solutions for the PEs

The solutions given by Theorem 1.2.1 are usually called weak solutions. We are now interested in strong solutions (and even more regular solutions in Section 1.4). We use here the same terminology as in fluid mechanics (incompressible Navier–Stokes equations): weak solutions are those in $L^\infty(L^2)$ and $L^2(H^1)$, strong solutions are those in $L^\infty(H^1)$ and $L^2(H^2)$. We notice that we cannot obtain directly the global existence of strong solutions for the PEs as, e.g., for the Navier–Stokes equations using a single a priori estimate (obtained by replacing $\tilde{U}$ by ΔU in (2.13)). Instead, to derive the necessary a priori estimates we proceed by steps: we successively derive estimates in $L^\infty(L^2)$ and $L^2(H^1)$ for u_z, u_x, v_z, v_x, ρ_z and ρ_x (here the subscripts t, x, z denote differentiation). Notice that the order in which we obtain these estimates cannot be changed in the calculations below.

Firstly, using (2.11) we rewrite (2.1a) as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p_{\mathrm{s}}}{\partial x} - \frac{1}{\varepsilon}\int_0^z \rho_x(x,\, z',\, t)\,\mathrm{d}z' = \nu_{\boldsymbol{v}}\Delta u + S_u. \tag{3.1}$$

We differentiate (3.1) with respect to z and we find, with $w_z = -u_x$:

$$u_{tz} + uu_{xz} + wu_{zz} - \frac{1}{\varepsilon}v_z - \frac{1}{\varepsilon}\rho_x - \nu_{\boldsymbol{v}}u_{xxz} - \nu_{\boldsymbol{v}}u_{zzz} = S_{u,\,z}\,,$$

where $S_{u,\,z} = \partial_z S_u = \partial S_u/\partial z$. After multiplying this equation by u_z and integrating over $\mathcal{M}$, we find:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + \nu_{\boldsymbol{v}}\|u_z\|^2 + \int_{\mathcal{M}} uu_zu_{xz}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} wu_zu_{zz}\,\mathrm{d}\mathcal{M}\\ - \frac{1}{\varepsilon}\int_{\mathcal{M}} v_zu_z\,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} \rho_xu_z\,\mathrm{d}\mathcal{M} = \int_{\mathcal{M}} u_zS_{u,\,z}\,\mathrm{d}\mathcal{M}.\end{aligned}$$

Integrating by parts and taking into account the periodicity and the conservation of mass equation (2.1d) we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + \nu_{\boldsymbol{v}}\|u_z\|^2 - \frac{1}{\varepsilon}\int_{\mathcal{M}} v_zu_z\,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} \rho_xu_z\,\mathrm{d}\mathcal{M} = \int_{\mathcal{M}} u_zS_{u,\,z}\,\mathrm{d}\mathcal{M}. \tag{3.2}$$

In all that follows $K(\varepsilon)$, $K'(\varepsilon)$, $K''(\varepsilon)$, ..., denote constants depending on ε and other data but not on U_0; we use the same symbol for different constants. We easily obtain the following estimates:

$$\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} v_zu_z\,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} vu_{zz}\,\mathrm{d}\mathcal{M}\Big| \leq K(\varepsilon)|v|^2_{L^2} + \frac{\nu_{\boldsymbol{v}}}{6}\|u_z\|^2,$$

$$\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} \rho_x u_z \,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} \rho\, u_{xz} \,\mathrm{d}\mathcal{M}\Big| \leq \frac{\nu_{\boldsymbol{v}}}{6}\|u_z\|^2 + K(\varepsilon)|\rho|^2_{L^2},$$

$$\Big|\int_{\mathcal{M}} S_{u,z} u_z \,\mathrm{d}\mathcal{M}\Big| = \Big|\int_{\mathcal{M}} S_u u_{zz} \,\mathrm{d}\mathcal{M}\Big| \leq \frac{\nu_{\boldsymbol{v}}}{6}\|u_z\|^2 + c_1'|S_u|^2_{L^2}\,;$$

applied to (3.2), these give:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + \nu_{\boldsymbol{v}}\,\|u_z\|^2 \leq K(\varepsilon)(|v|^2_{L^2} + |\rho|^2_{L^2}) + c_1'|S_u|^2_{L^2}. \tag{3.3}$$

We apply Poincaré's inequality (1.9) and we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} + c_0\nu_{\boldsymbol{v}}|u_z|^2_{L^2} \leq K(\varepsilon)(|v|^2_{L^2} + |\rho|^2_{L^2}) + c_1'|S_u|^2_{L^2}. \tag{3.4}$$

Using Gronwall's lemma, we infer from (3.4) that:

(3.5)

$$\begin{aligned}|u_z(t)|^2_{L^2} &\leq |u_z(0)|^2_{L^2}e^{-c_0\nu_{\boldsymbol{v}}t} + K(\varepsilon)e^{-c_0\nu_{\boldsymbol{v}}t}\int_0^t (|v(t')|^2_{L^2} + |\rho(t')|^2_{L^2})e^{c_0\nu_{\boldsymbol{v}}t'}\,\mathrm{d}t' + c_2'|S_u|^2_\infty \\ &\leq |u_z(0)|^2_{L^2}e^{-c_0\nu_{\boldsymbol{v}}t} + K'(\varepsilon)(1 - e^{-c_0\nu_{\boldsymbol{v}}t})(|v|^2_\infty + |\rho|^2_\infty) + c_2'|S_u|^2_\infty \\ &\leq |u_z(0)|^2_{L^2}e^{-c_0\nu_{\boldsymbol{v}}t} + K'(\varepsilon)(|v|^2_\infty + |\rho|^2_\infty) + c_2'|S_u|^2_\infty,\end{aligned}$$

where $|v|_\infty = |v|_{L^\infty(\mathbb{R}_+;\,L^2(\mathcal{M}))}$, and similarly for ρ and S_u. We obtain an explicit bound for the norm of u_z in $L^\infty(\mathbb{R}_+;\,H)$:

$$|u_z(t)|^2_{L^2} \leq |u_z(0)|^2_{L^2} + K'(\varepsilon)(|v|^2_\infty + |\rho|^2_\infty) + c_2'|S_u|^2_\infty. \tag{3.6}$$

For what follows, we recall here the uniform Gronwall lemma (see e.g., [10]):
If ξ, η and y are three positive locally integrable functions on $(t_1,\,\infty)$ such that y' is locally integrable on $(t_1,\,\infty)$ and which satisfy

$$\begin{aligned}&y' \leq \xi y + \eta,\\ &\int_t^{t+r}\xi(s)\,\mathrm{d}s \leq a_1,\ \int_t^{t+r}\eta(s)\,\mathrm{d}s \leq a_2,\ \int_t^{t+r}y(s)\,\mathrm{d}s \leq a_3, \quad \forall\, t \geq t_1,\end{aligned} \tag{3.7}$$

where r, a_1, a_2, a_3 are positive constants, then

$$y(t+r) \leq \Big(\frac{a_3}{r} + a_2\Big)\,\mathrm{e}^{a_1}, \quad t \geq t_1. \tag{3.8}$$

The bound (3.6) depends on the initial data U_0. In order to obtain a bound independent of U_0 we apply the uniform Gronwall lemma to the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_z|^2_{L^2} \leq K(\varepsilon)(|v|^2_{L^2} + |\rho|^2_{L^2}) + c_1'|S_u|^2_{L^2}. \tag{3.9}$$

to obtain

$$|u_z(t)| \leq K'(\varepsilon, r, r_0'), \quad \forall\, t \geq t_1', \tag{3.10}$$

where $t_1' = t_0(|U_0|_{L^2}) + r$ and $r > 0$ is fixed. Integrating equation (3.20) from t to $t+r$ with $r > 0$ as before, we also find:

$$\int_t^{t+r} \|u_z(s)\|^2 \,\mathrm{d}s \le K''(\varepsilon, r, r_0'), \qquad \forall\, t \ge t_1'. \tag{3.11}$$

We now derive the same kind of estimates for u_x: We differentiate (3.1) with respect to x and we obtain

$$\begin{aligned} u_{tx} + u_x^2 + u u_{xx} + w u_{xz} + w_x u_z - \frac{1}{\varepsilon} v_x + \frac{1}{\varepsilon} p_{s,xx} + \int_z^0 \rho_{xx}(z')\,\mathrm{d}z' \\ - \nu_{\boldsymbol{v}}\, u_{xxx} - \nu_{\boldsymbol{v}}\, u_{zzx} = S_{u,\,x}; \end{aligned} \tag{3.12}$$

multiplying this equation by u_x and integrating over $\mathcal{M}$ we find, using (2.1d):

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_x|_{L^2}^2 + \int_{\mathcal{M}} u_x^3 \,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w_x u_z u_x \,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} v_x u_x \,\mathrm{d}\mathcal{M} - \frac{1}{\varepsilon}\int_{\mathcal{M}} p_{s,\,x\,x} u_x \,\mathrm{d}\mathcal{M} \\ + \int_{\mathcal{M}} \Big(\int_z^0 \rho_{x\,x}(z')\,\mathrm{d}z' \Big) u_x \,\mathrm{d}\mathcal{M} + \nu_{\boldsymbol{v}} \|u_x\|^2 = \int_{\mathcal{M}} u_x S_{u,\,x} \,\mathrm{d}\mathcal{M}. \end{aligned} \tag{3.13}$$

Based on the Hölder, Sobolev and interpolation inequalities, we derive the following estimates:

$$\begin{aligned} \Big| \int_{\mathcal{M}} u_x^3 \,\mathrm{d}\mathcal{M} \Big| &\le |u_x|_{L^3(\mathcal{M})}^3 \le c_4' |u_x|_{H^{1/3}(\mathcal{M})}^3 \le c_5' |u_x|_{L^2}^2 \|u_x\| \\ &\le \frac{\nu_{\boldsymbol{v}}}{12} \|u_x\|^2 + c_6' |u_x|_{L^2}^4, \end{aligned}$$

$$\begin{aligned} \Big| \int_{\mathcal{M}} w_x u_z u_x \,\mathrm{d}\mathcal{M} \Big| &\le c_7' |w_x|_{L^2} |u_z|_{L^2}^{1/2} \|u_z\|^{1/2} |u_x|_{L^2}^{1/2} \|u_x\|^{1/2} \\ &\le c_8' |u_{xx}|_{L^2} |u_z|_{L^2}^{1/2} \|u_z\|^{1/2} |u_x|_{L^2}^{1/2} \|u_x\|^{1/2} \\ &\le \frac{\nu_{\boldsymbol{v}}}{12} \|u_x\|^2 + c_9' |u_z|_{L^2}^2 \|u_z\|^2 |u_x|_{L^2}^2, \end{aligned}$$

By the definition of V, and since p_{s} is independent of z, we find:

$$\frac{1}{\varepsilon} \Big| \int_{\mathcal{M}} p_{\mathrm{s},\,x\,x} u_x \,\mathrm{d}\mathcal{M} \Big| = \frac{1}{\varepsilon} \Big| \int_0^L p_{\mathrm{s},\,xx} \int_{-L_3/2}^{L_3/2} u_x \,\mathrm{d}z\,\mathrm{d}x \Big| = 0.$$

We can also prove the following estimates:

$$\frac{1}{\varepsilon} \Big| \int_{\mathcal{M}} v_x u_x \,\mathrm{d}\mathcal{M} \Big| \le \frac{\nu_{\boldsymbol{v}}}{12} \|u_x\|^2 + K'(\varepsilon) |v|_{L^2}^2,$$

$$\begin{aligned} \frac{1}{\varepsilon} \Big| \int_{\mathcal{M}} \Big(\int_z^0 \rho_{x\,x}(z')\,\mathrm{d}z' \Big) u_x \,\mathrm{d}\mathcal{M} \Big| &= \frac{1}{\varepsilon} \Big| \int_{\mathcal{M}} \Big(\int_z^0 \rho_x(z')\,\mathrm{d}z' \Big) u_{xx} \,\mathrm{d}\mathcal{M} \Big| \\ &\le \frac{\nu_{\boldsymbol{v}}}{12} \|u_x\|^2 + K''(\varepsilon) |\rho_x|_{L^2}^2, \end{aligned}$$

$$\Big|\int_{\mathcal{M}} u_x S_{u,x}\,\mathrm{d}\mathcal{M}\Big| \le \frac{\nu_{\boldsymbol{v}}}{12}\|u_x\|^2 + c'_{10}|S_u|^2_\infty.$$

With these relations (3.13) implies:

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_x|^2_{L^2} + \nu_{\boldsymbol{v}}\|u_x\|^2 \le \xi\,|u_x|^2_{L^2} + \eta, \tag{3.14}$$

where we denoted

$$\xi = \xi(t) = 2c'_6|u_x|^2_{L^2} + 2c'_9|u_z|^2_{L^2}\|u_z\|^2,$$

and

$$\eta = \eta(t) = 2K'(\varepsilon)|v|^2_{L^2} + 2K''(\varepsilon)|\rho_x|^2_{L^2} + 2c'_{10}|S_u|^2_\infty.$$

We easily conclude from (5.12) that

$$u_x \in L^\infty(0,\, t_\star;\, L^2) \cap L^2(0,\, t_\star;\, H^1), \quad \forall\, t_\star > 0. \tag{3.15}$$

However, for later purposes, (3.15) is not sufficient, and we need estimates uniform in time.

We will apply the uniform Gronwall lemma to (5.12) with $t_1 = t'_1$ as in (3.10). Noting that

$$\begin{aligned}\int_t^{t+r} \xi(t')\,\mathrm{d}t' &= \int_t^{t+r} [2c'_6|u_x|^2_{L^2} + 2c'_9|u_z(t')|^2_{L^2}\|u_z(t')\|^2]\,\mathrm{d}t' \\ &\le 2c'_6\int_t^{t+r} |u_x(t')|^2_{L^2}\,\mathrm{d}t' + 2c'_9|u_z|^2_\infty\int_t^{t+r} \|u_z(t')\|^2\,\mathrm{d}t' \\ &\le a_1, \qquad \forall\, t \ge t'_1,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\int_t^{t+r} \eta(t')\,\mathrm{d}t' &= \int_t^{t+r} [2K'(\varepsilon)|v|^2_{L^2} + 2K''(\varepsilon)|\rho_x|^2_{L^2} + 2c'_{10}|S_u|^2_\infty]\,\mathrm{d}t' \\ &\le K(\varepsilon) + 2c'_{10}r|S_u|^2_\infty \\ &= a_2, \qquad \forall\, t \ge t'_1,\end{aligned} \tag{3.17}$$

$$\int_t^{t+r} |u_x(t')|^2_{L^2}\,\mathrm{d}t' \le a_3, \qquad \forall\, t \ge t'_1, \tag{3.18}$$

(3.8) then yields:

$$|u_x(t)|^2_{L^2} \le \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \quad \forall\, t \ge t'_1 + r, \tag{3.19}$$

and thus

$$|u_x|_{L^2} \in L^\infty(\mathbb{R}_+). \tag{3.20}$$

Note that in (3.16)–(3.18) we can use bounds on $|u_z|_\infty$ (and other similar terms) independent of U_0, since $t \ge t_0(|U_0|_{L^2}) + r$. Integrating equation (3.14) from 0 to $t'_1 + r$ where $t'_1 = t'_1(|U_0|_{L^2})$, we obtain a bound for u_x in $L^2(0,\, t'_1 + r;\, H^1)$ which depends on $\|U_0\|$.

A bound independent of U_0 is obtained if we work with $t \geq t_1' + r = t_1'' = t_1''(|U_0|_{L^2})$: Integrating equation (5.12) from t to $t+r$ with r as before, we find:

$$\int_t^{t+r} \|u_x(s)\|^2 \, \mathrm{d}s \leq K(\varepsilon), \qquad \forall\, t \geq t_1''. \tag{3.21}$$

We perform similar computations for v_z: We differentiate (2.1b) with respect to z, multiply the resulting equation by v_z and integrate over $\mathcal{M}$. Using again the conservation of mass relation, we arrive at:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_z|_{L^2}^2 + \int_{\mathcal{M}} u_z v_x v_z \,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} w_z v_z^2 \,\mathrm{d}\mathcal{M} + \frac{1}{\varepsilon}\int_{\mathcal{M}} u_z v_z \,\mathrm{d}\mathcal{M} + \nu_{\boldsymbol{v}}\|v_z\|^2 \\ = \int_{\mathcal{M}} v_z S_{u,\,z} \,\mathrm{d}\mathcal{M}. \end{aligned} \tag{3.22}$$

We notice the following estimate:

$$\begin{aligned} \Big|\int_{\mathcal{M}} u_z v_x v_z \,\mathrm{d}\mathcal{M}\Big| &\leq c_{11}'|u_z|_{L^2}^{1/2}\|u_z\|^{1/2}|v_x|_{L^2}|v_z|_{L^2}^{1/2}\|v_z\|^{1/2} \\ &\leq \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + c_{12}'|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3}|v_z|_{L^2}^{2/3} \\ &\leq \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + c_{12}'|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3}(1+|v_z|_{L^2}^2). \end{aligned}$$

We also see that

$$\begin{aligned} \Big|\int_{\mathcal{M}} w_z v_z v_z \,\mathrm{d}\mathcal{M}\Big| = \Big|\int_{\mathcal{M}} u_x v_z v_z \,\mathrm{d}\mathcal{M}\Big| &\leq c_{13}'|u_x|_{L^2}^{1/2}\|u_x\|^{1/2}|v_z|_{L^2}^{3/2}\|v_z\|^{1/2} \\ &\leq \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + c_{14}'|u_x|_{L^2}^{2/3}\|u_x\|^{2/3}|v_z|_{L^2}^2, \end{aligned}$$

$$\frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} u_z v_z \,\mathrm{d}\mathcal{M}\Big| = \frac{1}{\varepsilon}\Big|\int_{\mathcal{M}} u\, v_{zz} \,\mathrm{d}\mathcal{M}\Big| \leq \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + K(\varepsilon)|u|_{L^2}^2,$$

$$\Big|\int_{\mathcal{M}} S_{v,\,z} v_z \,\mathrm{d}\mathcal{M}\Big| = \Big|\int_{\mathcal{M}} S_v v_{zz} \,\mathrm{d}\mathcal{M}\Big| \leq \frac{\nu_{\boldsymbol{v}}}{8}\|v_z\|^2 + c_{15}'|S_v|_{\infty}^2,$$

which gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}|v_z|_{L^2}^2 + \nu_{\boldsymbol{v}}\|v_z\|^2 \leq \xi|v_z|^2 + \eta, \tag{3.23}$$

where we denoted

$$\eta = \eta(t) = 2c_{12}'|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3} + 2K(\varepsilon)|u|^2 + 2c_{15}'|S_v|_{\infty}^2,$$

and

$$\xi = \xi(t) = 2c_{12}'|u_z|_{L^2}^{2/3}\|u_z\|^{2/3}|v_x|_{L^2}^{4/3} + 2c_{14}'|u_x|_{L^2}^{2/3}\|u_x\|^{2/3}.$$

From (3.23), using the estimates obtained before and applying the classical Gronwall lemma we obtain bounds depending on the initial data for v_z in $L^{\infty}_{\mathrm{loc}}(0, t_\star; L^2)$ and $L^2_{\mathrm{loc}}(0, t_\star; H^1)$, valid for any finite interval of time $(0, t_\star)$.

To obtain estimates valid for all time, we apply the uniform Gronwall lemma observing that:

$$\tag{3.24} \begin{aligned} \int_t^{t+r} \xi(t')\,\mathrm{d}t' &\le 2c'_{12}|u_z|_\infty^{2/3}\Big(\int_t^{t+r} \|u_z(t')\|\,\mathrm{d}t'\Big)^{1/3}\Big(\int_t^{t+r} |v_x(t')|^2_{L^2}\,\mathrm{d}t'\Big)^{2/3} \\ &\quad + 2c'_{14}|u_x|_\infty^{2/3}\int_t^{t+r} \|u_x(t')\|^{2/3}\,\mathrm{d}t' \\ &\le a_1, \quad \forall\, t \ge t''_1, \end{aligned}$$

$$\tag{3.25} \begin{aligned} \int_t^{t+r} \eta(t')\,\mathrm{d}t' &\le 2c'_{12}|u_z|_\infty^{2/3}\Big(\int_t^{t+r} \|u_z(t')\|\,\mathrm{d}t'\Big)^{1/3}\Big(\int_t^{t+r} |v_x(t')|^2_{L^2}\,\mathrm{d}t'\Big)^{2/3} \\ &\quad + 2K(\varepsilon)|u|^2_\infty r + 2c'_{15} r |S_v|^2_\infty \\ &\le a_2, \quad \forall\, t \ge t''_1, \end{aligned}$$

$$\tag{3.26} \int_t^{t+r} |v_z(t')|^2\,\mathrm{d}t' \le a_3, \quad \forall\, t \ge t''_1.$$

Then the uniform Gronwall lemma gives:

$$\tag{3.27} |v_z(t)|^2_{L^2} \le \Big(\frac{a_3}{r} + a_2\Big) e^{a_1}, \quad \forall\, t \ge t''_1 + r,$$

with a_1, a_2, a_3 as in (3.25), (3.24) and (3.26). Integrating equation (3.23) from t to $t+r$ with $r > 0$ as above and $t \ge t''_1 + r$, we find:

$$\tag{3.28} \int_t^{t+r} \|v_z(s)\|^2\,\mathrm{d}s \le K(\varepsilon), \quad \forall\, t \ge t''_1 + r.$$

The same methods apply to v_x, ρ_z and ρ_x, noticing that at each step we precisely use the estimates from the previous steps, so the order can not be changed in this calculations.

With these estimates, the Galerkin method as used for the proof of Theorem 1.2.1 gives the existence of strong solutions:

Theorem 1.3.1. *Given $U_0 \in V$ and $S \in L^\infty(\mathbb{R}_+; H)$, there is a unique solution U of equation (2.13) with $U(0) = U_0$ such that*

$$\tag{3.29} U \in L^\infty(\mathbb{R}_+; V) \cap L^2(0, t_\star; (\dot{H}^2(\mathcal{M}))^3),\ \forall\, t_\star > 0.$$

Proof: As we said, the existence of strong solutions follows from the previous estimates. It remains to prove the uniqueness.

Assume U_1 and U_2 are two solutions of problem (2.13) satisfying (3.29), and let $U = U_1 - U_2$. We write (2.13) for U_1 and U_2 with $\tilde{U} = U$; combining the resulting equations, we find:

$$\tag{3.30} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_H + a(U, U) + b(U_1, U_1, U) - b(U_2, U_2, U) = 0.$$

Using (2.15b) we obtain:

$$\tag{3.31} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_H + c_1\|U\|^2 + b(U, U_2, U) \le 0.$$

From Lemma (1.2.1) and using Young's inequality we find that:

$$\text{(3.32)}\qquad \begin{aligned} b(U, U_2, U) &\leq c_1'|U|_{L^2}\|U\|\|U_2\| + c_2'|U|_{L^2}^{1/2}\|U\|^{3/2}|U_2|_{V_2}^{1/2} \\ &\leq \frac{c_1}{2}\|U\|^2 + c_3'|U|_{L^2}^2\|U_2\|^2 + c_4'|U|_{L^2}^2|U_2|_{V_2}^2. \end{aligned}$$

Going back to (3.31) we find:

$$\text{(3.33)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 \leq c_5'|U|_H^2(\|U_2\|^2 + |U_2|_{V_2}^2).$$

Since U_2 satisfies (3.29) the function

$$t \longrightarrow \|U_2(t)\|^2 + |U_2(t)|_{V_2}^2 \text{ is integrable,}$$

and we can apply the Gronwall lemma which yields, since $U_1(0) = U_2(0)$,

$$\text{(3.34)}\qquad |U(t)|_H^2 \leq 0, \quad \forall\, t \in [0, t_\star].$$

From (3.34) we conclude that $U_1 = U_2$.

1.4 More Regular Solutions for the PEs

In this section we show how to obtain estimates on the higher order derivatives from which one can derive the existence of solutions of the PEs in $(\dot{H}^m(\mathcal{M}))^3$ for all $m \in \mathbb{N}$, $m \geq 2$ (hence up to $\mathcal{C}^\infty$ regularity). In all that follows we work with U_0 in $(\dot{H}^m_{\text{per}}(\mathcal{M}))^3$.

We set $|U|_m = \left(\sum_{[\alpha]=m} |D^\alpha U|_{L^2}^2\right)^{1/2}$. We fix $m \geq 2$ and, proceeding by induction, we assume that for all $0 \leq l \leq m-1$, we have shown that

$$\text{(4.1)}\qquad U \in L^\infty(\mathbb{R}_+;\, (\dot{H}^l(\mathcal{M}))^3) \cap L^2(0,\, t_\star;\, (\dot{H}^{l+1}(\mathcal{M}))^3), \quad \forall\, t_\star > 0,$$

with

$$\text{(4.2)}\qquad \int_t^{t+r} |U(t')|_{l+1}^2\,\mathrm{d}t' \leq a_l, \quad \forall\, t \geq t_l(U_0),$$

where a_l is a constant depending on the data (and l) but not on U_0, and $r > 0$ is fixed (the same as before). We then want to establish the same results for $l = m$.

In equation (2.13) we take $\tilde{U} = \Delta^m U(t)$ with $m \geq 2$ and t arbitrarily fixed, and we obtain:

$$\text{(4.3)}\qquad \begin{aligned} \Big(\frac{\mathrm{d}U}{\mathrm{d}t},\, \Delta^m U\Big)_{L^2} + a(U, \Delta^m U) + b(U, U, \Delta^m U) + \frac{1}{\varepsilon}e(U,\, \Delta^m U) \\ = (S,\, \Delta^m U)_{L^2}. \end{aligned}$$

Integrating by parts, using periodicity and the coercivity of a and the fact that $e(U, U) = 0$, we find:

$$\text{(4.4)}\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U(t)|_m^2 + c_1|U|_{m+1}^2 \leq |b(U, U, \Delta^m U)| + |(S,\, \Delta^m U)_{L^2}|.$$

We need to estimate the terms on the right hand side of (4.4). We first notice that

$$|(S,\, \Delta^m U)_{L^2}| \le c|S|^2_{m-1} + \frac{c_1}{2(m+3)}|U|^2_{m+1}, \tag{4.5}$$

and it remains to estimate $|b(U, U, \Delta^m U)|$.

By the definition of b we have:

$$\begin{aligned} b(U,U,\Delta^m U) = &\int_{\mathcal{M}} (uu_x + w(U)u_z)\Delta^m u \,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (uv_x + w(U)v_z)\Delta^m v \,\mathrm{d}\mathcal{M} \\ &+ \kappa \int_{\mathcal{M}} (u\rho_x + w(U)\rho_z)\Delta^m \rho \,\mathrm{d}\mathcal{M}. \end{aligned} \tag{4.6}$$

The computations are similar for all the terms, and, for simplicity, we shall only estimate the first integral on the right hand side of (4.6).

We notice that $b(U, U, \Delta^m U)$ is a sum of integrals of the type

$$\int_{\mathcal{M}} u\frac{\partial u}{\partial x} D_1^{2\alpha_1} D_3^{2\alpha_3} u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} w(U)\frac{\partial u}{\partial z} D_1^{2\alpha_1} D_3^{2\alpha_3} u \,\mathrm{d}\mathcal{M},$$

where $\alpha_i \in \mathbb{N}$ with $\alpha_1 + \alpha_3 = m$. By D_i we denoted the differential operator $\partial/\partial x_i$. Integrating by parts and using periodicity, the integrals take the form

$$\int_{\mathcal{M}} D^\alpha\Big(u\frac{\partial u}{\partial x}\Big) D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} D^\alpha\Big(w(U)\frac{\partial u}{\partial z}\Big) D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{4.7}$$

where $D^\alpha = D_1^{\alpha_1} D_3^{\alpha_3}$. Using Leibniz' formula, we see that the integrals are sums of integrals of the form

$$\int_{\mathcal{M}} u\, D^\alpha \frac{\partial u}{\partial x}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} w(U)\, D^\alpha \frac{\partial u}{\partial z}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{4.8}$$

and of integrals of the form

$$\int_{\mathcal{M}} \delta^k u\, \delta^{m-k} \frac{\partial u}{\partial x}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} \delta^k w(U)\, \delta^{m-k} \frac{\partial u}{\partial z}\, D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{4.9}$$

with $k = 1, ..., m$, where δ^k is some differential operator D^α with $[\alpha] = \alpha_1 + \alpha_3 = k$. For each α, after integration by parts we see that the sum of the two integrals in (4.8) is zero because of the mass conservation equation (2.1d). It remains to estimate the integrals of type (4.9). We use here the Sobolev and interpolation inequalities. For the first term in (4.9) we write:

$$\begin{aligned} \Big|\int_{\mathcal{M}} \delta^k u\, \delta^{m-k} \frac{\partial u}{\partial x}\, D^\alpha u \,\mathrm{d}\mathcal{M}\Big| &\le |\delta^k u|_{L^4} \Big|\delta^{m-k}\frac{\partial u}{\partial x}\Big|_{L^4} |D^\alpha u|_{L^2} \\ &\le c_1' |\delta^k u|^{1/2}_{L^2} |\delta^k u|^{1/2}_{H^1} \Big|\delta^{m-k}\frac{\partial u}{\partial x}\Big|^{1/2}_{L^2} \Big|\delta^{m-k}\frac{\partial u}{\partial x}\Big|^{1/2}_{H^1} |D^\alpha u|_{L^2} \\ &\le c_1' |U|^{1/2}_k |U|^{1/2}_{k+1} |U|^{1/2}_{m-k+1} |U|^{1/2}_{m-k+2} |U|_m, \end{aligned} \tag{4.10}$$

where $k = 1, ..., m$.

The second term from (4.9) is estimated as follows:

$$\begin{aligned}\Big|\int_{\mathcal{M}} \delta^k w(U)\, \delta^{m-k}\frac{\partial u}{\partial z}\, D^\alpha u \,\mathrm{d}\mathcal{M}\Big| &\leq |\delta^k w(U)|_{L^2}\Big|\delta^{m-k}\frac{\partial u}{\partial z}\Big|_{L^4}|D^\alpha u|_{L^4} \\ &\leq c_2'|\delta^k w(U)|_{L^2}\Big|\delta^{m-k}\frac{\partial u}{\partial z}\Big|_{L^2}^{1/2}\Big|\delta^{m-k}\frac{\partial u}{\partial z}\Big|_{H^1}^{1/2}|D^\alpha u|_{L^2}^{1/2}|D^\alpha u|_{H^1}^{1/2} \\ &\leq c_3'|U|_{k+1}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_m^{1/2}|U|_{m+1}^{1/2},\end{aligned} \tag{4.11}$$

where $k = 1, ..., m$.

From (4.10) and (4.11) we obtain that:

$$\begin{aligned}|b(U, U, \Delta^m U)| \leq c_3 &\sum_{k=1}^{m} |U|_k^{1/2}|U|_{k+1}^{1/2}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_m \\ &+ c_3\sum_{k=1}^{m} |U|_{k+1}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_m^{1/2}|U|_{m+1}^{1/2}.\end{aligned} \tag{4.12}$$

We now need to bound the terms on the right hand side of (4.12). The terms corresponding to $k = 2, ..., m-1$ in the first sum do not contain $|U|_{m+1}$ and we leave them as they are. For $k = 1$ and $k = m$, we apply Young's inequality and we obtain:

$$c_3|U|_1^{1/2}|U|_2^{1/2}|U|_m^{3/2}|U|_{m+1}^{1/2} \leq \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_4'|U|_1^{2/3}|U|_2^{2/3}|U|_m^2. \tag{4.13}$$

For the terms in the second sum in (4.12) we distinguish between $k = 1$, $k = m$ and $k = 2, ..., m-1$. The term corresponding to $k = 1$ is bounded by:

$$c_3|U|_2|U|_m|U|_{m+1} \leq \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_5'|U|_2^2|U|_m^2. \tag{4.14}$$

For $k = m$ we find:

$$c_3|U|_1^{1/2}|U|_2^{1/2}|U|_m^{1/2}|U|_{m+1}^{3/2} \leq \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_6'|U|_1^2|U|_2^2|U|_m^2. \tag{4.15}$$

For the terms corresponding to $k = 2, ..., m-1$ we apply Young's inequality in the following way:

$$\begin{aligned}&c_3|U|_{k+1}|U|_{m-k+1}^{1/2}|U|_{m-k+2}^{1/2}|U|_m^{1/2}|U|_{m+1}^{1/2} \\ &\qquad\leq \frac{c_1}{2(m+3)}|U|_{m+1}^2 + c_7'|U|_{k+1}^{4/3}|U|_{m-k+1}^{2/3}|U|_{m-k+2}^{2/3}|U|_m^{2/3}.\end{aligned} \tag{4.16}$$

Gathering all the estimates above we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_m^2 + c_1|U|_{m+1}^2 \leq \xi + \eta|U|_m^2,$$

where the expressions of ξ and η are easily derived from (4.4), (4.13), (4.14), (4.15) and (4.16). Using the Gronwall lemma and the induction hypotheses (4.1)–(4.2) we obtain a bound for U in $L^\infty(0, t_\star; H^m)$ and $L^2(0, t_\star; H^{m+1})$, for all fixed $t_\star > 0$, this bound depending also on $|U_0|_m$. We also see that, because of the induction hypotheses (4.1)–(4.2), we can apply the uniform Gronwall lemma and we obtain U bounded in $L^\infty(\mathbb{R}_+;\, H^m)$ with a bound independent of $|U_0|_m$ when $t \geq t_m(U_0)$; we also obtain an analogue of (4.2). The details regarding the way we apply the uniform Gronwall lemma and derive these bounds are similar to the developments in Section 1.3.

In summary we have proven the following result:

Theorem 1.4.1. *Given $m \in \mathbb{N}$, $m \geq 1$, $U_0 \in V \cap (\dot{H}^m_{\text{per}}(\mathcal{M}))^3$ and $S \in L^\infty(\mathbb{R}_+; H \cap (\dot{H}^{m-1}_{\text{per}}(\mathcal{M}))^3)$, equation (2.13) has a unique solution U such that*

$$U \in L^\infty(\mathbb{R}_+;\, (\dot{H}^m_{\text{per}}(\mathcal{M}))^3) \cap L^2(0, t_\star;\, (\dot{H}^{m+1}_{\text{per}}(\mathcal{M}))^3), \quad \forall\, t_\star > 0. \tag{4.17}$$

Remark 1.4.1. Since $\cap_{m\geq 0}\dot{H}^m_{\text{per}}(\mathcal{M}) = \dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M})$, given $U_0 \in (\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3$ and $S \in L^\infty(\mathbb{R}_+;\, (\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3)$, equation (2.13) has a unique solution U belonging to $L^\infty(\mathbb{R}_+;\, (\dot{H}^m_{\text{per}}(\mathcal{M}))^3)$ for all $m \in \mathbb{N}$; that is, U is in $L^\infty(\mathbb{R}_+; (\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3)$. Regularity (differentiability) in time can be also derived if S is also $\mathcal{C}^\infty$ in time. However the arguments above do not provide the existence of an absorbing set in $(\dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M}))^3$.

