Weak disorder for low dimensional polymers: The model of stable laws.

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Abstract

In this paper, we consider directed polymers in random environment with long range jumps in discrete space and time. We extend to this case some techniques, results and classifications known in the usual short range case. However, some properties are drastically different when the underlying random walk belongs to the domain of attraction of an α-stable law. For instance, we construct natural examples of directed polymers in random environment which experience weak disorder in low dimension.

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1 Introduction

Directed polymers in random environment can be viewed as random walks in a random potential, which is inhomogeneous both in time and space. We restrict here to the discrete case where the walk has discrete time and space $\mathbb{Z}^d, d \geq 1$. A number of motivations for considering these models are given in the physics litterature, in the context of growing random surfaces [20], of nonequilibrium steady states and phase transitions [21]. An increasing interest for these models is showing up in the mathematical community,

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and recent, striking results are the characterizations of the "weak disorder – strong disorder" and the "delocalization – localization" transitions given in [9] and [13]. We give precise definitions of these concepts in definition 3.1 and above corollary 6.4. Roughly, weak disorder and delocalization mean that the polymer behaves like the random walk, although strong disorder and localization mean that it is extremely influenced by the medium and it concentrates in just a few corridors where the medium is favorable. It is not known whether these two phase transitions coincide or not; However, a partial step is made for nearest neighbor walks in [11]. Heavy tailed environments are studied in [26], they cause a strong form of localization. Small dimensions are shown to be special [9], [13]: for nearest neighbor walks, strong disorder always in dimension 1 and 2. Moreover, it was recently proved [14] that the polymer is always localized in dimension 1.

One of the purposes of the present paper is to clarify the nature of the "weak disorder – strong disorder" transition. We show that strong disorder relates to an infinite number of meetings of two independent random walkers, for a variety of models. We also study the influence of the jump distribution on the "delocalization – localization" transition, and the interplay between jump tails, space dimension and existence of delocalized phase. An interesting contribution here is to construct natural, general examples of directed polymers in random environment which experience weak disorder in low dimension. The jumps have to be long-tailed. Since the pionneer work of Paul Lévy, the long-time behavior of such walks is known to be classified by stable laws, and our results will depend on the stable law which attracts the random walk.

Stable laws and Lévy flights model abnormal diffusion and mimic rapid turbulent transport. They also arise naturally from coarse-graining procedures for short range walks, e.g. hiting times. Lévy flights in a random potential are considered in [12] to analyze the $A+A \to \emptyset$ chemical reaction and explain the phenomenon of superfast reaction, when a small amount of potential disorder added to the turbulent fluid leads to an increase rate of the reaction. Dynamics of particle randomly moving along a disordered hetero-polymer subject to rapid conformal changes, lead to superdiffusive motion. A model is introduced in [8], corresponding to a Lévy flight in a random potential in chemical coordinates. Both these models have time-independent potential, but the time-dependent case simply corresponds to crossings in the presence of strong external fields. An instructive review of the ocurrence of Lévy processes in sciences as fluid mechanics, solid state physics, polymer chemistry and mathematical finance, is given in [27].

We will assume that the random walk belongs to the domain of attraction of an α-stable law for some $\alpha \in (0,2]$. This implies that random walk at large times n roughly scales like $n^{1/\alpha}$. In the case $\alpha < 2$ it also means that the tails $P(|\omega_1| > r)$ of individual jumps are of order $r^{-1/\alpha}$ for large r. The case $\alpha = 2$ includes the usual one where the walk is nearest neighbor. The medium is assumed to have finite exponential moments. We prove that weak disorder holds for $d = 1, \alpha < 1$ and $d = 2, \alpha < 2$, at least for high temperature. This is rather surprising in view of the results mentioned above in the simple random walk case. In dimension $d \geq 3$, our results here are not qualitatively different from those obtained for the simple random walk. For completeness we will state the results in all dimensions, but we emphasize low dimensions. All through, we assume that the environment has finite exponential moments.

The paper is organized as follows. In the next section, we introduce the model and recall some necessary facts on stable laws and their attraction domain. Then, the free energy is defined, together with the regimes of weak disorder and of strong disorder. We give sufficient conditions for weak disorder in section 4 together with some properties of the polymer there, and sufficient conditions for strong disorder in section 5. The last section is dedicated to the phase diagrams and localization properties. As already mentioned, we extend some constructions and results from nearest neighbor random walks to long range ones, we will not repeat proofs unless necessary but indicate precise references instead.

