Regular rapidly decreasing nonlinear generalized functions. Application to microlocal regularity

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Abstract

We present new types of regularity for nonlinear generalized functions, based on the notion of regular growth with respect to the regularizing parameter of the Colombeau simplified model. This generalizes the notion of $\mathcal{G}^\infty$-regularity introduced by M. Oberguggenberger. A key point is that these regularities can be characterized, for compactly supported generalized functions, by a property of their Fourier transform. This opens the door to microanalysis of singularities of generalized functions, with respect to these regularities. We present a complete study of this topic, including properties of the Fourier transform (exchange and regularity theorems) and relationship with classical theory, via suitable results of embeddings.

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1 Introduction

The various theories of nonlinear generalized functions are suitable frameworks to set and solve differential or integral problems with irregular operators or data. Even for linear problems, these theories are efficient to overcome some limitations of the distributional framework. We follow in this paper the theory introduced by J.-F. Colombeau [2], [3], [10], [19]. To be short, a special Colombeau type algebra is a factor space $\mathcal{G} = \mathcal{X}/\mathcal{N}$ of moderate modulo negligible nets. The moderateness (respectively the negligibility) of nets is defined by their asymptotic behavior when a real parameter ε tends to 0.

A local and microlocal analysis of singularities of nonlinear generalized functions has been developed during the last decade, based on the notion of $\mathcal{G}^\infty$-regularity [20]. A generalized function is $\mathcal{G}^\infty$-regular if it has uniform growth bounds, with respect to the regularization parameter ε, for all derivatives. In fact, this notion appears to be the exact generalization of the C^∞-regularity for distributions, in the sense given by the result of [20] which asserts that $\mathcal{G}^\infty \cap \mathcal{D}'$ is equal to C^∞.

In this paper we include $\mathcal{G}^\infty$ and $\mathcal{G}$ in a new framework of spaces of $\mathcal{R}$-*regular nonlinear generalized functions*, in which the growth bounds are defined with the help of spaces $\mathcal{R}$ of sequences satisfying natural conditions of stability. One main property of those spaces is that the elements with compact support can be characterized by a "$\mathcal{R}$-property" of their Fourier transform. (Those Fourier Transforms belong to some regular subspaces of spaces of *rapidly* *decreasing generalized functions* [8], [22].) Thus, the parallel is complete with the C^∞-regularity of compactly supported distributions. Moreover, from this characterization, we deduce that the microlocal behavior of a generalized function with respect to a given $\mathcal{R}$-regularity is completely similar to the one of a distribution with respect to the C^∞-regularity. In particular, we can handle the $\mathcal{R}$-wavefront of an element of $\mathcal{G}$ as the C^∞ one of a distribution. Finally, the $\mathcal{G}^\infty$-regularity for an element of $\mathcal{G}$ appears as a remarkable particular case.

With this new notion of $\mathcal{R}$-regularity, we enlarge the possibility for the study of the propagation of singularities through differential and pseudo differential operators, since we give a less restrictive framework than the $\mathcal{G}^\infty$-regularity. This will be studied in forthcoming papers.

Let us also quote that spaces of $\mathcal{R}$-regular generalized functions have been used in a problem of Schwartz kernel type theorem. More precisely, we showed in [4] that some nets of linear maps (parametrized by $\varepsilon \in (0,1]$), satisfying some growth conditions similar to those introduced for $\mathcal{R}$-regular spaces, give rise to linear maps between spaces of generalized functions. Moreover, those maps can be represented by generalized integral kernel on some special $\mathcal{R}$-regular subspaces of $\mathcal{G}(\Omega)$ in which the growth bounds are at most sublinear with respect to l. This reinforces the interest of this framework.

The paper is organized as follows. In section 2, we introduce the spaces of $\mathcal{R}$-regular generalized functions and we precise some classical results about the embedding of $\mathcal{D}'$ into these spaces. Section 3 is devoted to the study of the space $\mathcal{G}_\mathcal{S}$ of rapidly decreasing generalized functions. In particular, we show that $\mathcal{O}_C^{'}$, the space of rapidly decreasing distributions, is embedded in $\mathcal{G}_\mathcal{S}$. Thus, $\mathcal{G}_\mathcal{S}$ plays for $\mathcal{O}_C^{'}$ the role that $\mathcal{G}$ plays for $\mathcal{D}'$. Section 4 contains the material related to Fourier transform of elements of $\mathcal{G}_\mathcal{S}$ and especially an exchange theorem which is, in the context of $\mathcal{R}$-regularity, an analogon and a generalization of the classical exchange theorem between $\mathcal{O}_C'$ and $\mathcal{O}_M$. Section 5 gives the above mentioned characterization by Fourier transform of compactly supported $\mathcal{R}$-regular generalized functions whereas, in section 6, we present the $\mathcal{R}$-local and $\mathcal{R}$-microlocal analysis of generalized functions.

2 The sheaf of Colombeau simplified algebras and related subsheaves

2.1 Sheaves of regular generalized functions

Notation 1 *For two sequences $(N_1, N_2) \in \left(\mathbb{R}_+^\mathbb{N}\right)^2$, we say that N_1 is smaller or equal to N_2 and note $N_1 \preccurlyeq N_2$ iff $\forall n \in \mathbb{N} \quad N_1(n) \leq N_2(n)$.*

Definition 1 *We say that a subspace $\mathcal{R}$ of $\mathbb{R}_+^\mathbb{N}$ is* regular *if $\mathcal{R}$ is non empty and*
(i) $\mathcal{R}$ is "overstable" by translation and by maximum

$$\forall N \in \mathcal{R},\ \ \forall \left(k,k'\right) \in \mathbb{N}^2,\ \ \exists N' \in \mathcal{R},\ \ \forall n \in \mathbb{N} \qquad N(n+k) + k' \leq N'(n), \tag{1}$$

$$\forall N_1 \in \mathcal{R},\ \ \forall N_2 \in \mathcal{R},\ \ \exists N \in \mathcal{R},\ \ \forall n \in \mathbb{N} \qquad \max\left(N_1(n), N_2(n)\right) \leq N(n). \tag{2}$$

(ii) For all N_1 and N_2 in $\mathcal{R}$, there exists $N \in \mathcal{R}$ such that

$$\forall (l_1, l_2) \in \mathbb{N}^2 \qquad N_1(l_1) + N_2(l_2) \leq N(l_1 + l_2). \tag{3}$$