1.5 Appendix: Physical Background

The large-scale ocean equations considered in this article, also called the Primitive Equations (PEs), are derived from the general conservation laws of physics using the Boussinesq and hydrostatic approximations. They comprise: the conservation of horizontal momentum equation, the hydrostatic equation, the continuity equation, the equation for the temperature (conservation of energy), the equation of diffusion for the salinity and the equation of state (see, e.g., [7], [8] or [12]):

$$\frac{\partial \boldsymbol{v}^*}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)\boldsymbol{v}^* + w^* \frac{\partial \boldsymbol{v}^*}{\partial z^*} + fk \times \boldsymbol{v}^* + \frac{1}{\rho_{\text{ref}}}\nabla p^* = \mu^*_{\boldsymbol{v}}\, \Delta^*_{\text{h}} \boldsymbol{v}^* + \nu^*_{\boldsymbol{v}} \frac{\partial^2 \boldsymbol{v}^*}{\partial z^{*2}}, \tag{5.1a}$$

$$\frac{\partial p^*_{\text{full}}}{\partial z^*} = -\rho^*_{\text{full}}\, g, \tag{5.1b}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0, \tag{5.1c}$$

$$\frac{\partial T}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)T + w^* \frac{\partial T}{\partial z^*} = \mu_T\, \Delta^*_{\text{h}} T + \nu_T \frac{\partial^2 T}{\partial z^{*2}}, \tag{5.1d}$$

$$\frac{\partial S}{\partial t^*} + (\boldsymbol{v}^* \cdot \nabla^*)S + w^* \frac{\partial S}{\partial z^*} = \mu_S\, \Delta^*_{\text{h}} S + \nu_S \frac{\partial^2 S}{\partial z^{*2}}, \tag{5.1e}$$

$$\rho^*_{\text{full}} = \rho_{\text{ref}}\left[1 - \beta_T(T - T_{\text{ref}}) + \beta_S(S - S_{\text{ref}})\right]. \tag{5.1f}$$

Here $\boldsymbol{v}^* = (u^*, v^*)$ is the horizontal velocity, w^* the vertical velocity, p^*_{full} the (full) pressure, ρ^*_{full} the (full) density, T the temperature and S the salinity. Asterisks denote dimensional quantities, a notation which will be useful below when we non-dimensionalise. The constants ρ_{ref}, T_{ref}, S_{ref} denote reference (average) values respectively for the density, temperature and salinity; g is the gravitational acceleration and f the Coriolis parameter. The horizontal gradient and Laplacian operators are denoted by ∇^* and Δ_{h}^*, respectively.

We recall that in the hydrostatic approximation of the Boussinesq equation, the conservation of the vertical momentum equation is replaced by the hydrostatic equation corresponding to its leading terms (5.1b). We chose a linear equation of state (5.1f), but this is not essential; appropriate nonlinear equations could be handled similarly; also ρ_{ref}, T_{ref} and S_{ref} could be nonconstant with suitable changes in the following. Equations (5.1) correspond to the f-plane approximation of equations on the sphere, hence $f = \text{constant} = 2\Omega$, Ω being the angular velocity of the Earth in its rotation around the poles' axes.

A simplification of this system can be obtained if we assume that $\beta_T \nu_T = \beta_S \nu_S$ and $\beta_T \mu_T = \beta_S \mu_S$ so that (5.1d)–(5.1f) can be combined into a single equation for ρ, namely:

$$\frac{\partial \rho^*_{\mathrm{full}}}{\partial t^*} + u^* \frac{\partial \rho^*_{\mathrm{full}}}{\partial x^*} + v^* \frac{\partial \rho^*_{\mathrm{full}}}{\partial y^*} + w^* \frac{\partial \rho^*_{\mathrm{full}}}{\partial z^*} = \mu^*_\rho \, \Delta^*_{\mathrm{h}} \rho^*_{\mathrm{full}} + \nu^*_\rho \, \frac{\partial^2 \rho^*_{\mathrm{full}}}{\partial z^{*2}} \, . \tag{5.2}$$

We are interested in the case where the density ρ^*_{full} is of the form

$$\rho^*_{\mathrm{full}}(x,\, y,\, z,\, t) = \rho_{\mathrm{ref}} + \bar{\rho}(z) + \rho^*(x,\, y,\, z,\, t), \tag{5.3}$$

where $\bar{\rho} = \bar{\rho}(z)$ is a stratification profile of the density. Similarly, we write the pressure as,

$$p^*_{\mathrm{full}}(x,\, y,\, z,\, t) = p_{\mathrm{ref}} + \bar{p}(z) + p^*(x,\, y,\, z,\, t), \tag{5.4}$$

where $\partial p_{\mathrm{ref}}/\partial z = -g\rho_{\mathrm{ref}}$ and $\partial \bar{p}/\partial z^* = -g\bar{\rho}$. With this, (5.1b) reduces to

$$\frac{\partial p^*}{\partial z^*} = -g\rho^*. \tag{5.5}$$

We shall be interested in the physical regimes where $|\bar{\rho}| \ll |\rho_{\mathrm{ref}}|$ and $|\rho^*| \ll |\bar{\rho}|$, the first inequality meaning that the density profile $\bar{\rho}$ does not depart too much from a mean reference value ρ_{ref} and the second one meaning that the horizontal and temporal variations of the density surfaces are very small compared to the vertical stratification. Furthermore, we consider a part of the ocean where $\bar{\rho}(z)$ is a linear function of z and introduce the (constant) Brunt–Väisälä frequency N^*, defined by

$$(N^*)^2 = -\frac{g}{\rho_{\mathrm{ref}}} \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z} \, . \tag{5.6}$$

With this, the evolution equation for density (5.2) can be written as,

$$\frac{\partial \rho^*}{\partial t^*} + u^* \frac{\partial \rho^*}{\partial x^*} + v^* \frac{\partial \rho^*}{\partial y^*} + w^* \frac{\partial \rho^*}{\partial z^*} - \frac{\rho_{\mathrm{ref}}}{g} (N^*)^2 \, w^* = \mu_\rho \, \Delta^*_{\mathrm{h}} \rho^* + \nu_\rho \, \frac{\partial^2 \rho^*}{\partial z^{*2}} \, . \tag{5.7}$$

At this point, we have reduced the PEs to (5.1a), (5.5), (5.1c), and (5.7), with the dependent variables being $(\boldsymbol{v}^*, w^*, \rho^*, p^*)$. We now non-dimensionalise this set of equations by means of the following typical scales: For length and velocity, we write

$$x^* = Lx, \qquad y^* = Ly, \qquad z^* = Hz,$$
$$u^* = Uu, \qquad v^* = Uv, \qquad w^* = Ww,$$

where (x, y, z) and (u, v, w) are dimensionless variables. We also define the aspect ratio

$$\delta := H/L. \tag{5.8}$$

Since we are interested in the advective timescale, we write $t^* = Tt$ with t dimensionless, where

$$T = L/U, \tag{5.9}$$

and define the Rossby number as

$$\varepsilon = U/fL. \tag{5.10}$$

The (perturbation) pressure p^* is non-dimensionalised by

$$p^* = (U^2 \rho_{\text{ref}}/\varepsilon)\, p, \tag{5.11}$$

and the (perturbation) density ρ^* by

$$\rho^* = (U^2 \rho_{\text{ref}}/\varepsilon g H)\, \rho, \tag{5.12}$$

where again $p(x, y, z, t)$ and $\rho(x, y, z, t)$ are dimensionless. We define the "Burgers" number as

$$N = N^* H/fL. \tag{5.13}$$

Finally, we define the non-dimensional eddy viscosity coefficients (inverse Reynolds numbers) by

$$\begin{aligned} \mu_{\boldsymbol{v}} &= \mu^*_{\boldsymbol{v}}/UL, & \nu_{\boldsymbol{v}} &= \nu^*_{\boldsymbol{v}} L/UH^2, \\ \mu_\rho &= \mu^*_\rho/UL, & \nu_\rho &= \nu^*_\rho L/UH^2\,. \end{aligned}$$

We shall choose $\mu_{\boldsymbol{v}} = \nu_{\boldsymbol{v}}$ and $\mu_\rho = \nu_\rho$ for the sake of simplicity.

With these, we can write the PEs in the completely non-dimensional form,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}}\Delta_3 u + S_u\,, \tag{5.14a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u + \frac{1}{\varepsilon}\frac{\partial p}{\partial y} = \nu_{\boldsymbol{v}}\Delta_3 v + S_v\,, \tag{5.14b}$$

$$\frac{\partial p}{\partial z} = -\rho, \tag{5.14c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{5.14d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} - \frac{N^2}{\varepsilon}w = \nu_\rho\Delta_3\rho + S_\rho\,. \tag{5.14e}$$

Here the forcing and source terms S_u, S_v, S_ρ have been added to the right-hand sides for mathematical generality.

In this paper, we shall consider the case of two spatial dimensions by assuming that all functions are independent of y, but we allow v to be non-zero. We intend to study the three-dimensional case in a similar paper. The system (5.14) becomes now (2.1). We notice easily that if $u,\, v,\, \rho,\, w,\, p$ are solutions of (2.1) for $S = (S_u,\, S_v,\, S_\rho)$, then $\tilde{u},\, \tilde{v},\, \tilde{\rho},\, \tilde{w},\, \tilde{p}$ are solutions of (2.1) for $\tilde{S}_u,\, \tilde{S}_v,\, \tilde{S}_\rho$ where:

$$\begin{aligned}
\tilde{u}(x,\, z,\, t) &= u(x,\, -z,\, t), & \tilde{v}(x,\, z,\, t) &= v(x,\, -z,\, t),\\
\tilde{w}(x,\, z,\, t) &= -\, w(x,\, -z,\, t), & \tilde{p}(x,\, z,\, t) &= p(x,\, -z,\, t),\\
\tilde{\rho}(x,\, z,\, t) &= -\, \rho(x,\, -z,\, t), & &\\
\tilde{S}_u(x,\, z,\, t) &= S_u(x,\, -z,\, t), & \tilde{S}_v(x,\, z,\, t) &= S_v(x,\, -z,\, t),\\
\tilde{S}_\rho(x,\, z,\, t) &= -\, S_\rho(x,\, -z,\, t). & &
\end{aligned}$$

Therefore if we assume that $S_u,\, S_v$ are even in z and S_ρ is odd in z, then we can anticipate the existence of a solution of (2.1) such that:

$$u,\, v,\, w,\, p,\, \rho \text{ are periodic in } x \text{ and } z \text{ with periods } L_1 \text{ and } L_3,$$

and

$$u,\, v \text{ and } p \text{ are even in } z;\; w \text{ and } \rho \text{ are odd in } z,$$

provided the initial conditions satisfy the same symmetry properties. Our aim is to solve the problem (2.1) with the periodicity and symmetry properties above and with initial data

$$u = u_0,\; v = v_0,\; \rho = \rho_0 \text{ at } t = 0.$$

Hence the natural function spaces for this problem are as follows:

$$V = \{(u,\, v,\, \rho) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3,\; u,\, v \text{ even in } z \;,\, \rho \text{ odd in } z,\; u_{(k_1,\,0)} = 0\},$$
$$H = \text{closure of } V \text{ in } L^2(\mathcal{M})^3.$$

The motivations for considering periodic boundary conditions is that there are needed in studies on homogeneous turbulence of the atmosphere and also for the study of the renormalized equations considered in [9].

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Chapitre 2

Gevrey Class Regularity for the Primitive Equations in Space Dimension 2

Régularité de type Gevrey pour les Equations Primitives en deux dimensions

Ce chapitre est constitué de l'article **Gevrey Class Regularity for the Primitive Equations in Space Dimension 2**, article paru en 2004, dans le journal *Asympotic Analysis*, volume 39, numero 1, pages 1-13. Ce travail a pour but de montrer que les solutions des Equations Primitives en deux dimensions, avec conditions aux limites periodiques, appartiennent à un espace du type Gevrey, quand la force est une fonction analytique en temps et les données initiales sont dans l'espace de Sobolev H^1.

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Gevrey Class Regularity for the Primitive Equations in Space Dimension 2

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Abstract: The aim of this article is to prove results of space and time regularity of solutions for the Primitive Equations of the ocean in space dimension two with periodic boundary conditions. It is shown that these solutions belong to a certain Gevrey class of functions which is a subset of real analytic functions.

2.1 Introduction

In this article we consider the Primitive Equations for the ocean or for the atmosphere in space dimension two with periodic boundary conditions (for details regarding the form of the primitive equations see e.g. [8], [6] or [7]). The form of the equations used is in this article is close to that considered in [9], so for more details regarding the existence of the solutions for the primitive equations the reader is referred to [9]. In this article it is proved that, considering a forcing term which is an analytical function in time with values in some Gevrey space, the solutions of the Primitive Equations starting with initial data in the Sobolev space H^1 become, for some positive time, elements of a certain Gevrey class and the solutions are thus real analytic functions. One can show that the unique solution is restriction to the real time axis $t \geq 0$, of a complex function analytic in the temporal variable t in some complex neighborhood of the real time axis.

This article was inspired by the article by Foias and Temam [5] who proved similar results for the Navier-Stokes equations in space dimension two and three with periodical boundary conditions (see also [3]). We also mention here the works of Ferrari and Titi [2] who proved that the solutions of a certain class of nonlinear parabolic equations belong to a certain Gevrey class; also that of Cao, Rammaha and Titi [1] who established the Gevrey regularity for a certain class of analytic nonlinear parabolic equations on the two-dimensional sphere.

2.1.1 Preliminaries

We consider the PEs in their usual (dimensional) form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = \nu\,\Delta u + F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + fu = \nu\,\Delta v + F_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \mu\,\Delta T + F_T. \tag{1.1e}$$

Here $(u,\, v,\, w)$ are the three components of the velocity vector and, as usual, we denote respectively by p, ρ and T, the pressure, density and temperature deviations from a prescribed main value corresponding to the natural stratification. The relationship between ρ and T is $\rho = -\alpha\rho_0 T$. In general the temperature and the density are related by the equation of state $\rho = \rho_0(1 - \alpha(T - T_0))$ where ρ_0 and T_0 are the reference values for the density and the temperature, but in our case we already subtracted the average values from the actual values. The constant g is the gravitational acceleration and f the Coriolis parameter; ν and μ are the eddy diffusivity coefficients. This form of the PEs corresponds to the ocean, although the salinity has been omitted which does not raise any new mathematical difficulty; some minor changes, not done here, are necessary for the atmosphere.

We consider the following domain:

$$\mathcal{M} = (0,\, L_1) \times (-L_3/2,\, L_3/2), \tag{1.2}$$

and we assume space periodicity with period $\mathcal{M}$, that is, all functions are taken to satisfy $f(x + L_1, z, t) = f(x, z, t) = f(x, z + L_3, t)$ when extended to $\mathbb{R}^2$. Moreover, we assume, as in [9], that the following symmetries hold:

$$\begin{aligned} u(x,z,t) &= u(x,-z,t), & F_u(x,z,t) &= F_u(x,-z,t),\\ v(x,z,t) &= v(x,-z,t), & F_v(x,z,t) &= F_v(x,-z,t),\\ T(x,z,t) &= -T(x,-z,t), & F_T(x,z,t) &= -F_T(x,-z,t),\\ w(x,z,t) &= -w(x,-z,t), & p(x,z,t) &= p(x,-z,t), \end{aligned} \tag{1.3}$$

that is to say that we search for u, v, p even and w, T odd; the motivations for considering such solutions are described in [9]. Note that without the symmetry properties (1.7), space periodicity is not consistent with the equations (2.1).

The natural function spaces for this problem are as follows:

$$\begin{aligned} V = \{&(u,\, v,\, T) \in (\dot H^1_{\text{per}}(\mathcal{M}))^3,\\ &u,\, v \text{ even in } z,\ T \text{ odd in } z,\ \textstyle\int_{-L_3/2}^{L_3/2} u(x,z')\,\mathrm{d}z' = 0\}, \end{aligned} \tag{1.4}$$

$$H = \text{closure of } V \text{ in } (\dot L^2(\mathcal{M}))^3. \tag{1.5}$$

Here the dots above $\dot{H}^1_{\mathrm{per}}$ or $\dot{L}^2$ denote the functions with average in $\mathcal{M}$ equal to zero. These spaces are endowed with Hilbert scalar products; in H the scalar product is

$$(U,\,\tilde{U})_H = (u,\,\tilde{u})_{L^2} + (v,\,\tilde{v})_{L^2} + \kappa(T,\,\tilde{T})_{L^2}, \tag{1.6}$$

and in $\dot{H}^1_{\mathrm{per}}$ and V the scalar product is (using the same notation when there is no ambiguity):

$$((U,\,\tilde{U}))_V = ((u,\,\tilde{u})) + ((v,\,\tilde{v})) + \kappa((T,\,\tilde{T})); \tag{1.7}$$

here we have written $\mathrm{d}\mathcal{M}$ for $\mathrm{d}x\,\mathrm{d}z$, and

$$((\phi,\,\tilde{\phi})) = \int_{\mathcal{M}} \Big(\frac{\partial\phi}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial\tilde{\phi}}{\partial z}\Big)\,\mathrm{d}\mathcal{M}. \tag{1.8}$$

The relations above define the norms $|\cdot|_H$ and $\|\cdot\|_V$. The positive constant κ is chosen below. We have

$$|U|_H \le c_0\|U\|_V,\ \forall\, U \in V, \tag{1.9}$$

where $c_0 > 0$ is a positive constant related to κ and the Poincaré constant in $\dot{H}^1_{\mathrm{per}}(\mathcal{M})$.

The prognostic variables of the system are u, v and T and the diagnostic variables are w and p. We can express the diagnostic variables w and p in terms of the prognostic variables u, v, and T. For each $U = (u, v, T) \in V$ we can determine uniquely

$$w = w(U) = -\int_0^z u_x(x, z', t)\,\mathrm{d}z'. \tag{1.10}$$

Note that $w = 0$ at $z = 0$ and $L_3/2$ by the requirements on w (periodicity and anti-symmetry); see more details in [9]. By (2.2), the fact that $w = 0$ at $z = L_3/2$ gives the constraint on u:

$$\int_{-L_3/2}^{L_3/2} u_x\,\mathrm{d}z = 0. \tag{1.11}$$

As for the pressure, it can be determined uniquely in terms of T up to its value at $z = 0$, p_{s}, namely,

$$p(x, z, t) = p_{\mathrm{s}}(x, t) + \alpha\rho_0 \int_0^z T(x, z', t)\,\mathrm{d}z'.$$

Considering a test function $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\rho})$ in V, we multiply equation (2.1a) by $\tilde{u}$, (2.1b) by $\tilde{v}$ and (2.1e) by $\kappa\tilde{\rho}$. We obtain the variational formulation of the problem as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_H + a(U,\tilde{U}) + b(U,U,\tilde{U}) + e(U,\tilde{U}) = (F,\tilde{U})_H, \qquad \forall\,\tilde{U} \in V, \tag{1.12}$$

and we supplement this equation with the initial condition $U = U_0$.

Here we set

$$
\begin{aligned}
a(U,\tilde U) &= \nu((u,\tilde u)) + \nu((v,\tilde v)) + \kappa\mu((T,\tilde\rho)),\\
e(U,\tilde U) &= f\int_{\mathcal M}(u\tilde v - v\tilde u)\,\mathrm{d}\mathcal M - \alpha g\int_{\mathcal M} T\tilde w\,\mathrm{d}\mathcal M,\\
b(U,U^\sharp,\tilde U) &= \int_{\mathcal M}\Big(u\frac{\partial u^\sharp}{\partial x} + w(U)\frac{\partial u^\sharp}{\partial z}\Big)\tilde u\,\mathrm{d}\mathcal M + \int_{\mathcal M}\Big(u\frac{\partial v^\sharp}{\partial x} + w(U)\frac{\partial v^\sharp}{\partial z}\Big)\tilde v\,\mathrm{d}\mathcal M\\
&\qquad + \kappa\int_{\mathcal M}\Big(u\frac{\partial \rho^\sharp}{\partial x} + w(U)\frac{\partial \rho^\sharp}{\partial z}\Big)\tilde \rho\,\mathrm{d}\mathcal M.
\end{aligned}
$$

We notice that:

$$
a(U,U) + e(U,U) = \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - \alpha g\int_{\mathcal M} Tw(U)\,\mathrm{d}\mathcal M,
$$

and since

$$
\Big|\alpha g\int_{\mathcal M} Tw(U)\,\mathrm{d}\mathcal M\Big| \le \alpha g|T|_{L^2}|w(U)|_{L^2} \le c\alpha g\|T\|\|u\|,
$$

we find that

$$
a(U,U) + e(U,U) \ge \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - c\alpha g\|T\|\|u\|. \tag{1.13}
$$

From equation (1.24) we see that for κ large enough, more specifically for $\kappa \ge (g^2\alpha^2c^2)/(\nu\mu)$ the bilinear, continuous form $a+e$ is coercive on V, and

$$
a(U,U) + e(U,U) \ge \frac{\nu}{2}\|u\|^2 + \nu\|v\|^2 + \kappa\frac{\mu}{2}\|T\|^2 \ge c_1\|U\|_V^2. \tag{1.14}
$$

We also mention that the form b is trilinear continuous from $V\times V\times V_2$ where V_2 is defined as the closure of V in $(\dot H^2_{\mathrm{per}}(\mathcal M))^3$; for more details regarding the way we obtain these results, see e.g. [9].

Equation (1.12) is equivalent to an evolution equation of the form:

$$
\begin{aligned}
&\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U,U) + E(U) = F,\\
&U(0) = U_0,
\end{aligned} \tag{1.15}
$$

in the space V_2', which is the dual of V_2. For more details regarding the derivation of the variational and evolutional form for the Primitive Equations and also for the derivation of the properties of the forms a and b the reader is referred to [9]. In that article existence and uniqueness of solutions and regularity results in all Sobolev spaces H^m are derived for the non-dimensionalised Primitive Equations for the ocean in periodic space dimension two; though the equations are not absolutely identical to those considered here, one can, with minimal changes, derive similar results for the equations considered here.

All the functions being periodic, they admit Fourier series expansions. Hence, for instance, for U we write

$$
U = \sum_{(k_1,k_3)\in\mathbb Z^2} U_{(k_1,k_3)}e^{i(k_1'x + k_3'z)},
$$

where $k'_j = 2\pi k_j/L_j$. We also introduce the following notation:

$$[U_k]_\kappa^2 = |u_k|^2 + |v_k|^2 + \kappa |T_k|^2.$$

Considering the Laplacian $-\Delta$, we define the Gevrey class $D(e^{\tau(-\Delta)^s})$ as the set of functions U in H satisfying

$$|\mathcal{M}| \sum_{k\in\mathbb{Z}^2} e^{2\tau|k'|^{2s}}[U_k]^2 = |e^{\tau(-\Delta)^s}U|_H^2 < \infty. \tag{1.16}$$

The norm of the Hilbert space $D(e^{\tau(-\Delta)^s})$ is given by

$$|U|_{D(e^{\tau(-\Delta)^s})} = |e^{\tau(-\Delta)^s}U|_H, \text{ for } U \in D(e^{\tau(-\Delta)^s}), \tag{1.17}$$

and the associated scalar product is

$$(U,V)_{D(e^{\tau(-\Delta)^s})} = (e^{\tau(-\Delta)^s}U, e^{\tau(-\Delta)^s}V)_H, \text{ for } U, V \in D(e^{\tau(-\Delta)^s}). \tag{1.18}$$

Another Gevrey type space that we will use is $D((-\Delta)^{1/2}e^{\tau(-\Delta)^s})$, which is a Hilbert space when endowed with the inner product:

$$\begin{aligned}(U,V)_{D((-\Delta)^{1/2}e^{(-\Delta)^s})} &= ((-\Delta)^{1/2}e^{\tau(-\Delta)^s}U, (-\Delta)^{1/2}e^{\tau(-\Delta)^s}V)_H \\ &= ((e^{\tau(-\Delta)^s}U, e^{\tau(-\Delta)^s}V))_V,\end{aligned} \tag{1.19}$$

for U, V in $D((-\Delta)^{1/2}e^{\tau(-\Delta)^s})$; the norm of the space is given by

$$\begin{aligned}|U|^2_{D((-\Delta)^{1/2}e^{\tau(-\Delta)^s})} = |(-\Delta)^{1/2}e^{\tau(-\Delta)^s}U|_H^2 &= \|e^{\tau(-\Delta)^s}U\|_V^2 \\ &= |\mathcal{M}| \sum_{k\in\mathbb{Z}^2} |k'|^2 e^{2\tau|k'|^{2s}}[U_k]_\kappa^2.\end{aligned} \tag{1.20}$$

2.2 A Priori Estimates for the Real Case

As we already mentioned in the introduction, our aim is to prove that the solutions of the PEs are real functions analytic in time with values in Gevrey spaces and the restriction of some complex analytic functions in time in the neighborhood of a real positive interval. We start in this section by deriving some a priori estimates in the real case and then we consider the complex case.

We begin with the following technical result:

Lemma 2.2.1. *Let U, $U^\sharp$ and $\tilde{U}$ be given in $D(\Delta e^{\tau(-\Delta)^s})$ for $\tau \geq 0$. Then the following inequality holds:*

$$\begin{aligned}|(e^{\tau(-\Delta)^{1/2}}B(U,U^\sharp), e^{\tau(-\Delta)^{1/2}}\Delta\tilde{U})_H| &\leq c_2 |e^{\tau(-\Delta)^{1/2}}(-\Delta)^{1/2}U|^{1/2} \\ &|e^{\tau(-\Delta)^{1/2}}\Delta U|^{1/2} |e^{\tau(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\sharp|^{1/2} |e^{\tau(-\Delta)^{1/2}}\Delta U^\sharp|^{1/2} |e^{\tau(-\Delta)^{1/2}}\Delta\tilde{U}|.\end{aligned} \tag{2.1}$$

Proof: We start by writing the trilinear form b in Fourier modes. For that purpose we define, for each $j \in \mathbb{Z}^2$, δ_j as j_1'/j_3' when $j_3' \neq 0$ and as 0 when $j_3' = 0$. We obtain:

$$
\begin{aligned}
b(U, U^\sharp, \tilde{U}) = &\sum_{j+l+k=0} i(j_1' - j_3'\delta_l) u_l u_j^\sharp \tilde{u}_k + \sum_{j+l+k=0} i(j_1' - j_3'\delta_l) u_l v_j^\sharp \tilde{v}_k \\
&+ \kappa \sum_{j+l+k=0} i(j_1' - j_3'\delta_l) u_l T_j^\sharp \tilde{\rho}_k.
\end{aligned} \tag{2.2}
$$

We then compute:

$$
\begin{aligned}
&(e^{\tau(-\Delta)^{1/2}} B(U, U^\sharp), e^{\tau(-\Delta)^{1/2}} \Delta \tilde{U})_H = \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)|k'|^2 e^{2\tau|k'|} u_l u_j^\sharp \tilde{u}_k \\
&+ \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)|k'|^2 e^{2\tau|k'|} u_l v_j^\sharp \tilde{v}_k + \kappa \sum_{j+l+k=0} i(j_1' - j_3'\delta_l)|k'|^2 e^{2\tau|k'|} u_l T_j^\sharp \tilde{\rho}_k.
\end{aligned} \tag{2.3}
$$

We now associate to each u the function $\check{u}$ defined by:

$$
\check{u} = \sum_{j \in \mathbb{Z}^2} \check{u}_j e^{ij'\cdot x}, \text{ where } \check{u}_j = e^{\tau|j'|}|u_j|, \tag{2.4}
$$

and we also use similar notations for the other functions.

Using the notation above and the fact that $|k| - |l| - |j| \leq 0$ since $j + l + k = 0$, we continue to bound the right-hand side of relation (2.3) and we obtain:

$$
\begin{aligned}
&|(e^{\tau(-\Delta)^{1/2}} B(U, U^\sharp), e^{\tau(-\Delta)^{1/2}} \Delta \tilde{U})_H| \leq c \sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{u}_j^\sharp||\check{\tilde{u}}_k| \\
&\quad + c \sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{v}_j^\sharp||\check{\tilde{v}}_k| + \kappa c \sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{T}_j^\sharp||\check{\tilde{\rho}}_k|,
\end{aligned} \tag{2.5}
$$

where we also used the estimate $|j_1' - j_3'\delta_l| \leq c|j'||l'|$. Here and in the sequel c denotes a constant which may be different at different places.

We estimate the first term from the right-hand side of (2.5), the rest of the estimates being identical. For that purpose, we define the following functions:

$$
\xi(x) = \sum_{j \in \mathbb{Z}^2} |j'|\check{u}_j e^{ij'\cdot x}, \quad \psi(x) = \sum_{j \in \mathbb{Z}^2} |j'|\check{u}_j^\sharp e^{ij'\cdot x}, \quad \theta(x) = \sum_{j \in \mathbb{Z}^2} |j'|^2 \check{\tilde{u}}_j e^{ij'\cdot x},
$$

and we write:

$$
\begin{aligned}
&\sum_{j+l+k=0} |j'||l'||k'|^2 |\check{u}_l||\check{u}_j^\sharp||\check{\tilde{u}}_k| = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \xi(x)\psi(x)\theta(x)\, \mathrm{d}\mathcal{M} \leq c|\xi|_{L^4}|\psi|_{L^4}|\theta|_{L^2} \\
&\quad \leq c|\xi|_{L^2}^{1/2}\|\xi\|^{1/2}|\psi|_{L^2}^{1/2}\|\psi\|^{1/2}|\theta|_{L^2} \\
&\quad \leq c\|e^{t(-\Delta)^{1/2}}U\|^{1/2}|\Delta e^{t(-\Delta)^{1/2}}U|^{1/2}\|e^{t(-\Delta)^{1/2}}U^\sharp\|^{1/2}|\Delta e^{t(-\Delta)^{1/2}}U^\sharp|^{1/2}|\Delta e^{t(-\Delta)^{1/2}}\tilde{U}|.
\end{aligned}
$$

Using the same kind of arguments for the other terms we find the relation (3.8).

Decomposition of the solution

We want to derive the Gevrey regularity of the problem:

$$\text{(2.6)}\qquad \begin{aligned} &U' + AU + B(U,U) + EU = F, \text{ in } V_2', \\ &U(0) = U_0. \end{aligned}$$

In all that follows we assume that the forcing F is an analytic function in time with values in the Gevrey space $D(e^{\sigma_1(-\Delta)^{1/2}})$ for some $\sigma_1 > 0$. To obtain the desired a priori estimates, we can suppose that the natural way would be to apply the operator $e^{t(-\Delta)^{1/2}}$ to equation (2.10) and to take the scalar product with $-\Delta e^{t(-\Delta)^{1/2}}$ in H. But taking into account the inequality (2.1), we see that, unlike in [5] for the Navier-Stokes equations, we would obtain a weak estimate for the nonlinear term which would force us to work with small initial data. In order to avoid imposing such a restriction, we split the solution U into $U = U^\star + \tilde{U}$, where $U^\star$ is the solution of the linear problem:

$$\text{(2.7)}\qquad \begin{aligned} &\frac{\mathrm{d}U^\star}{\mathrm{d}t} + AU^\star + EU^\star = F, \\ &U^\star(0) = U_0, \end{aligned}$$

and $\tilde{U}$ is the solution of the nonlinear problem:

$$\text{(2.8)}\qquad \begin{aligned} &\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t} + A\tilde{U} + B(\tilde{U},\tilde{U}) + B(\tilde{U},U^\star) + B(U^\star,\tilde{U}) + E\tilde{U} = -B(U^\star,U^\star), \\ &\tilde{U}(0) = 0. \end{aligned}$$

We will derive estimates and existence results for the linear problem (2.9) and then for the nonlinear problem (2.8) which is equivalent to (1.15), taking (2.9) into account. We start treating the linear problem:

The linear problem

We suppose that U_0 is in $D((-\Delta)^{1/2})$ and F is a function analytic in time with values in $D(e^{\sigma_1(-\Delta)^{1/2}})$, for some $\sigma_1 > 0$. Setting $\varphi(t) = \min(t, \sigma_1)$, we apply the operator $e^{\varphi(t)(-\Delta)^{1/2}}$ to equation (2.9) and then take the scalar product with $-\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star$ in H.

With the same κ as in (1.24) we have:

$$\text{(2.9)}\qquad \begin{aligned} &(e^{\varphi(t)(-\Delta)^{1/2}}AU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H \\ &\quad + (e^{\varphi(t)(-\Delta)^{1/2}}EU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H \geq c_1|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2. \end{aligned}$$

The relation above holds because:

$$
\begin{aligned}
&(e^{\varphi(t)(-\Delta)^{1/2}}AU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H + (e^{\varphi(t)(-\Delta)^{1/2}}EU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star)_H\\
&\quad = ((-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}AU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star)_H\\
&\qquad + ((-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}EU^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star)_H\\
&\quad = a(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star)\\
&\qquad + e(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star),
\end{aligned}
$$

where we used that A and E commute with $-\Delta$ and the fact that for the κ chosen before, $a+e$ is coercive. The commutativity of the operators A and E can be easily established using, for example, the Fourier series expansions.

We also have:

$$
\begin{aligned}
&(e^{\varphi(t)(-\Delta)^{1/2}}(U^\star)'(t), e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U^\star(t))_H\\
&\quad = \Big(\frac{\mathrm{d}}{\mathrm{d}t}((-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star), (-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star\Big)_H\\
&\qquad - \varphi'(t)((-\Delta)e^{\varphi(t)(-\Delta)^{1/2}}U^\star, (-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star)_H\\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - \varphi'(t)(\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star, (-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star)_H\\
&\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - |\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H\|e^{\varphi(t)(-\Delta)^{1/2}}U^\star\|_V\\
&\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - \frac{c_1}{4}|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 - \frac{1}{c_1}\|e^{\varphi(t)(-\Delta)^{1/2}}U^\star\|_V^2.
\end{aligned}
$$

The term containing the force F is estimated using the Schwarz inequality:

$$
\begin{aligned}
(e^{\varphi(t)(-\Delta)^{1/2}}F, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U^\star)_H &\leq |e^{\varphi(t)(-\Delta)^{1/2}}F|_H|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U^\star|_H\\
&\leq \frac{1}{c_1}|e^{\varphi(t)(-\Delta)^{1/2}}F|_H^2 + \frac{c_1}{4}|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U^\star|_H^2.
\end{aligned}
\tag{2.10}
$$

Taking into account all the estimates above, we obtain:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 + c_1|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2\\
&\qquad \leq \frac{2}{c_1}|e^{\varphi(t)(-\Delta)^{1/2}}F|_H^2 + \frac{2}{c_1}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2.
\end{aligned}
\tag{2.11}
$$

Applying the Gronwall lemma to (2.11), it follows that:

$$
|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star|_H^2 \leq |(-\Delta)^{1/2}U_0|_H^2 e^{\frac{2}{c_1}t} + \sup_{0\leq s\leq t}|e^{\sigma_1(-\Delta)^{1/2}}F(s)|_H^2 e^{\frac{2}{c_1}t},
\tag{2.12}
$$

which gives a bound of $(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star$ in $L^\infty(0,t_\star;H)$ for all $t_\star > 0$. Returning to (2.11) and integrating, we find a bound of $\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star$ in $L^2(0,t_\star;H)$ for all $t_\star > 0$.