2 Long jumps polymers

We first need to state a few elementary facts on

2.1 Stable laws

These are all possible distributional limit of sums of i.i.d. random vectors up to renormalization. By definition, a stable law on $\mathbb{R}^d$ is such that, for all $n \geq 1$, if $X_1, \ldots X_n$ are i.i.d. with this law, there exist $a_n > 0$ and $b_n \in \mathbb{R}^d$ such that

$$\frac{X_1 + \ldots + X_n - b_n}{a_n} \quad \text{still has this law}$$

To avoid triviality we assume that the law is not a Dirac mass. It can be shown that there is a unique $\alpha \in (0, 2]$ such that for all n, the above a_n is $a_n = n^{1/\alpha}$. This exponent α is called the *index* (or characteristic exponent) of the stable law, and we also say that P is α-stable.
Except a few special cases, stable laws are complicated, they cannot be written in terms of simple functions, but their Fourier transforms are simple. An α-stable random vector S_α has characteristic functions

$$E(e^{iz\cdot S_\alpha}) = e^{\psi(z)} \ , \quad z \in \mathbb{R}^d$$

where the form of the exponent ψ depends on the index $\alpha \in (0, 2)$:

$$\text{for } \alpha = 2 \ , \quad \psi(z) = i\tau \cdot z - \tfrac{1}{2} z \cdot Az \tag{2.1}$$

with $\tau \in \mathbb{R}^d$ and A a $d \times d$ symmetric positive definite matrix;

$$\begin{aligned} \text{for } \alpha \neq 1, 2 \ , \quad \psi(z) &= \psi(z)_{\alpha,\tau,\sigma} \qquad (2.2) \\ &= i\tau \cdot z - \int_{\mathcal{S}^{d-1}} |z \cdot \xi|^\alpha \left(1 - i \tan \frac{\pi\alpha}{2} \operatorname{sgn}(z \cdot \xi)\right) \sigma(d\xi) \end{aligned}$$

with $\tau \in \mathbb{R}^d$ and σ a finite nonzero measure on the unit sphere $\mathcal{S}^{d-1}$ (the sign function is defined by sgn$(u) = 1$ for $u > 0$, sgn$(u) = -1$ for $u < 0$ and sgn(0)=0);

$$\begin{aligned} \text{for } \alpha = 1 \ , \quad \psi(z) &= \psi(z)_{\alpha,\tau,\sigma} \qquad (2.3) \\ &= i\tau \cdot z - \int_{\mathcal{S}^{d-1}} \left(|z \cdot \xi| + \frac{2i}{\pi} z \cdot \xi \log |z \cdot \xi|\right) \sigma(d\xi) \end{aligned}$$

with $\tau \in \mathbb{R}^d$ and σ a finite nonzero measure on the unit sphere $\mathcal{S}^{d-1}$. The vector τ (sometimes called the translate) and the measure σ (sometimes called the spherical part of the Lévy measure) are uniquely defined. They are location and asymmetry parameters. The law is invariant under rotations centered at some $x \in \mathbb{R}^d$ if and only if $x = \tau$ and σ a uniform measure on the sphere; The law is invariant under the central symmetry with center $x \in \mathbb{R}^d$ if and only if $x = \tau$ and σ is invariant under $\xi \mapsto -\xi$.
Here are some special cases where the density is simple. In the case $\alpha = 2$, the law is the d-dimensional Gaussian with mean τ and covariance matrix A, with density

$$x \mapsto (2\pi)^{-d/2} (\det A)^{-1/2} \exp\left\{-\frac{1}{2}(x - m)^* A^{-1}(x - m)\right\} \ .$$

For $c > 0$, $\tau \in \mathbb{R}^d$ and Γ the Euler function, the d-dimensional Cauchy law with density

$$x \mapsto \Gamma((d+1)/2) \frac{c}{\pi^{(d+1)/2}(|x-\tau|^2 + c^2)^{(d+1)/2}}$$

is stable with $\alpha = 1$, with characteristic exponent $\psi(z)_{1,\tau,\sigma}$ with σ the uniform measure of mass c.
A complete overview on stable laws and domains of attraction is given in the book [3], and a shorter presentation in [7]. For stable processes, we refer to the books [3] and [22].

2.2 The model

• *The random walk:* $(\{\omega_n\}_{n\geq 0}, P)$ is a random walk on $\mathbb{Z}^d$ starting from 0, ie, the variables $\omega_{k+1} - \omega_k (k = 1, 2, \ldots)$ are i.i.d. under P with $\omega_0 = 0$, and we denote by q their common law $q(x) := P(\omega_1 = x)$. We assume that q belongs to the domain of attraction of a stable law (on $\mathbb{R}^d$) with some index $\alpha \in (0, 2)$. More precisely, we assume that there exist $\alpha \in (0, 2], \tau \in \mathbb{R}^d, \sigma$ a finite nonzero measure on $\mathcal{S}^{d-1}$, and deterministic sequences $a_n > 0, b_n \in \mathbb{R}^d$, such that

$$P\left(\exp\{iz \cdot \frac{\omega_n - b_n}{a_n}\}\right) \longrightarrow \exp \psi_{\alpha,0,\sigma}(z) \tag{2.4}$$

for all z. To simplify our discussion, we will also assume that the limit is truly d-dimensional, ie, that it satisfies (4.17) below.
We now give a short account on our assumption (2.4), and recall some facts on the domain of attraction of stable laws, starting with the simpler case of dimension $d = 1$. The reader may also decide to skip these details in a first reading, and jump directly to the important example 2.1.