Example 1
(i) The set $\mathcal{B}$ of bounded sequences and the set $\mathcal{A}$ of affine sequences are regular subsets of $\mathbb{R}_+^\mathbb{N}$, which is itself regular.
(ii) The set $\mathcal{L}_{og} = \left\{N \in \mathbb{R}_+^\mathbb{N} \mid \exists b \in \mathbb{R}_+,\ N : n \mapsto \ln n + b\right\}$ is not regular ((3) is not satisfied),

whereas $\mathcal{L}_{og}^1 = \left\{ N \in \mathbb{R}_+^{\mathbb{N}} \,\middle|\, \exists\,(a,b) \in \mathbb{R}_+^2,\ N : n \mapsto a\ln n + b \right\}$ *is regular. ((3) comes, for example, from* $\ln x + \ln y \leq 2\ln(x+y)$*, for* $x > 0$ *and* $y > 0$*.)*

Let Ω be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$) and consider the algebra $\mathrm{C}^\infty(\Omega)$ of complex valued smooth functions, endowed with its usual topology. This topology can be described by the family of seminorms $(p_{K,l})_{K \Subset \Omega, l \in \mathbb{N}}$ defined by

$$p_{K,l}(f) = \sup_{x \in K, |\alpha| \leq l} |\partial^\alpha f(x)|.$$

For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, we set

$$\mathcal{X}^{\mathcal{R}}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \,\middle|\, \forall K \Subset \Omega,\ \exists N \in \mathcal{R},\ \forall l \in \mathbb{N}\ \ p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(l)}\right) \text{ as } \varepsilon \to 0 \right\}$$
$$\mathcal{N}^{\mathcal{R}}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \,\middle|\, \forall K \Subset \Omega,\ \forall m \in \mathcal{R},\ \forall l \in \mathbb{N}\ \ p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{m(l)}\right) \text{ as } \varepsilon \to 0 \right\}.$$

Proposition 1
(i) *For any regular subspace* $\mathcal{R}$ *of* $\mathbb{R}_+^{\mathbb{N}}$*, the functor* $\Omega \to \mathcal{X}^{\mathcal{R}}(\Omega)$ *defines a sheaf of differential algebras over the ring*

$$\mathcal{X}(\mathbb{C}) = \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]} \,\middle|\, \exists q \in \mathbb{N}\ \ |r_\varepsilon| = \mathrm{O}\left(\varepsilon^{-q}\right) \text{ as } \varepsilon \to 0 \right\}.$$

(ii) *The functor* $\mathcal{N}^{\mathcal{R}} : \Omega \to \mathcal{N}^{\mathcal{R}}(\Omega)$ *defines a sheaf of ideals of the sheaf* $\mathcal{X}^{\mathcal{R}}(\cdot)$*.*
(iii) *For any regular subspaces* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *of* $\mathbb{R}_+^{\mathbb{N}}$*, with* $\mathcal{R}_1 \subset \mathcal{R}_2$*, the sheaf* $\mathcal{X}^{\mathcal{R}_1}(\Omega)$ *is a subsheaf of the sheaf* $\mathcal{X}^{\mathcal{R}_2}(\Omega)$*.*

Proof. We split the proof in two parts.
(a) *Algebraical properties.*- Let us first show that for any open set $\Omega \subset \mathbb{R}^d$, $\mathcal{X}^{\mathcal{R}}(\Omega)$ is a subalgebra of $\mathrm{C}^\infty(\Omega)^{(0,1]}$. Take $(f_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ in $\mathcal{X}^{\mathcal{R}}(\Omega)$ and $K \Subset \Omega$. There exist $N_f \in \mathcal{R}$ and $N_g \in \mathcal{R}$ such that

$$\forall l \in \mathbb{N}\ \ p_{K,l}(h_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N_h(l)}\right) \text{ as } \varepsilon \to 0, \text{ for } h_\varepsilon = f_\varepsilon,\ g_\varepsilon.$$

We get immediately that $p_{K,l}(f_\varepsilon + g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\max\left(N_f(l), N_g(l)\right)}\right)$ as $\varepsilon \to 0$, with $\max(N_f, N_g) \preccurlyeq N$ for some $N \in \mathcal{R}$ according to (2). Then, $(f_\varepsilon + g_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}^{\mathcal{R}}(\Omega)$.

For $(c_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{C})$, there exists q_c such that $|c_\varepsilon| = \mathrm{O}(\varepsilon^{-q_c})$ as $\varepsilon \to 0$. Then $p_{K,l}(c_\varepsilon f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N_f(l) - q_c}\right)$. From (1), there exists $N \in \mathcal{R}$ such that $N_f + q_c \preccurlyeq N$. Thus, $(c_\varepsilon f_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}(\Omega)$. It follows that $\mathcal{X}^{\mathcal{R}}(\Omega)$ is a submodule of $\mathrm{C}^\infty(\Omega)^{(0,1]}$ over $\mathcal{X}(\mathbb{C})$.

Consider now $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = l$. By Leibniz' formula, we have, for all $\varepsilon \in (0,1]$ and $x \in K$,

$$|\partial^\alpha(f_\varepsilon g_\varepsilon)(x)| = \sum_{\gamma \leq \alpha} C_\alpha^\gamma \left|\partial^\gamma f_\varepsilon(x)\, \partial^{\alpha-\gamma} g_\varepsilon(x)\right| \leq \sum_{\gamma \leq \alpha} C_\alpha^\gamma p_{K,|\gamma|}(f_\varepsilon)\, p_{K,|\alpha-\gamma|}(g_\varepsilon),$$

where C_α^γ is the generalized binomial coefficient. We have, for all $\gamma \leq \alpha$,

$$p_{K,|\gamma|}(f_\varepsilon)\, p_{K,|\alpha-\gamma|}(g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N_f(|\gamma|) - Ng(|\alpha-\gamma|)}\right) \text{ as } \varepsilon \to 0.$$

As $\gamma \leq \alpha$, we get $|\gamma| + |\alpha - \gamma| = |\alpha| = l$. According to (3), there exists $N \in \mathcal{R}$ such that, for all k and $k' \leq k$ in $\mathbb{N}$, $N_f(k') + N_g(k - k') \leq N(k)$. Then

$$\sup_{x \in K} |\partial^\alpha(f_\varepsilon g_\varepsilon)(x)| = \mathrm{O}\left(\varepsilon^{-N(l)}\right) \text{ as } \varepsilon \to 0.$$

Thus $p_{K,l}(f_\varepsilon\, g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(l)}\right)$ as $\varepsilon \to 0$, and $(f_\varepsilon\, g_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}(\Omega)$.