The nonlinear problem

We now need to study the Gevrey regularity for the following nonlinear problem:

$$\begin{aligned} &\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t} + A\tilde{U} + B(\tilde{U},\tilde{U}) + B(\tilde{U},U^\star) + B(U^\star,\tilde{U}) + E\tilde{U} = -B(U^\star,U^\star),\\ &\tilde{U}(0) = 0, \end{aligned} \tag{2.13}$$

where $U^\star$ is the solution of the linear problem presented above.

As for the linear case, at a time t, we apply the operator $e^{\varphi(t)(-\Delta)^{1/2}}$ to each side of equation (2.13) and then we take the scalar product in H of the resulting equation with $e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)\tilde{U}$. The difference between this case and the linear case appears in the terms containing the operator B and, to estimate these terms, we use Lemma 4.3.1. Note that since the norm on H is equivalent to the usual norm on L^2, in the right hand side of (3.8) we can change the norm on L^2 with the norm on H, changing only the preceding constant. Thus, we obtain:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + c_1|\Delta e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2\\ &\quad\le f(t)|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(t)\\ &\qquad + c_1|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H|\Delta e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} f(t) &= c_1' + c_2'|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star(t)|_H^2|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star(t)|_H^2,\\ g(t) &= c_3'|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}U^\star(t)|_H^2|\Delta e^{\varphi(t)(-\Delta)^{1/2}}U^\star(t)|_H^2. \end{aligned}$$

We rewrite (2.14) as:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + (c_0 - c_1|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H)|\Delta e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2\\ &\qquad\qquad \le f(t)|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(t). \end{aligned} \tag{2.15}$$

Since $\tilde{U}(0) = 0$, we may assume that:

$$|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H \le \frac{c_1}{2c_2}, \text{ on some finite interval of time } (0,t_0). \tag{2.16}$$

On that interval the following estimate holds:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + \frac{c_1}{2}|\Delta e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2\\ &\quad\le f(t)|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(t). \end{aligned} \tag{2.17}$$

Taking into account the a priori estimates obtained for $U^\star$, we find that f and g are functions locally integrable. So, we can apply the Gronwall lemma and deduce the following estimate on $(0,t_0)$:

$$|(-\Delta)^{1/2}e^{\varphi(t)(-\Delta)^{1/2}}\tilde{U}|_H^2 \le \int_0^{t^\star} g(s)\exp(\int_s^{t^\star} f(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s. \tag{2.18}$$

Since f and g are locally integrable, we can define $t^\star = t(F, U_0, \sigma_1)$ as the first time for which:

$$\int_0^{t^\star} g(s) \exp(\int_s^{t^\star} f(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s = \frac{c_1}{2c_2}. \tag{2.19}$$

Then, on the interval $(0, t^\star)$ we find:

$$|(-\Delta)^{1/2} e^{\varphi(t)(-\Delta)^{1/2}} \tilde{U}(t)|_H \le c_0/2c_1.$$

Hence, on $(0, t^\star)$, with $t^\star$ defined by (2.19), the solution $\tilde{U}$ satisfies both (2.15) and (2.17).

2.3 Time Analyticity in Gevrey Spaces

As mentioned in the Introduction, the task of this article is to prove that the solutions of the Primitive Equations are analytic in time with values in some Gevrey spaces. In fact we show that the solution is the restriction to $\mathbb{R}_+$ of a complex analytic function in the temporal variable in a complex domain containing an interval $(0, t_1)$. In order to derive such a result, we use an already classical method (see e.g. [4] or [5]), the idea being to pass from the Primitive Equations written in real time to an extended equation in the complex time. To avoid too complicated notations and because there is no risk of confusion, for the extended spaces and operators we use the same notations as in the real case. In this way, equation (2.10) is rewritten as:

$$\frac{\mathrm{d}U}{\mathrm{d}\zeta} + AU + B(U, U) + EU = F, \tag{3.1}$$

where $\zeta \in \mathbb{C}$ is the complex time.

In all what follows, $\zeta = se^{i\theta}$, where $s > 0$ and $\cos\theta > 0$ so that the real part of ζ is positive. As for the real case, we need to split the solution of the equation (3.1) into $U^\star$ and $\tilde{U}$, where $U^\star$ is the solution of the linear equation:

$$\begin{aligned} &\frac{\mathrm{d}U^\star}{\mathrm{d}\zeta} + AU^\star + EU^\star = F, \\ &U^\star(0) = U_0, \end{aligned} \tag{3.2}$$

and $\tilde{U}$ is the solution of the nonlinear problem:

$$\begin{aligned} &\frac{\mathrm{d}\tilde{U}}{\mathrm{d}\zeta} + A\tilde{U} + B(\tilde{U}, \tilde{U}) + B(\tilde{U}, U^\star) + B(U^\star, \tilde{U}) + E\tilde{U} = -B(U^\star, U^\star), \\ &\tilde{U}(0) = 0. \end{aligned} \tag{3.3}$$

We start by deriving the a priori estimates for $U^\star$. For that purpose, we apply the operator $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}$ to equation (3.2) and then take the scalar product in H with $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)U^\star$, multiply by $e^{i\theta}$ and take the real part.

We notice that:

$$
\begin{aligned}
&\operatorname{Re} e^{i\theta}\Big(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\frac{\mathrm{d}U^\star}{\mathrm{d}\zeta}, \Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star\Big)_H \\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 \\
&\qquad - \varphi'(s\cos\theta)\cos\theta \operatorname{Re} e^{i\theta}(\Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star, (-\Delta)^{1/2}e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star)_H.
\end{aligned} \tag{3.4}
$$

Using the same constant κ as in (1.24), we find:

$$
\begin{aligned}
&\operatorname{Re} e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}AU^\star, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star)_H \\
&\qquad + \operatorname{Re} e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}EU^\star, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star)_H \\
&\qquad \geq c_1\cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2.
\end{aligned} \tag{3.5}
$$

From all the computations above we conclude that:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 + c_1\cos\theta\,|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2 \\
&\quad \leq \cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H \\
&\qquad + |e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}F|_H|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H.
\end{aligned}
$$

Restricting θ so that $\cos\theta \geq \sqrt{2}/2$ and making use of the Cauchy-Schwarz inequality, we find:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 + c_1\cos\theta\,|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2 \\
&\qquad \leq \frac{2\cos\theta}{c_1}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 + \frac{2}{c_1\cos\theta}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}F|_H^2.
\end{aligned} \tag{3.6}
$$

We can now apply the Gronwall lemma to (3.6) and obtain:

$$
\begin{aligned}
|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}U^\star|_H^2 \leq &|(-\Delta)^{1/2}U_0|_H^2 \exp\Big(\frac{1}{c_1}s\Big) \\
&+ 2|e^{\sigma_1(-\Delta)^{1/2}}F|_H^2 \exp\Big(\frac{1}{c_1}s\Big).
\end{aligned} \tag{3.7}
$$

Since $U_0 \in D((-\Delta)^{1/2})$, we deduce from (3.7) a bound on $U^\star(se^{i\theta})$ in $D(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2})$ for θ such that $\sqrt{2}/2 \leq \cos\theta \leq 1$ and for $s \leq t$, for all $t \geq 0$.

Integrating equation (3.7), one can see that

$$
\int_0^s |e^{\varphi(s'\cos\theta)(-\Delta)^{1/2}}\Delta U^\star|_H^2\,\mathrm{d}s' \leq C(s, F, U_0, \sigma_1), \text{ for all } s \geq 0. \tag{3.8}
$$

Having in mind these estimates, we start deriving estimates for the solution $\tilde{U}$ of equation (3.3).

The calculations for obtaining the a priori estimates are the same as for the linear case: we apply $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}$ to equation (3.3), take the scalar product in H with

$e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)U^\star$ and then multiply the resulting equation by $e^{i\theta}$ and take the real part. Using Lemma 4.3.1 in order to estimate the terms containing the B operator, we find:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + c_1\cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2 \\
&\quad\leq f(s)|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(s) \\
&\qquad + c_3|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2,
\end{aligned}\tag{3.9}$$

where

$$f(s) = \frac{1}{c_1} + c_1'|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star|_H^2|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U^\star|_H^2,$$
$$g(s) = \frac{\sqrt{2}}{c_1}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}F|_H^2.$$

We obtained the form of the functions f and g using the Cauchy-Schwarz inequality and restricting θ to $\sqrt{2}/2 \leq \cos\theta \leq 1$.

We can also write inequality (3.10) as:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + (c_1\frac{\sqrt{2}}{2} - c_3|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H) \\
&\quad\cdot|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2 \leq f(s)|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\tilde{U}|_H^2 + g(s).
\end{aligned}\tag{3.10}$$

Since $\tilde{U}(0) = 0$, we may assume that:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \leq \frac{c_1\sqrt{2}}{4c_3},$$

on some finite interval $(0, t_0)$ and, on this interval, $\tilde{U}$ satisfies the inequality:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U|_H^2 + c_1\frac{\sqrt{2}}{4}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U|_H^2 \\
&\quad\leq f(s)|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U|_H^2 + g(s).
\end{aligned}\tag{3.11}$$

Since f and g depend on the solution $U^\star$ of the linear problem and we already obtained a priori estimates on $U^\star$, we see that for all $\theta \in [-\pi/4, \pi/4]$, f and g are locally integrable functions. Thus we can apply the Gronwall lemma to (3.11) and we find the following estimate on $(0, t_0)$:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \leq \int_0^t g(s)\exp(\int_s^t f(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s.\tag{3.12}$$

Since f and g are locally integrable functions, we can define $t_1 = t(F, U_0, \sigma_1)$ as the time for which we have:

$$\int_0^{t_1} g(s)\exp(\int_s^{t_1} f(\tau)\,\mathrm{d}\tau)\,\mathrm{d}s = \frac{c_1\sqrt{2}}{4c_3}.\tag{3.13}$$

So on the interval $(0, t_1)$ we find:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \leq \frac{c_1\sqrt{2}}{4c_3}. \tag{3.14}$$

We define the region:

$$D(U_0, F, \sigma_1) = \{\zeta = se^{i\theta},\ |\theta| \leq \pi/4,\ 0 < s < t_1(U_0, F, \sigma_1)\}, \tag{3.15}$$

and from the previous estimates we obtain a bound on $U(\zeta)$ in $D((-\Delta)^{1/2}e^{\varphi(s\cos\theta)(-\Delta)^{1/2}})$:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{1/2}\tilde{U}|_H \leq \frac{c_1\sqrt{2}}{4c_3}, \text{ for } \zeta \in \bar{D}(U_0, F, \sigma_1). \tag{3.16}$$

We can now state the main result of this article:

Theorem 2.3.1. *Let U_0 be given in $D((-\Delta)^{1/2})$ and let F be a function analytic in time with values in $D(e^{\sigma_1(-\Delta)^{1/2}})$ for some $\sigma_1 > 0$. Then there exists t_1 depending on the initial data such that the function*

$$t \to (-\Delta)^{1/2}e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U(t),$$

is analytic on $(0, t_1)$, where $\varphi(t) = \min(t, \sigma_1)$ and t_1 is defined by relation (3.13).

Proof: In order to prove the existence of an analytic solution, we use the Galerkin approximation method based on the Fourier series, and the energy estimates obtained above. For the solutions of the Galerkin approximation the a priori estimates which are formally derived above hold rigorously and the bounds are independent of the order m of the Galerkin approximation. With these estimates we can pass to the limit $m \to \infty$ using classical theorems concerning convergence of analytic functions. From here follows Theorem 2.3.1.

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Chapitre 3

Sobolev and Gevrey regularity results for the primitive equations in three space dimensions

Régularité de type Sobolev et Gevrey pour les Equations Primitives en trois dimensions

Le but de ce chapitre, qui reproduit un travail en collaboration avec D. Wirosoetisno, est de prouver en dimension d'espace trois des résultats analogues à ceux obtenus dans les chapitres précédents. On montre ici l'existence et l'unicité, localement en temps, d'une solution très régulière, pour les Equations Primitives dans dimension trois, si les données initiales sont assez régulières. On etablit aussi la régularité de type Gevrey pour les solutions des Equations Primitives.

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Sobolev and Gevrey regularity results for the primitive equations in three space dimensions

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Abstract: The aim of this paper is to present a qualitative study of the Primitive Equations in a three dimensional domain, with periodical boundary conditions. We start by recalling some already existing results regarding the existence locally in time of weak solutions and existence and uniqueness of strong solutions, and we prove the existence of very regular solutions, up to $\mathcal{C}^\infty$-regularity. In the second part of the paper we prove that the solution of the Primitive Equations belongs to a certain Gevrey class of functions.

3.1 Introduction

In this article we consider the Primitive Equations for the ocean or for the atmosphere in 3 space dimensions, with periodic boundary conditions. The general form of the equations governing the movement of the oceans and atmosphere is derived from the basic conservation laws, but the resulting equations are too difficult to handle. That is why, using scale analysis methods and physical observations, the equations are usually approximated by different models, having simpler forms (in principle), one of them being the Primitive Equations (for more details on the form of the Primitive Equations and their derivation, see e.g., [7], [8], [9]).

As we already mentioned, in this article we consider the 3D Primitive Equations with space periodicity and start by recalling the known results of existence, uniqueness and regularity of solutions, in the usual H^1 Sobolev space (see [7], [8], [15]). We then prove a regularity result in higher order Sobolev spaces; for a similar result for the Primitive Equations in space dimension 2, see [11]. We also study the Gevrey regularity for the PEs; in fact, we show that considering a forcing term which is analytic in time with values in some Gevrey space, the solutions of the PEs starting with initial data in the Sobolev space H^1 instantly become elements of a certain Gevrey class and remain there for a certain interval of time. The study of the Gevrey regularity for the solutions was inspired by the article of Foias and Temam [4] who proved this type of results for the Navier–Stokes equations in 2 and 3 space dimensions with periodic boundary conditions. We also mention the works of Promislow [12], of Ferrari and Titi [3], who obtained Gevrey regularity results for a certain class of nonlinear parabolic equations; also, Cao, Rammaha and Titi [2] established the Gevrey regularity for a certain class of analytic nonlinear parabolic equations on the sphere. The Gevrey regularity of the Primitive Equations in 2 space dimensions was proven in [10].

The Primitive Equations in their dimensional form read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + v\frac{\partial u}{\partial x_2} + w\frac{\partial u}{\partial x_3} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x_1} = \nu\Delta u + F_u, \tag{1.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_1} + v\frac{\partial v}{\partial x_2} + w\frac{\partial v}{\partial x_3} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial x_2} = \nu\Delta v + F_v, \tag{1.1b}$$

$$\frac{\partial p}{\partial x_3} = -\rho g, \tag{1.1c}$$

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0, \tag{1.1d}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x_1} + v\frac{\partial T}{\partial x_2} + w\frac{\partial T}{\partial x_3} = \mu\Delta T + F_T. \tag{1.1e}$$

In the system above, $(u,\, v,\, w)$ are the three components of the velocity vector and p, ρ and T are respectively the perturbations of the pressure, of the density and of the temperature from the reference (average) constant state p_0, ρ_0, and T_0. The relation between the temperature and the density is given by the equation of state, and we consider here a version of this equation linearized around the reference state ρ_0 and T_0,

$$\rho_{\text{full}} = \rho_0(1 - \beta_T(T - T_0)), \tag{1.2}$$

so that for the perturbations ρ and T:

$$\rho = -\beta_T\rho_0 T. \tag{1.3}$$

The constant g is the gravitational acceleration and f the Coriolis parameter, ν and μ are the eddy diffusivity coefficients, (F_u, F_v) represent body forces per unit of mass and F_T represents a heating source. In applications F_u, F_v vanish for the ocean but we consider here nonzero forces for mathematical generality. When required, we denote by F the vector (F_u, F_v, F_T).

We work in a limited domain:

$$\mathcal{M} = (0, L_1) \times (0, L_2) \times (-L_3/2, L_3/2), \tag{1.4}$$

and we assume space periodicity with period $\mathcal{M}$, meaning that all functions are taken to satisfy:

$$f(x_1, x_2, x_3, t) = f(x_1 + L_1, x_2, x_3, t) = f(x_1, x_2 + L_2, x_3, t) = f(x_1, x_2, x_3 + L_3, t), \tag{1.5}$$

when extended to $\mathbb{R}^3$.

All functions being periodic, they admit Fourier series, hence we can write:

$$f(x_1, x_2, x_3, t) = \sum_{k \in \mathbb{R}^3} f_k(t) e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3)}, \tag{1.6}$$

where, for notational conciseness, we set $k_j' = 2\pi k_j / L_j$ for $j = 1, 2, 3$.

Moreover, we assume as in [10], [11], that the following symmetries hold:

$$\begin{aligned} u(x_1, x_2, x_3, t) &= u(x_1, x_2, -x_3, t), & F_u(x_1, x_2, x_3, t) &= F_u(x_1, x_2, -x_3, t), \\ v(x_1, x_2, x_3, t) &= v(x_1, x_2, -x_3, t), & F_v(x_1, x_2, x_3, t) &= F_v(x_1, x_2, -x_3, t), \\ T(x_1, x_2, x_3, t) &= -T(x_1, x_2, -x_3, t), & F_T(x_1, x_2, x_3, t) &= -F_T(x_1, x_2, -x_3, t), \\ w(x_1, x_2, x_3, t) &= -w(x_1, x_2, -x_3, t), & p(x_1, x_2, x_3, t) &= p(x_1, x_2, -x_3, t); \end{aligned} \tag{1.7}$$

in other words, u, v, p are even and w, T odd in x_3. These conditions are often used in numerical studies of rotating stratified turbulence (see e.g. [1]). Note that without these symmetry properties, space periodicity is not consistent with (1.1).

The following function spaces are used:

$$V = \{U = (u,\, v,\, T) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3,\ u,\, v \text{ even in } x_3,\ T \text{ odd in } x_3, \\ \int_{-L_3/2}^{L_3/2} (u_{x_1}(x_1, x_2, x_3') + v_{x_2}(x_1, x_2, x_3'))\, \mathrm{d}x_3' = 0\}, \tag{1.8}$$

$$H = \text{closure of } V \text{ in } (\dot{L}^2(\mathcal{M}))^3. \tag{1.9}$$

Here the dots above $\dot{H}^1_{\text{per}}$ and $\dot{L}^2$ denote the functions with zero average over $\mathcal{M}$.

These spaces are endowed with the following scalar products: on H we consider the scalar product

$$(U,\, \tilde{U})_H = (u,\, \tilde{u})_{L^2} + (v,\, \tilde{v})_{L^2} + \kappa (T,\, \tilde{T})_{L^2}, \tag{1.10}$$

and on V the scalar product is

$$((U,\, \tilde{U}))_V = ((u,\, \tilde{u})) + ((v,\, \tilde{v})) + \kappa ((T,\, \tilde{T})), \tag{1.11}$$

where we have written

$$((\Phi,\, \tilde{\Phi})) = \int_{\mathcal{M}} \nabla \Phi \cdot \nabla \tilde{\Phi}\, \mathrm{d}\mathcal{M}. \tag{1.12}$$

The positive constant κ will be chosen below. Since we assumed that all functions have zero average, a generalized Poincaré inequality holds, meaning that we have:

$$|U|_H \leq c_0 \|U\|_V, \quad \forall U \in V, \tag{1.13}$$

which guarantees that $\|\cdot\|$ is indeed a norm on V equivalent to the usual H^1 norm.

In system (1.1), the unknown functions are regrouped in two sets: the prognostic variables u, v and T for which an initial value problem will be defined, and the diagnostic variables ρ, w and p which can be defined, at each instant of time, as functions of the prognostic variables, using the equations and the boundary conditions. The density ρ is already expressed in terms of the temperature T by the equation of state (1.3). Given the prognostic variable $U = (u, v, T) \in V$, we can uniquely determine the vertical velocity w from the conservation of mass equation as:

$$w(U) = w(x_1, x_2, x_3, t) = -\int_0^{x_3} (u_{x_1} + v_{x_2})\,\mathrm{d}x_3', \tag{1.14}$$

where we used $w(x_1, x_2, 0, t) = 0$, since w is odd in x_3. Using (1.1d), the fact that w is periodic gives the constraint

$$\int_{-L_3/2}^{L_3/2} (u_{x_1} + v_{x_2})\,\mathrm{d}x_3 = 0. \tag{1.15}$$

From equation (1.1c), the pressure can be determined uniquely in terms of T, up to its value p_s at $x_3 = 0$, namely,

$$p(x_1, x_2, x_3, t) = p_s(x_1, x_2, 0, t) + \beta_T \rho_0 \int_0^{x_3} T(x_1, x_2, x_3', t)\,\mathrm{d}x_3'. \tag{1.16}$$

In fact, we fully determine the Fourier coefficients p_k of the pressure p for $k_3 \neq 0$ but not for $k_3 = 0$. That means that the part of the pressure we can not determine is the average of the pressure in the vertical direction:

$$p_\star(x_1, x_2) = \frac{1}{L_3} \int_{-L_3/2}^{L_3/2} p(x_1, x_2, x_3, t)\,\mathrm{d}x_3 = \sum_{k,\,k_3=0} p_k(t) e^{i(k_1' x_1 + k_2' x_2)}. \tag{1.17}$$

We deduce below the relation between the average of the pressure in the vertical direction and the surface pressure:

(1.18)

$$\begin{aligned}
p(x_1, x_2, x_3, t) &= p_s(x_1, x_2, 0, t) + \beta_T \rho_0 \int_0^{x_3} \sum_k T_k(t) e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3')}\,\mathrm{d}x_3 \\
&= p_s(x_1, x_2, t) + \beta_T \rho_0 \sum_{k,\,k_3 \neq 0} \frac{T_k(t)}{i k_3'} e^{i(k_1' x_1 + k_2' x_2)} (e^{i k_3' x_3} - 1) \\
&= \sum_{(k_1,k_2)} (p_{s,\,(k_1,k_2)} - \beta_T \rho_0 \sum_{k_3 \neq 0} \frac{T_k(t)}{i k_3'}) e^{i(k_1' x_1 + k_2' x_2)} + \beta_T \rho_0 \sum_{k,\,k_3 \neq 0} \frac{T_k(t)}{i k_3'} e^{i k' \cdot x} \\
&= \sum_{(k_1,k_2)} p_{\star,\,(k_1,k_2)} e^{i(k_1' x_1 + k_2' x_2)} + \beta_T \rho_0 \sum_{k,\,k_3 \neq 0} \frac{T_k(t)}{i k_3'} e^{i k' \cdot x},
\end{aligned}$$

where $p_\star$ is the average of p in the vertical direction. Then:

$$p_{\star,\,(k_1,k_2)} = p_{s,\,(k_1,k_2)} - \beta_T \rho_0 \sum_{k_3 \neq 0} \frac{T_k(t)}{i k_3'}. \tag{1.19}$$

The variational formulation of the problem

In order to obtain the variational formulation of this problem, we consider a test function $U^\flat = (u^\flat, v^\flat, T^\flat) \in V$, multiply (1.1a) by $u^\flat$, (1.1b) by $v^\flat$, and (1.1e) by $\kappa T^\flat$, and integrate over $\mathcal{M}$. Using the integration by parts and the space periodicity, we find that system (1.1) is formally equivalent to the following problem:

To find $U : [0, t_0] \to V$, such that,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(U, U^\flat)_H + a(U, U^\flat) + b(U, U, U^\flat) + e(U, U^\flat) = (F, U^\flat)_H, \quad \forall\, U^\flat \in V,\\
&U(0) = U_0.
\end{aligned}
\tag{1.20}
$$

In (1.20) we introduced the bilinear, continuous form $a : V \times V \to \mathbb{R}$ as:

$$
a(U, U^\flat) = \nu((u, u^\flat)) + \nu((v, v^\flat)) + \kappa\mu((T, T^\flat)),
\tag{1.21}
$$

the trilinear form b as:

$$
\begin{aligned}
b(U, U^\sharp, U^\flat) = &\int_{\mathcal{M}} (u\frac{\partial u^\sharp}{\partial x}u^\flat + v\frac{\partial u^\sharp}{\partial y}u^\flat + w(U)\frac{\partial u^\sharp}{\partial z}u^\flat)\,\mathrm{d}\mathcal{M}\\
&+ \int_{\mathcal{M}} (u\frac{\partial v^\sharp}{\partial x}v^\flat + v\frac{\partial v^\sharp}{\partial y}v^\flat + w(U)\frac{\partial v^\sharp}{\partial z}v^\flat)\,\mathrm{d}\mathcal{M}\\
&+ \kappa\int_{\mathcal{M}} (u\frac{\partial T^\sharp}{\partial x}T^\flat + v\frac{\partial T^\sharp}{\partial y}T^\flat + w(U)\frac{\partial T^\sharp}{\partial z}\tilde{T})\,\mathrm{d}\mathcal{M},
\end{aligned}
\tag{1.22}
$$

and the bilinear form e, $e : V \times V \to \mathbb{R}$ which is continuous:

$$
e(U, U^\flat) = f\int_{\mathcal{M}} (uv^\flat - vu^\flat)\,\mathrm{d}\mathcal{M} - g\beta_T\int_{\mathcal{M}} Tw(U^\flat)\,\mathrm{d}\mathcal{M}.
\tag{1.23}
$$

We note that

$$
a(U, U) + e(U, U) = \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - g\beta_T\int_{\mathcal{M}} Tw(U)\,\mathrm{d}\mathcal{M}.
\tag{1.24}
$$

We then estimate:

$$
|g\beta_T\int_{\mathcal{M}} Tw(U)\,\mathrm{d}\mathcal{M}| \le g\beta_T|T|_{L^2}|w(U)|_{L^2} \le cg\beta_T(\|u\| + \|v\|)\|T\|;
\tag{1.25}
$$

here we used (1.14) and the Poincaré inequality. We find:

$$
a(U, U) + e(U, U) \ge \nu\|u\|^2 + \nu\|v\|^2 + \kappa\mu\|T\|^2 - cg\beta_T\|u\|\|T\| - cg\beta_T\|v\|\|T\|.
\tag{1.26}
$$

From equation (1.26), we see that choosing κ large enough, more specifically $\kappa \ge 2(cg\beta_T)^2/(\nu\mu)$, the bilinear continuous form $a + e$ is coercive on V, and

$$
a(U, U) + e(U, U) \ge \frac{\nu}{2}\|u\|^2 + \frac{\nu}{2}\|v\|^2 + \frac{\kappa\mu}{2}\|T\|^2 \ge c_1\|U\|_V^2,\ c_1 = \min(\nu, \mu).
\tag{1.27}
$$

In order to study the properties of the form b, we introduce the space V_2, defined as:

$$
V_2 = \text{ the closure of } V \cap (\dot{H}^2_{\text{per}}(\mathcal{M}))^3 \text{ in } (\dot{H}^2_{\text{per}}(\mathcal{M}))^3.
\tag{1.28}
$$

We have the following result on b:

Lemma 3.1.1. *The form b is trilinear continuous from $V \times V_2 \times V$ into $\mathbb{R}$ and from $V \times V \times V_2$ into $\mathbb{R}$, and*

$$|b(U, U^\sharp, U^\flat)| \le c_2 \|U\| |U^\sharp|_H^{1/2} \|U^\sharp\|^{1/2} \|U^\flat\|_{V_2}, \quad \forall\, U, U^\sharp \in V,\ U^\flat \in V_2. \tag{1.29}$$

Furthermore,

$$b(U, U^\flat, U^\flat) = 0 \quad \forall\, U \in V,\ U^\flat \in V_2,$$

and

$$b(U, U^\flat, U^\sharp) = -b(U, U^\sharp, U^\flat), \quad \forall\, U, U^\flat, U^\sharp \in V \text{ with } U^\flat \text{ or } U^\sharp \in V_2.$$

Proof: The proof is based on appropriate estimates for the terms of $b(U, U^\sharp, U^\flat)$; Hölder, Sobolev and interpolation inequalities are used. For more details on how this type of results is derived, see [7], [11] or [15].

We can now write (1.20) as an evolution equation in the Hilbert space V_2', which is the dual space of V_2. For that purpose we observe that we can associate the following operators to the forms a, b and e above:

A linear continuous from V into V', defined by $\langle AU, U^\flat\rangle = a(U, U^\flat), \quad \forall\, U, U^\flat \in V$,

B bilinear, continuous from $V \times V$ into V_2', defined by
$$\langle B(U, U^\flat),\, U^\sharp\rangle = b(U, U^\flat, U^\sharp) \quad \forall\, U, U^\flat \in V,\ \forall\, U^\sharp \in V_2,$$

E linear continuous from V into V', defined by $\langle EU, U^\flat\rangle = e(U, U^\flat), \quad \forall\, U, U^\flat \in V$.

Then equation (1.20) is equivalent to the following operator evolution equation in V_2':

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U, U) + EU = F, \\ &U(0) = U_0. \end{aligned} \tag{1.30}$$

In the second section we present some existence, uniqueness and Sobolev regularity results for the Primitive Equations, that is (1.20) or (1.30). We start by recalling the existence of weak solutions (result already available thanks to [7]), the existence and uniqueness of strong solutions (result already available, see [15]) and we conclude by proving the existence of more regular solutions, up to $\mathcal{C}^\infty$ regularity. For these high regularity results we use periodic boundary conditions; for a similar result in two space dimensions, see [11].

In the third section we prove that the solutions of the Primitive Equations in space dimension three are real functions analytic in time with values in some Gevrey space.

3.2 Sobolev regularity results

As we mentioned before, we start by recalling some results already available and then we prove the existence of very regular solutions.

Theorem 3.2.1. *Given $U_0 \in H$ and $F \in L^\infty(\mathbb{R}_+; H)$, there exists at least one solution U of problem (1.20) such that:*

$$U \in L^\infty(\mathbb{R}_+; H) \cap L^2(0,T;V), \quad \forall T > 0. \tag{2.1}$$

Proof: The proof of this theorem is based on the a priori estimates given below which, by classical methods, lead to (2.1); we briefly recall these calculations needed below.

Taking $U^\flat = U(t)$ in equation (1.20), for an arbitrary fixed $t > 0$, we obtain after some basic computations:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + c_1\|U\|_H^2 \le c_1'|F|_\infty^2, \quad \frac{\mathrm{d}}{\mathrm{d}t}|U|_H^2 + \frac{c_1}{c_0}|U|_H^2 \le c_1'|F|_\infty^2, \tag{2.2}$$

where $|F|_\infty$ is the norm of F in $L^\infty(\mathbb{R}_+; H)$. Using Gronwall inequality, from (2.2) we find

$$|U(t)|_H^2 \le |U(0)|_H^2 e^{-\frac{c_1 t}{c_0}} + \frac{c_1' c_0}{c_1}(1 - e^{-\frac{c_1 t}{c_0}})|F|_\infty^2. \tag{2.3}$$

Inequality (2.3) implies:

$$\limsup_{t\to\infty} |U(t)|_H^2 \le \frac{c_1' c_0}{c_1}|F|_\infty^2 =: r_0^2. \tag{2.4}$$

After these a priori estimates of U in $L^\infty(\mathbb{R}_+; H)$, we integrate (2.3) and find:

$$\int_0^{t_1} \|U\|^2 \,\mathrm{d}t \le K(U_0, F, t_1), \quad \forall t_1 > 0, \tag{2.5}$$

where $K(U_0, F, t_1)$ denotes a constant depending on the initial data U_0 and on the time $t_1 > 0$. These estimates are at the basis of the proof of existence in Theorem 3.2.1 (for more details, see [7]).

We also note that for a forcing independent on t, $F(t) \equiv F \in H$, inequality (2.4) implies that any ball $\mathcal{B}(0, r_0')$ in H, with $r_0' > r_0$ is an absorbing ball.

The existence and uniqueness of a strong solution is given by the following theorem (see e.g., [5], [15]):

Theorem 3.2.2. *Given $U_0 \in V$ and $F \in L^2(0,T;H)$, there exists $t_\star > 0$, $t_\star = t_\star(\|U_0\|)$ and a unique solution $U = U(t)$ of (1.20) on $(0, t_\star)$, such that:*

$$U \in \mathcal{C}(0, t_\star; V) \cap L^2(0, t_\star; (\dot{H}^2_{\mathrm{per}}(\mathcal{M}))^3).$$

Proof: The proof is based, as usual, on some a priori estimates for the solution U, obtained by taking $U^\flat = -\Delta U$ in (1.20). First of all, let us note that the "standard" treatment of the bilinear term gives the estimate,

$$|(B(U,U),\, \Delta U)_H| \le c\|U\|_V^{1/2}|\Delta U|_H^{5/2}, \tag{2.6}$$

The term $|\Delta U|_H^{5/2}$ is too strong to be dominated, meaning it cannot be majorized by $|\Delta U|^2$ on the left-hand side.

In order to overcome this difficulty, the idea is to use an anisotropic treatment for the terms in $b(U, \tilde{U}, U^\sharp)$ which contain $w(U)$. This gives the following result, which is proved in [15] (see also [5]):

Lemma 3.2.1. *The trilinear form b is continuous from $V_2 \times V_2 \times H$ into $\mathbb{R}$, and:*

$$|b(U,\tilde{U},U^\sharp)| \le c_3(\|U\|_V \|\tilde{U}\|_V^{1/2}\|\tilde{U}\|_{V_2}^{1/2} + \|U\|_V^{1/2}\|U\|_{V_2}^{1/2}\|\tilde{U}\|_V^{1/2}\|\tilde{U}\|_{V_2}^{1/2})|U^\sharp|_H, \tag{2.7}$$

for all U, $\tilde{U}$ in V_2 and $U^\sharp$ in H.