Dimension 1: In one dimension, this assumption can be described in terms of the tails of q. We follow the presentation of section 8.3 in [3]. The cases $\alpha \in (0, 2)$ and $\alpha = 2$ being different, we start with

1. Case $\alpha \in (0, 2)$. We let R_0 be the space of slowly varying functions in the sense of Karamata, i.e. of functions $\ell : [0, \infty) \mapsto [0, \infty)$ such that

$$\ell(sr)/\ell(r) \to 1 \quad (r \to \infty) \quad \forall s > 0.$$

Examples of such functions are constants, $\ln r$ or $\exp[\ln r/\ln\ln r]$. Assumption (2.4) is equivalent to

$$P(|\omega_1| \geq r) = r^{-\alpha}\ell(r) \tag{2.5}$$

for some $\ell \in R_0$, and

$$\frac{P(\omega_1 \leq -r)}{P(|\omega_1| \geq r)} \to q_* \ , \quad \frac{P(\omega_1 \geq r)}{P(|\omega_1| \geq r)} \to p_* \quad (r \to \infty) \tag{2.6}$$

where we note that $p_* + q_* = 1$. Then, the sequence a_n is of the form

$$a_n = n^{1/\alpha}\ell'(n)$$

with a slowly varying function $\ell' \in R_0$ which can be taken such that

$$P^2(e^{iz(\omega_1-\tilde{\omega}_1)/a_n})^n \to e^{-2|z|^\alpha} \quad (n \to \infty) \quad \forall z \in \mathbb{R}$$

In this case, the limit in (2.4) has

$$\sigma(+1) = p_* \ , \quad \sigma(-1) = q_*$$

(note that $S^0 = \{-1, +1\}$). Moreover, we can take

$$b_n = 0 \quad \text{for } \alpha < 1 \ ,$$

$$b_n = nP(\omega_1) \quad \text{for } \alpha \in (1, 2) \ ,$$

in which case ω_1 is integrable, and

$$b_n = nP[\psi(\omega_1/a_n)] \ , \quad \psi(t) = t(1+t^2)^{-1} \quad \text{for } \alpha = 1 \ .$$

The reader is refered to [3], pp. 343-347, for further details.

2. $\alpha = 2$: the assumption (2.4) is then equivalent to the function $r \mapsto P(|\omega_1|^2 : |\omega_1| \leq r)$ being slowly varying. A sufficient condition is existence of a second moment of ω_1, and (2.4) is simply the standard central limit theorem.

Dimension $d \geq 2$: Let $\varphi(z) = P(e^{iz\cdot\omega_1})$. The characterization of assumption (2.4) is known in terms of the characteristic function φ of the law q and of slowly varying functions (see theorem 2.6.5 in [19] and [1], corollary 1–2 in section 2). Assumption (2.4) with a truly d-dimensional limit in the sense of (4.17) is equivalent, for $\alpha \in (0,2) \setminus \{1\}$, to

$$\log \varphi(z) = \begin{cases} \phi_{\alpha,0,\sigma}(z)\ell(1/|z|) + iz \cdot \tau + o(|z|^\alpha \ell(1/|z|)) & \text{if } \alpha > 1 \,, \\ \phi_{\alpha,0,\sigma}(z)\ell(1/|z|) + o(|z|^\alpha \ell(1/|z|)) & \text{if } \alpha < 1 \,. \end{cases}$$

This is for $\alpha \neq 1$, the case $\alpha = 1$ being more complicated. We simply mention that, for a *symmetric* law q and $\alpha = 1$, assumption (2.4) with (4.17), is equivalent to

$$\ln \varphi(z) = -\int_{\mathcal{S}^{d-1}} |z \cdot \xi| \sigma(d\xi) \times \ell(1/|z|) + o(|z|\ell(1/|z|)) \,,$$

and we refer to corollary 2 in section 2 of [1], for the somewhat cumbersome general case.
We give a generic example where the assumption holds.

Example 2.1 *Let $d = 1$ or 2, and q be symmetric with*

$$q(x) = b(|x|^{-d-\alpha} + \varepsilon(x)) \quad \text{as } |x| \to \infty \,, \; x \in Z^d \tag{2.7}$$

with $\alpha \in (0,2)$. Then, the assumption (2.4) hold true with $\tau = 0$ and σ uniform, using [1],[19]), and estimates of the characteristic function (see e.g. (2.13) and (3.11) in [18]).

• *The random environment:* $\eta = \{\eta(n,x) : n \in \mathbb{N},\ x \in \mathbb{Z}^d\}$ is a sequence of r.v.'s which are real valued, non-constant, and i.i.d.(independent identically distributed) r.v.'s defined on a probability space $(\Omega_\eta, \mathcal{G}, Q)$ such that

$$Q[\exp(\beta\eta(n,x))] < \infty \quad \text{for all } \beta \in \mathbb{R}.$$

We then let $\lambda(\beta) = \ln Q[\exp(\beta\eta(n,x))]$.