For the properties related to $\mathcal{N}^{\mathcal{R}}(\Omega)$, we have the following fundamental lemma:

Lemma 2 *The set $\mathcal{N}^{\mathcal{R}}(\Omega)$ is equal to Colombeau's ideal*

$$\mathcal{N}(\Omega)=\left\{(f_\varepsilon)_\varepsilon\in \mathrm{C}^\infty(\Omega)^{(0,1]}\mid \forall K\Subset\Omega,\ \forall l\in\mathbb{N},\ \forall m\in\mathbb{N}\ \ p_{K,l}(f_\varepsilon)=\mathrm{O}(\varepsilon^m)\ \text{ as }\varepsilon\to 0\right\}.$$

Indeed, take $(f_\varepsilon)_\varepsilon\in\mathcal{N}^{\mathcal{R}}(\Omega)$. For any $K\Subset\Omega$, $l\in\mathbb{N}$ and $m\in\mathbb{N}$, choose $N\in\mathcal{R}$. According to (1) there exists $N'\in\mathcal{R}$ such that $N+m\preccurlyeq N'$. Thus, $p_{K,l}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{N'(l)}\right)=\mathrm{O}(\varepsilon^m)$ as $\varepsilon\to 0$ and $(f_\varepsilon)_\varepsilon\in\mathcal{N}(\Omega)$. Conversely, given $(f_\varepsilon)_\varepsilon\in\mathcal{N}(\Omega)$ and $N\in\mathcal{R}$, we have $p_{K,l}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{N(l)}\right)$ as $\varepsilon\to 0$, since this estimates holds for all $m\in\mathbb{N}$.

(*b*) *Sheaf properties*.- The proof follows the same lines as the one of Colombeau simplified algebras. (See for example [10], theorem 1.2.4.) First, the definition of restriction (by the mean of the restriction of representatives) is straightforward as in Colombeau's case. For the sheaf properties, we have to replace Colombeau's usual estimates by $\mathcal{X}^{\mathcal{R}}$-estimates. But, at each place this happens, we have only to consider a finite number of terms, by compactness properties. Thus, the stability by maximum of $\mathcal{R}$ (property (2)) induces the result. Furthermore, lemma 2 shows that nothing changes for estimates dealing with the ideal. Finally, point (*iii*) of the proposition follows directly from the obvious inclusion $\mathcal{X}^{\mathcal{R}_1}(\Omega)\subset\mathcal{X}^{\mathcal{R}_2}(\Omega)$. ■

Definition 2 *For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the sheaf of algebras*

$$\mathcal{G}^{\mathcal{R}}(\,\cdot\,)=\mathcal{X}^{\mathcal{R}}(\,\cdot\,)/\mathcal{N}^{\mathcal{R}}(\,\cdot\,)$$

is called the sheaf of $\mathcal{R}$-regular algebras of (nonlinear) generalized functions.

According to lemma 2, we have $\mathcal{N}^{\mathcal{R}}=\mathcal{N}$. From now on, we shall write all algebras with Colombeau's ideal $\mathcal{N}$. The sheaf $\mathcal{G}^{\mathcal{R}}(\cdot)$ turns out to be a sheaf of differential algebras and a sheaf of modules over the factor ring $\overline{\mathbb{C}}=\mathcal{X}(\mathbb{C})/\mathcal{N}(\mathbb{C})$ with

$$\mathcal{N}(\mathbb{C})=\left\{(r_\varepsilon)_\varepsilon\in\mathbb{C}^{(0,1]}\mid\forall p\in\mathbb{N}\ \ |r_\varepsilon|=\mathrm{O}(\varepsilon^p)\ \text{ as }\varepsilon\to 0\right\}.$$

Example 2 *Taking $\mathcal{R}=\mathbb{R}_+^{\mathbb{N}}$, we recover the sheaf of* Colombeau simplified *or* special algebras.

Notation 2 *In the sequel, we shall write $\mathcal{G}(\Omega)$ (resp. $\mathcal{X}_M(\Omega)$) instead of $\mathcal{G}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$ (resp. $\mathcal{X}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$). For $(f_\varepsilon)_\varepsilon$ in $\mathcal{X}_M(\Omega)$ or $\mathcal{X}^{\mathcal{R}}(\Omega)$, $[(f_\varepsilon)_\varepsilon]$ will be its class in $\mathcal{G}(\Omega)$ or in $\mathcal{G}^{\mathcal{R}}(\Omega)$, since these classes are obtained modulo the same ideal. (We consider $\mathcal{G}^{\mathcal{R}}(\Omega)$ as a subspace of $\mathcal{G}(\Omega)$.)*

Example 3 *Taking $\mathcal{R}=\mathcal{B}$, introduced in example 1, we obtain the sheaf of $\mathcal{G}^\infty$-generalized functions [20].*

Example 4 *Take a in $[0,+\infty]$ and set*

$$\mathcal{R}_0=\left\{N\in\mathbb{R}_+^{\mathbb{N}}\ \middle|\ \lim_{l\to+\infty}(N(l)/l)=0\right\}\ ;\ \textit{For } a>0:\mathcal{R}_a=\left\{N\in\mathbb{R}_+^{\mathbb{N}}\ \middle|\ \limsup_{l\to+\infty}(N(l)/l)<a\right\}.$$

For any a in $[0,+\infty]$, $\mathcal{R}_a$ is a regular subset of $\mathbb{R}_+^{\mathbb{N}}$. The corresponding sheaves $\mathcal{G}^{\mathcal{R}_a}(\cdot)$ are the sheaves of algebras of generalized functions with slow growth introduced in [4]. Note that, for a in $(0,+\infty]$, a sequence N is in $\mathcal{R}_a$ iff there exists $(a',b)\in(\mathbb{R}^+)^2$ with $a'<a$ such that $N(l)\leq a'l+b$. The growth of the sequence N is at most linear.

Remark 1 *We have the obvious sheaves inclusions $\mathcal{X}^\infty(\,\cdot\,)\subset\mathcal{X}^{\mathcal{R}_a}(\,\cdot\,)\subset\mathcal{X}(\,\cdot\,)$.*

For a given regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the notion of support of a section $f \in \mathcal{G}^{\mathcal{R}}(\Omega)$ (Ω open subset of $\mathbb{R}^d$) makes sense since $\mathcal{G}^{\mathcal{R}}(\cdot)$ is a sheaf. The following definition will be sufficient for this paper.

Definition 3 *The support of a generalized function $f \in \mathcal{G}^{\mathcal{R}}(\Omega)$ is the complement in Ω of the largest open subset of Ω where f is null.*

Notation 3 *We denote by $\mathcal{G}_C^{\mathcal{R}}(\Omega)$ the subset of $\mathcal{G}^{\mathcal{R}}(\Omega)$ of elements with compact support.*

Lemma 3 *Every $f \in \mathcal{G}_C^{\mathcal{R}}$ has a representative $(f_\varepsilon)_\varepsilon$, such that each f_ε has the same compact support.*

We shall not prove this lemma here, since lemma 12 below gives the main ideas of the proof.