We return to the proof of the theorem. Using Lemma 3.2.1, we can estimate the trilinear term as:

$$|(B(U,U),\Delta U)_H| \le c_4\|U\|_V|\Delta U|_H^2. \tag{2.8}$$

This estimate allows us to obtain some a priori estimates, but since the estimate is a weak one (the term $|\Delta U|_H$ has power 2), a direct estimate would force us to work with small initial data. In order to avoid imposing such a restriction, we split the solution U of equation (1.30) into $U = U^\star + \tilde{U}$, where $U^\star$ is the solution of the linear problem (as in [5], [15]):

$$\begin{aligned} &\frac{\mathrm{d}U^\star}{\mathrm{d}t} + AU^\star + EU^\star = F, \\ &U^\star(0) = U_0, \end{aligned} \tag{2.9}$$

and $\tilde{U}$ is the solution of the following nonlinear problem, in which $U^\star$ is now known:

$$\begin{aligned} &\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t} + A\tilde{U} + B(\tilde{U},\tilde{U}) + B(\tilde{U},U^\star) + B(U^\star,\tilde{U}) + E\tilde{U} = -B(U^\star,U^\star), \\ &\tilde{U}(0) = 0. \end{aligned} \tag{2.10}$$

We start by deriving a priori estimates for $U^\star$. We take the scalar product of (2.9) with $-\Delta U^\star$ in H and we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U^\star\|_V^2 + c_1|\Delta U^\star|_H^2 \le \frac{c_1}{2}|\Delta U^\star|_H^2 + \frac{1}{2c_1}|F|_H^2, \tag{2.11}$$

which leads to:

$$\sup_{0\le t\le t_1} \|U^\star(t)\|_V^2 \le \frac{1}{2c_1}|F|^2_{L^2(0,t_1;H)} + \|U_0\|_V^2, \tag{2.12}$$

and

$$\int_0^{t_1} |\Delta U^\star(t)|_H^2 \,\mathrm{d}t \le \frac{1}{2{c_1}^2}|F|^2_{L^2(0,t_1;H)} + \frac{1}{c_1}\|U_0\|_V^2. \tag{2.13}$$

Using the following a priori estimates and classical methods (e.g. Galerkin's method), we prove the existence of a solution of (2.10) on some interval $(0,t_\star)$, where $t_\star$ is also determined below. Assuming that $U^\star$ is known in $L^\infty(0,t_1;V) \cap L^2(0,t_1;H^2)$ for all $t_1 > 0$, we take the scalar product of (2.10) with $\Delta\tilde{U}$ in H and we use Lemma 3.2.1. We have the following estimates:

$$|(B(\tilde{U},U^\star),\Delta\tilde{U})_H| = |b(\tilde{U},U^\star,\Delta\tilde{U})| \le \frac{c_1}{8}|\Delta\tilde{U}|_H^2 + c\|\tilde{U}\|_V^2(1+\|U^\star\|_{H^2}^2), \tag{2.14}$$

(2.15) $$|(B(U^\star,\tilde U),\Delta\tilde U)_H| = |b(U^\star,\tilde U,\Delta\tilde U)| \le \frac{c_1}{8}|\Delta\tilde U|_H^2 + c\|U^\star\|_V^2\|U^\star\|_{H^2}^2\|\tilde U\|_V^2,$$

and

(2.16) $$|(B(U^\star,U^\star),\Delta\tilde U)_H| = |b(U^\star,U^\star,\Delta\tilde U)| \le \frac{c_1}{8}|\Delta\tilde U|_H^2 + c\|U^\star\|_V^2\|U^\star\|_{H^2}^2.$$

Taking into account all these estimates, (2.10) leads to the following estimate:

(2.17) $$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde U\|_V^2 + (c_1 - c_4\|\tilde U\|_V)|\Delta\tilde U|_H^2 \le \gamma(t)\|\tilde U\|_V^2 + \eta(t),$$

with

$$\gamma(t) = c(1+\|U^\star\|_{H^2}^2 + \|U^\star\|_V^2\|U^\star\|_{H^2}^2),$$
$$\eta(t) = c\|U^\star\|_V^2\|U^\star\|_{H^2}^2.$$

Using (2.12) and (2.13), we see that the functions γ and η are integrable on any interval $(0,t_1)$. Since $\tilde U(0)=0$, we may assume that:

(2.18) $$\|\tilde U\|_V \le \frac{c_1}{2c_4},\ \text{on some finite interval of time } (0,t_0).$$

On that interval, we can write (2.17) as:

(2.19) $$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde U\|_V^2 + \frac{c_1}{2}|\Delta\tilde U|_H^2 \le \gamma(t)\|\tilde U\|_V^2 + \eta(t).$$

Applying the Gronwall lemma to (2.19), we deduce the following estimate on $(0,t_0)$:

(2.20) $$\|\tilde U\|_V^2 \le \int_0^t \eta(s)\exp\Big(\int_s^t \gamma(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

Since the functions γ and η are integrable on $(0,T)$, we can define $t_\star = t_\star(F,U_0)$ as the first time for which

(2.21) $$\int_0^{t_\star}\eta(s)\exp\Big(\int_s^{t_\star}\gamma(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s = \Big(\frac{c_1}{2c_4}\Big)^2.$$

Then, on the interval $(0,t_\star)$ we find $\|\tilde U\|_V \le c_1/2c_4$. Hence, on $(0,t_\star)$ the solution $\tilde U$ satisfies both (2.17) and (2.18).

We have then the necessary a priori estimates in order to deduce, using the Fourier–Galerkin method, the existence of a solution U of (1.20) such that:

(2.22) $$U \in L^\infty(0,t_\star;V)\cap L^2(0,t_\star;(\dot H^2_{\mathrm{per}}(\mathcal{M}))^3).$$

The continuity of U from $[0,t_\star]$ into V is proved using an interpolation argument, see e.g. [6] or [14].

The uniqueness of the solution is easily obtained by classical methods, meaning we consider two solutions U_1, U_2 of (1.30) which satisfy (2.22) and estimate $U=U_1-U_2$ in the H norm, we find that the solutions coincide.

As we mentioned at the beginning of this section, we now prove the existence and uniqueness of the regular solution of the Primitive Equations, up to $\mathcal{C}^\infty$ regularity. We have the following result:

Theorem 3.2.3. *Given $m \in \mathbb{N}$, $m \geq 2$, $U_0 \in V \cap (\dot{H}^m_{\text{per}}(\mathcal{M}))^3$ and $F \in L^\infty(0,T;H \cap (\dot{H}^{m-1}_{\text{per}}(\mathcal{M}))^3)$, there exists $t_{\star\star} = t_{\star\star}(F, U_0)$ and a unique solution U of equation (1.30) on $[0, t_{\star\star}]$ such that:*

$$U \in \mathcal{C}(0, t_{\star\star}; (\dot{H}^m_{\text{per}}(\mathcal{M}))^3) \cap L^2(0, t_{\star\star}; (\dot{H}^{m+1}_{\text{per}}(\mathcal{M}))^3). \tag{2.23}$$

Moreover, if $U_0 \in V$ and $F \in L^\infty(0,T;H \cap (\dot{H}^{m-1}_{\text{per}}(\mathcal{M}))^3)$, then the solution U of equation (1.30) belongs to $\mathcal{C}((0, t_{\star\star}]; \dot{H}^m_{\text{per}}(\mathcal{M}))^3)$.

Proof: The proof is based on a priori estimates on the higher order derivatives.

We set $|U|_m = (\sum_{[\alpha]=m} |D^\alpha U|^2_H)^{1/2}$, where D^α is the differential operator $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$, $D_i = \partial/\partial x_i$; α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{N}$ and $[\alpha] = \alpha_1 + \alpha_2 + \alpha_3$. In equation (1.20) we take $\tilde{U} = (-\Delta)^m U(t)$, with $m \geq 2$ and t arbitrarily fixed; we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(U, (-\Delta)^m U)_H + a(U, (-\Delta)^m U) + b(U, U, (-\Delta)^m U) + e(U, (-\Delta)^m U) \\ = (F, (-\Delta)^m U)_H.\end{aligned} \tag{2.24}$$

We also note that:

$$a(U, (-\Delta)^m U) + e(U, (-\Delta)^m U) = (a+e)((-\Delta)^{m/2} U, (-\Delta)^{m/2} U) \geq c_1 |U(t)|^2_{m+1},$$

where we used the coercivity of $a + e$.

Integrating by parts and using the periodicity, we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|U(t)|^2_m + c_1 |U(t)|^2_{m+1} \leq |b(U, U, (-\Delta)^m U)| + |(F, (-\Delta)^m U)_H|. \tag{2.25}$$

We need to estimate the terms on the right hand side of (2.25). The last term can be easily estimated as:

$$|(F, (-\Delta)^m U)_H| \leq c|F|^2_{m-1} + \frac{c_1}{2(2m+3)}|U|^2_{m+1}. \tag{2.26}$$

In order to estimate the term $b(U, U, (-\Delta)^m U)$, we note that the integrals we need to consider are of the types:

$$\begin{aligned}&\int_{\mathcal{M}} u \frac{\partial u}{\partial x} D_1^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u \, \mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} v \frac{\partial u}{\partial y} D_1^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u \, \mathrm{d}\mathcal{M}, \\ &\int_{\mathcal{M}} w(U) \frac{\partial u}{\partial z} D_1^{2\alpha_1} D_2^{2\alpha_2} D_3^{2\alpha_3} u \, \mathrm{d}\mathcal{M},\end{aligned} \tag{2.27}$$

where, as before, $\alpha_i \in \mathbb{N}$ with $[\alpha] = \alpha_1 + \alpha_2 + \alpha_3 = m$. Integrating by parts and using periodicity, the integrals become:

$$\int_{\mathcal{M}} D^\alpha \big(u \frac{\partial u}{\partial x}\big) D^\alpha u \, \mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} D^\alpha \big(v \frac{\partial u}{\partial y}\big) D^\alpha u \, \mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} D^\alpha \big(w(U) \frac{\partial u}{\partial z}\big) D^\alpha u \, \mathrm{d}\mathcal{M}. \tag{2.28}$$

Using Leibniz' formula, we see that the integrals can be written as sums of integrals of the form

$$\int_{\mathcal{M}} uD^\alpha \frac{\partial u}{\partial x} D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} vD^\alpha \frac{\partial u}{\partial y} D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} w(U)D^\alpha \frac{\partial u}{\partial z} D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{2.29}$$

and of integrals of the form

$$\int_{\mathcal{M}} \delta^k u \delta^{m-k} \frac{\partial u}{\partial x} D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} \delta^k v \delta^{m-k} \frac{\partial u}{\partial y} D^\alpha u \,\mathrm{d}\mathcal{M}, \quad \int_{\mathcal{M}} \delta^k w(U) \delta^{m-k} \frac{\partial u}{\partial z} D^\alpha u \,\mathrm{d}\mathcal{M}, \tag{2.30}$$

where $k = 1, ..., m$ and δ^k is some differential operator D^α with $[\alpha] = k$.

Note that for each α, after integration by parts, the sum of the integrals of type (2.29) is zero because of the mass conservation equation (1.1d). It remains to estimate the integrals of type (2.30). The first two integrals in (2.30) lead to the same kind of estimates, so in fact we only need to estimate the first and last integrals, which we do using Sobolev and interpolation inequalities. For the first integral, we write:

$$\begin{aligned} \Big| \int_{\mathcal{M}} \delta^k u \delta^{m-k} \frac{\partial u}{\partial x} D^\alpha u \,\mathrm{d}\mathcal{M} \Big| &\le |\delta^k u|_{L^3} \Big|\delta^{m-k} \frac{\partial u}{\partial x}\Big|_{L^6} |D^\alpha u|_{L^2} \\ &\le c|U|_k^{1/2} |U|_{k+1}^{1/2} |U|_{m-k+2} |U|_m, \end{aligned} \tag{2.31}$$

where $k = 1, ..., m$.

For the last integral we write, when $k < m$:

$$\begin{aligned} \Big| \int_{\mathcal{M}} \delta^k w(U) \delta^{m-k} \frac{\partial u}{\partial z} D^\alpha u \,\mathrm{d}\mathcal{M} \Big| &\le |\delta^k w(U)|_{L^2} \Big|\delta^{m-k} \frac{\partial u}{\partial z}\Big|_{L^3} |D^\alpha u|_{L^6} \\ &\le c|U|_{k+1} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_{m+1}, \end{aligned} \tag{2.32}$$

and when $k = m$:

$$\Big| \int_{\mathcal{M}} \delta^m w(U) \frac{\partial u}{\partial z} D^\alpha u \,\mathrm{d}\mathcal{M} \Big| \le |\delta^m w(U)|_{L^2} \Big|\frac{\partial u}{\partial z}\Big|_{L^6} |D^\alpha u|_{L^3} \le c|U|_2 |U|_m^{1/2} |U|_{m+1}^{3/2}. \tag{2.33}$$

Gathering relations (2.31), (2.32) and (2.33), we find:

$$\begin{aligned} |b(U, U, (-\Delta)^m U)| &\le c \sum_{k=1}^{m} |U|_{m-k+2} |U|_k^{1/2} |U|_{k+1}^{1/2} |U|_m \\ &\quad + c \sum_{k=1}^{m-1} |U|_{k+1} |U|_{m-k+1}^{1/2} |U|_{m-k+2}^{1/2} |U|_{m+1} + c|U|_2 |U|_m^{1/2} |U|_{m+1}^{3/2}. \end{aligned} \tag{2.34}$$

We now need to bound the terms from the right-hand side of (2.34):

For the case when $m > 2$, we notice that not all terms on the right hand side of (2.34) contain $|U|_{m+1}$. From the first sum, only terms corresponding to $k = 1$ and to $k = m$ contain $|U|_{m+1}$, and we estimate them as:

$$\begin{aligned} &|U|_{m+1} |U|_1^{1/2} |U|_2^{1/2} |U|_m \le \frac{c_1}{2(m+3)} |U|_{m+1}^2 + c_1' |U|_1 |U|_2 |U|_m^2; \\ &|U|_2 |U|_m^{3/2} |U|_{m+1}^{1/2} \le \frac{c_1}{2(m+3)} |U|_{m+1}^2 + c_2' |U|_2^{4/3} |U|_m^2. \end{aligned}$$

Terms from the second sum corresponding to $k = 2, ..., m-1$, are estimated as:

$$c|U|_{k+1}|U|^{1/2}_{m-k+1}|U|^{1/2}_{m-k+2}|U|_{m+1} \leq \frac{c_1}{2(m+3)}|U|^2_{m+1} + c'_3|U|^2_{k+1}|U|_{m-k+1}|U|_{m-k+2}, \tag{2.35}$$

while for the term for $k = 1$, as well as for the last term in (2.34), we have:

$$c|U|_2|U|^{1/2}_m|U|^{3/2}_{m+1} \leq \frac{c_1}{2(m+3)}|U|^2_{m+1} + c'_4|U|^4_2|U|^2_m. \tag{2.36}$$

Gathering all the estimates above, we obtain the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_m + c_1|U|^2_{m+1} \leq \theta + \varphi|U|^2_m, \tag{2.37}$$

where the expressions of the functions $\theta = \theta(t)$ and $\varphi = \varphi(t)$ can be easily derived from the estimates above. The functions θ and φ are formed from sums involving the terms $|U|_k$, with $k \leq m$.

We also note that for $m = 2$ we obtain, using the Young inequality, the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}|U|^2_2 + c_1|U|^2_3 \leq c|F|^2_1 + c|U|_1|U|^3_2 + c|U|^4_1|U|^2_2 + c|U|^{10/3}_2. \tag{2.38}$$

Inequality (2.38) can be also written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(1 + |U|^2_2) \leq K(|F|^2_{L^\infty(H^1)}, |U|_{L^\infty(0,t_\star;H^1)})(1 + |U|^2_2)^{5/3}, \tag{2.39}$$

and we obtain that there exists a time $t_{\star\star} \leq t_\star$ depending on F, $|U_0|_2$ and on $t_\star$ of Theorem 3.2.2, such that:

$$|U(t)|_2 \leq K(U_0), \quad \forall\, 0 \leq t \leq t_{\star\star}. \tag{2.40}$$

Using the Gronwall lemma, we find that for each $m \geq 2$, we have a bound for U in $L^\infty(0, t_{\star\star}; \dot{H}^m_{\text{per}}(\mathcal{M}))$ and $L^2(0, t_{\star\star}; \dot{H}^{m+1}_{\text{per}}(\mathcal{M}))$, where $t_{\star\star}$ was defined above. From this result, the first part of the theorem easily follows.

For the second part of the theorem we notice that since U_0 belongs to V, the solution U of problem (1.30) belongs, according to Theorem 3.2.2, to $L^2(0, t_\star; \dot{H}^2_{\text{per}})$. This means that $U(t) \in \dot{H}^2_{\text{per}}(\mathcal{M})$ almost everywhere on $(0, t_\star)$, so there exists a t_1 arbitrarily small such that $U(t_1) \in \dot{H}^2_{\text{per}}(\mathcal{M})$. Using now the first part of the theorem we obtain that the solution U is such that:

$$U \in \mathcal{C}([t_1, t_{\star\star}); \dot{H}^2_{\text{per}}(\mathcal{M})) \cap L^2(t_1, t_{\star\star}; \dot{H}^3_{\text{per}}(\mathcal{M})).$$

Using the same argument as before, we find a t_2 belonging to the interval $[t_1, t_{\star\star}]$, arbitrarily close to t_1, such that $U(t_2) \in \dot{H}^3_{\text{per}}(\mathcal{M})$. Applying the result deduced before, in the first part of the theorem, we obtain that the solution U is such that:

$$U \in \mathcal{C}([t_2, t_{\star\star}]; \dot{H}^3_{\text{per}}(\mathcal{M})) \cap L^2(t_2, t_{\star\star}; \dot{H}^4_{\text{per}}(\mathcal{M})).$$

Recurrently we arrive at:

$$U \in \mathcal{C}([t_{m-1}, t_{\star\star}]; \dot{H}^m_{\text{per}}(\mathcal{M})) \cap L^2(t_{m-1}, t_{\star\star}; \dot{H}^{m+1}_{\text{per}}(\mathcal{M})).$$

where t_{m-1} is arbitrarily close to zero. From this relation, the result follows immediately:

$$U \in \mathcal{C}((0, t_{\star\star}]; \dot{H}^m_{\text{per}}(\mathcal{M}))$$

Remark 3.2.1. Note here that $t_{\star\star}$ is independent of m; in fact $t_{\star\star} = t(F, |U_0|_2, t_\star)$ is the time for which $|U|_2 \in L^\infty(0, t_{\star\star})$. Then, for each $m > 2$, the functions θ and φ from (2.37) are locally integrable on $(0, t_{\star\star})$ so, by the Gronwall lemma, we obtain a bound of $|U(t)|_m$ on the same interval $(0, t_{\star\star})$.

As a consequence of the above remark, we also deduce the following result:

Remark 3.2.2. Given $U_0 \in (\dot{\mathcal{C}}^\infty(\bar{\mathcal{M}}))^3$ and $F \in L^\infty(0, t, (\dot{\mathcal{C}}^\infty(\bar{\mathcal{M}}))^3)$, Theorem 3.2.3 gives also the existence of a solution U continuous from $(0, t_{\star\star})$ into $\cap_{m\geq 0} \dot{H}^m_{\text{per}}(\mathcal{M}) = \dot{\mathcal{C}}^\infty_{\text{per}}(\mathcal{M})$.

If $F \in \mathcal{C}^\infty(\bar{\mathcal{M}} \times [0, t])$, estimates on the time derivatives of U can be also obtained as e.g. in [13] for the case of Navier-Stokes equations, so that U is finally $\mathcal{C}^\infty$ in space and time on $(0, t_{\star\star})$.

3.3 Gevrey regularity results

As mentioned in the introduction, the aim of this paper is also to prove that the solutions of the PEs are real functions analytic in time with values in a Gevrey space; in fact we prove that the solutions are the restriction to a positive real interval of some complex analytic function in time. We start this section by introducing some notations and defining the Gevrey spaces we will consider.

We introduce the following notation:

$$[U_k]^2_\kappa = |u_k|^2 + |v_k|^2 + \kappa |T_k|^2.$$

Considering the Laplacian Δ, we define the Gevrey class $D(e^{\tau(-\Delta)^{1/2}})$, $\tau > 0$, as the set of functions U in H satisfying

$$|\mathcal{M}| \sum_{k\in\mathbb{Z}^3} e^{2\tau|k'|} [U_k]^2_\kappa = |e^{\tau(-\Delta)^{1/2}} U|^2_H < \infty. \tag{3.1}$$

The norm of the Hilbert space $D(e^{\tau(-\Delta)^{1/2}})$ is given by

$$|U|_\tau := |U|_{D(e^{\tau(-\Delta)^{1/2}})} = |e^{\tau(-\Delta)^{1/2}} U|_H, \text{ for } U \in D(e^{\tau(-\Delta)^{1/2}}), \tag{3.2}$$

and the associated scalar product is

$$(U, V)_\tau := (U, V)_{D(e^{\tau(-\Delta)^{1/2}})} = (e^{\tau(-\Delta)^{1/2}} U, e^{\tau(-\Delta)^{1/2}} V)_H, \text{ for } U, V \in D(e^{\tau(-\Delta)^{1/2}}). \tag{3.3}$$

Another Gevrey space that we will use is $D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})$, $m \geq 1$ integer, which is a Hilbert space when endowed with the inner product:

$$(U,V)_{D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})} = ((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}U, (-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}V)_H; \tag{3.4}$$

the norm of the space is given by

$$|U|^2_{D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})} = |(-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}U|^2_H = |\mathcal{M}| \sum_{k\in\mathbb{Z}^3} |k'|^{2m} e^{2\tau|k'|}[U_k]^2_\kappa. \tag{3.5}$$

Estimate on b:

We start with the following estimate on b, following the idea of Foias and Temam for the Navier-Stokes equations [4]:

Lemma 3.3.1. *Let U, $U^\sharp$ and $U^\flat$ be given in $D((-\Delta)^{3/2}e^{\tau(-\Delta)^{1/2}})$, for $\tau \geq 0$. Then the following inequality holds:*

$$\begin{aligned} |((-\Delta)^{1/2}B(U,U^\sharp),(-\Delta)^{3/2}U^\flat)_\tau| &\leq c|\Delta U|_\tau |\Delta U^\sharp|_\tau^{1/2} |(-\Delta)^{3/2}U^\sharp|_\tau^{1/2} |(-\Delta)^{3/2}U^\flat|_\tau \\ &+ c|\Delta U|_\tau^{1/2} |(-\Delta)^{3/2}U|_\tau^{1/2} |\Delta U^\sharp|_\tau |(-\Delta)^{3/2}U^\flat|_\tau. \end{aligned} \tag{3.6}$$

Proof: We first write the trilinear form b in Fourier modes. In order to simplify the writing, we define, for each $j \in \mathbb{Z}^3$, $\delta_{j,n}$ as j'_n/j'_3 when $j'_3 \neq 0$ and as 0 when $j'_3 = 0$, for $n = 1, 2$. With obvious notations, the trilinear form is then written as:

$$\begin{aligned} b(U,U^\sharp,U^\flat) &= \sum_{j+l+k=0} i(l'_1u_j + l'_2v_j + l'_3w_j)u^\sharp_l u^\flat_k \\ &\quad + \sum_{j+l+k=0} i(l'_1u_j + l'_2v_j + l'_3w_j)v^\sharp_l v^\flat_k + \sum_{j+l+k=0} i(l'_1u_j + l'_2v_j + l'_3w_j)T^\sharp_l T^\flat_k \\ &= \text{using the fact that, by definition, } w_j = 0 \text{ for } j_3 = 0 \ (w \text{ is odd in } x_3) \\ &= \sum_{j+l+k=0} i[(l'_1 - \delta_{j,1}l'_3)u_j + (l'_2 - \delta_{j,2}l'_3)v_j](u^\sharp_l u^\flat_k + v^\sharp_l v^\flat_k + \kappa T^\sharp_l T^\flat_k). \end{aligned} \tag{3.7}$$

We then compute:

$$\begin{aligned} &((-\Delta)^{1/2}B(U,U^\sharp),(-\Delta)^{3/2}U^\flat)_\tau \\ &\quad = \sum_{j+l+k=0} i[(l'_1 - \delta_{j,1}l'_3)u_j + (l'_2 - \delta_{j,2}l'_3)v_j]e^{2\tau|k'|}|k'|^4(u^\sharp_l u^\flat_k + v^\sharp_l v^\flat_k + \kappa T^\sharp_l T^\flat_k). \end{aligned} \tag{3.8}$$

We associate to each function u, a function $\check{u}$ defined by:

$$\check{u} = \sum_{j\in\mathbb{Z}^3} \check{u}_j e^{i(j'_1x + j'_2y + j'_3z)}, \text{ where } \check{u}_j = e^{\tau|j'|}|u_j|; \tag{3.9}$$

we also use similar notations for the other functions.

Since all the terms are similar, we need only to estimate the first sum from (3.8), denoted by I. We find:

$$|I| \leq \sum_{j+k+l=0} |l'||j'||k'|^4 e^{2\tau|k'|}|u_j||u_l^\sharp||u_k^\flat|, \tag{3.10}$$

where we used the estimate $|l'_1 - \delta_{j',1} l'_3| \leq (L_3/2\pi)|j'||l'|$. Since $j+k+l=0 \Longleftrightarrow j'+k'+l'=0$, we find $|k'|-|l'|-|j'| \leq 0$ and we have:

$$\begin{aligned} |I| &\leq \sum_{j+k+l=0} |l'||j'|(|l'|+|j'|)|k'|^3 \check{u}_j \check{u}_l^\sharp \check{u}_k^\flat \\ &\leq \sum_{j+k+l=0} |j'||l'|^2|k'|^3 \check{u}_j \check{u}_l^\sharp \check{u}_k^\flat + \sum_{j+k+l=0} |j'|^2|l'||k'|^3 \check{u}_j \check{u}_l^\sharp \check{u}_k^\flat \\ &= \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} q_1(x) q_2^\sharp(x) q_3^\flat(x)\, \mathrm{d}\mathcal{M} + \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} q_2(x) q_1^\sharp(x) q_3^\flat(x)\, \mathrm{d}\mathcal{M}, \end{aligned} \tag{3.11}$$

where we wrote:

$$\begin{aligned} &q_1(x) = \sum_{j\in\mathbb{Z}^3} |j'|\check{u}_j e^{i(j'_1x+j'_2y+j'_3z)}, \quad q_2(x) = \sum_{j\in\mathbb{Z}^3} |j'|^2\check{u}_j e^{i(j'_1x+j'_2y+j'_3z)}, \\ &q_3(x) = \sum_{j\in\mathbb{Z}^3} |j'|^3\check{u}_j e^{i(j'_1x+j'_2y+j'_3z)}, \end{aligned} \tag{3.12}$$

and the definitions for $q_i^\sharp$ and $q_i^\flat$ for $i=1,2,3$ are the similar ones.

Using the Hölder and the imbedding inequalities, we find:

$$\begin{aligned} |I| &\leq |q_1|_{L^6}|q_2^\sharp|_{L^3}|q_3^\flat|_{L^2} + |q_2|_{L^3}|q_1^\sharp|_{L^6}|q_3^\flat|_{L^2} \\ &\leq c|q_1|_{H^1}|q_2^\sharp|_{L^2}^{1/2}|q_2^\sharp|_{H^1}^{1/2}|q_3^\flat|_{L^2} + c|q_2|_{L^2}^{1/2}|q_2|_{H^1}^{1/2}|q_1^\sharp|_{H^1}|q_3^\flat|_{L^2} \\ &\leq c|\Delta U|_\tau|\Delta U^\sharp|_\tau^{1/2}|(-\Delta)^{3/2}U^\sharp|_\tau^{1/2}|(-\Delta)^{3/2}U^\flat|_\tau \\ &\qquad + c|\Delta U|_\tau^{1/2}|(-\Delta)^{3/2}U|_\tau^{1/2}|\Delta U^\sharp|_\tau|(-\Delta)^{3/2}U^\flat|_\tau. \end{aligned} \tag{3.13}$$

Analogue estimates for the other terms, yield Lemma 3.3.1.

A priori estimates for the real case

We first derive some a priori estimates in the real-time case and then we consider the complex-time case. In all that follows we assume that the forcing term F is analytic in time with values in the Gevrey space $D(e^{\sigma_1(-\Delta)^{1/2}}(-\Delta)^{1/2})$, for some $\sigma_1 > 0$, and $U_0 \in D(-\Delta)$. Setting $\varphi(t) = \min(t, \sigma_1)$, we apply the operator $e^{\varphi(t)(-\Delta)^{1/2}}\Delta$ to equation (1.30), then we take the scalar product in H with $e^{\varphi(t)(-\Delta)^{1/2}}\Delta U$.

Since $a+e$ is coercive, we have:

$$\begin{aligned} &(e^{\varphi(t)(-\Delta)^{1/2}}\Delta AU, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H + (e^{\varphi(t)(-\Delta)^{1/2}}\Delta EU, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H \\ &\quad = a(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U) + e(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U) \\ &\quad \geq c_1|e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U|_H^2. \end{aligned}$$

For the bilinear term, we apply Lemma 3.3.1 and find:

$$\begin{aligned} |(e^{\varphi(t)(-\Delta)^{1/2}}\Delta B(U,U), e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H| &\leq c_2|\Delta U|^{3/2}_{\varphi(t)}|(-\Delta)^{3/2}U|^{3/2}_{\varphi(t)} \\ &\leq \frac{c_1}{4}|(-\Delta)^{3/2}U|^2_{\varphi(t)} + c_3|\Delta U|^6_{\varphi(t)}. \end{aligned} \tag{3.14}$$

For the term containing the time derivative of U, we have:

$$\begin{aligned} &(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U'(t), e^{\varphi(t)(-\Delta)^{1/2}}\Delta U(t))_H \\ &\quad = \big(\frac{\mathrm{d}}{\mathrm{d}t}(e^{\varphi(t)(-\Delta)^{1/2}}\Delta U), e^{\varphi(t)(-\Delta)^{1/2}}\Delta U\big)_H \\ &\qquad - \varphi'(t)(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U)_H \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U|^2_H - \varphi'(t)(e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U, e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)U)_H \\ &\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U|^2_H - |e^{\varphi(t)(-\Delta)^{1/2}}(-\Delta)^{3/2}U|_H|e^{\varphi(t)(-\Delta)^{1/2}}\Delta U|_H \\ &\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|^2_{\varphi(t)} - \frac{c_1}{4}|(-\Delta)^{3/2}U|^2_{\varphi(t)} - \frac{1}{c_1}|\Delta U|^2_{\varphi(t)}. \end{aligned}$$

The term containing the force F is estimated as:

$$\begin{aligned} (e^{\varphi(t)(-\Delta)^{1/2}}\Delta F, e^{\varphi(t)(-\Delta)^{1/2}}\Delta U)_H &\leq |(-\Delta)^{1/2}F|_{\varphi(t)}|(-\Delta)^{3/2}U|_{\varphi(t)} \\ &\leq \frac{1}{c_1}|(-\Delta)^{1/2}F|^2_{\varphi(t)} + \frac{c_1}{4}|(-\Delta)^{3/2}U|^2_{\varphi(t)}. \end{aligned} \tag{3.15}$$

Gathering all these estimates, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|^2_{\varphi(t)} + c_1|(-\Delta)^{3/2}U|^2_{\varphi(t)} \leq \frac{2}{c_1}|\Delta U|^2_{\varphi(t)} + c_2'|\Delta U|^6_{\varphi(t)} + c_3'|(-\Delta)^{1/2}F|^2_{\varphi(t)}. \tag{3.16}$$

We consider the function $y(t) = 1 + |\Delta U|^2_{\varphi(t)}$. Since

$$|(-\Delta)^{1/2}F|^2_{\varphi(t)} \leq |(-\Delta)^{1/2}F|^2_{\sigma_1},$$

we find, for any $t_1 > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) \leq c_4 y^3(t), \qquad 0 < t < t_1, \tag{3.17}$$

where c_4 is a constant depending on the norm of F in $L^\infty(0,t_1;D((-\Delta)^{1/2}e^{\sigma_1(-\Delta)^{1/2}}))$.

We easily deduce that there exists a time $t'_\star$, $0 < t'_\star \leq t_1$, $t'_\star = t'_\star(F,U_0) = 3/8y^2(0)c_4$, such that $y(t) \leq 2y(0)$ for all $0 \leq t \leq t'_\star(F,U_0)$. We then obtain the following a priori estimate:

$$|\Delta U(t)|^2_{\varphi(t)} \leq 1 + 2|\Delta U_0|^2_H, \quad \forall\, t \leq t'_\star(F,U_0). \tag{3.18}$$

A priori estimates for the complex case

In order to prove that the solution is analytic in time and coincides with the restriction of a complex function in time to a real positive interval, we consider equation (1.30) with a complex time $\zeta \in \mathbb{C}$, and U a complex function. We take the complexified spaces H and V denoted $H_{\mathbb{C}}$ and $V_{\mathbb{C}}$ [1], so equation (1.30) is rewritten as:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}\zeta} + AU + B(U,U) + E(U) = F, \\ &U(0) = U_0, \end{aligned} \tag{3.19}$$

where $\zeta \in \mathbb{C}$ is the complex time.

We consider ζ of the form $\zeta = se^{i\theta}$, where $s > 0$ and $\cos\theta > 0$ so that the real part of ζ is positive. We apply $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta$ to equation (3.19) and take the scalar product in $H_{\mathbb{C}}$ with $e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U$. We then multiply the resulting equation by $e^{i\theta}$ and take the real part. We find:

$$\begin{aligned} &\mathrm{Re}\, e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta\frac{\mathrm{d}U}{\mathrm{d}\zeta}, \Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U)_H \\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U|_H^2 \\ &\qquad + \varphi'(s\cos\theta)\cos\theta\,\mathrm{Re}\, e^{i\theta}(\Delta e^{\varphi(s\cos\theta)(-\Delta)^{3/2}}U, \Delta e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}U)_H \\ &\quad \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\Delta U|^2_{\varphi(s\cos\theta)} - \cos\theta|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)}|\Delta U|_{\varphi(s\cos\theta)}. \end{aligned} \tag{3.20}$$

Since $a + e$ is coercive for our choice of κ, we also find:

$$\begin{aligned} &\mathrm{Re}\, e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta AU, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U)_H \\ &\quad + \mathrm{Re}\, e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta EU, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U)_H \\ &\quad \geq c_1\cos\theta|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}(-\Delta)^{3/2}U|_H^2 = c_1\cos\theta|(-\Delta)^{3/2}U|^2_{\varphi(s\cos\theta)}. \end{aligned} \tag{3.21}$$

For the forcing term, we write:

$$\begin{aligned} &|\mathrm{Re}\, e^{i\theta}(e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta F, e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U)_H| \leq |(-\Delta)^{1/2}F|_{\varphi(s\cos\theta)}|(-\Delta)^{3/2}U|_{\varphi(s\cos\theta)} \\ &\qquad \leq \frac{c_1}{6}\cos\theta|(-\Delta)^{3/2}U|^2_{\varphi(s\cos\theta)} + \frac{1}{c_1\cos\theta}|(-\Delta)^{1/2}F|^2_{\varphi(s\cos\theta)}. \end{aligned} \tag{3.22}$$

For the bilinear term B we use Lemma 3.3.1 and the Young inequality:

$$\begin{aligned} |\mathrm{Re}\, e^{i\theta}(\Delta B(U,U), \Delta U)_{\varphi(s\cos\theta)}| &\leq c_2|\Delta U|^{3/2}_{\varphi(s\cos\theta)}|(-\Delta)^{3/2}U|^{3/2}_{\varphi(s\cos\theta)} \\ &\leq \frac{c_1}{6}\cos\theta|(-\Delta)^{3/2}U|^2_{\varphi(s\cos\theta)} + \frac{c_3}{(\cos\theta)^3}|\Delta U|^6_{\varphi(s\cos\theta)}. \end{aligned} \tag{3.23}$$

[1]For the scalar products and the norms we use the same notations as in the real case.