• *The polymer measure:* For any $n > 0$, define the probability measure μ_n on the path space $(\Omega_\omega, \mathcal{F})$ by

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp\{\beta H_n(\omega)\}\, P(d\omega), \tag{2.8}$$

where $\beta > 0$ is a parameter (the inverse temperature), where

$$H_n(\omega) = H_n(\eta,\omega) = \sum_{1 \le j \le n} \eta(j, \omega_j) \tag{2.9}$$

and

$$Z_n = Z_n(\beta, \eta) = P\left[\exp\left(\beta \sum_{1\le j\le n} \eta(j, \omega_j)\right)\right] \tag{2.10}$$

is the the partition function.

3 Free energy, and the natural martingale

The partition function is random, but it is self-averaging as n increases.

Proposition 3.1 *Let $\alpha \in (0, 2]$ arbitrary. As $n \to \infty$, the quenched free energy converges to a deterministic constant:*

$$\frac{1}{n} \ln Z_n \longrightarrow p(\beta) := \lim_{m\to\infty} \frac{1}{m} Q \ln Z_m \tag{3.11}$$

$Q-$a.s. and in L^q $(1 \le q < \infty)$. Moreover, we have the annealed bound

$$p(\beta) \le \lambda(\beta) \tag{3.12}$$

□ The proof in the case of a simple random walk P (proof of prop. 2.5 in [13], pp. 720–722) covers the general case of $\alpha \in (0, 2]$ without change. The last inequality comes from Jensen inequality, which writes

$$\frac{1}{m} Q \ln Z_m \le \frac{1}{m} \ln QZ_m = \lambda(\beta)$$

■

The sequence $(W_n, n \ge 1)$ defined by

$$W_n = Z_n \exp(-n\lambda(\beta)) \tag{3.13}$$

is a positive, mean 1, martingale with respect to the environmental filtration $(\mathcal{G}_n) = \sigma(\eta(t, x), t \le n, x \in \mathbb{Z})$. This was noticed first by Bolthausen [6]. By the martingale convergence theorem, the limit

$$W_\infty = \lim_{n\nearrow\infty} W_n$$

exists Q-a.s. It is clear that the event $\{W_\infty = 0\}$ is measurable with respect to the tail σ-field $\bigcap_{n\ge1} \sigma[\eta(j, x)\ ;\ j \ge n,\ x \in \mathbb{Z}^d]$. By Kolmogorov's zero-one law every event in the tail σ-field has probability 0 or 1. Hence, there are only two possibilities for the positivity of the limit

$$Q\{W_\infty > 0\} = 1 \,, \tag{3.14}$$

or

$$Q\{W_\infty = 0\} = 1 \,. \tag{3.15}$$

Definition 3.1 *The above situations (3.14) and (3.15) will be called the* **weak disorder** *phase and the* **strong disorder** *phase, respectively. In the first case, $p = \lambda$.*

In corollary 6.3 at the end of the paper we explain why it is important to decide if the inequality in (3.12) is an equality or not.

4 Existence of weak disorder, properties

We need now to consider on the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, the probability measure $P^{\otimes 2} = P^{\otimes 2}(d\omega, d\widetilde{\omega})$, that we will view as the distribution of the couple $(\omega, \widetilde{\omega})$ with $\widetilde{\omega} = (\widetilde{\omega}_k)_{k\geq 0}$ an independent copy of $\omega = (\omega_k)_{k\geq 0}$.
When P satisfies (2.7), we see that the random walk $\omega - \tilde{\omega}$ is attracted by the symmetric α-stable law. Precisely, with a_n from (2.7), we have

$$P^{\otimes 2}\left(\exp\{iz \cdot \frac{\omega_n - \tilde{\omega}_n}{a_n}\}\right) \longrightarrow \exp\left(\psi_{\alpha,0,\sigma}(z) + \psi_{\alpha,0,\sigma}(z)\right) = \exp \psi_{\alpha,0,\sigma'}(z) \tag{4.16}$$

with $\sigma'(B) = \sigma(B) + \sigma(-B)$ for all Borel subset B of $\mathcal{S}^{d-1}$.
For later purposes, it is essential to observe that the difference $\omega - \tilde{\omega}$ is a transient random walk – i.e., $N_\infty := \sum_{n=1}^\infty \mathbf{1}_{\omega_n = \tilde{\omega}_n} < \infty$ a.s. – in the three following cases:

(i) $d = 1$ and $\alpha \in (0, 1)$,

(ii) $d = 2$ and $\alpha \neq 2$,

(iii) $d \geq 3$ and $\alpha \in (0, 2]$, provided the limit is truly d-dimensional. This extra assumption for $\alpha \in (0,2)$ means that the linear space spanned by the support of the measure σ is $\mathbb{R}^d$, and for $\alpha = 2$ that the covariance matrix A is non-degenerate,

$$\text{Vect}(\operatorname{supp}\sigma) = \mathbb{R}^d \quad \text{or} \quad \text{rank}(A) = d\ , \tag{4.17}$$

according to the case $\alpha < 2$ or $\alpha = 2$. Indeed, with $\phi(z) = P^{\otimes 2}[\exp\{iz\cdot(\omega_1 - \tilde{\omega}_1)\}]$, (4.16) amounts to

$$\phi(z) = \exp\left\{\psi_{\alpha,0,\sigma'}(z)\ell\Big(\frac{z}{|z|}, \frac{1}{|z|}\Big)\right\} \tag{4.18}$$

with $\ell(\xi,\cdot)$ a slowly varying function depending continuously on $\xi \in \mathcal{S}^{d-1}$. Since it holds, under (4.17),

$$\psi_{\alpha,0,\sigma'}(z) \le C|z|^\alpha\ ,$$

we have

$$\int_{[-\pi,\pi[^d} \frac{dz}{1-\phi(z)} \quad \begin{cases} = \infty & \text{if} \quad \alpha > d \\ < \infty & \text{if} \quad \alpha < d \end{cases} \tag{4.19}$$

Applying the Chung-Fuchs criterion (P1 in section 8 of [24]), we see that the walk $\omega - \tilde{\omega}$ is transient in the second case, and then

$$\pi(p) := P^{\otimes 2}(\exists n \ge 1\ \omega_n - \tilde{\omega}_n = 0) < 1$$

Remark 4.1 *The walk $\omega - \tilde{\omega}$ is recurrent when $\alpha > d$. The border case $\alpha = d$ is more subtle: Transience may hold or may not hold depends on the slowly varying term in (4.18). In the positive, the validity of the the next two theorems will extend to critical cases $\alpha = d = 1$, $\alpha = d = 2$.*

Theorem 4.1 Weak disorder region in dimension 1, 2. *In addition to (4.17), assume either (i) $d = 1$ and $\alpha \in (0,1)$, or (ii) $d = 2$, $\alpha \neq 2$, or (iii) $d \ge 3$ and $\alpha \in (0,2]$. Then, for all β such that*

$$\lambda(2\beta) - 2\lambda(\beta) < \ln 1/\pi(p)\ , \tag{4.20}$$

we have $W_\infty > 0$ Q-a.s.

The result may come as a surprise, since for P the simple random walk, it was proved that $W_\infty = 0$ Q-a.s. [9] [13], and even that $p < \lambda$ [14]. The method used in the last reference is based on comparisons with polymers models on trees. It is impossible to extend it to long range jumps, although related ideas can be –and will be– in the sequel, see (5.24).
Following the techniques of [4] using a conditional second moment, one could extend the validity of the result to a domain in β larger than (4.20).

□ Following [6] we compute the L^2-norm of the martingale W_n. To do so, we represent W_n^2 in terms of an independent couple $(\omega, \widetilde{\omega})$ introduced above.

$$\begin{aligned} Q[W_n^2] &= Q\left[P^{\otimes 2}\prod_{t=1}^n e^{\beta[\eta(t,\omega_t)+\eta(t,\tilde{\omega}_t)]-2\lambda(\beta)}\right] \\ &= P^{\otimes 2}\left[\prod_{t=1}^n \left(e^{\lambda(2\beta)-2\lambda(\beta)}\mathbf{1}_{\omega_t=\tilde{\omega}_t}+\mathbf{1}_{\omega_t\neq\tilde{\omega}_t}\right)\right] \\ &= P^{\otimes 2}\left[e^{\gamma_1 N_n}\right] , \end{aligned}$$

with $\gamma_1 = \lambda(2\beta) - \lambda(\beta)$, and N_n the number of intersections of the paths $\omega, \tilde{\omega}$ up to time n,

$$N_n = N_n(\omega, \tilde{\omega}) = \sum_{t=1}^n \mathbf{1}_{\omega_t=\tilde{\omega}_t} \tag{4.21}$$

As $n \to \infty$, $N_n \nearrow N_\infty$, and by monotone convergence $Q[W_n^2] \nearrow P^{\otimes 2}\left[e^{CN_\infty}\right]$. In the cases under consideration, the random variable N_∞ is geometrically distributed

$$P^{\otimes 2}(N_\infty = n) = \pi(\sigma)^n[1-\pi(\sigma)] , \quad n \geq 0 ,$$

with $\pi(p)$ the probability of return defined above the theorem. Hence it has finite exponential moments

$$P^{\otimes 2}[\exp \gamma_1 N_\infty] < \infty \iff \gamma_1 < \ln 1/\pi(p)$$

Therefore, when $\lambda(2\beta) - 2\lambda(\beta) < \ln 1/\pi(p)$, the martingale W_n is bounded in L^2, and by the classical L^2-convergence theorem, it converges in L^2 to a limit, which is necessarily equal to W_∞. So $QW_\infty = \lim_n QW_n = 1$, which excludes the possibility that the limit vanishes. ■

Inside the subset of the weak disorder region determined by the condition (4.20), the fluctuations of the path remain similar to those of P.