2.2 Some embeddings

For any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ and any Ω open subset of $\mathbb{R}^d$, $\mathrm{C}^\infty(\Omega)$ is embedded into $\mathcal{G}^{\mathcal{R}}(\Omega)$ by the canonical embedding

$$\sigma : \mathrm{C}^\infty(\Omega) \to \mathcal{G}^{\mathcal{R}}(\Omega) \quad f \to [(f_\varepsilon)_\varepsilon] \text{ with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1].$$

We refine here the well known result about the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. Following the ideas of [19], we consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\textstyle\int \rho(x)\,\mathrm{d}x = 1, \qquad \int x^m \rho(x)\,\mathrm{d}x = 0 \text{ for all } m \in \mathbb{N}^d \setminus \{0\}.$$

(Such a map can be chosen as the Fourier transform of a function of $\mathcal{D}(\mathbb{R}^d)$ equal to 1 on a neighborhood of 0.) We now choose $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $\overline{B(0,1)}$ and $\chi \equiv 0$ on $\mathbb{R}^d \backslash B(0,2)$. We define

$$\forall \varepsilon \in (0,1], \ \forall x \in \mathbb{R}^d \quad \theta_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)\chi(|\ln \varepsilon|\, x).$$

Finally, consider $(\kappa_\varepsilon)_\varepsilon \in \left(\mathcal{D}(\mathbb{R}^d)\right)^{(0,1]}$ such that

$$\forall \varepsilon \in (0,1), \ 0 \le \kappa_\varepsilon \le 1 \quad \kappa_\varepsilon \equiv 1 \text{ on } \left\{ x \in \Omega \;\middle|\; d(x, \mathbb{R}^d \backslash \Omega) \ge \varepsilon \text{ and } d(x,0) \le 1/\varepsilon \right\}.$$

With these ingredients, the map

$$\iota : \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega) \quad T \mapsto (\kappa_\varepsilon T * \theta_\varepsilon)_\varepsilon + \mathcal{N}(\Omega) \tag{4}$$

is an embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ such that $\iota_{|\mathrm{C}^\infty(\Omega)} = \sigma$.

The proof is mainly based on the following property of $(\theta_\varepsilon)_\varepsilon$:

$$\textstyle\int \theta_\varepsilon(x)\,\mathrm{d}x = 1 + \mathrm{O}\left(\varepsilon^k\right) \text{ as } \varepsilon \to 0, \quad \forall m \in \mathbb{N}^d \setminus \{0\} \ \int x^m \theta_\varepsilon(x)\,\mathrm{d}x = \mathrm{O}\left(\varepsilon^k\right) \text{ as } \varepsilon \to 0. \tag{5}$$

Set

$$\mathcal{R}_1 = \left\{ N \in \mathbb{R}_+^{\mathbb{N}} \mid \exists b \in \mathbb{R}_+, \forall l \in \mathbb{R} \ \ N(l) \le l + b \right\}. \tag{6}$$

One can verify that the set $\mathcal{R}$ is regular and we set $\mathcal{G}^{(1)}(\cdot) = \mathcal{G}^{\mathcal{R}_1}(\cdot)$.

Proposition 4 *The image of $\mathcal{D}'(\Omega)$ by the embedding, defined by (4), is included in $\mathcal{G}^{(1)}(\Omega)$.*

Proof. The proof is a refinement of the classical proof (see [5]), and we shall focus on the estimation of the growth of $(\kappa_\varepsilon T * \theta_\varepsilon)_\varepsilon$ which is the main novelty. We shall do it for the case $\Omega = \mathbb{R}^d$, for which the additional cutoff by $(\kappa_\varepsilon)_\varepsilon$ is not needed. Then, for a given $T \in \mathcal{D}'(\mathbb{R}^d)$, we have $\iota(T) = (T * \theta_\varepsilon)_\varepsilon + \mathcal{N}(\mathbb{R}^d)$, with $(T * \theta_\varepsilon)_\varepsilon = \langle T, \theta_\varepsilon(y - \cdot)\rangle$.

Let us fix a compact set K and consider W an open subset of $\mathbb{R}^d$ such that $K \subset W \subset \overline{W} \Subset \mathbb{R}^d$. With the above definitions, the function $x \mapsto \theta_\varepsilon(y - x)$ belongs to $\mathcal{D}(W)$ for all $y \in K$ and ε small enough, since the support of θ_ε shrinks to $\{0\}$ when ε tends to 0. Therefore, for $\beta \in \mathbb{N}^d$ and ε small enough, we have

$$\begin{aligned} \forall y \in K \quad \partial^\beta (T * \theta_\varepsilon)(y) &= \left\langle T, \partial^\beta \{x \mapsto \theta_\varepsilon(y - x)\}\right\rangle \\ &= \left\langle T_{|W}, \partial^\beta \{x \mapsto \theta_\varepsilon(y - x)\}\right\rangle \\ &= (-1)^{|\beta|} \left\langle T_{|W}, \left\{x \mapsto \left(\partial^\beta \theta_\varepsilon\right)(y - x)\right\}\right\rangle. \end{aligned}$$

By using the local structure of distributions [24], we can write $T_{|W} = \partial_x^\alpha f$ where f is a compactly supported continuous function having its support included in W. It follows

$$\begin{aligned} \forall y \in K \quad \partial^\beta (T * \theta_\varepsilon)(y) &= (-1)^{|\beta|} \left\langle \partial_x^\alpha f, \left(\partial^\beta \theta_\varepsilon\right)(y - \cdot)\right\rangle \\ &= (-1)^{|\alpha|+|\beta|} \left\langle f, \left(\partial^{\alpha+\beta} \theta_\varepsilon\right)(y - \cdot)\right\rangle \\ &= (-1)^{|\alpha|+|\beta|} \int_W f(x)\, \partial^{\alpha+\beta} \theta_\varepsilon(y - x)\, \mathrm{d}x. \end{aligned}$$