Gathering all the estimates above, we find the following differential inequality:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\Delta U|^2_{\varphi(s\cos\theta)} + \frac{c_1}{2}\cos\theta|(-\Delta)^{3/2}|^2_{\varphi(s\cos\theta)} &\leq \frac{1}{c_1\cos\theta}|(-\Delta)^{1/2}F|^2_{\varphi(s\cos\theta)} \\
&+ \frac{\cos\theta}{c_1}|\Delta U|^2_{\varphi(s\cos\theta)} + \frac{c_3}{(\cos\theta)^3}|\Delta U|^6_{\varphi(s\cos\theta)}.
\end{aligned}
\tag{3.24}
$$

We restrict θ such that $\sqrt{2}/2 \leq \cos\theta \leq 1$ (in fact we can restrict θ to any domain such that $\cos\theta \geq c > 0$). Writing

$$y(s) = 1 + |\Delta U(s)|^2_{\varphi(s\cos\theta)},$$

the differential inequality (3.24) becomes:

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s} \leq c(F)y^3(s), \quad 0 < s < t_1, \tag{3.25}$$

where $c(F)$ is a constant depending as before on the forcing term F. Therefore, by similar reasoning as for the real case, we find that there exists a time $t'_\star$, $0 < t'_\star \leq t_1$, $t'_\star = t'_\star(F, U_0)$ such that:

$$|e^{\varphi(s\cos\theta)(-\Delta)^{1/2}}\Delta U(se^{i\theta})|^2_H \leq 1 + 2|\Delta U_0|^2_H, \quad \forall\, 0 \leq s \leq t'_\star(F, U_0). \tag{3.26}$$

Considering the complex region

$$\mathcal{D}(U_0, F, \sigma_1) = \{\zeta = se^{i\theta},\ |\theta| \leq \pi/4,\ 0 < s < t'_\star(F, U_0)\}, \tag{3.27}$$

estimate (3.26) gives us a bound for $U(\zeta)$, when $\zeta \in \mathcal{D}(U_0, F, \sigma_1)$.

We can now state the main result of this section:

Theorem 3.3.1. *Let U_0 be given in $\dot{H}^2_{\text{per}}(\mathcal{M})$ and let F be a given function analytic in time with values in $D(e^{\sigma_1(-\Delta)^{1/2}}(-\Delta)^{1/2})$ for some $\sigma_1 > 0$. Then there exists $t'_\star > 0$ depending on the data, including U_0, and a unique solution U of (1.30) on $(0, t'_\star)$ such that the function*

$$t \rightarrow \Delta e^{\varphi(t)(-\Delta)^{1/2}}U(t),$$

is analytic from $(0, t'_\star)$ with values in H, where $\varphi(t) = \min(t, \sigma_1)$ and $t'_\star$ was defined above.

Proof: The proof is based on the a priori estimates obtained above and the use of the Galerkin–Fourier method; see e.g. [4]. The solutions of the Galerkin approximation satisfy rigorously the estimates formally derived above, and the bounds are independent of the order m of the Galerkin approximation. We can then pass to the limit $m \rightarrow \infty$, using classical results on the convergence of analytic functions.

Remark 3.3.1. Taking into account the second part of Theorem 3.2.3, we see that the result of Theorem 3.3.1 still holds true while starting with initial data $U_0 \in V$, since at arbitrarily small time t the solution U satisfies $U(t) \in \dot{H}^2_{\text{per}}(\mathcal{M})$ and $U \in \mathcal{C}((0, t_{\star\star}]; \dot{H}^2_{\text{per}}(\mathcal{M}))$.

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Chapitre 4

Renormalization Group Method Applied to the Primitive Equations

La méthode de la renormalisation appliquée aux Equations Primitives

Ce chapitre est constitué de l'article **Renormalization Group Method Applied to the Primitive Equations**, ecrit en collaboration avec R. Temam et D. Wirosoetisno, article paru en 2005 dans le *Journal of Differential Equations*, volume 208, pages 215-257. Dans cet article on étudie le comportement asymptotique des Equations Primitives en dimension deux, quand le nombre de Rossby tend vers zero. Le modèle considéré ici est le modèle deja apparu dans le premier chapitre. En utilisant une méthode de renormalisation decrite dans le corps de l'article, on montre que l'on peut éliminer les oscillations de la solution exacte et on obtient ainsi une bonne approximation pour la solution du système original.

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Renormalization Group Method Applied to the Primitive Equations

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Abstract: In this article we study the limit, as the Rossby number ε goes to zero, of the Primitive Equations of the atmosphere and the ocean. From the mathematical viewpoint we study the averaging of a penalisation problem displaying oscillations generated by an antisymmetric operator and by the presence of two time scales.

4.1 Introduction

The study of the limit, as the Rossby number ε goes to zero, of the equations of the atmosphere and the oceans is a major physical and computational problem to which much effort has been devoted. In a more mathematical context, this problem is related to the averaging of oscillations using renormalization and other averaging procedures.

In the mathematical literature, an important contribution is due to Schochet [18] who tackled similar problems by studying an asymptotics in the fast time variable; such problems have also been studied in the general framework of wave equations by Joly, Rauch, Metivier [9], Grenier [8], and Gallagher [7]. For the equations of the atmosphere and the ocean, mathematical work includes the following: Embid and Majda [6], Babin, Mahalov, and Nicolaenko (see, e.g., [1], [3]) or Warn, Bokhove, Shepherd, and Vallis [23]. Many more articles on the subject are available in the physics and mathematical literature.

In the mathematical physics literature, a number of averaging problems and procedures have been studied or proposed; see e.g., the article [10] by van Kampen on the elimination of fast variables, or the averaging procedure by Bogolyubov and Mitropolsky [4]. Our work follows more closely the approach, based on renormalization theory, of Chen, Oono, and Goldenfeld [5] revisited by Ziane [25]. Here we also extend to infinite dimension part of the work by Temam and Wirosoetisno [22] valid in finite dimension.

As we said, the renormalization method that we use here was introduced in [5] and [25]. It was then applied to different types of partial differential equations by Moise, Temam, and Ziane (see [14], [15]); the method was also applied to ordinary differential equations (see e.g., [13], [21], [25]).

This article is organized as follows: In the first part of Section 4.2 (Subsection 4.2.1), we present the PEs and recall a few facts on their mathematical setting, some well-known, and some borrowed from a companion paper [17]. In the second part of Section 4.2 (Subsection 4.2.2), we recall a few facts about renormalization following [5], [15], [25], [21]. In Section 4.3 we study the properties of the renormalized system, starting with the existence of weak solutions and ending the section with the existence of very regular solutions. In Section 4.4 we show that we can approximate the exact solution of the primitive equations by an asymptotic solution which exists for all times and we estimate the difference between the exact and asymptotic solutions. We end the paper with three appendices: in Section 4.5 we give the details of the derivation of the renormalized system, in Section 4.6.1 we give a result of number theory needed in Section 4.4 to bound some small denominators necessary for the error estimates, and in Section 4.6.2 we present an alternate method for bounding the small denominators.

4.2 The Initial and Renormalized Problems

In Section 4.2.1 we recall the Primitive Equations in a form suitable for our study. In Section 4.2.2 we recall a few facts about renormalization.

4.2.1 The PEs in Space Dimension Two

We work in the two-dimensional space and consider the domain

$$\mathcal{M} = (0,\, L_1) \times (-L_3/2,\, L_3/2),$$

$0x$ being the west–east direction, and $0z$ being the vertical direction. All the quantities depend only on x, z and t. We consider the PEs written in the nondimensional form (2.1) below; a description of the derivation of these equations and a study concerning the existence and regularity of their solutions is given in Petcu, Temam and Wirosoetisno

[17]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} - \frac{1}{\varepsilon}v + \frac{1}{\varepsilon}\frac{\partial p}{\partial x} = \nu_{\boldsymbol{v}}\,\Delta u + S_u, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} + \frac{1}{\varepsilon}u = \nu_{\boldsymbol{v}}\,\Delta v + S_v, \tag{2.1b}$$

$$\frac{\partial p}{\partial z} = -N\rho, \tag{2.1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.1d}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} - \frac{N}{\varepsilon}w = \nu_{\rho}\,\Delta\rho + S_{\rho}. \tag{2.1e}$$

Here u, v, w are the non-dimensional components of the three dimensional velocity vector, p is the pressure, ρ is the density and ε is the Rossby number. In the more physical situation, the source terms S_u, S_v, and S_ρ usually vanish; they are introduced here for mathematical generality. Here $\nu_{\boldsymbol{v}}$ and ν_ρ are the non-dimensional eddy viscosity coefficients, N is the Burgers number, and we set $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$. In the physical problem, the total pressure is

$$p_{\text{full}} = p_{\text{ref}} + \bar{p} + p',$$

and the total density is

$$\rho_{\text{full}} = \rho_{\text{ref}} + \bar{\rho} + \rho'.$$

Here p_{ref} is a hydrostatic pressure corresponding to the reference value of the density ρ_{ref}, $\bar{\rho}$ is the density stratification profile which is linear in z and $\bar{p}$ is the pressure in hydrostatic equilibrium with it; p' and ρ' are perturbations from these states. In (2.1) we do not work with the total pressure and the total density but with the perturbations p' and ρ' where the primes were dropped and ρ' has been replaced by ρ'/N. See [17] for more details regarding the derivation of this system.

We also assume that all the unknown functions are $\mathcal{M}$-periodic. The prognostic variables of this system are u, v, ρ and the diagnostic variables are p, w; as we will see below, p and w can, at each instant of time, be (essentially) determined in terms of the prognostic variables.

We recall that an $\mathcal{M}$-periodic function

$$u = \sum_{(k_1,\, k_3)\in\mathbb{Z}^2} u_{(k_1,\, k_3)} e^{2\pi i(k_1 x/L_1 + k_3 z/L_3)},$$

is in $H^m_{\text{per}}(\mathcal{M})$, $m > 0$, if and only if

$$\sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^m |u_k|^2 < \infty,$$

where we denoted by k the pair $(k_1,\, k_3)$. We denote by $\dot{H}^m_{\text{per}}(\mathcal{M})$ the functions from $H^m_{\text{per}}(\mathcal{M})$ with average zero on $\mathcal{M}$. In order to simplify the writing we will also set $k_i' = 2\pi k_i/L_i$. We easily notice that if (u, v, ρ, w, p) is a solution of (2.1) for $S = (S_u,\, S_v,\, S_\rho)$, then $(\tilde{u}, \tilde{v}, \tilde{\rho}, \tilde{w}, \tilde{p})$ is also a solution of (2.1) for $\tilde{S}_u$, $\tilde{S}_v$, $\tilde{S}_\rho$ where:

$$\begin{aligned} \tilde{u}(x,\, z,\, t) &= u(x,\, -z,\, t), & \tilde{p}(x,\, z,\, t) &= p(x,\, -z,\, t),\\ \tilde{v}(x,\, z,\, t) &= v(x,\, -z,\, t), & \tilde{S}_u(x,\, z,\, t) &= S_u(x,\, -z,\, t),\\ \tilde{w}(x,\, z,\, t) &= -w(x,\, -z,\, t), & \tilde{S}_v(x,\, z,\, t) &= S_v(x,\, -z,\, t),\\ \tilde{\rho}(x,\, z,\, t) &= -\rho(x,\, -z,\, t), & \tilde{S}_\rho(x,\, z,\, t) &= -S_\rho(x,\, -z,\, t). \end{aligned}$$

Hence, assuming that S_u and S_v are even in z, and that S_ρ is odd in z,

$$\begin{aligned} S_u(x,z,t) &= S_u(x,-z,t),\\ S_v(x,z,t) &= S_v(x,-z,t),\\ S_\rho(x,z,t) &= -S_\rho(x,-z,t), \end{aligned}$$

it is natural to look for a solution where u, v and p are even in z and ρ, w odd in z,

$$\begin{aligned} u(x,z,t) &= u(x,-z,t), & w(x,z,t) &= -w(x,-z,t),\\ v(x,z,t) &= v(x,-z,t), & p(x,z,t) &= p(x,-z,t),\\ \rho(x,z,t) &= -\rho(x,-z,t). \end{aligned}$$

For more details regarding the motivation of this choice (symmetry and periodicity) we refer the reader to [17].

In accordance with these requirements of symmetry and periodicity, we introduce the following function spaces:

$$\begin{aligned} \boldsymbol{V} &= \{(u,\, v,\, \rho) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3;\; u,\, v \text{ even in } z,\, \rho \text{ odd in } z,\, u_{(k_1,0)} = 0,\, \forall\, k_1 \in \mathbb{Z}\},\\ \boldsymbol{H} &= \text{ the closure of } \boldsymbol{V} \text{ in } (\dot{L}^2(\mathcal{M}))^3,\\ \boldsymbol{V}_2 &= \text{ the closure of } \boldsymbol{V} \cap (\dot{H}^2_{\text{per}}(\mathcal{M}))^3 \text{ in } (\dot{H}^2_{\text{per}}(\mathcal{M}))^3. \end{aligned}$$

The condition $u_{(k_1,0)} = 0$, $\forall\, k_1$, expresses the condition (2.3) appearing below.

We can express the diagnostic variables w and p in terms of the prognostic variables u, v, and ρ. For each $U = (u, v, \rho) \in \boldsymbol{V}$ we can determine uniquely

$$w = w(U) = -\int_0^z u_x(x, z', t)\, \mathrm{d}z'. \tag{2.2}$$

Note that $w = 0$ at $z = 0$ and $L_3/2$ by the requirements of w (periodicity and antisymmetry); see more details in [17]. By (2.2), the fact that $w = 0$ at $z = L_3/2$ gives the constraint on u

$$\int_{-L_3/2}^{L_3/2} u_x\, \mathrm{d}z = 0. \tag{2.3}$$

As for the pressure, it can be determined uniquely in terms of ρ up to p_{s}, writing

$$p(x,z,t) = p_{\mathrm{s}}(x,t) - \int_0^z \rho(x, z', t)\, \mathrm{d}z'.$$

For U, $\tilde{U} \in \boldsymbol{V}$, we set

$$((U,\tilde{U})) = ((u,\tilde{u})) + ((v,\tilde{v})) + ((\rho,\tilde{\rho})), \quad \|U\| = ((U,U))^{1/2}. \tag{2.4}$$

where we have written $\mathrm{d}\mathcal{M}$ for $\mathrm{d}x\,\mathrm{d}z$, and

$$((\phi,\, \tilde{\phi})) = \int_{\mathcal{M}} \Big(\frac{\partial\phi}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial\tilde{\phi}}{\partial z}\Big)\,\mathrm{d}\mathcal{M}. \tag{2.5}$$

By the Poincaré inequality,

$$|U|_{L^2} \leq c_0\|U\|, \quad \forall\, U \in \boldsymbol{V}, \tag{2.6}$$

so that $\|\cdot\|$ is a Hilbert norm on $\boldsymbol{V}$.

The space $\boldsymbol{H}$ is endowed with the usual scalar product of $(L^2(\mathcal{M}))^3$.

Variational Formulation of the Problem

We introduce the following forms:

$$\begin{aligned}
a(U,\tilde{U}) &= \nu_v((u,\,\tilde{u})) + \nu_v((v,\,\tilde{v})) + \nu_\rho((\rho,\,\tilde{\rho})),\\
e(U,\tilde{U}) &= \int_{\mathcal{M}} (-v\,\tilde{u} + u\,\tilde{v})\,\mathrm{d}\mathcal{M} + N\int_{\mathcal{M}} \rho\tilde{w}\,\mathrm{d}\mathcal{M} - N\int_{\mathcal{M}} w\tilde{\rho}\,\mathrm{d}\mathcal{M},\\
b(U,U^\sharp,\tilde{U}) &= \int_{\mathcal{M}}\Big(u\frac{\partial u^\sharp}{\partial x} + w(U)\frac{\partial u^\sharp}{\partial z}\Big)\tilde{u}\,\mathrm{d}\mathcal{M} + \int_{\mathcal{M}}\Big(u\frac{\partial v^\sharp}{\partial x} + w(U)\frac{\partial v^\sharp}{\partial z}\Big)\tilde{v}\,\mathrm{d}\mathcal{M}\\
&\quad + \int_{\mathcal{M}}\Big(u\frac{\partial \rho^\sharp}{\partial x} + w(U)\frac{\partial \rho^\sharp}{\partial z}\Big)\tilde{\rho}\,\mathrm{d}\mathcal{M}.
\end{aligned}$$

The variational form of the problem is obtained by multiplying (2.1a), (2.1b), (2.1e), by u, v and ρ respectively, integrating over $\mathcal{M}$ and adding the resulting equations. After some easy calculations we arrive at this problem:

Given $t_\star > 0$ arbitrary, $U_0 \in \boldsymbol{H}$ and $S = (S_u,\, S_v,\, S_\rho) \in L^2(0,\, t_\star; \boldsymbol{H})$, we look for a function U from $(0,\, t_\star)$ into $\boldsymbol{V}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U})_{\boldsymbol{H}} + a(U,\tilde{U}) + b(U,U,\tilde{U}) + \frac{1}{\varepsilon}e(U,\tilde{U}) = (S,\tilde{U})_{\boldsymbol{H}}, \quad \forall\, \tilde{U} \in \boldsymbol{V}, \tag{2.7}$$

and

$$U(0) = U_0. \tag{2.8}$$

We also define the linear operators

$$A : \boldsymbol{V} \to \boldsymbol{V}', \quad \langle AU,\tilde{U}\rangle_{\boldsymbol{V}',\boldsymbol{V}} = a(U,\tilde{U}), \quad \forall\, U,\tilde{U} \in \boldsymbol{V}', \tag{2.9}$$

$$L : \boldsymbol{V} \to \boldsymbol{V}', \quad \langle LU,\tilde{U}\rangle_{\boldsymbol{V}',\boldsymbol{V}} = e(U,\tilde{U}), \quad \forall\, U,\tilde{U} \in \boldsymbol{V}, \tag{2.10}$$

and the bilinear form

$$B : \boldsymbol{V} \times \boldsymbol{V}_2 \to \boldsymbol{V}', \quad \langle B(U,\tilde{U}),U^\sharp\rangle_{\boldsymbol{V}',\boldsymbol{V}} = b(U,\tilde{U},U^\sharp), \quad \forall\, U,U^\sharp \in \boldsymbol{V},\, \tilde{U} \in \boldsymbol{V}_2, \tag{2.11}$$

where $\boldsymbol{V}'$ denotes the dual space of $\boldsymbol{V}$; it is shown in [17] that b is trilinear continuous on $\boldsymbol{V} \times \boldsymbol{V}_2 \times \boldsymbol{V}$ and $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}_2$ so that B is bilinear continuous from $\boldsymbol{V} \times \boldsymbol{V}_2$ into $\boldsymbol{V}'$ and from $\boldsymbol{V} \times \boldsymbol{V}$ into $\boldsymbol{V}'_2$.

Then problem (2.7) with initial condition (2.8) is equivalent to the abstract evolution equation:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U,U) + \frac{1}{\varepsilon}LU = S, \quad \text{in } \boldsymbol{V}'_2, \\ &U(0) = U_0. \end{aligned} \tag{2.12}$$

Regarding the existence and uniqueness of solutions of (2.7) we recall from [17] the following result:

Theorem 4.2.1. *Given $U_0 \in \boldsymbol{H}$ and $S \in L^\infty(\mathbb{R}_+; \boldsymbol{H})$, there exists at least one solution U of equation (2.7) with initial condition (2.8) such that*

$$U \in L^\infty(\mathbb{R}_+; \boldsymbol{H}) \cap L^2(0,\, t_\star;\, \boldsymbol{V}), \ \text{for all } t_\star > 0. \tag{2.13}$$

If $U_0 \in \boldsymbol{V}$ and $S \in L^\infty(\mathbb{R}_+; \boldsymbol{H})$, there exists a unique solution U of (2.7)–(2.8) such that

$$U \in L^\infty(\mathbb{R}_+; \boldsymbol{V}) \cap L^2(0,\, t_\star;\, (\dot{H}^2_{\mathrm{per}}(\mathcal{M}))^3), \quad \forall\, t_\star > 0.$$

Moreover, for all $m \in \mathbb{N}$, $m \geq 2$, if $U_0 \in (\dot{H}^m_{\mathrm{per}}(\mathcal{M}))^3$ and $S \in L^\infty(\mathbb{R}_+; (\dot{H}^{m-1}_{\mathrm{per}}(\mathcal{M}))^3)$, then $U \in L^\infty(\mathbb{R}_+; (\dot{H}^m_{\mathrm{per}}(\mathcal{M}))^3) \cap L^2(0,\, t_\star;\, (\dot{H}^{m+1}_{\mathrm{per}}(\mathcal{M}))^3)$, $\forall\, t_\star > 0$.

4.2.2 Asymptotics and Renormalization Group Method

The aim of this article is to present an application of the renormalization group method (RG) to the 2D primitive equations described above. The RG method gives us an algorithm for finding approximate (averaged) solutions for a general equation of the form:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{1}{\varepsilon}LU = \mathcal{F}(U), \\ &U(0) = U_0, \end{aligned} \tag{2.14}$$

where $\varepsilon > 0$ is a small parameter and L is an antisymmetric operator, so that the solutions of (2.14) display large oscillations for ε small. We assume that L is a diagonalizable, antisymmetric linear operator (not necessarily bounded) and $\mathcal{F}$ is a nonlinear operator. Two natural time scales (at least) are present in (2.14), the slow time t, and the fast time $s = t/\varepsilon$. To implement the RG method, we imagine a formal asymptotic expansion for equation (2.14) written in the fast time variable:

$$\begin{aligned} &\frac{\mathrm{d}\check{U}}{\mathrm{d}s} + L\check{U} = \varepsilon\mathcal{F}(\check{U}), \\ &\check{U}(0) = U_0, \end{aligned} \tag{2.15}$$

where we have set $\check{U}(s) = U(\varepsilon s)$. In what follows we drop the checks and the formal expansion is written:

$$U = U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + \cdots . \tag{2.16}$$

We formally substitute (2.16) into (2.15) and we find:

$$\frac{\mathrm{d}U^0}{\mathrm{d}s} + L\,U^0 = 0, \tag{2.17}$$

$$\frac{\mathrm{d}U^1}{\mathrm{d}s} + L\,U^1 = \mathcal{F}(U^0), \tag{2.18}$$

$$\frac{\mathrm{d}U^2}{\mathrm{d}s} + L\,U^2 = \nabla_U \mathcal{F}(U^0)\cdot U^1, \tag{2.19}$$

and so on.

The solution of (2.17) can be written

$$U^0(s) = e^{-Ls}U(0).$$

For equation (2.18) we apply the variation of constants formula and we obtain:

$$U^1(s) = e^{-Ls}\int_0^s e^{Ls'}\mathcal{F}(e^{-Ls'}U_0)\,\mathrm{d}s'. \tag{2.20}$$

For U^1 we choose the initial data to be zero, but other choices may be appropriate (see [21]).

We set $F(s,\cdot) = e^{Ls}\mathcal{F}(e^{-Ls}\cdot)$ and we split F into two parts: the resonant part $F_r(\cdot)$ corresponding to the time-independent part of $F(s,\cdot)$ and the remaining non-resonant part $F_n(s,\cdot)$. In our applications, $\mathcal{F}$ will be polynomial[1] in U and the definition of the time-independent part of F is not problematic. We thus have:

$$F(s,U) = F_r(U) + F_n(s,U), \tag{2.21}$$

and we define the primitive of the non-resonant part by

$$F_{np}(s,U) = \int_0^s F_n(s',U)\,\mathrm{d}s'. \tag{2.22}$$

Substituting these relations in (2.20) we find:

$$U^1(s) = e^{-Ls}\{sF_r(U_0) + F_{np}(s,U_0)\}. \tag{2.23}$$

The first order RG equation, as discussed in [21], is of the form:

$$\begin{aligned} \frac{\mathrm{d}\bar{U}}{\mathrm{d}s} &= \varepsilon F_r(\bar{U}), \\ \bar{U}(0) &= U_0. \end{aligned} \tag{2.24}$$

For the details, see e.g., [15], [18] and [21]. The first order approximate solution is defined by

$$\tilde{U}^1(s) = e^{-Ls}\{\bar{U}(s) + \varepsilon F_{np}(s,\bar{U}(s))\}, \tag{2.25}$$

[1] Here we call polynomial function a function of the form $\mathcal{F}(U) = \sum_{j=0}^{n}\mathcal{F}_j(U,\cdots,U)$, where n is finite arbitrary, and $\mathcal{F}_j$ is j-linear continuous on a suitable function space.

and it is shown, e.g., in [18], that $\tilde{U}^1 - U$ is of order ε in an interval of time s of order $\mathcal{O}(1/\varepsilon)$ and in an interval of time t of order $\mathcal{O}(1)$.

The renormalized system (2.24)–(2.25) gives us an $\mathcal{O}(\varepsilon)$ approximation to the exact solution over a timescale $t \sim \mathcal{O}(1)$ without having to solve an oscillatory differential equation. Because of the computational difficulties, in this article we only derive the first-order approximate solution but we can apply the method to higher-order approximate solutions as described in [21] in the context of ordinary differential equations.

In this article, the polynomial $\mathcal{F}$ is taken to be of the form:

$$\mathcal{F}(U) = S - A(U) - B(U,U),$$

where S is an external force, A is a linear coercive operator and B is a bilinear operator. In Section 4.5, we explicitly construct the resonant parts of A and B. We will see that the resonant parts of A and B have the same properties as the original operators; this does not seem to happen at higher orders. In Section 4.6.1 and Section 4.6.2 we give two different methods to handle the small denominators, one result being a typical number theory result and the other is a more particular result, the method following [3].

4.3 Description of the Renormalized System

We start this section by writing the initial system (2.1) in Fourier modes and by introducing a change of variables to facilitate the computation of the renormalized equation (Section 4.3.1). In the subsequent subsections we prove the existence of weak solutions (Section 4.3.2), of strong solutions (Section 4.3.3) and of even more regular solutions for the renormalized system (Section 4.3.4).

4.3.1 The Original Equations in Fourier Modes

We introduce the fast time $s = t/\varepsilon$ in the system (2.1). Abusing the notation, new functions depending on x, z and s are denoted in the same way as before. We obtain the following system:

$$\begin{aligned}
&\frac{\partial u}{\partial s} + \varepsilon u \frac{\partial u}{\partial x} + \varepsilon w \frac{\partial u}{\partial z} - v + \frac{\partial p}{\partial x} = \varepsilon \nu_{\boldsymbol{v}} \Delta u + \varepsilon S_u,\\
&\frac{\partial v}{\partial s} + \varepsilon u \frac{\partial v}{\partial x} + \varepsilon w \frac{\partial v}{\partial z} + u = \varepsilon \nu_{\boldsymbol{v}} \Delta v + \varepsilon S_v,\\
&\frac{\partial p}{\partial z} = -N\rho,\\
&u_x + w_z = 0,\\
&\frac{\partial \rho}{\partial s} + \varepsilon u \frac{\partial \rho}{\partial x} + \varepsilon w \frac{\partial \rho}{\partial z} - Nw = \varepsilon \nu_\rho \Delta \rho + \varepsilon S_\rho.
\end{aligned} \tag{3.1}$$

All the functions being periodic, they admit Fourier series expansions. Hence, for instance, for u we write

$$u = \sum_{(k_1,\, k_3) \in \mathbb{Z}^2} u_{(k_1,k_3)} e^{i(k'_1 x + k'_3 z)},$$

where $k'_j = 2\pi k_j/L_j$. Note here that, by periodicity of w, integration of the fourth equation of (3.1) yields

$$\int_{-L_3/2}^{L_3/2} u_x \, \mathrm{d}z = 0. \tag{3.2}$$

In Fourier series, this is equivalent to the condition $u_{(k_1,0)} = 0$ for all $k_1 \in \mathbb{Z}$, which appears in the definition of the space $\boldsymbol{V}$. The fact that w is odd in z implies that $w_{(k_1,0)} = 0$, for all k_1. We use these properties in what follows.

We hereby assume that S_u, S_v, S_ρ are functions independent of time.

With primes denoting $\partial/\partial s$, we can write the system (3.1) in Fourier modes as follows

$$\begin{aligned}
&u'_k + \varepsilon \sum_{j+l=k} (il'_1 u_j u_l + il'_3 w_j u_l) - v_k + ik'_1 p_k = -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 u_k + \varepsilon S_{u,k},\\
&v'_k + \varepsilon \sum_{j+l=k} (il'_1 u_j v_l + il'_3 w_j v_l) + u_k = -\varepsilon \nu_{\boldsymbol{v}} |k'|^2 v_k + \varepsilon S_{v,k},\\
&ik'_3 p_k = -N\rho_k,\\
&k'_1 u_k + k'_3 w_k = 0,\\
&\rho'_k + \varepsilon \sum_{j+l=k} (il'_1 u_j \rho_l + il'_3 w_j \rho_l) - N w_k = -\varepsilon \nu_\rho |k'|^2 \rho_k + \varepsilon S_{\rho,k}.
\end{aligned} \tag{3.3}$$

The zeroth order system

We now make explicit for our problem the solution of the linear zeroth order equation (2.17), whose solution will be used later on in the variation of constants formulas and in particular in the analogue of (2.20). With the same notation as before and with $U = (u, v, \rho)$, we have

$$\begin{aligned}
&u'_k - v_k + ik'_1 p_k = 0,\\
&v'_k + u_k = 0,\\
&ik'_3 p_k = -N\rho_k,\\
&k'_1 u_k + k'_3 w_k = 0,\\
&\rho'_k - N w_k = 0.
\end{aligned} \tag{3.4}$$

For $k_3 = 0$, we have $u_{(k_1,0)} = 0$, $w_{(k_1,0)} = 0$ and $\rho_{(k_1,0)} = 0$ from the definition of the space $\boldsymbol{V}$, so only the first two lines of system (3.4) are nontrivial:

$$\begin{aligned}
&- v_k + ik'_1 p_k = 0,\\
&v'_k = 0.
\end{aligned} \tag{3.5}$$

This gives us $v_{(k_1,0)}(s) = v_{(k_1,0)}(0)$ and (3.5) allows us to express p_k in terms of v_k.

For $k_3 \neq 0$ we can express the k-component of the diagnostic variables in terms of the prognostic variables:

$$p_k = -\frac{N}{ik'_3} \rho_k, \tag{3.6}$$

$$w_k = -\delta_k u_k, \tag{3.7}$$

where for notational conciseness we have set

$$\delta_k = \frac{k_1'}{k_3'} \text{ if } k_3' \neq 0, \text{ and } \delta_k = 0 \text{ if } k_3' = 0. \tag{3.8}$$

Substituting (3.6) and (3.7) in (3.4) we find:

$$\begin{aligned} &u_k' - v_k - \delta_k N \rho_k = 0, \\ &v_k' + u_k = 0, \\ &\rho_k' + \delta_k N u_k = 0. \end{aligned} \tag{3.9}$$

To solve this system we introduce the following change of unknowns suggested by the diagonalization of system (3.9). We set:

$$n_k = \frac{1}{\beta_k} v_k + \frac{\delta_k N}{\beta_k} \rho_k = (v_k,\ \rho_k) \cdot \vec{\phi}_k, \tag{3.10}$$

where we denoted

$$\beta_k = \left(1 + \delta_k^2 N^2\right)^{1/2}, \tag{3.11}$$

and

$$\vec{\phi}_k = \Big(\frac{1}{\beta_k},\ \frac{\delta_k N}{\beta_k}\Big). \tag{3.12}$$

We also define the following vector:

$$\vec{\gamma}_k = \Big(-\frac{\delta_k N}{\beta_k},\ \frac{1}{\beta_k}\Big). \tag{3.13}$$

and we set $m_k = (v_k,\ \rho_k) \cdot \vec{\gamma}_k$. For notational conciseness we also set

$$\vec{m}_k = m_k \vec{\gamma}_k, \quad \vec{n}_k = n_k \vec{\phi}_k. \tag{3.14}$$

Note that $\vec{\phi}_k = (1,0)$ and $\vec{\gamma}_k = (0,1)$ when $k_3 = 0$.

Conversely, given m_k and n_k, the initial unknowns can be recovered using $v_k = (m_k, n_k) \cdot \vec{\gamma}_k$ and $\rho_k = (m_k, n_k) \cdot \vec{\phi}_k$.

In the new variables u_k, n_k, m_k, the system (3.9) for $k_3 \neq 0$ can now be written as:

$$\begin{aligned} &u_k' - \beta_k n_k = 0, \\ &n_k' + \beta_k u_k = 0, \\ &m_k' = 0, \end{aligned} \tag{3.15}$$

and this system is easy to solve.

Weak formulation (in the new variables)

We denote by n and m the functions

$$n(x,z,s) = \textstyle\sum_k n_k(s) e^{i(k_1' x + k_3' z)}, \quad m(x,z,s) = \textstyle\sum_k m_k(s) e^{i(k_1' x + k_3' z)},$$

where here and elsewhere $\sum_k$ means the summation over $k = (k_1, k_3) \in \mathbb{Z}^2 \backslash \{\mathbf{0}\}$.

We also consider S_n and S_m similarly defined by their Fourier series. Here we have set $S_{m,k} = (S_{v,k},\, S_{\rho,k}) \cdot \vec{\gamma}_k$ and $S_{n,k} = (S_{v,k},\, S_{\rho,k}) \cdot \vec{\phi}_k$.

As we saw before, $m_{(k_1,0)} = 0$. This motivates us to introduce the following spaces:

$$\begin{aligned} \tilde{\boldsymbol{V}} &= \{(u,n,m) \in (\dot{H}^1_{\text{per}}(\mathcal{M}))^3 : u_{(k_1,0)} = 0,\ u,\ n \text{ are even in } z,\ m \text{ is odd in } z\}, \\ \tilde{\boldsymbol{H}} &= \text{ the closure of } \tilde{\boldsymbol{V}} \text{ in } (\dot{L}^2(\mathcal{M}))^3. \end{aligned}$$

Notice that technically the space $\tilde{\boldsymbol{V}}$ is the same as $\boldsymbol{V}$ but the components play different roles.