Theorem 4.2 *Assume either (i) $d = 1$ and $\alpha \in (0,1)$, or (ii) $d = 2$, $\alpha \neq 2$, or (iii) $d = 3$ and $\alpha \in (0,2]$, and assume (4.17). When (4.20) holds, we have for all bounded continuous function $g : \mathbb{R} \to \mathbb{R}$,*

$$\mu_n \left[g\left(\frac{\omega_n - b_n}{a_n}\right)\right] \to \nu(g)$$

in Q-probability as $n \to \infty$, where ν is the α-stable law with characteristic function $\psi_{\alpha,0,\sigma}$.

□ We let $\nu_n(\cdot) = P[(\omega_n - b_n)/a_n \in \cdot]$.

$$\begin{aligned}
& Q\left(\left|\mu_n\left[g\left(\frac{\omega_n - b_n}{a_n}\right)\right] - \nu_n(g)\right|^2 W_n^2\right) \\
= \; & P^2 Q\left(e^{\beta H_n(\omega) + \beta H_n(\tilde{\omega}) - 2n\lambda}\left[g\left(\frac{\omega_n - b_n}{a_n}\right) - \nu_n(g)\right]\left[g\left(\frac{\tilde{\omega}_n - b_n}{a_n}\right) - \nu_n(g)\right]\right) \\
= \; & P^2\left(e^{\gamma_1 N_n}\left[g\left(\frac{\omega_n - b_n}{a_n}\right) - \nu_n(g)\right]\left[g\left(\frac{\tilde{\omega}_n - b_n}{a_n}\right) - \nu_n(g)\right]\right) \qquad (4.22)
\end{aligned}$$

We know that, under P^2, the r.v. N_n converges to N_∞ a.s., and that $(\omega_n - b_n)/a_n$ – and similarly $(\tilde{\omega}_n - b_n)/a_n$ – converges to ν in law. Now, we claim that, under P^2, the triple

$$(N_n, (\omega_n - b_n)/a_n, (\tilde{\omega}_n - b_n)/a_n) \xrightarrow{\text{law}} (N, S, \tilde{S}) \qquad (4.23)$$

with $(N, S, \tilde{S})$ an independent triple where N has the same law as N_∞, S and $\tilde{S}$ have the law ν. The proof of this fact makes use of the observation that

$$\sup_{n \geq m} P^2(N_n \neq N_m) \to 0 \text{ as } n \to \infty$$

since $N_n \nearrow N_\infty < \infty$ a.s. Fix $m \geq 1$ and $f, g, \tilde{g}$ continuous and bounded. For all $n \geq m$, we write

$$\begin{aligned}
& P^2\left[f(N_n) g(\frac{\omega_n - b_n}{a_n}) \tilde{g}(\frac{\omega_n - b_n}{a_n})\right] \\
= \; & P^2\left[f(N_n) g(\frac{\omega_n - b_n}{a_n}) \tilde{g}(\frac{\omega_n - b_n}{a_n}) \mathbf{1}_{N_n = N_m}\right] + \varepsilon(n,m) \\
= \; & P^2\left[f(N_m) g(\frac{\omega_n - b_n}{a_n}) \tilde{g}(\frac{\omega_n - b_n}{a_n}) \mathbf{1}_{N_n = N_m}\right] + \varepsilon(n,m)
\end{aligned}$$

$$
\begin{aligned}
&= P^2\left[f(N_m)g(\frac{\omega_n-\omega_m-b_n}{a_n})\tilde{g}(\frac{\omega_n-\omega_m-b_n}{a_n})\mathbf{1}_{N_n=N_m}\right]+\varepsilon'(n,m)\\
&= P^2\left[f(N_m)g(\frac{\omega_n-\omega_m-b_n}{a_n})\tilde{g}(\frac{\omega_n-\omega_m-b_n}{a_n})\right]+\varepsilon"(n,m)\\
&= P^2[f(N_m)]\times P\left[g(\frac{\omega_n-\omega_m-b_n}{a_n})\right]\times P\left[\tilde{g}(\frac{\omega_n-\omega_m-b_n}{a_n})\right]+\varepsilon"(n,m)\ ,
\end{aligned}
$$

which equalities define the terms $\varepsilon(n,m), \varepsilon'(n,m), \varepsilon''(n,m)$ on their first occurence. Here,

$$
|\varepsilon(n,m)| \leq \|f\|_\infty \|g\|_\infty \|\tilde{g}\|_\infty P(N_n \neq N_m)
$$

tends to 0 as $m \to \infty$ uniformly in $n \geq m$, $\varepsilon'(n,m) - \varepsilon(n,m) \to 0$ as $n \to \infty$ for all fixed m, and $\sup_{n\geq m} \varepsilon''(n,m) \to 0$ as $m \to \infty$. The last equality comes from independence in the increments of the random walks, and of the two random walks ω and $\tilde{\omega}$. Hence, letting $n \to \infty$ and then $m \to \infty$, we get

$$
P^2\left[f(N_n)g(\frac{\omega_n-b_n}{a_n})\tilde{g}(\frac{\omega_n-b_n}{a_n})\right] \to P^2[f(N_\infty)]\times\nu[g]\times\nu[\tilde{g}]
$$

which proves (4.23). Coming back to (4.22), and since $P^2(e^{\gamma N_n}) < \infty$ for some small enough $\gamma > \gamma_1$, (4.23) implies that

$$
Q\left(\left|\mu_n\left[g\left(\frac{\omega_n-b_n}{a_n}\right)\right]-\nu_n(g)\right|^2 W_n^2\right) \to P^2(e^{\gamma_1 N_\infty})\left[\nu(g)-\nu(g)\right]^2 = 0
$$