Using the definition of $(\theta_\varepsilon)_\varepsilon$, we get

$$\begin{aligned} \forall \xi \in \mathbb{R} \quad \partial^{\alpha+\beta} \theta_\varepsilon(\xi) &= \sum_{\gamma \le \alpha+\beta} \mathrm{C}_{\alpha+\beta}^\gamma\, \partial^\gamma \rho_\varepsilon(\xi)\ \partial^{\alpha+\beta-\gamma}(\chi(\xi\,|\ln\varepsilon|)) \ \ (\text{with } \rho_\varepsilon(\cdot) = \frac{1}{\varepsilon^d}\rho\left(\frac{\cdot}{\varepsilon}\right)) \\ &= \sum_{\gamma \le \alpha+\beta} \mathrm{C}_{\alpha+\beta}^\gamma\, \varepsilon^{-d-|\gamma|}\, |\ln\varepsilon|^{|\alpha+\beta-\gamma|}\ (\partial^\gamma \rho)\left(\frac{\xi}{\varepsilon}\right) \left(\partial^{\alpha+\beta-\gamma}\chi\right)(\xi\,|\ln\varepsilon|), \end{aligned}$$

with $|\alpha + \beta - \gamma| = |\alpha| + |\beta| - |\gamma|$ since $\gamma \le \alpha + \beta$. For all $\gamma \le \alpha + \beta$, we have

$$\varepsilon^{-d-|\gamma|}\, |\ln\varepsilon|^{|\alpha+\beta-\gamma|} = \mathrm{O}\left(\varepsilon^{-d-1-|\alpha|-|\beta|}\right) \ \text{ as } \varepsilon \to 0.$$

As ρ and χ are bounded, as well as their derivatives, there exists $C_1 > 0$, depending on (α, β), such that

$$\forall \xi \in \mathbb{R} \quad \left|\partial^{\alpha+\beta}(\theta_\varepsilon(\xi))\right| \le C_1\, \varepsilon^{-d-1-|\alpha|-|\beta|}.$$

We get

$$\forall y \in K \quad \left|\partial^\beta (T * \theta_\varepsilon)(y)\right| \le C_1 \sup_{\xi \in \overline{W}} |f(\xi)|\ \mathrm{vol}\left(\overline{W}\right)\ \varepsilon^{-d-1-|\alpha|-|\beta|}.$$

From this last inequality, we deduce that for any $l \in \mathbb{N}$, there exists some constant $C_2 > 0$, depending on (l, K), such that

$$p_{K,l}((T * \theta_\varepsilon)) \le C_2 \varepsilon^{-d-1-|\alpha|-l} = C_2\, \varepsilon^{-N(l)} \text{ with } N(l) = l + d + 1 + |\alpha|,$$

which shows that $(T * \theta_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}_1}(\mathbb{R}^d)$. ■

We can summarize these results in the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{C}^\infty(\Omega) & \longrightarrow & \mathcal{D}'(\Omega) & & \\ \downarrow \sigma & & \downarrow \iota & & \\ \mathcal{G}^\infty(\Omega) & \longrightarrow & \mathcal{G}^{(1)}(\Omega) & \longrightarrow & \mathcal{G}(\Omega). \end{array} \qquad (7)$$

3 Rapidly decreasing generalized functions

3.1 Definition and first properties

Spaces of rapidly decreasing generalized functions have been introduced in the literature ([8], [22], [23]), notably in view of the definition of the Fourier transform in convenient spaces of nonlinear generalized functions. We give here a more complete description of this type of space in the framework of $\mathcal{R}$-regular spaces.

Definition 4 *We say that a subspace $\mathcal{R}'$ of the space $\mathbb{R}_+^{\mathbb{N}^2}$ of maps from $\mathbb{N}^2$ to $\mathbb{R}_+$ is* regular *if*
(i) $\mathcal{R}'$ is "overstable" by translation and by maximum

$$\forall N \in \mathcal{R}',\ \forall \left(k,k',k''\right) \in \mathbb{N}^3,\ \exists N' \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2 \quad N\left(q+k, l+k'\right) + k'' \leq N'(q,l), \quad (8)$$

$$\forall N_1 \in \mathcal{R}',\ \forall N_2 \in \mathcal{R}',\ \exists N \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2 \qquad \max\left(N_1(q,l), N_2(q,l)\right) \leq N(q,l). \quad (9)$$

(ii) For any N_1 and N_2 in $\mathcal{R}'$, there exists $N \in \mathcal{R}'$ such that

$$\forall (q_1,q_2,l_1,l_2) \in \mathbb{N}^4 \qquad N_1(q_1,l_1) + N_2(q_2,l_2) \leq N(q_1+q_2, l_1+l_2). \quad (10)$$

Example 5
(i) The set $\mathcal{B}'$ of bounded maps from $\mathbb{N}^2$ to $\mathbb{R}_+$ is a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$.
(ii) The set $\mathbb{R}_+^{\mathbb{N}^2}$ of all maps from $\mathbb{N}^2$ to $\mathbb{R}_+$ is a regular set.

We consider Ω an open subset of $\mathbb{R}^d$ and the space $\mathcal{S}(\Omega)$ of rapidly decreasing functions defined on Ω, endowed with the family of seminorms $\mathcal{Q}(\Omega) = (\mu_{q,l})_{(q,l)\in\mathbb{N}^2}$ defined by

$$\mu_{q,l}(f) = \sup_{x\in\Omega, |\alpha|\leq l} (1+|x|)^q \left|\partial^\alpha f(x)\right|.$$

Let $\mathcal{R}'$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$ and set

$$\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{S}(\Omega)^{(0,1]} \;\middle|\; \exists N \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2\ \ \mu_{q,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(q,l)}\right) \text{ as } \varepsilon \to 0 \right\},$$
$$\mathcal{N}_{\mathcal{S}}^{\mathcal{R}'}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{S}(\Omega)^{(0,1]} \;\middle|\; \forall m \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2\ \ \mu_{q,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{m(q,l)}\right) \text{ as } \varepsilon \to 0 \right\}.$$

As for lemma 2, we have $\mathcal{N}_{\mathcal{S}}^{\mathcal{R}'}(\Omega) = \mathcal{N}_{\mathcal{S}}(\Omega)$, with

$$\mathcal{N}_{\mathcal{S}}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{S}(\Omega)^{(0,1]} \;\middle|\; \forall (q,l) \in \mathbb{N}^2,\ \forall m \in \mathbb{N}\quad \mu_{q,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^m) \text{ as } \varepsilon \to 0 \right\}.$$

Proposition 5
(i) For any regular subspace $\mathcal{R}'$ of $\mathbb{R}_+^{\mathbb{N}^2}$, the functor $\Omega \to \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$ defines a presheaf (it allows restrictions) of differential algebras over the ring $\mathcal{X}(\mathbb{C})$.
(ii) The functor $\mathcal{N}_{\mathcal{S}} : \Omega \to \mathcal{N}_{\mathcal{S}}(\Omega)$ defines a presheaf of ideals of the presheaf $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)$.
(iii) For any regular subspaces $\mathcal{R}'_1$ and $\mathcal{R}'_2$ of $\mathbb{R}_+^{\mathbb{N}^2}$, with $\mathcal{R}'_1 \subset \mathcal{R}'_2$, the presheaf $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'_1}(\Omega)$ is a subpresheaf of the presheaf $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'_2}(\Omega)$.