We also introduce the space

$$\tilde{\boldsymbol{V}}_2 = \text{ the closure of } \tilde{\boldsymbol{V}} \cap (\dot{H}^2_{\text{per}}(\mathcal{M}))^3 \text{ in } (\dot{H}^2_{\text{per}}(\mathcal{M}))^3. \tag{3.16}$$

We now define the linear operators $\tilde{A}$, $\tilde{L}$ from $\tilde{\boldsymbol{V}}$ into the dual $\tilde{\boldsymbol{V}}'$ of $\tilde{\boldsymbol{V}}$, and the bilinear operator $\tilde{B}$ from $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}$ into $\tilde{\boldsymbol{V}}_2'$. These operators are the expressions of A and B in the new variables. With $V = (u,n,m)$, they are defined by their Fourier series components $\tilde{A}_k$, $\tilde{B}_k$ as follows,

$$\begin{aligned} \tilde{A}V &= \textstyle\sum_k \tilde{A}_k(V) e^{i(k_1'x + k_3'z)}, \\ \tilde{B}(V,V^\flat) &= \textstyle\sum_k \tilde{B}_k(V,V^\flat) e^{i(k_1'x + k_3'z)}, \\ \tilde{L}V &= \textstyle\sum_k \tilde{L}_k(V) e^{i(k_1'x + k_3'z)}. \end{aligned}$$

More explicitly, for $\tilde{A}_k$ we have

$$\tilde{A}_k V_k = \begin{pmatrix} |k'|^2 \nu_{\boldsymbol{v}} u_k \\ |k'|^2 \nu_{\boldsymbol{v}} n_k + (\nu_\rho - \nu_{\boldsymbol{v}})|k'|^2 (N\delta_k/\beta_k)(m_k, n_k) \cdot \vec{\phi}_k \\ |k'|^2 \nu_{\boldsymbol{v}} m_k + |k'|^2 (1/\beta_k)(\nu_\rho - \nu_{\boldsymbol{v}})(m_k, n_k) \cdot \vec{\phi}_k \end{pmatrix} \qquad \text{for all } k,$$

while for $\tilde{L}_k$ we have

$$\begin{aligned} \tilde{L}_k &= 0 && \text{for } k_3 = 0, \\ \tilde{L}_k &= \begin{pmatrix} 0 & -\beta_k & 0 \\ \beta_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{for } k_3 \neq 0, \end{aligned}$$

and similarly for $\tilde{B}$,

$$\begin{aligned} \tilde{B}_k(V,V^\flat) &= \begin{pmatrix} 0 \\ i \sum^k k_1' u_j (\vec{m}_l^\flat + \vec{n}_l^\flat) \cdot \vec{\phi}_k \\ 0 \end{pmatrix} && \text{for } k_3 = 0, \\ \tilde{B}_k(V,V^\flat) &= \begin{pmatrix} i \sum^k (l_1' - l_3'\delta_j) u_j u_l^\flat \\ i \sum^k (l_1' - l_3'\delta_j) u_j (\vec{m}_l^\flat + \vec{n}_l^\flat) \cdot \vec{\phi}_k \\ i \sum^k (l_1' - l_3'\delta_j) u_j (\vec{m}_l^\flat + \vec{n}_l^\flat) \cdot \vec{\gamma}_k \end{pmatrix} && \text{for } k_3 \neq 0. \end{aligned}$$

Here and elsewhere in this paper $\sum^k$ means that the sum is taken over j, l in $\mathbb{Z}^2\backslash\{\mathbf{0}\}$, for $j+l=k$.

The resulting system from this change of variables can be written in the form

$$V' + \tilde{L}V = \varepsilon\tilde{\mathcal{G}}(V), \tag{3.17}$$

where $\tilde{S} = (S_u, S_n, S_m)$ and

$$\tilde{\mathcal{G}}(V) = -\tilde{A}V - \tilde{B}(V,V) + \tilde{S}.$$

We also define the bilinear forms $\tilde{a}(V,V^\flat) = \langle\tilde{A}V, V^\flat\rangle_{\tilde{\boldsymbol{V}}',\tilde{\boldsymbol{V}}}$ and $\tilde{e}(V,V^\flat) = \langle\tilde{L}V, V^\flat\rangle_{\tilde{\boldsymbol{V}}',\tilde{\boldsymbol{V}}}$ where V and $V^\flat$ belong to $\tilde{\boldsymbol{V}}$. We also introduce $\tilde{b}(V,V^\flat,V^\sharp) = \langle\tilde{B}(V,V^\flat),V^\sharp\rangle_{\tilde{\boldsymbol{V}}',\tilde{\boldsymbol{V}}}$ where V, $V^\sharp$ belong to $\tilde{\boldsymbol{V}}$ and $V^\flat$ belongs to $\tilde{\boldsymbol{V}}_2$. Writing explicitly the trilinear form $\tilde{b}$ we find:

$$\begin{aligned}\tilde{b}(V,\, V^\flat,\, V^\sharp) = i\sum\nolimits^{\mathrm{c}}&(l_1'-l_3'\delta_j)u_j u_l^\flat u_k^\sharp \\ &+ i\sum\nolimits^{\mathrm{c}}(l_1'-l_3'\delta_j)u_j(\vec{m}_l^\flat+\vec{n}_l^\flat)\cdot(\vec{m}_k^\sharp+\vec{n}_k^\sharp).\end{aligned}$$

Here and elsewhere in this paper, $\sum^{\mathrm{c}}$ means that the sum is taken over j, l, k, for $j+l+k=0$.

The variational formulation of the problem in the new variables now reads:

Given $t_\star > 0$ arbitrary, $V_0 \in \tilde{\boldsymbol{H}}$ and $\tilde{S} = (S_u,\, S_n,\, S_m) \in L^2(0,t_\star;\tilde{\boldsymbol{H}})$, we look for a function V from $(0,t_\star)$ into $\tilde{\boldsymbol{V}}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(V,V^\flat)_{\tilde{\boldsymbol{H}}} + \tilde{a}(V,V^\flat) + \tilde{b}(V,V,V^\flat) + \frac{1}{\varepsilon}\tilde{e}(V,V^\flat) = (\tilde{S},V^\flat)_{\tilde{\boldsymbol{V}}}, \quad \forall\, V^\flat \in \tilde{\boldsymbol{V}}, \tag{3.18}$$

and

$$V(0) = V_0. \tag{3.19}$$

The first order system

We write the full nonlinear system (3.3) in terms of the new variables.

For $k_3 \neq 0$, the system (3.3) in the new variables reads:

$$\begin{aligned}
u_k' - \beta_k n_k &= -\varepsilon\nu_v|k'|^2u_k - i\varepsilon\sum\nolimits^k(l_1'-l_3'\delta_j)u_ju_l + \varepsilon S_{u,k},\\
n_k' + \beta_k u_k &= -\varepsilon\nu_{\boldsymbol{v}}|k'|^2n_k - \varepsilon|k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})\frac{\delta_kN}{\beta_k}(m_k,n_k)\cdot\vec{\phi}_k\\
&\qquad - i\varepsilon\sum\nolimits^k(l_1'-l_3'\delta_j)u_j(\vec{m}_l+\vec{n}_l)\cdot\vec{\phi}_k + \varepsilon S_{n,k},\\
m_k' &= -\varepsilon\nu_{\boldsymbol{v}}|k'|^2m_k - \varepsilon|k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})\frac{1}{\beta_k}(m_k,n_k)\cdot\vec{\phi}_k\\
&\qquad - i\varepsilon\sum\nolimits^k(l_1'-l_3'\delta_j)u_j(\vec{m}_l+\vec{n}_l)\cdot\vec{\gamma}_k + \varepsilon S_{m,k}.
\end{aligned} \tag{3.20}$$

For the case $k_3 = 0$ we note that $u_k = 0$ and $m_k = 0$ because of the definitions of the spaces.

Study of the new variational problem

We can see, after some elementary computations, that $\tilde{a}$ is a bilinear and coercive form on $\tilde{\boldsymbol{V}}$, so it remains to prove the properties of $\tilde{b}$.

Lemma 4.3.1. *The form $\tilde{b}$ is trilinear continuous from $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}_2 \times \tilde{\boldsymbol{V}}$ to $\mathbb{R}$ and from $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}_2$ to $\mathbb{R}$, and*

$$\begin{aligned} &\tilde{b}(V, V^\flat, V^\flat) = 0, && \forall\, V \in \tilde{\boldsymbol{V}}, \quad \forall\, V^\flat \in \tilde{\boldsymbol{V}}_2, \\ &\tilde{b}(V, V^\flat, V^\sharp) = -\tilde{b}(V, V^\sharp, V^\flat) && \forall\, V,\, V^\flat,\, V^\sharp \in \tilde{\boldsymbol{V}}, \ \text{with } V^\flat \text{ or } V^\sharp \in \tilde{\boldsymbol{V}}_2. \end{aligned} \tag{3.21}$$

Furthermore:

$$|\tilde{b}(V, V^\flat, V^\sharp)| \le c\,|V|_{H^1} |V^\flat|_{H^1}^{1/2} |V^\flat|_{H^2}^{1/2} |V^\sharp|_{L^2}^{1/2} |V^\sharp|_{H^1}^{1/2}, \tag{3.22}$$

for all V, $V^\sharp$ in $\tilde{\boldsymbol{V}}$ and $V^\flat$ in $\tilde{\boldsymbol{V}}_2$.

Proof: To prove the continuity of the bilinear form and (3.22), we estimate for example the second term of $\tilde{b}(V, V^\flat, V^\sharp)$, the estimates being similar for all the terms:

$$\begin{aligned} &\Big| i \sum\nolimits^{c} (l_1' - l_3'\delta_j) u_j (\vec{m}_l^\flat + \vec{n}_l^\flat) \cdot (\vec{m}_k^\sharp + \vec{n}_k^\sharp) \Big| \\ &\qquad \le \sum\nolimits^{c} |l'||j'||u_j| (|m_l^\flat| + |n_l^\flat|)(|m_k^\sharp| + |n_k^\sharp|) \\ &\qquad \le \int_{\mathcal{M}} \eta_1 \eta_2 \eta_3 \, \mathrm{d}\mathcal{M} \le |\eta_1|_{L^2} |\eta_2|_{L^4} |\eta_3|_{L^4} \\ &\qquad \le c |\eta_1|_{L^2} ||\eta_2|_{L^2}^{1/2} |\eta_2|_{H^1}^{1/2} |\eta_3|_{L^2}^{1/2} |\eta_3|_{H^1}^{1/2} \\ &\qquad \le c |V|_{H^1} |V^\flat|_{H^1}^{1/2} |V^\flat|_{H^2}^{1/2} |V^\sharp|_{L^2}^{1/2} |V^\sharp|_{H^1}^{1/2}; \end{aligned}$$

here we wrote:

$$\begin{aligned} &\eta_1 = \sum_j |j'||u_j| e^{i(x\cdot j')}, \quad \eta_2 = \sum_j |j'| (|m_j^\flat| + |n_j^\flat|) e^{i(x\cdot j')}, \\ &\eta_3 = \sum_j (|m_j^\sharp| + |n_j^\sharp|) e^{i(x\cdot j')} \end{aligned}$$

It remains to prove the orthogonality property (3.21). For $V^\flat = V^\sharp$ we have:

$$\begin{aligned} \tilde{b}(V, V^\flat, V^\flat) = i \sum\nolimits^{c} (l_1' - l_3'\delta_j) u_j u_l^\flat u_k^\flat \\ + i \sum\nolimits^{c} (l_1' - l_3'\delta_j) u_j (\vec{m}_l^\flat + \vec{n}_l^\flat) \cdot (\vec{m}_k^\flat + \vec{n}_k^\flat). \end{aligned} \tag{3.23}$$

Interchanging k and l and adding the resulting equations to (3.23), we find:

$$\begin{aligned} \tilde{b}(V, V^\flat, V^\flat) &= \frac{i}{2} \sum\nolimits^{c} [l_1' + k_1' - (l_3' + k_3')\delta_j] u_j u_l^\flat u_k^\flat \\ &\quad + \frac{i}{2} \sum\nolimits^{c} [l_1' + k_1' - (l_3' + k_3')\delta_j] u_j (\vec{m}_l^\flat + \vec{n}_l^\flat) \cdot (\vec{m}_k^\flat + \vec{n}_k^\flat) \\ &= 0. \end{aligned}$$

We have used here the fact that

$$l_1' + k_1' - (l_3' + k_3')\delta_j = -j_1' + j_3'\frac{j_1'}{j_3'} = 0.$$

Remark 4.3.1. Because of the algebraic way we changed the variables and the conservation of the properties for the linear and bilinear operators, we have exactly the same result as Theorem 4.2.1 for the new system.

4.3.2 The Renormalized Equation. Existence of Weak Solutions

We turn now to the renormalized system [the analogue of (2.24) for (3.17)],

$$\frac{\mathrm{d}\bar{V}}{\mathrm{d}t} + \tilde{A}_r(\bar{V}) + \tilde{B}_r(\bar{V}, \bar{V}) = \tilde{S}_r. \tag{3.24}$$

The computation of $\tilde{A}_r$, $\tilde{B}_r$ and $\tilde{S}_r$ is given in Section 4.5. It is established there that $\tilde{a}_r(V, V^\sharp) = \langle \tilde{A}_r V, V^\sharp \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ is a bilinear continuous form in $\tilde{\boldsymbol{V}}$ satisfying

$$\tilde{a}_r(\bar{V}, \bar{V}) \geq c_1 \|\bar{V}\|^2, \tag{3.25}$$

and that $\tilde{b}_r(V, V^\sharp, V^\flat) = \langle \tilde{B}_r(V, V^\sharp), V^\flat \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$ is trilinear continuous on $\tilde{\boldsymbol{V}} \times \tilde{\boldsymbol{V}}_2 \times \tilde{\boldsymbol{V}}$ satisfying

$$\tilde{b}_r(\bar{V}, \bar{V}, \bar{V}) = 0. \tag{3.26}$$

The variational formulation of the renormalized problem (3.24)
Given $t_\star > 0$ arbitrary and

$$\bar{V}_0 \in \tilde{\boldsymbol{H}},\ \tilde{S}_r \in \tilde{\boldsymbol{H}},$$

we look for a function $\bar{V}$ from $(0, t_\star)$ into $\tilde{\boldsymbol{V}}$, such that, for every test function $V^\sharp \in \tilde{\boldsymbol{V}}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\bar{V}, V^\sharp\right) + \tilde{a}_r(\bar{V}, V^\sharp) + \tilde{b}_r(\bar{V}, \bar{V}, V^\sharp) = (\tilde{S}_r, V^\sharp), \tag{3.27}$$

with

$$\bar{V}(0) = \bar{V}_0. \tag{3.28}$$

As usual, in order to solve this problem we need to obtain some a priori estimates. For that purpose, for arbitrary fixed $t > 0$, we set $V^\sharp = \bar{V}(t)$ in equation (3.27). Taking into account the coercivity (3.25) and orthogonality (3.26) properties, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{V}|^2_{L^2} + c_1\|\bar{V}\|^2 \leq (\tilde{S}_r, \bar{V})_{L^2} \leq \frac{c_1}{2}\|\bar{V}\|^2 + c_1'|\tilde{S}_r|^2_{L^2}.$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{V}|_{L^2}^2 + c_1\|\bar{V}\|^2 \le 2c_1'|\tilde{S}_r|_{L^2}^2. \tag{3.29}$$

Applying Poincaré's inequality (2.6) we find,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{V}|_{L^2}^2 + c_1c_0|\bar{V}|_{L^2}^2 \le 2c_1'|\tilde{S}_r|_{L^2}^2, \tag{3.30}$$

and, using the Gronwall lemma,

$$|\bar{V}(t)|_{L^2}^2 \le e^{-c_1c_0t}|\bar{V}(0)|_{L^2}^2 + \frac{2c_1'}{c_0c_1}|\tilde{S}_r|_{L^2}^2(1-e^{-c_0c_1t}). \tag{3.31}$$

This bounds $\bar{V}(t)$ for all t by its initial data,

$$|\bar{V}(t)|_{L^2}^2 \le |\bar{V}(0)|_{L^2}^2 + \frac{2c_1'}{c_0c_1}|\tilde{S}_r|_{L^2}^2.$$

Eq. (3.31) also gives us a bound on $\bar{V}(t)$ independent of the initial data: Setting $r_0^2 := (2c_1'/c_0c_1)|\tilde{S}_r|_{L^2}^2$, we obtain by classical computations (see e.g., [20]) that any ball $B(0, r_0')$ with $r_0' > r_0$ is an absorbing ball and that $|\bar{V}(t)|_{L^2}^2 \le {r_0'}^2$ for all $t \ge t_0(|\bar{V}_0|_{L^2})$.

Using the previous estimates and the Galerkin method we can establish the existence of weak solutions of (3.27) and (3.28) exactly as for the original problem (Theorem 4.2.1):

Theorem 4.3.1. *Given $t_\star > 0$, $\tilde{S}_r \in \tilde{\boldsymbol{H}}$ and $\bar{V}_0 \in \tilde{\boldsymbol{H}}$, the problem (3.27)–(3.28) has at least one solution*

$$\bar{V} \in L^\infty(\mathbb{R}_+; \tilde{\boldsymbol{H}}) \cap L^2(0,\, t_\star; \tilde{\boldsymbol{V}}).$$

4.3.3 Strong Solutions for the Renormalized Equation

We derive the appropriate a priori estimates. Setting $V^\sharp = \Delta\bar{V}(t)$ in (3.27) with $t > 0$ arbitrary, we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{V}\|^2 + c_1|\Delta\bar{V}|_{L^2}^2 \le |\tilde{b}_r(\bar{V},\, \bar{V},\, \Delta\bar{V})| + c_3'|\tilde{S}_r|_{L^2}^2 + \frac{c_1}{4}|\Delta\bar{V}|_{L^2}^2. \tag{3.32}$$

Bounding the trilinear form on the r.h.s. using Lemma 4.5.1,

$$|\tilde{b}_r(\bar{V}, \bar{V}, \Delta\bar{V})| \le 2c_2|\bar{V}|_{L^2}^{1/2}\|\bar{V}\||\Delta\bar{V}|_{L^2}^{3/2},$$

we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{V}\|^2 + c_1|\Delta\bar{V}|_{L^2}^2 &\le 2c_2|\bar{V}|_{L^2}^{1/2}\|\bar{V}\||\Delta\bar{V}|_{L^2}^{3/2} + c_3'|\tilde{S}_r|_{L^2}^2 + \frac{c_1}{4}|\Delta\bar{V}|_{L^2}^2 \\ &\le c_4'|\bar{V}|_{L^2}^2\|\bar{V}\|^4 + c_3'|\tilde{S}_r|_{L^2}^2 + \frac{c_1}{2}|\Delta\bar{V}|_{L^2}^2,\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{V}\|^2 + c_1|\Delta\bar{V}|_{L^2}^2 \le 2c_4'|\bar{V}|_{L^2}^2\|\bar{V}\|^4 + 2c_3'|\tilde{S}_r|_{L^2}^2. \tag{3.33}$$

Existence follows from applying the classical Gronwall lemma, giving us a bound on $\bar{V}$ in $L^\infty(0, t_\star; H^1)$.

A bound uniform in time is obtained in the following manner: We pick $r > 0$ arbitrary and integrate (3.29) from t to $t + r$ with $t \geq 0$,

$$c_1 \int_t^{t+r} \|\bar{V}(t')\|^2 \, \mathrm{d}t' \leq r c_2' |\tilde{S}_r|_{L^2}^2 + |\bar{V}(t)|_{L^2}^2. \tag{3.34}$$

This and the fact that $|\bar{V}|_{L^2}$ is bounded in $L^\infty(\mathbb{R}_+)$ allows us to apply the uniform Gronwall lemma to (3.33) (as in [20]). Computations similar to those in [17] give us estimates uniform in time and we have that $\|\bar{V}\|$ is bounded in $L^\infty(\mathbb{R}_+)$.

Integrating (3.33) from 0 to $t_\star$ we obtain a bound of $\bar{V}$ in $L^2(0,\, t_\star;\, \tilde{\boldsymbol{V}} \cap (H^2_{\text{per}}(\mathcal{M}))^3)$. For later purposes, we note that integrating (3.33) from t to $t + r$ gives us

$$\int_t^{t+r} |\Delta \bar{V}(t')|_{L^2}^2 \, \mathrm{d}t' \leq k(r,\, \tilde{S}_r), \quad \forall\, t \geq t_1(|\bar{V}_0|_{L^2}, r). \tag{3.35}$$

These a priori estimates give the following:

Theorem 4.3.2. *Given $\tilde{S}_r \in \tilde{\boldsymbol{H}}$ and $\bar{V}_0 \in \tilde{\boldsymbol{V}}$, the problem (3.27) has a unique solution*

$$\bar{V} \in L^\infty(\mathbb{R}_+; \tilde{\boldsymbol{V}}) \cap L^2(0, t_\star; \tilde{\boldsymbol{V}} \cap (H^2_{\text{per}}(\mathcal{M}))^3), \quad \forall\, t_\star > 0. \tag{3.36}$$

Remark 4.3.2. (i) Uniqueness in Theorem 4.3.2 is proved in a classical way.

(ii) The proof Theorem 4.3.2 for the renormalized system (3.24)–(3.28) is simpler than for the original system (2.7) due to the fact that the analogue of (3.33) for the latter is of the form (see [17]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 + c_1 |\Delta U|_{L^2}^2 \leq c_1' |\Delta U|_{L^2}^2 \|U\| + c_2' \|U\|^2, \tag{3.37}$$

which does not lead immediately to the appropriate estimates in $L^\infty(0, t_1; \boldsymbol{V})$. The difference between the r.h.s. of (3.34) and (3.37) arises because the renormalized system does not contain problematic terms that are present in the original system.

4.3.4 More Regular Solutions for the Renormalized System

It is desirable to establish the existence of more regular solutions for the renormalized equation. We do this by induction. For simplicity we take the forcing S independent of time and S, $\bar{V}_0 \in \bigcap_m \dot{H}^m_{\text{per}}$.

Suppose that for a fixed arbitrary $m \in \mathbb{N}$, $m \geq 2$, we have

$$\begin{gathered} \bar{V} \in L^\infty(\mathbb{R}_+; \tilde{\boldsymbol{V}} \cap (H^{m-1}_{\text{per}}(\mathcal{M}))^3), \\ \int_t^{t+r} |\bar{V}(t')|_{H^m}^2 \, \mathrm{d}t' \leq K_m, \end{gathered} \tag{3.38}$$

for all $t > t_{m-1}(\bar{V}_0)$, where by K_m we denote as before a constant independent of the initial condition.

We seek to prove that

$$\bar{V} \in L^\infty(\mathbb{R}_+; \tilde{\boldsymbol{V}} \cap (H^m_{\text{per}}(\mathcal{M}))^3),$$
$$\int_t^{t+r} |\bar{V}(t')|^2_{H^{m+1}} \,\mathrm{d}t' \le K_{m+1}.$$

First we derive the a priori estimates: We set in (3.27)

$$\bar{V}_1 = (-\Delta)^m \bar{V}(t) = \sum_{k\in\mathbb{Z}^2} |k'|^{2m} \bar{V}_k(t)\, e^{i(k'\cdot x)},$$

with $t > 0$ fixed, to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{V}|^2_{H^m} + c_1|\bar{V}|^2_{H^{m+1}} \le |\tilde{b}_r(\bar{V},\bar{V},\Delta^m\bar{V})| + |(\tilde{S}_r,\, \Delta^m\bar{V})_{L^2}|. \tag{3.39}$$

We estimate $|\tilde{b}_r(\bar{V},\bar{V},\Delta^m\bar{V})|$ which, using (5.28), reads

$$\begin{aligned}
\tilde{b}_r(\bar{V},\bar{V},(-\Delta)^m\bar{V}) = &-\frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} k_1'|k_1'|^{2m}(\bar{n}_l\bar{u}_j - \bar{u}_l\bar{n}_j)\bar{n}_k\vec{\phi}_l\cdot\vec{\phi}_k \\
&- \frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3 l_3 k_3\neq 0\\ \beta_j=\beta_k}} |k'|^{2m}(l_1'-l_3'\delta_j)\bar{u}_k\bar{n}_j\bar{m}_l\vec{\phi}_k\cdot\vec{\gamma}_l - \frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3\neq 0,\, l_3=0\\ \beta_j=\beta_k}} l_1'|k_1'|^{2m}\bar{n}_j\bar{n}_l\bar{u}_k\vec{\phi}_l\cdot\vec{\phi}_k \\
&+ \frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3 l_3 k_3\neq 0\\ \beta_j=\beta_k}} |k'|^{2m}(l_1'-l_3'\delta_j)\bar{u}_j\bar{m}_l\bar{n}_k\vec{\gamma}_l\cdot\vec{\phi}_k + \frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3\neq 0,\, l_3=0\\ \beta_j=\beta_k}} l_1'|k_1'|^{2m}\bar{u}_j\bar{n}_l\bar{n}_k\vec{\phi}_l\cdot\vec{\phi}_k \\
&+ \frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3 l_3 k_3\neq 0\\ \beta_j=\beta_l}} |k'|^{2m}(l_1'-l_3'\delta_j)(\bar{n}_l\bar{u}_j - \bar{n}_j\bar{u}_l)\bar{m}_k\vec{\phi}_l\cdot\vec{\gamma}_k.
\end{aligned} \tag{3.40}$$

The first term of (3.40) is bounded as:

$$\begin{aligned}
&\left|\frac{i}{2}\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} k_1'|k_1'|^{2m}(\bar{n}_l\bar{u}_j - \bar{u}_l\bar{n}_j)\bar{n}_k\,\vec{\phi}_l\cdot\vec{\phi}_k\right| \\
&\quad\le c_1'\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} |k'|^{2m+1}(|\bar{n}_l||\bar{u}_j| + |\bar{n}_j||\bar{u}_l|)|\bar{n}_k| \\
&\quad\le c_2'\sideset{}{^{\mathrm{c}}}\sum_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} (|\bar{n}_l||\bar{u}_j| + |\bar{n}_j||\bar{u}_l|)|\bar{n}_k|(|j'|^m + |l'|^m)|k'|^{m+1} \\
&\quad\le c_3'\int_{\mathcal{M}} q_1\,q_2\,q_3\,\mathrm{d}\mathcal{M} + c_3'\int_{\mathcal{M}} q_3\,q_4\,q_5\,\mathrm{d}\mathcal{M} \\
&\quad\le c_3'|q_1|_{L^4}|q_2|_{L^4}|q_3|_{L^2} + c_3'|q_3|_{L^2}|q_4|_{L^4}|q_5|_{L^4} \\
&\quad\le c_4'|q_1|^{1/2}_{L^2}\|q_1\|^{1/2}|q_2|^{1/2}_{L^2}\|q_2\|^{1/2}|q_3|_{L^2} + c_4'|q_3|_{L^2}|q_4|^{1/2}_{L^2}\|q_4\|^{1/2}|q_5|^{1/2}_{L^2}\|q_5\|^{1/2} \\
&\quad\le c_5'|\bar{V}|^{1/2}_{L^2}\|\bar{V}\|^{1/2}|\bar{V}|^{1/2}_{H^m}|\bar{V}|^{3/2}_{H^{m+1}},
\end{aligned}$$

where we wrote:

$$q_1 = \sum_{j \in \mathbb{Z}^2} |\bar{u}_j||j'|^m e^{i(x \cdot j')}, \quad q_2 = \sum_{j \in \mathbb{Z}^2} |\bar{n}_j| e^{i(x \cdot j)}, \quad q_3 = \sum_{j \in \mathbb{Z}^2} |\bar{n}_j||j'|^{m+1} e^{i(x \cdot j')},$$
$$q_4 = \sum_{j \in \mathbb{Z}^2} |\bar{n}_j||j'|^m e^{i(x \cdot j')}, \quad q_5 = \sum_{j \in \mathbb{Z}^2} |\bar{u}_j| e^{i(x \cdot j')}.$$

Estimating similarly the other terms, we finally obtain:

Lemma 4.3.2. *There exists a constant $c_3 > 0$ depending only on L_1 and L_3 such that, for all $\bar{V}$ in $\tilde{\boldsymbol{V}} \cap (H^{2m}_{\text{per}}(\mathcal{M}))^3$,*

$$\tilde{b}_r(\bar{V}, \bar{V}, \Delta^m \bar{V}) \leq c_3 |\bar{V}|^{1/2} \|\bar{V}\|^{1/2} |\bar{V}|^{1/2}_{H^m} |\bar{V}|^{3/2}_{H^{m+1}}. \tag{3.41}$$

Returning to (3.39) and using Young's inequality, we find:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\bar{V}|^2_{H^m} + c_1|\bar{V}|^2_{H^{m+1}} &\leq c_3|\bar{V}|^{1/2}\|\bar{V}\|^{1/2}|\bar{V}|^{1/2}_{H^m}|\bar{V}|^{3/2}_{H^{m+1}} + |(\tilde{S}_r,\, \Delta^m \bar{V})_{L^2}| \\ &\leq \frac{c_1}{2}|\bar{V}|^2_{H^{m+1}} + c_1'|\tilde{S}_r|^2_{H^{m-1}} + c_2'|\bar{V}|^2_{L^2}|\bar{V}|^2_{H^1}|\bar{V}|^2_{H^m},\end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}|\bar{V}|^2_{H^m} + c_1|\bar{V}|^2_{H^{m+1}} \leq 2c_1'|\tilde{S}_r|^2_{H^{m-1}} + 2c_2'|\bar{V}|^2_{L^2}|\bar{V}|^2_{H^1}|\bar{V}|^2_{H^m}. \tag{3.42}$$

Applying the classical Gronwall lemma to (3.42) we obtain estimates in $L^\infty(0, t_\star;\, H^m)$ for all $t_\star > 0$, with the bounds depending on the initial data.

Bounds uniform in time, $\bar{V} \in L^\infty(\mathbb{R}_+;\, H^m)$, can be obtained by using the induction hypothesis and applying the Gronwall lemma to (3.42). The bound thus obtained is independent of $|U_0|_m$ when $t \geq t_m(U_0)$ but the bound of $\bar{V}$ in $L^\infty(0, t_m(U_0);\, H^m)$ depends of course on $|U_0|_m$.

Applying classical methods (see, e.g., [11], [20]) to the above a priori estimates, we find:

Theorem 4.3.3. *For any $m \in \mathbb{N}, m \geq 2$, given $\bar{V}_0 \in (H^m_{\text{per}}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$ and $\tilde{S}_r \in (H^{m-1}_{\text{per}}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$, there exists a unique solution $\bar{V}$ of (3.27) in $L^\infty(\mathbb{R}_+;\, (H^m_{\text{per}}(\mathcal{M}))^3)$.*

4.4 First-Order Error Estimates

We introduce as in Section 4.2 the first-order approximate solution $V^1(s)$

$$V^1(s) = e^{-s\tilde{L}}[\bar{V}(s) + \varepsilon G_{np}(\bar{V},\, s)]. \tag{4.1}$$

Here $\bar{V}(s)$ is the solution of the renormalized equation,

$$\begin{aligned}\frac{\mathrm{d}\bar{V}}{\mathrm{d}s} &= \varepsilon G_r(\bar{V}), \\ \bar{V}(0) &= V_0.\end{aligned} \tag{4.2}$$

Our aim in this section is to compare the approximate solution $V^1(s)$ to the exact solution $V(s)$, which satisfies

$$\begin{aligned} &\frac{\mathrm{d}V}{\mathrm{d}s} + \tilde{L}V = \varepsilon \mathcal{G}(V), \\ &V(0) = V_0. \end{aligned} \tag{4.3}$$

The notations we have used are as follows:

$$\begin{aligned} \mathcal{G}(V) &:= -\tilde{A}V - \tilde{B}(V,V) + \tilde{S}, \\ G(s,V) &:= e^{\tilde{L}s}\mathcal{G}(e^{-\tilde{L}s}V). \end{aligned}$$

The resonant and non-resonant parts of $G(s,V)$ are defined as in (2.21),

$$G(s,V) = G_r(V) + G_n(s,V), \tag{4.4}$$

and the primitive $G_{np}(s,V)$ of $G_n(s,V)$ is defined as in (2.22).

Denoting the error by

$$W(s) = V^1(s) - V(s) = e^{-s\tilde{L}}[\bar{V}(s) + \varepsilon G_{np}(\bar{V}(s), s)] - V(s), \tag{4.5}$$

we find after straightforward computations that it satisfies:

$$\begin{aligned} &\frac{\mathrm{d}W}{\mathrm{d}s} + \tilde{L}W + \varepsilon\tilde{A}W + \varepsilon\tilde{B}(W,W) + \varepsilon\tilde{B}(V^1,W) + \varepsilon\tilde{B}(W,V^1) = \varepsilon^2 R_\varepsilon, \\ &W(0) = 0, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} R_\varepsilon = -\tilde{A}e^{-s\tilde{L}}G_{np}(s,\bar{V}) &- \tilde{B}(e^{-s\tilde{L}}\bar{V}, e^{-s\tilde{L}}G_{np}(s,\bar{V})) \\ &- \tilde{B}(e^{-s\tilde{L}}G_{np}(s,\bar{V}), e^{-s\tilde{L}}\bar{V}) - \varepsilon\tilde{B}(e^{-s\tilde{L}}G_{np}(s,\bar{V}),\ e^{-s\tilde{L}}G_{np}(s,\bar{V})) \\ &- e^{-s\tilde{L}}\nabla_{\bar{V}}G_{np}(s,\bar{V})\cdot G_r(\bar{V}). \end{aligned} \tag{4.7}$$

We take the scalar product of (4.6) with W in $(L^2(\mathcal{M}))^3$ and, using the coercivity and orthogonality properties, we obtain,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|W|_{L^2}^2 + \varepsilon c_1|W|_{H^1}^2 &\le \varepsilon|\tilde{b}(W,V^1,W)| + \varepsilon^2|(R_\varepsilon,W)_{L^2}| \\ &\le \varepsilon|\tilde{b}(W,V^1,W)| + \varepsilon^2 c_0|R_\varepsilon|_{L^2}|W|_{H^1}. \end{aligned} \tag{4.8}$$

The first term on the r.h.s. is bounded using Lemma 4.3.1,

$$|\tilde{b}(W,V^1,W)| \le c|W|_{L^2}^{1/2}|V^1|_{H^1}^{1/2}|V^1|_{H^2}^{1/2}|W|_{H^1}^{3/2}; \tag{4.9}$$

applying Young's inequality to this and to $|R_\varepsilon|_{L^2}|W|_{H^1}$, we find:

$$\frac{\mathrm{d}}{\mathrm{d}s}|W|_{L^2}^2 + \varepsilon c_1|W|_{H^1}^2 \le \varepsilon^2 c'\,|R_\varepsilon|_{L^2}^2 + \varepsilon c\,|W|_{L^2}^2|V^1|_{H^1}^2|V^1|_{H^2}^2. \tag{4.10}$$

It remains to estimate R_ε and V^1.

Estimates for R_ε

We start with

$$\begin{aligned}|R_\varepsilon|_{L^2} \le c\,|e^{-\tilde{L}s}G_{np}(\bar{V},s)|_{H^2} + |e^{-\tilde{L}s}\nabla_{\bar{V}}G_{np}(\bar{V},s)\cdot G_r(\bar{V})|_{L^2}\\ + |\tilde{B}(e^{-\tilde{L}s}\bar{V}, e^{-\tilde{L}s}G_{np}(\bar{V},s))|_{L^2} + |\tilde{B}(e^{-\tilde{L}s}G_{np}(\bar{V},s), e^{-\tilde{L}s}\bar{V})|_{L^2}\\ + \varepsilon\,|\tilde{B}(e^{-\tilde{L}s}G_{np}(\bar{V},s), e^{-\tilde{L}s}G_{np}(\bar{V},s))|_{L^2}\end{aligned} \tag{4.11}$$

Note that since the eigenvalues of the matrix $\tilde{L}_k$ are purely imaginary for all $k\in\mathbb{Z}^2$,

$$|e^{-\tilde{L}s}V| \le |V|, \tag{4.12}$$

where here $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^3$.