Since W_n^{-2} converges to a finite limit, it is bounded in probability, this yields the desired convergence in probability. ■

Remark 4.2 *(i) In the case when P is the nearest neighbor simple random walk, the condition (4.20) implies a quenched central limit theorem, ie, that central limit theorem holds for a.e. realization of the environment [6]. Our result here is weaker. Due to the lack of moments for the long jumps here, the natural martingales which can be used in the standard case are not defined in the present setup.*
(ii) In the case when P is the nearest neighbor simple random walk, it was shown in [17] that central limit theorem holds (in a weak form at least) as soon as $W_\infty > 0$. Then it is questionable whether in the model of the present

paper, weak disorder implies convergence of the renormalized position of the polymer to an α-stable law. We leave the question open.
(iii) When P is the simple random walk, many other results are known under condition (4.20), for instance:

1. *Local limit theorem for the polymer measure [23], [25];*
2. *How does the polymer depends on the environment ? (This question is answered in [5] by computing the random corrections to gaussian for cumulants of the polymer position.)*

We leave open the question of which is the counterpart of these results for long range random walks we consider here ($\alpha \in (0,2)$).

We end this section with a model where weak disorder holds at all temperature and all dimension. Viewed as a growing random surface, it does not have a roughening transition, and consequently in this respect, it does not belong to the Kardar-Parisi-Zhang (KPZ) class [20].

Example 4.1 *Bernoulli environment. The case when $\eta(t,x) = 1$ or 0 with probability q and $1-q$ respectively, is remarkable since weak disorder may hold at all temperature. Here we find $\lambda = \ln[pe^{\beta} + (1-q)]$, and we see from direct computations that*

$$\lim_{\beta \nearrow \infty} \gamma_1(\beta) = -\ln(q).$$

Hence, (4.20) holds for all $\beta \geq 0$ if $q > \pi(p)$. Theorem 4.1 shows that, in this case, weak disorder holds for all $\beta \geq 0$, and Theorem 4.2 shows that the polymer position at time n still fluctuates at order $n^{1/\alpha}$.

5 Existence of strong disorder

The next result gives a sufficient condition for $p < \lambda$, which implies strong disorder.

Proposition 5.1 *Let $\alpha \in (0,2]$ and d arbitrary. If*

$$\beta\lambda'(\beta) - \lambda(\beta) > -\sum_{k \in \mathbb{Z}} q(k) \ln q(k) \tag{5.24}$$

then $p < \lambda$.

We note that the important quantity is here the entropy $-\sum_x q(x)\ln q(x)$ of the walk, which does not directly relates to the recurrence/transience behavior of the walk. The entropy is always finite under our assumptions on q.

Example 5.1 *Gaussian environment. If η is standard gaussian $\mathcal{N}(0,1)$, then $\gamma_1(\beta)=\beta^2$ and hence (4.20) holds if $\beta<\sqrt{\ln(1/\pi_d)}$, though (5.24) holds for all $\beta>-\sum_x q(x)\ln q(x)$. In this case, a phase transition takes place between weak and strong disorder.*

□ Note that

$$Z_n=\sum_{x\in\mathbb{Z}} q(x)e^{\beta\eta(1,x)}Z_{1,n}^x\ , \tag{5.25}$$

where $Z_{1,n}(x)$ has the same law as Z_{n-1}. Let $\theta\in(0,1)$. By the subadditive estimate

$$(u+v)^\theta\le u^\theta+v^\theta\ ,\quad u,v>0\ ,$$

we get

$$Z_n^\theta\le\sum_{x\in\mathbb{Z}} q(x)^\theta e^{\beta\theta\eta(1,x)}(Z_{1,n}^x)^\theta$$

Since $Z_{1,n}^x$ has the same law as Z_{n-1}, we obtain a bound on $u_n=QZ_n^\theta$:

$$\begin{aligned} u_n &\le \left[\sum_{x\in\mathbb{Z}} q(x)^\theta\exp\{\lambda(\beta\theta)\}\right]u_{n-1}\\ &\le \left[\sum_{x\in\mathbb{Z}} q(x)^\theta\exp\{\lambda(\beta\theta)\}\right]^n \end{aligned}$$

by induction. Now, observe that

$$Q\frac{1}{n}\ln Z_n=Q\frac{1}{n\theta}\ln Z_n^\theta\le\frac{1}{n\theta}\ln QZ_n^\theta$$

which, combined with the previous bound on $u_n=QZ_n^\theta$, yields

$$p\le\inf_{\theta\in(0,1)}\left\{\frac{1}{\theta}v(\theta)\right\}\ ,\quad v(\theta)=\left[\lambda(\beta\theta)+\ln\sum_{x\in\mathbb{Z}} q(x)^\theta\right]\ .$$