Proof.
(a) *Algebraical properties*.- Let us first prove that $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$ is a subalgebra of $\mathcal{S}(\Omega)^{(0,1]}$. The proof that $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$ is a sublinear space of $\mathrm{C}^\infty(\Omega)^{(0,1]}$ goes along the same lines as in proposition

1. For the product, take $(f_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ in $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$. According to the definitions, there exist $N_f \in \mathcal{R}'$ and $N_g \in \mathcal{R}'$ such that

$$\forall (q,l) \in \mathbb{N}^2 \quad \mu_{q,l}(h_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N_h(q,l)}\right) \text{ as } \varepsilon \to 0, \text{ for } h_\varepsilon = f_\varepsilon,\, g_\varepsilon.$$

Consider $(q,l) \in \mathbb{N}^2$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = l$. By Leibniz' formula, we have

$$\forall \varepsilon \in (0,1] \qquad \partial^\alpha (f_\varepsilon g_\varepsilon) = \sum_{\gamma \leq \alpha} C_\alpha^\gamma \, \partial^\gamma f_\varepsilon \, \partial^{\alpha-\gamma} g_\varepsilon.$$

Thus

$$\sup_{x\in\Omega} (1+|x|)^q \left|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)\right| \leq \sum_{\gamma \leq \alpha} C_\alpha^\gamma \, \mu_{q,|\gamma|}(f_\varepsilon)\, \mu_{0,|\alpha-\gamma|}(f_\varepsilon),$$

with, for all $\gamma \leq \alpha$, $\mu_{q,|\gamma|}(f_\varepsilon)\,\mu_{0,|\alpha-\gamma|}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N_f(q,|\gamma|)-N_g(0,|\alpha-\gamma|)}\right)$ as $\varepsilon \to 0$. Since $\gamma \leq \alpha$, we get $|\gamma| + |\alpha - \gamma| = |\alpha| = l$. According to (10), there exists $N \in \mathcal{R}'$ such that, for all k and $k' \leq k$ in $\mathbb{N}$, $N_1(q,k') + N_2(0,k-k') \leq N(q,k)$. Then

$$\sup_{x\in\Omega} \left|(1+|x|)^q \, \partial^\alpha (f_\varepsilon g_\varepsilon)(x)\right| = \mathrm{O}\left(\varepsilon^{-N(q,l)}\right) \text{ as } \varepsilon \to 0.$$

Thus $\mu_{q,l}(f_\varepsilon\, g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(q,l)}\right)$ as $\varepsilon \to 0$ and $(f_\varepsilon\, g_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$.

The same kind of estimates shows also that $\mathcal{N}_{\mathcal{S}}(\Omega)$ (or $\mathcal{N}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$) is an ideal of $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$.

(b) *Presheaf properties*.- Take Ω_1 and Ω_2 two open subsets of $\mathbb{R}^d$, with $\Omega_1 \subset \Omega_2$, $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega_2)$ (*resp.* $\mathcal{N}_{\mathcal{S}}(\Omega_2)$). As $u_\varepsilon \in \mathrm{C}^\infty(\Omega_2)$ for all $\varepsilon \in (0,1]$, it admits a restriction $u_{\varepsilon|\Omega_1}$, and obviously $\left(u_{\varepsilon|\Omega_1}\right)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega_1)$ (*resp.* $\mathcal{N}_{\mathcal{S}}(\Omega_2)$). So, restrictions are well defined in $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)$ and $\mathcal{N}_{\mathcal{S}}(\cdot)$. ■

Definition 5 *The presheaf* $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\cdot) = \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)/\mathcal{N}_{\mathcal{S}}(\cdot)$ *is called the presheaf of* $\mathcal{R}'$-regular rapidly decreasing generalized functions.

As for the case of $\mathcal{G}^{\mathcal{R}}(\cdot)$, the presheaf $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)$ is a presheaf of differential algebras and a sheaf of modules over the factor ring $\overline{\mathbb{C}} = \mathcal{X}(\mathbb{C})/\mathcal{N}(\mathbb{C})$.

Example 6 *Taking* $\mathcal{R}' = \mathbb{R}_+^{\mathbb{N}^2}$*, we obtain the presheaf of algebras of rapidly decreasing generalized functions ([8], [22], [23]).*

Notation 4 *In the sequel, we shall note* $\mathcal{G}_{\mathcal{S}}(\Omega)$ *(resp.* $\mathcal{X}_{\mathcal{S}}(\Omega)$*) instead of* $\mathcal{G}_{\mathcal{S}}^{\mathbb{R}_+^{\mathbb{N}^2}}(\Omega)$ *(resp.* $\mathcal{X}_{\mathcal{S}}^{\mathbb{R}_+^{\mathbb{N}^2}}(\Omega)$*). For all regular subset* $\mathcal{R}'$ *and* $(f_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$, $[(f_\varepsilon)_\varepsilon]_{\mathcal{S}}$ *denotes its class in* $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$.

Example 7 *Taking* $\mathcal{R}' = \mathcal{B}'$*, we obtain the presheaf of* $\mathcal{G}_{\mathcal{S}}^{\infty}$ *generalized functions or of* regular rapidly decreasing generalized functions.

Set

$$\mathcal{N}_{\mathcal{S}_*}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \mid \forall m \in \mathbb{N},\ \forall q \in \mathbb{N} \ \ \mu_{q,0}(f_\varepsilon) = \mathrm{O}(\varepsilon^m) \text{ as } \varepsilon \to 0 \right\}. \tag{11}$$

We have the exact analogue of theorems 1.2.25 and 1.2.27 of [10].

Lemma 6 *If the open set* Ω *is a box, i.e. the product of* d *open intervals of* $\mathbb{R}$ *(bounded or not) then* $\mathcal{N}_{\mathcal{S}}(\Omega)$ *is equal to* $\mathcal{N}_{\mathcal{S}_*}(\Omega) \cap \mathcal{X}_{\mathcal{S}}(\Omega)$.