By arguments similar to those used in the proof of Lemmas 4.3.1 and 4.3.2, one can show that, for all $p\in\mathbb{N}$,

$$|\tilde{B}(V,V^\flat)|_{H^p} \le c_4|V|_{H^{p+2}}|V^\flat|_{H^{p+2}}, \quad \forall\, V,\, V^\flat \in \tilde{\boldsymbol{V}}\cap(H^{p+2}_{\text{per}}(\mathcal{M}))^3. \tag{4.13}$$

Using this and (4.12) in (4.11), we have

$$\begin{aligned}|R_\varepsilon|_{L^2} \le c\,|G_{np}(\bar{V},s)|_{H^2} + 2c_4\,|\bar{V}|_{H^2}|G_{np}(\bar{V},s)|_{H^2} + \varepsilon c_4\,|G_{np}(\bar{V},s)|^2_{H^2}\\ + c|\nabla_{\bar{V}}G_{np}(\bar{V},s)\cdot G_r(\bar{V})|_{L^2}.\end{aligned} \tag{4.14}$$

To continue we need to estimate $|G_{np}(s,\bar{V})|_{H^2}$ and $|\nabla_{\bar{V}}G_{np}(s,\bar{V})\cdot G_r(\bar{V})|_{L^2}$.

Estimates for $G_{np}(\bar{V},s)$

We recall from Section 4.5 that $G_n = \tilde{A}_n + \tilde{B}_n + \tilde{S}_n$, with $\tilde{A}_n$, $\tilde{B}_n$ and $\tilde{S}_n$ being defined in (**??**), (5.22), (5.24) and (5.25). To estimate

$$G_{np}(s,\bar{V}) = \int_0^s G_n(s,\bar{V})\,\mathrm{d}s,$$

we shall need to bound terms of the forms:

$$I_1(j) = \frac{e^{s\alpha\beta_j}-1}{\alpha\beta_j}, \tag{4.15}$$

$$I_2(j,l) = \frac{e^{s(\alpha_1\beta_j+\alpha_2\beta_l)}-1}{\alpha_1\beta_j+\alpha_2\beta_l}, \qquad \text{where } \beta_j-\beta_l\neq 0, \tag{4.16}$$

$$I_3(j,l,k) = \frac{e^{s(\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k)}-1}{\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k}, \tag{4.17}$$

In these expressions, the αs can take on the values of $\pm i$ and the βs are real and not less than 1 [cf. (3.11)].

We now obtain bounds for the denominators in (4.16) and (4.17). It turns out that, provided that the Burgers number N does not lie in a set of measure zero, $\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k\neq 0$. Similarly, it can also be shown [cf. Sect. 4.6.2] that, when N lies outside a small set, the denominators can be bounded from below.

$I_1(j)$ is easily estimated:

(4.18) $$|I_1(j)| = \Big|\frac{e^{\alpha s \beta_j} - 1}{\alpha\beta_j}\Big| = \frac{\sqrt{2(1-\cos s\beta_j)}}{\beta_j} \le 2.$$

To estimate $I_2(j,l)$, $\beta_j \neq \beta_l$, we distinguish two cases:
(i) For $\alpha_1 = \alpha_2$, we obtain $|I_2(j,l)| = 2/|\beta_j + \beta_l| \le 1$.
(ii) For $\alpha_1 = -\alpha_2$, we need a bound for $2/|\beta_j - \beta_l|$. We assume without loss of generality that $\beta_j > \beta_l$; writing $N' = N^2(L_3/L_1)^2$ we find,

$$\begin{aligned}|I_2(j,l)| &= \frac{2}{\beta_j - \beta_l} = \frac{2(\beta_j+\beta_l)}{\beta_j^2 - \beta_l^2} = \frac{2(\beta_j+\beta_l)}{N'(j_1/j_3)^2 - N'(l_1/l_3)^2} = \frac{2}{N'}\frac{\beta_j+\beta_l}{j_1^2 l_3^2 - j_3^2 l_1^2} j_3^2 l_3^2 \\ &\le \frac{2}{N'}(\beta_j+\beta_l) j_3^2 l_3^2 \le \frac{2}{N'}\big(\sqrt{1+N'(j_1/j_3)^2} + \sqrt{1+N'(l_1/l_3)^2}\big)\, j_3^2 l_3^2 \\ &\le c(N')|j|^2|l|^2.\end{aligned}$$

To estimate $I_3(j,k,l)$ we also consider two cases:
(i) All α_i have the same sign, which immediately leads to $|I_3| \le 2/3$.
(ii) $\alpha_1 = \alpha_2 = -\alpha_3$, for which we compute

$$\begin{aligned}|I_3| &\le \frac{2}{|\beta_j + \beta_l - \beta_k|} \\ &= \frac{2|(\beta_j+\beta_l+\beta_k)(-\beta_j+\beta_l+\beta_k)(-\beta_l+\beta_j+\beta_k)|}{|(\beta_j+\beta_l+\beta_k)(\beta_j+\beta_l-\beta_k)(-\beta_j+\beta_l+\beta_k)(\beta_j-\beta_l+\beta_k)|} \\ &\le \frac{|J_1|}{|J_2|}\end{aligned}$$

where

$$\begin{aligned}J_1 &= 2(\beta_j+\beta_l+\beta_k)(-\beta_j+\beta_l+\beta_k)(-\beta_l+\beta_j+\beta_k)\; j_3^4 l_3^4 k_3^4, \\ J_2 &= 3 j_3^4 l_3^4 k_3^4 + 2N'(j_1^2 l_3^4 k_3^4 j_3^2 + l_1^2 l_3^2 j_3^4 k_3^4 + k_1^2 k_3^2 j_3^4 l_3^4) \\ &\quad + N'^2(2 j_1^2 j_3^2 l_1^2 l_3^2 k_3^4 + 2 j_1^2 j_3^2 k_1^2 k_3^2 l_3^4 + 2 l_1^2 l_3^2 k_1^2 k_3^2 j_3^4 - j_1^4 l_3^4 k_3^4 - l_1^4 j_3^4 k_3^4 - k_1^4 j_3^4 l_3^4).\end{aligned}$$

Setting

$$\begin{aligned}\sigma_1 &= 2 j_1^2 j_3^2 l_1^2 l_3^2 k_3^4 + 2 j_1^2 j_3^2 k_1^2 k_3^2 l_3^4 + 2 l_1^2 l_3^2 k_1^2 k_3^2 j_3^4 - j_1^4 l_3^4 k_3^4 - k_1^4 j_3^4 k_3^4 - k_1^4 j_3^4 l_3^4\,, \\ \sigma_2 &= 2(j_1^2 l_3^4 k_3^4 j_3^2 + l_1^2 l_3^2 j_3^4 k_3^4 + k_1^2 k_3^2 j_3^4 l_3^4), \\ \sigma_3 &= 3 j_3^4 l_3^4 k_3^4\,,\end{aligned}$$

we need to estimate $1/|{N'}^2\sigma_1 + N'\sigma_2 + \sigma_3|$. For this we recall from [19][2]:

For any $\delta > 0$ and for almost all $v \in \mathbb{R}$, there exists a constant K depending on v and δ such that

(4.19) $$|v^2 q + vp + r| > K(v,\delta)(|q| + |p| + |r|)^{-(2+\delta)}, \quad \forall\, p,q,r \in \mathbb{Z}.$$

[2]Pointed out to us by Yann Bugeaud (personal communication).

For the convenience of the reader, we provide in Section 4.6.1 an elementary proof of a weaker result in which the power $2+\delta$ is replaced by $3+\delta$.

Choosing N' such that (4.19) holds (almost all real numbers satisfy this property), we estimate I_3 as:

$$|I_3| \le J_1 K(N',\delta)(|\sigma_1|+|\sigma_2|+|\sigma_3|)^{2+\delta} \le K(N',\delta)|j|^{12+4\delta}|l|^{12+4\delta}|k|^{12+4\delta}. \tag{4.20}$$

We note that this result implies that the denominator $\alpha_1\beta_j+\alpha_2\beta_l+\alpha_3\beta_k$ in (4.17) is never zero for almost all $N'\in\mathbb{R}$.

We are now ready to estimate $|G_{np}(s,\bar V)|_{H^2}$: Taking into account (5.14) and (5.16), we see that $\tilde A_{np}(\bar V,\, s)$ only contains terms of type I_1 and we have:

$$\begin{aligned} |\tilde A_{np}(s,\bar V)|_{H^2} &\le c'\left[\textstyle\sum_k (|I_1(k)|\,|k'|^2|\bar m_k|)^2(1+|k'|^2)^2\right]^{1/2} \\ &\quad + c''\left[\textstyle\sum_k (|I_1(k)|\,|k'|^2(|\bar u_k|+|\bar n_k|))^2(1+|k'|^2)^2\right]^{1/2} \\ &\le c\,|\bar V|_{H^4}\,. \end{aligned} \tag{4.21}$$

Next, we estimate $\tilde B_{np}(s,\bar V)$. From (5.22) and (5.24), the most problematic terms (imposing the highest regularity on $\bar V$) are those which, after integration, are of type I_3. We only estimate the typical term $M_{1,\,2,\,np}$ (see Appendix for details on $M_{1,\,2,\,n}$), which we bound using (4.20):

$$\begin{aligned} &\Big|\sum_k -\frac{i}{8}\sum_{j_3 l_3\neq 0}^{\alpha,k} I_3(j,l,k)\,(l_1'-l_3'\delta_j)X_{\alpha_2,j}(\bar V)\,X_{\alpha_3,\,l}(\bar V)e^{i(k_1'x+k_3'z)}\begin{pmatrix}1\\-\alpha_1\end{pmatrix}\Big|_{H^2} \\ &\quad\le c(N',\delta)\sum_k\Big[\textstyle\sum^k |j'|^{13+4\delta}|l'|^{13+4\delta}|k'|^{12+4\delta}(|\bar u_j|+|\bar n_j|)(|\bar u_l|+|\bar n_l|)(1+|k'|^2)^2\Big]^{1/2} \\ &\quad\le c(N',\delta)\,|q_1^2|_{H^{14+4\delta}} \le c(N',\delta)\,|q_1|^2_{H^{14+4\delta}} \le c(N',\delta)\,|\bar V|^2_{H^{27+8\delta}}, \end{aligned}$$

where $q_1 := \sum_j |j'|^{13+4\delta}(|u_j|+|n_j|)e^{i(j_1'x+j_3'z)}$, and we have used $|l_1'-l_3'(j_1'/j_3')|\le|j'|\,|l'|$.

We can now write

$$|\tilde B_{np}(\bar V,\, s)|_{H^2} \le c(N',\delta)|\bar V|^2_{H^{27+8\delta}}. \tag{4.22}$$

Finally, noting that,

$$|\tilde S_{np}|_{H^2} \le |\tilde S|_{H^2},$$

we obtain the following estimate:

$$|G_{np}(s,\bar V)|_{H^2} \le c_1(N',\delta)|\bar V|_{H^4} + c_2(N',\delta)|\bar V|^2_{H^{27+8\delta}} + c_3(N',\delta)|\tilde S|_{H^2}\,, \tag{4.23}$$

valid, as (4.19) tells us, for almost every $N'\in\mathbb{R}$.

Estimates for $|\nabla_{\bar V}G_{np}(s,\bar V)\cdot G_r(\bar V)|_{L^2}$

We consider the bilinear form

$$\tilde B_{np}(s,\bar V,V^\sharp) = \begin{pmatrix}\tilde B^{(1)}_{np}(s,\bar V,V^\sharp)\\ \tilde B^{(2)}_{np}(s,\bar V,V^\sharp)\\ \tilde B^{(3)}_{np}(s,\bar V,V^\sharp)\end{pmatrix},$$

whose Fourier components are:
For $k_3 = 0$, $\tilde{B}^{(1)}_{np,\,k}(s, \bar{V}, V^\sharp) = 0$, $\tilde{B}^{(3)}_{np,\,k}(s, \bar{V}, V^\sharp) = 0$, and

$$\begin{aligned}\tilde{B}^{(2)}_{np,\,k}(s, \bar{V},\, V^\sharp) = \frac{ik'_1}{2} \sum_{j_3 l_3 \neq 0}^{\alpha,k} I_1(j) X_{\alpha_1,\,j}(\bar{V}) m^\sharp_l \vec{\gamma}_l \cdot \vec{\phi}_k \\ + \frac{ik'_1}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \alpha_1\beta_j + \alpha_2\beta_l \neq 0}}^{\alpha,k} I_2(j, l)\alpha_2 X_{\alpha_1,j}(\bar{V}) X_{\alpha_2,\,l}(V^\sharp) \vec{\phi}_l \cdot \vec{\phi}_k,\end{aligned}$$

For $k_3 \neq 0$:

$$\tilde{B}_{np,\,k}(s, \bar{V},\, V^\sharp) = \begin{pmatrix} M^k_{1,2,np}(s, \bar{V},\, V^\sharp) \\ M^k_{3,np}(s, \bar{V},\, V^\sharp) \end{pmatrix},$$

where

$$\begin{aligned}M^k_{1,\,2,\,np}(s,\bar{V}, V^\sharp) = \frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} I_3(j, l, k)(l'_1 - l'_3\delta_j) X_{\alpha_2,\,j}(\bar{V}) X_{\alpha_3,\,l}(V^\sharp) \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ + \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \alpha_1\beta_k + \alpha_2\beta_j \neq 0}}^{\alpha,k} I_2(j,\, k)(l'_1 - l'_3\delta_j)\alpha_1 X_{\alpha_2,\,j}(\bar{V}) m^\sharp_l \vec{\phi}_k \cdot \vec{\gamma}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ + \frac{i}{8} \sum_{j_3 l_3 \neq 0}^{\alpha,k} I_3(j, l, k)(l'_1 - l'_3\delta_j)\alpha_1\alpha_3 X_{\alpha_2,\,j}(\bar{V}) X_{\alpha_3,\,l}(V^\sharp) \vec{\phi}_k \cdot \vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ + \frac{i}{4} \sum_{\substack{l_3 = 0 \\ \alpha_1\beta_k + \alpha_2\beta_j \neq 0}}^{\alpha,k} I_2(j,\, k) l'_1 \alpha_1 X_{\alpha_2,\,j}(\bar{V}) n^\sharp_l \vec{\phi}_k \cdot \vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}M^k_{3,\,np}(s, \bar{V},\, V^\sharp) = \frac{i}{2} \sum_{l_3 = 0}^{\alpha,k} I_1(j) l'_1 X_{\alpha_1,\,j}(\bar{V}) n^\sharp_l \vec{\phi}_l \cdot \vec{\gamma}_k \\ + \frac{i}{2} \sum_{j_3 l_3 \neq 0}^{\alpha,k} (l'_1 - l'_3\delta_j) I_1(j) X_{\alpha_1,\,j}(\bar{V}) m^\sharp_l \vec{\gamma}_l \cdot \vec{\gamma}_k \\ + \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \alpha_1\beta_j + \alpha_2\beta_l \neq 0}}^{\alpha,k} I_2(j,\, l)(l'_1 - l'_3\delta_j)\alpha_2 X_{\alpha_1,\,j}(\bar{V}) X_{\alpha_2,\,l}(V^\sharp) \vec{\phi}_l \cdot \vec{\gamma}_k.\end{aligned}$$

Since $G_{np}(s, \bar{V}) = \tilde{A}_{np}(s, \bar{V}) + \tilde{B}_{np}(s, \bar{V}) + \tilde{S}_{np}$, we have

$$\begin{aligned}\nabla_{\bar{V}} G_{np}(\bar{V},\, s) \cdot G_r(\bar{V}) &= \nabla_{\bar{V}} \tilde{A}_{np}(\bar{V},\, s) \cdot G_r(\bar{V}) + \nabla_{\bar{V}} \tilde{B}_{np}(\bar{V},\, s) \cdot G_r(\bar{V}) \\ &= \tilde{A}_{np}(G_r(\bar{V}),\, s) + \tilde{B}_{np}(\bar{V},\, G_r(\bar{V}),\, s) + \tilde{B}_{np}(G_r(\bar{V}),\, \bar{V},\, s).\end{aligned}$$

Using the same type of argument as before, we have the estimates:

$$\begin{aligned}|\tilde{A}_{np}(G_r(\bar{V}),\, s)|_{L^2} &\leq c|G_r(\bar{V})|_{H^2}, \\ |\tilde{B}_{np}(\bar{V},\, G_r(\bar{V}),\, s)|_{L^2} &\leq K(N', \delta)|\bar{V}|_{H^{27+8\delta}} |G_r(\bar{V})|_{H^{27+8\delta}}, \\ |\tilde{B}_{np}(G_r(\bar{V}),\, \bar{V},\, s)|_{L^2} &\leq K(N', \delta)|\bar{V}|_{H^{27+8\delta}} |G_r(\bar{V})|_{H^{27+8\delta}}.\end{aligned}$$

We bound $G_r(\bar{V}) = -\tilde{A}_r(\bar{V}) - \tilde{B}_r(\bar{V},\, \bar{V}) + \tilde{S}_r$ using

$$\begin{aligned} |\tilde{S}_r|_{H^m} &\leq |\tilde{S}|_{H^m}, \\ |\tilde{B}_r(\bar{V},\, \bar{V})|_{H^m} &\leq c|\bar{V}|^2_{H^{m+2}}, \\ |\tilde{A}_r(\bar{V})|_{H^m} &\leq c|\bar{V}|_{H^{m+2}}, \end{aligned}$$

for all $m \in \mathbb{N}$. Finally, we find:

$$|\nabla_{\bar{V}} G_{np}(\bar{V},\, s) \cdot G_r(\bar{V})|_{L^2} \leq K(N', \delta, |\bar{V}|_{H^{29+8\delta}}, |\tilde{S}|_{H^{27+8\delta}}). \tag{4.24}$$

Putting the estimates we have just derived into (4.14), we have

$$|R_\varepsilon|_{L^2} \leq K(N', \delta, |\bar{V}|_{H^{29+8\delta}},\, |\tilde{S}|_{H^{27+8\delta}}). \tag{4.25}$$

Using Theorem 4.3.3, we can write this in terms of the initial conditions:

$$|R_\varepsilon|_{L^2} \leq K(N', \delta, |V_0|_{H^{29+8\delta}},\, |\tilde{S}|_{H^{28+8\delta}}). \tag{4.26}$$

Estimates for $W(s)$

Note that $V^1(s) = e^{s\tilde{L}}[\bar{V}(s) + \varepsilon G_{np}(s, \bar{V}(s))]$ has been bounded by (4.23),

$$|V^1(s)|_{H^2} \leq K(N', \delta, |\bar{V}|_{H^{27+8\delta}}, |\tilde{S}|_{H^2}), \quad \forall\, s > 0, \tag{4.27}$$

or, using Theorem 4.3.3 again,

$$|V^1(s)|_{H^2} \leq K(N', \delta, |V_0|_{H^{27+8\delta}}, |\tilde{S}|_{H^{26+8\delta}}), \quad \forall\, s > 0. \tag{4.28}$$

Putting this into (4.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}|W|^2_{L^2} + \varepsilon c_1 |W|^2_{H^1} \leq \varepsilon^2 \kappa_1 + \varepsilon \kappa_2 |W|^2_{L^2}, \tag{4.29}$$

where κ_1 and κ_2 are constants depending on N', δ, $|\bar{V}|_{H^{29+8\delta}}$ and $|\tilde{S}|_{H^{28+8\delta}}$. The desired bound on $W(s)$ follows from this using the classical Gronwall lemma:

$$|W(s)|^2_{L^2} \leq \varepsilon^2 \frac{\kappa_1}{\kappa_2}\, e^{\varepsilon \kappa_2 s}, \quad \forall\; s \geq 0. \tag{4.30}$$

Taking $\delta = 1/8$ and collecting the results in this section, we have the following:

Theorem 4.4.1. *For any L_1 and L_3, and for almost all Burgers numbers $N \in \mathbb{R}$, given $V_0 \in (H^{30}_{\rm per}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$, and $\tilde{S} \in (H^{29}_{\rm per}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$, the difference between the solution V of the original system (3.20) and the approximate solution V^1 given by (4.1) satisfies*

$$|V^1(t) - V(t)|^2_{L^2} \leq \varepsilon^2 \kappa' e^{\kappa'' t}, \quad \forall\; t \geq 0, \tag{4.31}$$

where κ' and κ'' are constants depending on N, L_1, L_3, V_0 and $\tilde{S}$.

Remark 4.4.1. We can redo the above estimates, using the bounds on I_3 given in Appendix 3 instead, to arrive at the following:

Theorem 4.4.2. *Let $\mu > 0$, L_1 and L_3 be fixed. Take $V_0 \in (H^{11}_{\mathrm{per}}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$ and $\tilde{S} \in (H^{10}_{\mathrm{per}}(\mathcal{M}))^3 \cap \tilde{\boldsymbol{V}}$. Then there exists a set $\Theta_3^\mu(L_1, L_3)$ having a Lebesgue measure $\operatorname{mes}\Theta_3^\mu(L_1, L_3) \leq \mu$ such that, for all Burgers numbers $N \notin \Theta_3^\mu(L_1, L_3)$, the difference between the solution V of the original system (3.20) and the approximate solution V^1 given by (4.1) satisfies,*

$$|V^1(t) - V(t)|^2_{L^2} \leq \varepsilon^2 \kappa' e^{\kappa'' t}, \quad \forall\, t \geq 0, \tag{4.32}$$

where κ' and κ'' are constants depending on N, L_1, L_3, μ, V_0 and $\tilde{S}$.

4.5 Appendix: Derivation of the renormalized equation

Following the algorithm briefly explained in Section 4.2.2, we start by solving the linear system obtained from (3.20) by dropping all order-ε terms (zeroth order approximation).

For $k_3 = 0$ we find:

$$u_{(k_1,0)} = 0, \quad m_{(k_1,0)} = 0, \tag{5.1}$$

and $n'_{(k_1,0)} = 0$ which implies that $n_{(k_1,0)}(s) = n_{(k_1,0)}(0)$.

For $k_3 \neq 0$ we find, as we already saw, the system (3.15):

$$\begin{aligned} u'_k - \beta_k n_k &= 0, \\ n'_k + \beta_k u_k &= 0, \\ m'_k &= 0. \end{aligned} \tag{5.2}$$

Setting $V_k = (u_k, n_k, m_k)$, this system of ordinary differential equations can be written as

$$V'_k + \tilde{L}_k V_k = 0 \quad \text{where } \tilde{L}_k = \begin{pmatrix} 0 & -\beta_k & 0 \\ \beta_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.3}$$

Its solution is $V_k(s) = e^{-s\tilde{L}_k} V_k(0)$; with

$$e^{-s\tilde{L}_k} = \begin{pmatrix} \frac{1}{2}\sum^\alpha e^{s\alpha\beta_k} R_\alpha & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{where} \;\; R_\alpha = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$$

and $\alpha = \pm i$, we have explicitly,

$$V_k(s) = \begin{pmatrix} \frac{1}{2}\sum^\alpha e^{s\alpha\beta_k}\big(u_k(0) - \alpha n_k(0)\big) \\ \frac{1}{2}\sum^\alpha \alpha e^{s\alpha\beta_k}\big(u_k(0) - \alpha n_k(0)\big) \\ m_k(0) \end{pmatrix}. \tag{5.4}$$

Denoting $X_{\alpha,k}(V) := u_k - \alpha n_k$, (5.4) reads

$$
\begin{aligned}
u_k(s) &= \tfrac{1}{2}\sum\nolimits^{\alpha} e^{s\alpha\beta_k} X_{\alpha,k}(V_0),\\
n_k(s) &= \tfrac{1}{2}\sum\nolimits^{\alpha} \alpha e^{s\alpha\beta_k} X_{\alpha,k}(V_0),\\
m_k(s) &= m_k(0).
\end{aligned} \tag{5.5}
$$

Here and throughout this paper, $\sum^{\alpha}$ always range over $\alpha = \pm i$; similarly for α_i.

For the $\mathcal{O}(\varepsilon)$ approximation, we need to separate the r.h.s. $G(s,V)$ into its resonant and non-resonant parts,

$$
G(s,V) = e^{s\tilde{L}}\mathcal{G}(e^{-s\tilde{L}}V) = G_r(V) + G_n(s,V), \tag{5.6}
$$

and then compute the primitive G_{np} of G_n. As usual, we analyse separately the cases $k_3 = 0$ and $k_3 \neq 0$.

The case $k_3 = 0$

In this case, the equations of motion (3.3) read

$$
\begin{aligned}
u_k &= 0,\\
n_k' &= -\varepsilon\nu_{\boldsymbol{v}}|k'|^2 n_k - \varepsilon i \sum\nolimits^{k} k_1' u_j(m_l\vec{\gamma}_l + n_l\vec{\phi}_l)\cdot\vec{\phi}_k + \varepsilon\, S_{n,k},\\
m_k &= 0,
\end{aligned} \tag{5.7}
$$

where the superscript k in $\sum^k$ means that it is taken over $j + l = k$ with k fixed. Since here the fast linear operator vanishes, $\tilde{L}_{(k_1,0)} = 0$, we have

$$
\begin{aligned}
e^{s\tilde{L}_k}\tilde{S}_k &= \tilde{S}_k,\\
\{e^{s\tilde{L}_k}\tilde{A}e^{-s\tilde{L}_k}\}_k &= \tilde{A}_k = \tilde{A}_{r,k},\\
\{e^{s\tilde{L}_k}\tilde{B}(e^{-s\tilde{L}}V, e^{-s\tilde{L}}V)\}_k &= \tilde{B}_k(e^{-s\tilde{L}}V, e^{-s\tilde{L}}V).
\end{aligned} \tag{5.8}
$$

The u and m components of $\tilde{B}_k$ vanish, so we only need to compute

(5.9)

$$
\begin{aligned}
\tilde{B}_k^{(n)} &= i\sum\nolimits^{k} k_1' u_j(m_l\vec{\gamma}_l + n_l\vec{\phi}_l)\cdot\vec{\phi}_k\\
&= ik_1'\sum\nolimits^{k}\Big[m_l(0)\vec{\gamma}_l + \tfrac{1}{2}\sum\nolimits^{\alpha_2} X_{\alpha_2,l}(V_0)\alpha_2 e^{s\alpha_2\beta_l}\vec{\phi}_l\Big]\cdot\vec{\phi}_k\Big[\tfrac{1}{2}\sum\nolimits^{\alpha_1} X_{\alpha_1,j}(V_0)e^{s\alpha_1\beta_j}\Big].
\end{aligned}
$$

The resonant part (i.e. the s-independent part) of this expression obtains when $\alpha_1\beta_j + \alpha_2\beta_l = 0$, which only happens when $\alpha_1 = -\alpha_2$ and $\beta_j = \beta_l$; this gives us,

$$
\begin{aligned}
\tilde{B}_{r,k}^{(n)}(V,V) &= \frac{ik_1'}{4}\sum_{\substack{j_3l_3\neq 0\\ \beta_j=\beta_l}}^{\alpha,k} \alpha X_{-\alpha,j}(V)X_{\alpha,l}(V)\vec{\phi}_l\cdot\vec{\phi}_k\\
&= \frac{ik_1'}{2}\sum_{\substack{j_3l_3\neq 0\\ \beta_j=\beta_l}}^{k} (n_l u_j - n_j u_l)\vec{\phi}_l\cdot\vec{\phi}_k\,.
\end{aligned} \tag{5.10}
$$

The non-resonant part of $\tilde{B}_k^{(n)}$ is,

$$
\begin{aligned}
\tilde{B}_{n,k}^{(n)}(s,V,V) &= \frac{ik'_1}{2}\sum_{j_3 l_3\neq 0}^{\alpha,k} e^{s\alpha_1\beta_j} m_l X_{\alpha_1,j}(V)\vec{\gamma}_l\cdot\vec{\phi}_k \\
&\quad + \frac{ik'_1}{4}\sum_{\substack{j_3 l_3\neq 0\\ \alpha_1\beta_j+\alpha_2\beta_l\neq 0}}^{\alpha,k} e^{s(\alpha_1\beta_j+\alpha_2\beta_l)}\alpha_2 X_{\alpha_1,j}(V)X_{\alpha_2,l}(V)\vec{\phi}_l\cdot\vec{\phi}_k\,.
\end{aligned}
\tag{5.11}
$$

The case $k_3\neq 0$

We begin with the linear operator A_k [cf. (3.20)],

$$
\tilde{A}_k V_k = \begin{pmatrix} |k'|^2\nu_{\boldsymbol{v}} u_k \\ |k'|^2\nu_{\boldsymbol{v}} n_k + (\nu_\rho-\nu_{\boldsymbol{v}})|k'|^2(N\delta_k/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \\ |k'|^2\nu_{\boldsymbol{v}} m_k + |k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})(1/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \end{pmatrix},
\tag{5.12}
$$

which we split into its diagonal and off-diagonal parts,

$$
\begin{aligned}
\tilde{A}_{1,\,k}V_k &= \nu_{\boldsymbol{v}}|k'|^2 V_k\,, \\
\tilde{A}_{2,\,k}V_k &= \begin{pmatrix} 0 \\ |k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})(N\delta_k/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \\ |k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})(1/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \end{pmatrix}.
\end{aligned}
\tag{5.13}
$$

Since $\tilde{A}_{1,k}$ is diagonal, it is completely resonant. To find the resonant part of $\tilde{A}_{2,k}$, we compute, using $V_k = e^{s\tilde{L}_k}V_0$,

$$
\begin{aligned}
e^{s\tilde{L}_k}\tilde{A}_{2,k}V_k &= \begin{pmatrix} \frac{1}{2}\sum^{\alpha} R_{-\alpha} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ |k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})(N\delta_k/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \\ |k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})(1/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}\sum^{\alpha}\alpha|k'|^2(N\delta_k/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k e^{s\alpha\beta_k} \\ \dfrac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}\sum^{\alpha}|k'|^2(N\delta_k/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k e^{s\alpha\beta_k} \\ |k'|^2(\nu_\rho-\nu_{\boldsymbol{v}})(1/\beta_k)(m_k,n_k)\cdot\vec{\phi}_k \end{pmatrix}.
\end{aligned}
\tag{5.14}
$$

Continuing the computations for $e^{s\tilde{L}_k}\tilde{A}_{2,k}V_k$, we obtain:

$$
\begin{pmatrix} \dfrac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}\sum^{\alpha_1}\alpha_1|k'|^2\dfrac{N\delta_k}{\beta_k}\Big[m_k(0)\dfrac{1}{\beta_k}+\dfrac{N\delta_k}{2\beta_k}\sum^{\alpha_2}\alpha_2 X_{\alpha_2,k}(V_0)e^{s\alpha_2\beta_k}\Big]e^{s\alpha_1\beta_k} \\ \dfrac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}\sum^{\alpha_1}|k'|^2\dfrac{N\delta_k}{\beta_k}\Big[m_k(0)\dfrac{1}{\beta_k}+\dfrac{N\delta_k}{2\beta_k}\sum^{\alpha_2}\alpha_2 X_{\alpha_2,\,k}(V_0)e^{s\alpha_2\beta_k}\Big]e^{s\alpha_1\beta_k} \\ (\nu_\rho-\nu_{\boldsymbol{v}})|k'|^2\dfrac{1}{\beta_k}\Big[m_k(0)\dfrac{1}{\beta_k}+\dfrac{N\delta_k}{2\beta_k}\sum^{\alpha}\alpha X_{\alpha,\,k}(V_0)e^{s\alpha\beta_k}\Big] \end{pmatrix},
\tag{5.15}
$$

where (5.5) has been used for the last equation. Using the fact that $\sum^\alpha X_{\alpha,k}(V_0) = 2u_k(0)$ and $\sum^\alpha \alpha X_{\alpha,k}(V_0) = 2n_k(0)$, we obtain from the last expression the resonant part of $\tilde{A}_k$:

$$\{\tilde{A}_{2,r}V\}_k = \begin{pmatrix} \dfrac{\nu_\rho - \nu_{\boldsymbol{v}}}{2}|k'|^2(N\delta_k/\beta_k)^2 u_k \\ \dfrac{\nu_\rho - \nu_{\boldsymbol{v}}}{2}|k'|^2(N\delta_k/\beta_k)^2 n_k \\ (\nu_\rho - \nu_{\boldsymbol{v}})|k'|^2(1/\beta_k)^2 m_k \end{pmatrix}. \tag{5.16}$$

The nonresonant part of $\tilde{A}_2$ is the remaining part, depending explicitly on s.