Since the function v is convex and positive at 0, there are only possibilities for the infimum. If the derivative of $v(\theta)/\theta$ is positive at $\theta=1$, the infimum is acheived at some $\theta \in (0,1)$, and is strictly less than $\lambda(\beta)$ (which is the value at 1). On the contrary, if the derivative of $v(\theta)/\theta$ is less or equal to 0, the infimum is for $\theta \to 1$ and the value is $\lambda(\beta)$. Finally, we compute

$$\frac{d}{d\theta}\frac{v(\theta)}{\theta}_{|\theta=1} = \beta\lambda'(\beta) - \lambda(\beta) + \sum_{k\in\mathbb{Z}} q(k)\ln q(k) > 0$$

which proves the claim. ∎

6 Phase diagram, transitions and localization

The sum in (5.24) is always finite with our choice of q. For unbounded η's, one easily checks that $\lim \beta\lambda'(\beta) - \lambda(\beta) = +\infty$ as $\beta \to +\infty$, so that this condition (5.24) will be checked for large β. On the other hand, the set of β's such that $p = \lambda$ is an interval (possibly with length 0). Indeed it is readily checked that the argument in Th. 3.2.(b) in [17] for the case $\alpha = 2$, extends to all values of $\alpha \in (0,2]$. To summarize,

Theorem 6.1 Phase diagram. *The exists a $\beta_c \in [0,\infty)$ such that $p = \lambda$ for $\beta \in [0,\beta_c]$ and $p < \lambda$ for $\beta > \beta_c$. Moreover, under the assumption (4.17), we have*

$$\beta_c > 0$$

in the following cases: (i) $d = 1$ and $\alpha \in (0,1)$, or (ii) $d = 2$, $\alpha \neq 2$, or (iii) $d = 3$ and $\alpha \in (0,2]$.

When $\alpha = 2$ it is well known [9] [13] that the discrepancy between p and λ relates to localization property of the polymer. In fact the computations in the special case $\alpha = 2$ still work for all α (e.g. [13], th. 2.1 and proof pp. 711–715). Then, we have

Theorem 6.2 *Let $\beta \neq 0$, α and d arbitrary. Define*

$$I_n = \mu_{n-1}^{\otimes 2}(\omega_n = \widetilde{\omega}_n)$$

Then,

$$\{W_\infty = 0\} = \Big\{\sum_{n\geq 1} I_n = \infty\Big\}, \quad Q\text{-a.s.} \tag{6.26}$$

Moreover, if $Q\{W_\infty = 0\} = 1$, there exist $c_1, c_2 \in (0,\infty)$ such that Q-a.s.,

$$c_1 \sum_{1\leq k\leq n} I_k \leq -\ln W_n \leq c_2 \sum_{1\leq k\leq n} I_k \qquad \text{for large enough } n\text{'s}. \tag{6.27}$$

We define the mass J_n of the favourite exit point for the polymer,

$$J_n = \max_{x \in \mathbb{Z}^d} \mu_{n-1}(\omega_n = x)$$

We view $J_n \in [0,1]$ as an index of localization of the polymer. When J_n vanishes, the polymer is delocalized in the sense that it spreads over all sites; This is the case for $\beta = 0$. On the other hand, when J_n does not vanish, the polymer is strongly localized in the sense it has a significant probability to go through a few special sites. More precisely,

Definition 6.1 *The polymer is delocalized if*

$$\lim_{n\to\infty} J_n = 0 \qquad Q-\text{a.s.}\ ,$$

and localized if

$$\text{Cesaro}-\liminf_{n\to\infty} J_n > 0 \qquad Q-\text{a.s.}$$

where the liminf is taken in the Cesaro sense, i.e.

$$\text{Cesaro}-\liminf_{n\to\infty} J_n = \liminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} J_t\ .$$

In view of the relations

$$J_n^2 \le I_n \le J_n\ ,$$

and following [13], it is not difficult to derive, from Theorem 6.2 and Proposition 3.1, the following corollary.

Corollary 6.3 *We have the equivalences*

$$p = \lambda \iff \text{delocalization}$$

and

$$p < \lambda \iff \text{localization}$$

In particular, for all β, either delocalization occurs or localization occurs. In other words, there is another dychotomy: for all fixed β, either J_n vanishes for almost every environment, or for almost every environment, $\liminf J_n$ is positive in the Cesaro sense.

Now, from Theorem 4.1 and Proposition 5.1 we derive

Corollary 6.4 *(a) Assume (4.17), and either (i) $d = 1$ and $\alpha \in (0,1)$, or (ii) $d = 2$, $\alpha \neq 2$, or (iii) $d \ge 3, \alpha \in (0,2]$. Then, delocalization holds for all β with (4.20).*
(b) Under the condition (5.24), localization holds.

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