Proof. We use similar technics as in the proof of Theorem 1.2.27 in [10]. It suffices to show the result for $\mathcal{X}_{\mathcal{S}}(\Omega)$ (which is the biggest subspace of $\mathcal{S}(\Omega)^{(0,1]}$ we may have to consider) and for real valued functions. We prove here that a derivative $\partial_j = \partial/\partial x_j$ $(1 \leq j \leq d)$ satisfies the 0-order estimate of the definition of $\mathcal{N}_{\mathcal{S}_*}$ (11). The proof for higher order derivatives, which goes by induction, is left to the reader. Let $(f_\varepsilon)_\varepsilon$ be in $\mathcal{N}_{\mathcal{S}_*}(\Omega) \cap \mathcal{X}_{\mathcal{S}}(\Omega)$, q in $\mathbb{N}$ and m in $\mathbb{N}$. As $(f_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{S}}(\Omega)$, there exists N such that

$$\forall q' \in \{0, ..., q\} \qquad \sup_{x\in\Omega} (1+|x|)^{q'} \left|\partial_j^2 f_\varepsilon(x)\right| = \mathrm{O}\left(\varepsilon^{-N}\right). \tag{12}$$

As $(f_\varepsilon)_\varepsilon$ is in $\mathcal{N}_{\mathcal{S}_*}(\Omega)$, we get

$$\forall q' \in \mathbb{N} \qquad \sup_{x\in\Omega} (1+|x|)^{q'} |f_\varepsilon(x)| = \mathrm{O}\left(\varepsilon^{N+2m}\right). \tag{13}$$

Since the open set Ω is a box, for ε sufficiently small (but independent of x), either the segment $\left[x, x+\varepsilon^{N+m}e_j\right]$ or $\left[x, x-\varepsilon^{N+m}e_j\right]$ is included in Ω. Suppose it is $\left[x, x+\varepsilon^{N+m}e_j\right]$. Taylor's theorem gives the existence of $\theta \in (0,1)$ such that

$$\partial_j f_\varepsilon(x) = \left(f_\varepsilon\left(x+\varepsilon^{N+m}e_j\right) - f_\varepsilon(x)\right)\varepsilon^{-N-m} - (1/2)\,\partial_j^2 f_\varepsilon(x_\theta)\,\varepsilon^{N+m}, \qquad x_\theta = x + \theta\varepsilon^{N+m}e_j.$$

This gives

$$(1+|x|)^q |\partial_j f_\varepsilon(x)| \leq \underbrace{(1+|x|)^q \left|f_\varepsilon\left(x+\varepsilon^{N+m}e_j\right)\right| \varepsilon^{-N-m}}_{*} + \underbrace{(1+|x|)^q |f_\varepsilon(x)|\,\varepsilon^{-N-m}}_{**} + \underbrace{\varepsilon^{N+m}(1+|x|)^q \left|\partial_j^2 f_\varepsilon(x_\theta)\right|}_{***}.$$

From (13), we get immediately that $(**)$ is of order $\mathrm{O}(\varepsilon^m)$. For $(*)$, we have

$$(1+|x|)^q \leq \left(1+\left|x+\varepsilon^{N+m}e_j\right| + \varepsilon^{N+m}\right)^q \leq \sum_{k=0}^{q} \mathrm{C}_q^k \left(1+\left|x+\varepsilon^{N+m}e_j\right|\right)^{q-k} \varepsilon^{(N+m)k}.$$

Then

$$(1+|x|)^q \left|f_\varepsilon\left(x+\varepsilon^{N+m}e_j\right)\right| \varepsilon^{-N-m} \leq \sum_{k=0}^{q} \mathrm{C}_q^k \left(1+\left|x+\varepsilon^{N+m}e_j\right|\right)^{q-k} \left|f_\varepsilon\left(x+\varepsilon^{N+m}e_j\right)\right| \varepsilon^{(N+m)(k-1)},$$

and (13) implies that $(*)$ is of order $\mathrm{O}(\varepsilon^m)$. Finally, the same method shows that $(***)$ is also of order $\mathrm{O}(\varepsilon^m)$. ■

3.2 Embeddings

3.2.1 The natural embeddings of $\mathcal{S}\left(\mathbb{R}^d\right)$ and $\mathcal{O}'_C\left(\mathbb{R}^d\right)$ into $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$

The embedding of $\mathcal{S}\left(\mathbb{R}^d\right)$ into $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ is done by the canonical injective map

$$\sigma_{\mathcal{S}} : \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \quad f \mapsto [(f_\varepsilon)_\varepsilon]_{\mathcal{S}} \text{ with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1].$$

In fact, the image of $\sigma_{\mathcal{S}}$ is included in $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}\left(\mathbb{R}^d\right)$ for any regular subset of $\mathcal{R}' \subset \mathbb{R}_+^{\mathbb{N}^2}$.

For the embedding of $\mathcal{O}'_C\left(\mathbb{R}^d\right)$, we consider $\rho \in \mathcal{S}\left(\mathbb{R}^d\right)$ which satisfies

$$\textstyle\int \rho(x)\,\mathrm{d}x = 1, \qquad \int x^m \rho(x)\,\mathrm{d}x = 0 \text{ for all } m \in \mathbb{N}^d \backslash \{0\} \tag{14}$$

Set

$$\forall \varepsilon \in (0,1]\,,\ \ \forall x \in \mathbb{R}^d \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\,\rho\left(\frac{x}{\varepsilon}\right). \tag{15}$$

Note that, contrary to the case of the embedding of $\mathcal{D}'(\Omega)$ in $\mathcal{G}(\Omega)$, we don't need an additional cutoff.

Theorem 7 *The map*

$$\iota_\mathcal{S} : \mathcal{O}'_C\left(\mathbb{R}^d\right) \to \mathcal{G}_\mathcal{S}\left(\mathbb{R}^d\right) \qquad u \mapsto \left[(u * \rho_\varepsilon)_\varepsilon\right]_\mathcal{S} \tag{16}$$

is a linear embedding which commutes with partial derivatives.

Proof. Take $u \in \mathcal{O}'_C\left(\mathbb{R}^d\right)$. First, as $u \in \mathcal{O}'_C\left(\mathbb{R}^d\right)$ and $\rho_\varepsilon \in \mathcal{S}\left(\mathbb{R}^d\right)$, $u_\varepsilon * \rho_\varepsilon$ is in $\mathcal{S}\left(\mathbb{R}^d\right)$ for all $\varepsilon \in (0,1]$. (This result is classical: Our proof, with a slight adaptation, shows also this result.)