Next, we treat the bilinear from $\tilde{B}$:

$$\begin{aligned} e^{s\tilde{L}_k}\tilde{B}_k(e^{-s\tilde{L}}V_0, e^{-s\tilde{L}}V_0) &= \begin{pmatrix} \frac{1}{2}\sum^\alpha e^{s\alpha\beta_k}R_{-\alpha} & 0 \\ 0 & I \end{pmatrix} \cdot \tilde{B}_k(e^{-s\tilde{L}}V_0, e^{-s\tilde{L}}V_0) \\ &= \begin{pmatrix} M^k_{1,2} \\ M^k_3 \end{pmatrix}, \end{aligned} \tag{5.17}$$

where we denoted by $M^k_{1,2}$ the u and n components of the resulting column and by M^k_3 the m component. We have

$$\begin{aligned} M^k_{1,2} &= \frac{i}{2}\sum^\alpha e^{s\alpha\beta_k}\sum^k (l'_1 - l'_3\delta_j)u_j u_l \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \\ &+ \frac{i}{2}\sum^\alpha \alpha e^{s\alpha\beta_k}\sum^k (l'_1 - l'_3\delta_j)u_j(m_l\vec{\gamma}_l + n_l\vec{\phi}_l)\cdot\vec{\phi}_k \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}, \end{aligned} \tag{5.18}$$

or, using (5.5),

$$\begin{aligned} M^k_{1,2} = \frac{i}{8}\sum_{j_3l_3\neq 0}^{\alpha,k} & e^{s(\alpha_1\beta_k+\alpha_2\beta_j+\alpha_3\beta_l)}(l'_1 - l'_3\delta_j)X_{\alpha_2,j}(V_0)X_{\alpha_3,l}(V_0)\begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ &+ \frac{i}{4}\sum_{j_3l_3\neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j)}(l'_1 - l'_3\delta_j)\alpha_1 X_{\alpha_2,j}(V_0)m_l(0)\vec{\gamma}_l\cdot\vec{\phi}_k \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ &+ \frac{i}{8}\sum_{j_3l_3\neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j+\alpha_3\beta_l)}(l'_1 - l'_3\delta_j)\alpha_1\alpha_3 X_{\alpha_2,j}(V_0)X_{\alpha_3,l}(V_0)\vec{\phi}_k\cdot\vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix} \\ &+ \frac{i}{4}\sum_{l_3=0}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j)}l'_1\alpha_1 X_{\alpha_2,j}(V_0)n_{(l_1,0)}(0)\vec{\phi}_l\cdot\vec{\phi}_k \begin{pmatrix} 1 \\ -\alpha_1 \end{pmatrix}. \end{aligned}$$

The resonant part of this expression obtains when $\alpha_1\beta_k + \alpha_2\beta_j = 0$ (implying that $\alpha_1 = -\alpha_2$ and $\beta_k = \beta_j$), or when $\alpha_1\beta_k + \alpha_2\beta_j + \alpha_3\beta_l = 0$. As shown in Section 4.4, the latter scenario does not happen if the Burgers number N lies outside a set of measure zero. Assuming the generic situation, the resonant part of $M^k_{1,2}$ is

$$\begin{aligned} M^k_{1,2,r} &= \frac{i}{4}\sum_{\substack{j_3l_3\neq 0 \\ \beta_k=\beta_j}}^{\alpha,k} (l'_1 - l'_3\delta_j)\alpha X_{-\alpha,j}(V)m_l\vec{\phi}_k\cdot\vec{\gamma}_l \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} \\ &+ \frac{i}{4}\sum_{\substack{l_3=0 \\ \beta_k=\beta_j}}^{\alpha,k} l'_1\alpha X_{-\alpha,j}(V)n_l\vec{\phi}_k\cdot\vec{\phi}_l \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}. \end{aligned} \tag{5.19}$$

After some elementary computations we obtain:

$$M_{1,r}^k = -\frac{i}{2}\sum_{\substack{j_3 l_3\neq 0\\ \beta_k=\beta_j}}^{k}(l_1' - l_3'\delta_j)n_j m_l \vec{\phi}_k\cdot\vec{\gamma}_l - \frac{i}{2}\sum_{\substack{l_3=0\\ \beta_k=\beta_j}}^{k} l_1' n_j n_l \vec{\phi}_k\cdot\vec{\phi}_l, \tag{5.20}$$

$$M_{2,r}^k = \frac{i}{2}\sum_{\substack{j_3 l_3\neq 0\\ \beta_k=\beta_j}}^{k}(l_1' - l_3'\delta_j)u_j m_l \vec{\phi}_k\cdot\vec{\gamma}_l + \frac{i}{2}\sum_{\substack{l_3=0\\ \beta_k=\beta_j}}^{k} l_1' u_j n_l \vec{\phi}_k\cdot\vec{\phi}_l. \tag{5.21}$$

Similarly, the nonresonant part of $M_{1,2}^k$ is

(5.22)

$$\begin{aligned} M_{1,2,n}^k = &\frac{i}{8}\sum_{j_3 l_3\neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j+\alpha_3\beta_l)}(l_1' - l_3'\delta_j)X_{\alpha_2,j}(V)X_{\alpha_3,l}(V)\begin{pmatrix}1\\-\alpha_1\end{pmatrix}\\ &+\frac{i}{4}\sum_{\substack{j_3 l_3\neq 0\\ \alpha_1\beta_k+\alpha_2\beta_j\neq 0}}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j)}(l_1' - l_3'\delta_j)\alpha_1 X_{\alpha_2,j}(V) m_l\vec{\phi}_k\cdot\vec{\gamma}_l\begin{pmatrix}1\\-\alpha_1\end{pmatrix}\\ &+\frac{i}{8}\sum_{j_3 l_3\neq 0}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j+\alpha_3\beta_l)}(l_1' - l_3'\delta_j)\alpha_1\alpha_3 X_{\alpha_2,j}(V)X_{\alpha_3,l}(V)\vec{\phi}_k\cdot\vec{\phi}_l\begin{pmatrix}1\\-\alpha_1\end{pmatrix}\\ &+\frac{i}{4}\sum_{\substack{l_3=0\\ \alpha_1\beta_k+\alpha_2\beta_j\neq 0}}^{\alpha,k} e^{s(\alpha_1\beta_k+\alpha_2\beta_j)} l_1'\alpha_1 X_{\alpha_2,j}(V) n_l\vec{\phi}_k\cdot\vec{\phi}_l\begin{pmatrix}1\\-\alpha_1\end{pmatrix}. \end{aligned}$$

We turn now to the m component of M,

$$\begin{aligned} M_3^k &= i\sum_{j_3\neq 0}^{k}(l_1' - l_3'\delta_j)u_j(m_l\vec{\gamma}_l + n_l\vec{\phi}_l)\cdot\vec{\gamma}_k\\ &= \frac{i}{2}\sum_{l_3=0}^{\alpha,k} e^{\alpha_1 s\beta_j} l_1' X_{\alpha_1,j}(V_0)n_l(0)\vec{\phi}_l\cdot\vec{\gamma}_k\\ &\quad+\frac{i}{2}\sum_{j_3 l_3\neq 0}^{k} e^{\alpha_1 s\beta_j}(l_1' - l_3'\delta_j)X_{\alpha_1,j}(V_0)m_l(0)\vec{\gamma}_l\cdot\vec{\gamma}_k\\ &\quad+\frac{i}{4}\sum_{j_3 l_3\neq 0}^{\alpha,k} e^{s(\alpha_1\beta_j+\alpha_2\beta_l)}(l_1' - l_3'\delta_j)\alpha_2 X_{\alpha_1,j}(V_0)X_{\alpha_2,l}(V_0)\vec{\phi}_l\cdot\vec{\gamma}_k, \end{aligned}$$

where we have use (5.5) for the last equality. Its resonant part is,

$$\begin{aligned} M_{3,r}^k &= \frac{i}{4}\sum_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}}^{\alpha,k}(l_1' - l_3'\delta_j)\alpha X_{-\alpha,j}(V)X_{\alpha,l}(V)\vec{\phi}_l\cdot\vec{\gamma}_k\\ &= \frac{i}{2}\sum_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}}^{k}(l_1' - l_3'\delta_j)(u_j n_l - n_j u_l)\vec{\phi}_l\cdot\vec{\gamma}_k, \end{aligned} \tag{5.23}$$

while its nonresonant part is,

$$
\begin{aligned}
M_{3,n}^k = {} & \frac{i}{2} \sum_{l_3=0}^{\alpha,k} e^{\alpha_1 s \beta_j} l_1' X_{\alpha_1,\, j}(V) n_{(l_1,\, 0)} \vec{\phi}_l \cdot \vec{\gamma}_k \\
& + \frac{i}{2} \sum_{j_3 l_3 \neq 0}^{\alpha,k} e^{\alpha_1 s \beta_j} (l_1' - l_3' \delta_j) X_{\alpha_1,\, j}(V) m_l \vec{\phi}_l \cdot \vec{\gamma}_k \\
& + \frac{i}{4} \sum_{\substack{j_3 l_3 \neq 0 \\ \alpha_1 \beta_j + \alpha_2 \beta_l \neq 0}}^{\alpha,k} e^{s(\alpha_1 \beta_j + \alpha_2 \beta_l)} \alpha_2 (l_1' - l_3' \delta_j) X_{\alpha_1,j}(V) X_{\alpha_2,l}(V) \vec{\phi}_l \cdot \vec{\gamma}_k.
\end{aligned}
\tag{5.24}
$$

Finally, we compute

$$
\begin{aligned}
\{e^{\tilde{L}s} \tilde{S}_k\}_k &= \begin{pmatrix} \frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} R_{-\alpha} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S_{u,k} \\ S_{n,k} \\ S_{m,k} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} (S_{u,k} + \alpha S_{n,k}) \\ -\frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} \alpha (S_{u,k} + \alpha S_{n,k}) \\ S_{m,k} \end{pmatrix},
\end{aligned}
$$

whence we find:

$$
\tilde{S}_{r,k} = \begin{pmatrix} 0 \\ 0 \\ S_{m,k} \end{pmatrix} \quad \text{and} \quad \tilde{S}_{n,k} = \begin{pmatrix} \frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} (S_{u,k} + \alpha S_{n,k}) \\ -\frac{1}{2} \sum^{\alpha} e^{s\alpha\beta_k} \alpha (S_{u,k} + \alpha S_{n,k}) \\ 0 \end{pmatrix}.
\tag{5.25}
$$

The renormalized system

We have now computed all the terms in the renormalized system,

$$
\frac{\mathrm{d}V}{\mathrm{d}t} + \tilde{A}_r V + \tilde{B}_r(V,\, V) = \tilde{S}_r,
\tag{5.26}
$$

written here in the slow time t. Explicitly, we have in Fourier modes for $k = (k_1, 0)$:

$$
\begin{aligned}
\frac{\mathrm{d}u_k}{\mathrm{d}t} &= 0, \\
\frac{\mathrm{d}m_k}{\mathrm{d}t} &= 0, \\
\frac{\mathrm{d}n_k}{\mathrm{d}t} &= -\nu_{\boldsymbol{v}} {k'}_1^2 n_{(k_1,\, 0)} - \frac{i}{2} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_j = \beta_l}}^{k} k_1' (n_l u_j - n_j u_l) \vec{\phi}_l \cdot \vec{\phi}_k + S_{n,\, k}.
\end{aligned}
\tag{5.27}
$$

For $k_3 \neq 0$, we have

$$
\frac{\mathrm{d}u_k}{\mathrm{d}t} = -\nu_{\boldsymbol{v}} |k'|^2 u_k - \frac{\nu_\rho - \nu_{\boldsymbol{v}}}{2} |k'|^2 \frac{N^2 \delta_k^2}{\beta_k^2} u_k + \frac{i}{2} \sum_{\substack{j_3 l_3 \neq 0 \\ \beta_k = \beta_j}}^{k} (l_1' - l_3' \delta_j) n_j m_l \vec{\gamma}_l \cdot \vec{\phi}_k
$$

$$
\begin{aligned}
&+\frac{i}{2}\sum_{\substack{l_3=0\\ \beta_k=\beta_j}}^{k} l_1' n_j n_l \vec{\phi}_k \cdot \vec{\phi}_l,\\
\frac{\mathrm{d}n_k}{\mathrm{d}t} &= -\nu_{\boldsymbol{v}}|k'|^2 n_k - \frac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}|k'|^2\frac{N^2\delta_k^2}{\beta_k^2}n_k - \frac{i}{2}\sum_{\substack{j_3 l_3\neq 0\\ \beta_k=\beta_j}}^{k}(l_1'-l_3'\delta_j)u_j m_l \vec{\phi}_k\cdot\vec{\gamma}_l\\
&-\frac{i}{2}\sum_{\substack{l_3=0\\ \beta_k=\beta_j}}^{k} l_1' u_j n_l \vec{\phi}_k\cdot\vec{\phi}_l,\\
\frac{\mathrm{d}m_k}{\mathrm{d}t} &= -\nu_{\boldsymbol{v}}|k'|^2 m_k - (\nu_\rho-\nu_{\boldsymbol{v}})|k'|^2\frac{1}{\beta_k^2}m_k\\
&-\frac{i}{2}\sum_{\substack{j_3 l_3\neq 0\\ \beta_j=\beta_l}}^{k}(l_1'-l_3'\delta_j)(u_j n_l - n_j u_l)\vec{\phi}_l\cdot\vec{\gamma}_k + S_{m,k}.
\end{aligned}
$$

Properties of the renormalized system

As mentioned in the Introduction, the renormalized linear operator $\tilde{A}_r$ and bilinear operator $\tilde{B}_r$ in (5.26) enjoy some properties of their original counterparts, as we now show:

$$
\begin{aligned}
\tilde{a}_r(V,V) &= \langle \tilde{A}_r V,\, V\rangle_{\tilde{\boldsymbol{V}}',\tilde{\boldsymbol{V}}}\\
&= \nu_{\boldsymbol{v}}\sum_k |k'|^2|n_k|^2 + \nu_{\boldsymbol{v}}\sum_k |k'|^2|u_k|^2\\
&\quad+\frac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}\sum_k |k'|^2\frac{N^2\delta_k^2}{\beta_k^2}|u_k|^2 + \frac{\nu_\rho-\nu_{\boldsymbol{v}}}{2}\sum_k |k'|^2\frac{N^2\delta_k^2}{\beta_k^2}|n_k|^2\\
&\quad+\nu_{\boldsymbol{v}}\sum_k |k'|^2|m_k|^2 + (\nu_\rho-\nu_{\boldsymbol{v}})\sum_k |k'|^2\frac{1}{\beta_k^2}|m_k|^2.
\end{aligned}
$$

After some elementary computations we have

$$
\tilde{a}_r(V,V) \geq \min(\nu_{\boldsymbol{v}},\nu_\rho)(\|u\|^2+\|n\|^2+\|m\|^2),
$$

thus proving the coercivity of a_r in $\tilde{\boldsymbol{V}}$.

We turn now to the trilinear form $\tilde{b}_r(V, V^\flat, V^\sharp) = \langle \tilde{B}_r(V, V^\flat), V^\sharp \rangle_{\tilde{\boldsymbol{V}}', \tilde{\boldsymbol{V}}}$,

$$
\begin{aligned}
\tilde{b}_r(V, V^\flat, V^\sharp) = &-\frac{i}{2} \sum_{\substack{j_3 \neq 0,\, k_3 = 0 \\ \beta_j = \beta_l}}^{\mathrm{c}} k_1' (n_l^\flat u_j - u_l^\flat n_j) n_k^\sharp \vec{\phi}_l \cdot \vec{\phi}_k \\
&- \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} (l_1' - l_3' \delta_j) u_k^\sharp n_j m_l^\flat \vec{\phi}_k \cdot \vec{\gamma}_l \\
&- \frac{i}{2} \sum_{\substack{j_3 \neq 0,\, l_3 = 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} l_1' n_j n_l^\flat u_k^\sharp \vec{\phi}_l \cdot \vec{\phi}_k + \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} (l_1' - l_3' \delta_j) u_j m_l^\flat n_k^\sharp \vec{\phi}_k \cdot \vec{\gamma}_l \\
&+ \frac{i}{2} \sum_{\substack{j_3 \neq 0,\, l_3 = 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} l_1' u_j n_k^\sharp n_l^\flat \vec{\phi}_l \cdot \vec{\phi}_k + \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_l}}^{\mathrm{c}} (l_1' - l_3' \delta_j)(u_j n_l^\flat - u_l^\flat n_j) m_k^\sharp \vec{\gamma}_k \cdot \vec{\phi}_l.
\end{aligned}
\tag{5.28}
$$

Interchanging k with l and using the elementary relation

$$
l_1' + k_1' - (k_3' + l_3')(j_1'/j_3') = -j_1' + j_3'(j_1'/j_3') = 0 \quad (\text{since } j + l + k = 0),
$$

we now compute

$$
\begin{aligned}
\tilde{b}_r(V,\, V^\flat,\, V^\flat) = &- \frac{i}{2} \sum_{\substack{j_3 \neq 0,\, l_3 = 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} l_1' (n_k^\flat u_j - u_k^\flat n_j) n_l^\flat \vec{\phi}_l \cdot \vec{\phi}_k \\
&- \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} (l_1' - l_3' \delta_j) u_k^\flat n_j m_l^\flat \vec{\gamma}_l \cdot \vec{\phi}_k \\
&- \frac{i}{2} \sum_{\substack{j_3 \neq 0,\, l_3 = 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} l_1' n_j u_k^\flat n_l^\flat \vec{\phi}_l \cdot \vec{\phi}_k + \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} (l_1' - l_3' \delta_j) u_j m_l^\flat n_k^\flat \vec{\phi}_k \cdot \vec{\gamma}_l \\
&+ \frac{i}{2} \sum_{\substack{j_3 \neq 0,\, l_3 = 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} l_1' u_j n_k^\flat n_l^\flat \vec{\phi}_l \cdot \vec{\phi}_k \\
&+ \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} (l_1' - l_3' \delta_j)(n_l^\flat u_j - n_j u_l^\flat) m_k^\flat \vec{\phi}_l \cdot \vec{\gamma}_k \\
= &- \frac{i}{2} \sum_{\substack{j_3 l_3 k_3 \neq 0 \\ \beta_j = \beta_k}}^{\mathrm{c}} \Big(l_1' + k_1' - l_3' \frac{j_1'}{j_3'} + k_3' \frac{j_1'}{j_3'} \Big) (u_l^\flat n_j - u_j n_l^\flat) m_k^\flat \vec{\phi}_l \cdot \vec{\gamma}_k = 0.
\end{aligned}
\tag{5.29}
$$

We have thus proved that the orthogonality of $b(V, V^\sharp, V^\flat)$ is preserved in the renormalized system.

Lemma 4.5.1. *There exists a constant $c_2 > 0$ such that for all $V = (u, n, m)$, $V^\flat = (u^\flat, n^\flat, m^\flat)$, $V^\sharp = (v^\sharp, n^\sharp, m^\sharp)$, with V, $V^\sharp \in \tilde{\boldsymbol{V}}$ and $V^\flat \in \tilde{\boldsymbol{V}}_2$, we have*

$$
\begin{aligned}
|\tilde{b}_r(V,\, V^\flat,\, V^\sharp)| \leq\, & c_2 \|V\|^{1/2} |\Delta V|_{L^2}^{1/2} |V^\flat|_{L^2}^{1/2} \|V^\flat\|^{1/2} |V^\sharp|_{L^2} \\
&+ c_2 |V|_{L^2}^{1/2} \|V\|^{1/2} \|V^\flat\|^{1/2} |\Delta V^\flat|_{L^2}^{1/2} |V^\sharp|_{L^2},
\end{aligned}
\tag{5.30}
$$

(5.31) $$|\tilde{b}_r(V, V^\flat, V^\sharp)| \le c_2 \|V\| \|V^\flat\| \|V^\sharp\|.$$

Proof: We need to estimate each term of $\tilde{b}_r(V, V^\flat, V^\sharp)$. In order to facilitate the computations we write:

$$u_1 = \sum_{j=(j_1,j_3)\in\mathbb{Z}^2} |u_j| e^{i(xj_1'+zj_3')}, \quad u_2 = \sum_{j=(j_1,j_3)\in\mathbb{Z}^2} |j'||u_j| e^{i(xj_1'+zj_3')},$$

and similarly for n and m. We estimate $|l_1' - l_3'(j_1'/j_3')|$ taking into account the summation conditions $\beta_j = \beta_k \Leftrightarrow |j_1'/j_3'| = |k_1'/k_3'|$: When $j_1'/j_3' = k_1'/k_3'$, we have from $j + l + k = 0$ that $|l_1' - l_3'(j_1'/j_3')| = 0$. When $j_1'/j_3' = -k_1'/k_3'$, we write $j_1' = -sk_1'$, $j_3' = sk_3'$, and using $j + l + k = 0$ again we have $|l_1' - l_3'(j_1'/j_3')| = 2|k_1'| \le 2(|j'| + |l'|)$. We also have $|k_1' - k_3'(j_1'/j_3')| = 2|k_1'| \le 2(|j'| + |l'|)$.

We can now proceed and estimate $|\tilde{b}_r(V,\, V^\flat,\, V^\sharp)|$:

$$\begin{aligned}
&\left| \frac{i}{2} \sum^{c}_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} k_1'(n_l^\flat u_j - u_l^\flat n_j)\, n^\sharp_{(k_1,\,0)}\, \vec{\phi}_l \cdot \vec{\phi}_k \right| \\
&\quad \le c \sum^{c}_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} (|u_j||n_l^\flat||n_k^\sharp| + |n_j||u_l^\flat||n_k^\sharp|)(|j'| + |l'|) \\
&\quad \le c\int_{\mathcal{M}} u_2 n_1^\flat n_1^\sharp \, \mathrm{d}\mathcal{M} + c\int_{\mathcal{M}} u_1^\flat n_2 n_1^\sharp \, \mathrm{d}\mathcal{M} + c\int_{\mathcal{M}} u_1 n_1^\sharp n_2^\flat \, \mathrm{d}\mathcal{M} + c\int_{\mathcal{M}} u_2^\flat n_1 n_1^\sharp \, \mathrm{d}\mathcal{M} \\
&\quad \le c|u_2|_{L^4(\mathcal{M})}|n_1^\sharp|_{L^4(\mathcal{M})}|n_1^\flat|_{L^2(\mathcal{M})} + c|u_1^\flat|_{L^4(\mathcal{M})}|n_2|_{L^4(\mathcal{M})}|n_1^\sharp|_{L^2(\mathcal{M})} \\
&\qquad + c|n_2^\flat|_{L^4(\mathcal{M})}|u_1|_{L^4(\mathcal{M})}|n_1^\sharp|_{L^2(\mathcal{M})} + c|u_2^\flat|_{L^4(\mathcal{M})}|n_1|_{L^4(\mathcal{M})}|n_1^\sharp|_{L^2(\mathcal{M})}.
\end{aligned}$$

Using the fact that $|u|_{L^4(\mathcal{M})} \le c|u|_{H^{1/2}(\mathcal{M})}$ in space dimension two, we find:

$$\begin{aligned}
&\left| \frac{i}{2} \sum^{c}_{\substack{j_3\neq 0,\, k_3=0\\ \beta_j=\beta_l}} k_1'(n_l^\flat u_j - u_l^\flat n_j)\, n^\sharp_{(k_1,\,0)}\, \vec{\phi}_l \cdot \vec{\phi}_k \right| \\
&\quad \le c\|V\|^{1/2}|\Delta V|^{1/2}_{L^2}|V^\flat|^{1/2}_{L^2}\|V^\flat\|^{1/2}|V^\sharp|_{L^2} + c|V|^{1/2}_{L^2}\|V\|^{1/2}\|V^\flat\|^{1/2}|\Delta V^\flat|^{1/2}_{L^2}|V^\sharp|_{L^2}.
\end{aligned}$$

All the other terms can be estimated in the same manner, giving us (5.30). The proof of (5.31) follows using the same type of argument.

4.6 Auxiliary Results

4.6.1 A Result in Number Theory

In this section we prove for interested readers a (weaker) analogue of the small denominator estimate (4.19) used in Section 4.4.

Lemma 4.6.1. *For any $\delta > 3$ and for almost every $\xi \in (0, R)$, where R is an arbitrarily natural number , there exists a constant $\gamma > 0$ such that $|p+q\xi+r\xi^2| > \gamma|p^2+q^2+r^2|^{-\delta/2}$ for all $(p, q, r) \in \mathbb{Z}^3\backslash\{\mathbf{0}\}$.*

Proof: We need to show that the set

$$\Omega = \{\xi \in (0, R) : \forall\, \gamma > 0\; \exists\, (p, q, r) \in \mathbb{Z}^3\backslash\{\mathbf{0}\} \text{ with } |p + q\xi + r\xi^2| \leq \gamma|p^2 + q^2 + r^2|^{-\delta/2}\}$$

has measure zero.

We first split $\mathbb{Z}^3\backslash\{\mathbf{0}\}$ into $Z_1 + Z_2 + Z_3 + Z_4$, where

$$\begin{aligned} Z_1 &= \{(p, q, r) : r\xi^2 + q\xi + p = 0 \text{ has no solution in } \mathbb{R}\},\\ Z_2 &= \{(p, q, r) : r\xi^2 + q\xi + p = 0 \text{ has a double root } |\xi_*| \leq 2R\},\\ Z_3 &= \{(p, q, r) : r\xi^2 + q\xi + p = 0 \text{ has two simple roots}\}, \end{aligned}$$

and Z_4 covers the other cases which do not concern us. Noting that

$$\Omega = \bigcap_{\gamma} \bigcup_{p,q,r} \Omega_\gamma(p, q, r),$$

we fix γ and (p, q, r), and compute the measure of the set

$$\Omega_\gamma(p, q, r) = \{\xi \in (0, R) : |p + q\xi + r\xi^2| \leq \gamma|p^2 + q^2 + r^2|^{-\delta/2}\}. \tag{6.1}$$

We now consider Z_1, Z_2 and Z_3 in turn.

$(p, q, r) \in Z_1$: $\operatorname{mes}\Omega_\gamma(p, q, r) = 0$ for $\gamma < 1/4$, because

$$\min_{\xi\in\mathbb{R}} |r\xi^2 + q\xi + p| = \frac{|q^2 - 4pr|}{4|r|} \geq \gamma|p^2 + q^2 + r^2|^{-1}$$

and $|q^2 - 4pr| \geq 1$ in this case.

$(p, q, r) \in Z_2$: in this case $|r| \geq 1$ and $q^2 - 4pr = 0$, which implies $pr \geq 0$. We then have,

$$\operatorname{mes}\Omega_\gamma(p, q, r) \leq \sqrt{\gamma/|r|}\, |p^2 + q^2 + r^2|^{-\delta/4}. \tag{6.2}$$

Since the root $|\xi_*| \leq 2R$, $q^2 \leq 8r^2$ and (using $4pr = q^2$) also $p^2 \leq 4r^2R^4$. Therefore $\sqrt{|r|} \geq C(R)|p^2 + q^2 + r^2|^{1/4}$ and

$$\operatorname{mes}\Omega_\gamma(p, q, r) \leq \sqrt{\gamma}\, C(R)\, |p^2 + q^2 + r^2|^{-(\delta+1)/4}. \tag{6.3}$$

Since $q^2 = 4pr$, this is equivalent to (allowing us to sum over $\mathbb{Z}^2\backslash\{\mathbf{0}\}$ in (6.6) below)

$$\operatorname{mes}\Omega_\gamma(p, q, r) \leq \sqrt{\gamma}\, C(R)\, |p^2 + r^2|^{-(\delta+1)/4}. \tag{6.4}$$

$(p, q, r) \in Z_2$: as before, we assume that $r \geq 1$; the case $r \leq -1$ is similar, and the "linear" case $r = 0$ is easy. We denote $\eta = \gamma|p^2 + q^2 + r^2|^{-\delta/2}$, $\Delta = q^2 - 4pr$,

$\Delta_- = q^2 - 4p(r-\eta)$ and $\Delta_+ = q^2 - 4p(r+\eta)$. Considering the neighborhood of a root, and noting that $\Delta_- > 0$ whenever $\gamma < 1/4$, we have

$$\operatorname{mes}\{\xi : |r\xi^2 + q\xi + p| \leq \eta\} \leq \frac{\sqrt{\Delta_+} - \sqrt{\Delta_-}}{2r} = \frac{8\eta}{\sqrt{\Delta_+} + \sqrt{\Delta_-}} \leq \frac{8\eta}{\sqrt{\Delta}} \leq 8\eta.$$

Regardless of where the roots lie, we thus have

$$\Omega_\gamma(p,q,r) \leq 16\gamma|p^2 + q^2 + r^2|^{-\delta/2}. \tag{6.5}$$

Putting together the results of the three cases, we have

$$\operatorname{mes}\Omega_\gamma \leq 16\gamma \textstyle\sum_{p,q,r} |p^2 + q^2 + r^2|^{-\delta/2} + \sqrt{\gamma}\, C(R) \textstyle\sum_{p,r} |p^2 + r^2|^{-(\delta+1)/4} \tag{6.6}$$

where the first sum is taken over $\mathbb{Z}^3 \backslash \{\mathbf{0}\}$ and the second over $\mathbb{Z}^2 \backslash \{\mathbf{0}\}$. Both sums converge when $\delta > 3$, giving us

$$\operatorname{mes}\Omega_\gamma \leq \sqrt{\gamma}\, C(\delta, R), \tag{6.7}$$

valid for $\gamma < 1/4$, whence it follows that $\operatorname{mes}\Omega = 0$.

4.6.2 Another Estimate for Small Denominators

In this section, following an alternate approach due to Babin, Mahalov, and Nicolaenko (see [3]), we present another way of estimating the three-wave resonances. In a sense the method is an improvement of that used in Section 4.4 because we require less regularity on the initial data. On the other hand, it is weaker because it is valid only for Burgers numbers belonging to a certain quasi-resonant set.

Recall that $\beta_k = [1 + N^2(k_1'/k_3')^2]^{1/2}$. As in Section 4.4, we need to estimate the term

$$I_3 = \frac{e^{s(\alpha_1\beta_j + \alpha_2\beta_l + \alpha_3\beta_k)} - 1}{\alpha_1\beta_j + \alpha_2\beta_l + \alpha_3\beta_k}, \tag{6.8}$$

where $\alpha_1\beta_j + \alpha_2\beta_l + \alpha_3\beta_k \neq 0$, α_1, α_2, $\alpha_3 = \pm i$ and $j + l + k = 0$.

The problem is nontrivial only when the α_i are not of the same sign; with no loss of generality, we suppose that $\alpha_1 = \alpha_2 = -\alpha_3$. In estimating $|\beta_j + \beta_l - \beta_k|^{-1}$, we have two cases:

Case 1. If $|\beta_l - \beta_k| \leq \beta_j/2$, then $|\beta_j + \beta_l - \beta_k|^{-1} \leq 2/\beta_j \leq 2$ and we are done.

Case 2. If $|\beta_l - \beta_k| \geq \beta_j/2$, some work is needed. We estimate

$$\begin{aligned} |I_3| &\leq \frac{2}{|\beta_j + \beta_l - \beta_k|} \\ &= \frac{2|(\beta_j + \beta_l + \beta_k)(-\beta_j + \beta_l + \beta_k)(-\beta_l + \beta_j + \beta_k)|}{|(\beta_j + \beta_l + \beta_k)(\beta_j + \beta_l - \beta_k)(-\beta_j + \beta_l + \beta_k)(\beta_j - \beta_l + \beta_k)|} \\ &=: 2I_3'. \end{aligned} \tag{6.9}$$

Denoting $\lambda = N^2$ and $\chi_k = (k_1'/k_3')^2$, we have

$$I_3' = \frac{|(\beta_j + \beta_l + \beta_k)(-\beta_j + \beta_l + \beta_k)(-\beta_l + \beta_j + \beta_k)|}{|P(\lambda)|}, \tag{6.10}$$

where

$$P(\lambda) = \lambda^2(\chi_k^2 + \chi_j^2 + \chi_l^2 - 2\chi_k\chi_j - 2\chi_j\chi_l - 2\chi_k\chi_l) - 2\lambda(\chi_j + \chi_l + \chi_k) - 3. \tag{6.11}$$

The discriminant of this quadratic polynomial is

$$\Delta = 2[(\chi_j - \chi_l)^2 + (\chi_l - \chi_k)^2 + (\chi_k - \chi_j)^2] \geq 0. \tag{6.12}$$

Since $P(\lambda) = 0$ has no more than two solutions for each fixed (j,l), the set of Burgers numbers N for which $\beta_j + \beta_l - \beta_k = 0$ is at most countable. We denote the solutions of $P(\lambda) = 0$ by $\lambda_\pm(j,l)$.

We define the three-wave quasi-resonant set $\Theta_3^\mu(L_1, L_3)$:

Given $\mu > 0$ and a sequence of positive numbers $\{\xi_{(j,l)}\}$ with $\sum_{(j,l)} \xi_{(j,l)} \leq 1$, we define the three-wave quasi-resonant set $\Theta_3^\mu(L_1, L_3)$ as:

$$\Theta_3^\mu(L_1, L_3) = \bigcup_{(j,l)\in\mathbb{Z}^2} \{N :\ 2|N - N^\star(j,l,L_1,L_3)| \leq \mu\xi_{(j,l)}\}, \tag{6.13}$$

where $N^\star(j,l,L_1,L_3) := \sqrt{\lambda_\pm(j,l,L_1,L_3)}$. It is obvious that the Lebesgue measure

$$\mathrm{mes}\, \Theta_3^\mu(L_1, L_3) \leq \mu,$$

for all L_1 and L_3.

For j, l, L_1 and L_3 given, the set $\{N : 2|N - N^\star(j,l,L_1,L_3)| \leq \mu\xi_{(j,l)}\}$ can be defined approximately by $|P(\lambda)| \leq \delta$. For δ small, we have

$$\begin{aligned} \delta &\simeq \left|\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}(0)\right|^{-1} |\lambda(\delta) - \lambda_\pm(j,l,L_1,L_3)| \\ &\simeq 2N_\pm(j,l,L_1,L_3)|N - N_\pm(j,l,L_1,L_3)| \left|\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}(0)\right|^{-1}, \end{aligned} \tag{6.14}$$

where

$$\left|\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}(0)\right| = \frac{1}{\sqrt{\Delta}} = \frac{1}{\sqrt{2[(\chi_j - \chi_l)^2 + (\chi_l - \chi_k)^2 + (\chi_k - \chi_j)^2]}}. \tag{6.15}$$

or, using $\beta_j^2 - \beta_k^2 = N^2(\chi_j - \chi_k)$,

$$\left|\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}(0)\right| = \frac{N^2}{\sqrt{2[(\beta_j^2 - \beta_l^2)^2 + (\beta_l^2 - \beta_k^2)^2 + (\beta_k^2 - \beta_j^2)^2]}} \leq \frac{N^2}{2\sqrt{2}}. \tag{6.16}$$

Since $\beta_k^2 \leq \max(1, N^2)|k'|^2$, for $N \notin \Theta_3^\mu(L_1, L_3)$, we have, using (6.14) that:

$$\begin{aligned} \frac{1}{|\beta_j + \beta_l - \beta_k|} &\leq C(N)\frac{(|k'| + |l'| + |j'|)^3}{|P(\lambda)|} \\ &\leq C(N, L_1, L_3)\frac{(|k'| + |j'| + |l'|)^3}{\mu\xi_{(j,l)}}. \end{aligned} \tag{6.17}$$

We now choose $\xi_{(j,l)}$: For any $\eta > 0$ we can take

$$\xi_{(j,l)} = c(\eta)|j'|^{-2-\eta}|l'|^{-2-\eta}, \tag{6.18}$$

where $c(\eta) = \left(\sum_{j,l\in\mathbb{Z}^2} |j'|^{-2-\eta}|l'|^{-2-\eta}\right)^{-1}$. Substituting this into (6.17), we obtain the following bound:

$$\frac{1}{|\beta_j + \beta_l - \beta_k|} \leq C(N, L_1, L_3, \eta)\frac{(|k'| + |j'| + |l'|)^3}{\mu}|l'|^{2+\eta}|j'|^{2+\eta}, \quad N \notin \Theta_3^\mu(L_1, L_3).$$

We can now conclude with the following result:

Lemma 4.6.2. *Let $\eta > 0$ and $\mu > 0$; then for every L_1, $L_3 \in \mathbb{R}$ and $N \notin \Theta_3^\mu(L_1, L_3)$ we have $\beta_j + \beta_l - \beta_k \neq 0$ for all j, l, k with $j + l + k = 0$, and*

$$\frac{1}{|\beta_j + \beta_l - \beta_k|} \leq \max\Big(2,\ C(N, L_1, L_3, \eta)\frac{(|k'| + |j'| + |l'|)^3}{\mu}|l'|^{2+\eta}|j'|^{2+\eta}\Big). \tag{6.19}$$

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