Consider $q \in \mathbb{N}$. Using the structure of elements of $\mathcal{O}'_C\left(\mathbb{R}^d\right)$ [24], we can find a finite family $(f_j)_{1\le j\le l(q)}$ of continuous functions such that $(1+|x|)^q f_j$ is bounded (for $1 \le j \le l(q)$), and $(\alpha_j)_{1\le j\le l(q)} \in \left(\mathbb{N}^d\right)^{l(q)}$ such that $u = \sum_{j=1}^{l(q)} \partial^{\alpha_j} f_j$. In order to simplify notations, we shall suppose that this family is reduced to one element f, that is $u = \partial^\alpha f$. Take now $\beta \in \mathbb{N}^d$. We have

$$\begin{aligned}\forall x \in \mathbb{R}^d \quad \partial^\beta (u * \rho_\varepsilon)(x) &= \partial^\beta (\partial^\alpha f * \rho_\varepsilon)(x) = \left(f * \partial^{\alpha+\beta}(\rho_\varepsilon)\right)(x),\\ &= \int f(x-y)\,\partial^{\alpha+\beta}(\rho_\varepsilon)(y)\,\mathrm{d}y.\end{aligned}$$

As $\rho_\varepsilon(y) = \varepsilon^{-d}\rho(y/\varepsilon)$, we have $\partial^{\alpha+\beta}(\rho_\varepsilon)(y) = \varepsilon^{-d-|\alpha|-|\beta|}\left(\partial^{\alpha+\beta}\rho\right)(y/\varepsilon)$ and

$$\forall x \in \mathbb{R}^d \quad \partial^\beta (u * \rho_\varepsilon)(x) = \varepsilon^{-|\alpha|-|\beta|} \int f(x-\varepsilon v)\,\partial^{\alpha+\beta}\rho(v)\,\mathrm{d}v.$$

On one hand, there exists a constant $C_1 > 0$ such that

$$\forall (x,v) \in \mathbb{R}^{2d} \quad |f(x-\varepsilon v)| \le C_1 (1+|x-\varepsilon v|)^{-q}.$$

On the other hand, as ρ is rapidly decreasing, there exists $C_2 > 0$ such that

$$\partial^{\alpha+\beta}\rho(v) \le C_2 (1+|v|)^{-q-d-1}.$$

These estimates imply the existence of a constant C_3 such that, for all $x \in \mathbb{R}^d$,

$$\left|\partial^\beta (u * \rho_\varepsilon)(x)\right| \le C_3\, \varepsilon^{-|\alpha|-|\beta|} \int \left((1+|x-\varepsilon v|)(1+|v|)\right)^{-q} (1+|v|)^{-d-1}\,\mathrm{d}v.$$

We have $(1+|x-\varepsilon v|) \ge (1+||x|-\varepsilon|v||)$ and a short study of the family of functions $\phi_{|x|,\varepsilon} : t \mapsto (1+||x|-\varepsilon t|)(1+t)$ for positive t shows that $\phi_{|x|,\varepsilon}(t) \ge 1+|x|$. Consequently

$$\begin{aligned}\forall x \in \mathbb{R}^d \quad \left|\partial^\beta (u * \rho_\varepsilon)(x)\right| &\le C_3\, \varepsilon^{-|\alpha|-|\beta|} (1+|x|)^{-q} \int (1+|v|)^{-d-1}\,\mathrm{d}v\\ &\le C_4\, \varepsilon^{-|\alpha|-|\beta|} (1+|x|)^{-q} \quad (C_4 \text{ positive constant}).\end{aligned}$$

It follows that $\mu_{q,l}(u * \rho_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(q,l)}\right)$ as $\varepsilon \to 0$ with $N(q,l) = |\alpha| + l$. (α may depends on q) This shows that $(u * \rho_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_\mathcal{S}\left(\mathbb{R}^d\right)$.

Finally, it is clear that $(u * \rho_\varepsilon)_\varepsilon \in \mathcal{N}_\mathcal{S}\left(\mathbb{R}^d\right)$ implies that $u_\varepsilon * \rho_\varepsilon \to 0$ in $\mathcal{S}'$, as $\varepsilon \to 0$. As $u_\varepsilon * \rho_\varepsilon \to u$ as $\varepsilon \to 0$, u is therefore null. ■

Theorem 8 *We have:* $\iota_{\mathcal{S}|\mathcal{S}(\mathbb{R}^d)} = \sigma_{\mathcal{S}}$.

Proof. We shall prove this assertion in the case $d = 1$, the general case only differs by more complicate algebraic expressions.

Let f be in $\mathcal{S}(\mathbb{R})$ and set $\Delta = \iota_{\mathcal{S}}(f) - \sigma(f)$. One representative of Δ is given by

$$\Delta_\varepsilon : \mathbb{R} \to \mathcal{F}(\mathrm{C}^\infty(\mathbb{R})) \qquad y \mapsto (f * \rho_\varepsilon)(y) - f(y) = \int f(y-x)\rho_\varepsilon(x)\,\mathrm{d}x - f(y).$$

Using the fact that $\int \rho_\varepsilon(x)\,\mathrm{d}x = 1$, we get

$$\forall y \in \mathbb{R} \qquad \Delta_\varepsilon(y) = \int (f(y-x) - f(y))\,\rho_\varepsilon(x)\,\mathrm{d}x.$$

Let m be an integer. Taylor's formula gives

$$\forall (x,y) \in \mathbb{R}^2 \qquad f(y-x) - f(y) = \sum_{j=1}^{m} \frac{(-x)^j}{j!} f^{(i)}(y) + \frac{(-x)^{m+1}}{m!} \int_0^1 f^{(m+1)}(y-ux)(1-u)^m\,\mathrm{d}u,$$

and, for all $y \in \mathbb{R}$,

$$\Delta_\varepsilon(y) = \sum_{j=1}^{m} \frac{(-1)^j}{j!} f^{(i)}(y) \int x^i \rho_\varepsilon(x)\,\mathrm{d}x + \underbrace{\int \frac{(-x)^{m+1}}{m!} \left(\int_0^1 f^{(m+1)}(y-ux)(1-u)^m\,\mathrm{d}u \right) \rho_\varepsilon(x)\,\mathrm{d}x}_{R_\varepsilon(m,y)}.$$

According to the choice of mollifiers, we have $\int x^i \rho_\varepsilon(x)\,\mathrm{d}x = 0$ for $\varepsilon \in (0,1]$ and $j \in \{0,\ldots,m\}$ and consequently $\Delta_\varepsilon(y) = R_\varepsilon(m,y)$.

Setting $v = x/\varepsilon$, we get

$$\forall y \in \mathbb{R} \qquad \Delta_\varepsilon(y) = \varepsilon^{m+1} \int \frac{(-v)^{m+1}}{m!} \left(\int_0^1 f^{(m+1)}(y - \varepsilon uv)(1-u)^m\,\mathrm{d}u \right) \rho(v)\,\mathrm{d}v.$$

Consider $q \in \mathbb{N}$. As ρ (*resp.* f) is in $\mathcal{S}(\mathbb{R})$, we get a constant $C_1 > 0$ (*resp.* $C_2 > 0$) such that $|\rho(t)| \leq C_1 (1+|t|)^{-m-3-q}$ (respectively $\left|f^{(m+1)}(t)\right| \leq C_2 (1+|t|)^{-q}$) for all $t \in \mathbb{R}$. Thus, there exists a constant $C_3 > 0$ such that

$$\forall y \in \mathbb{R} \qquad |\Delta_\varepsilon(y)| \leq C_3 \frac{\varepsilon^{m+1}}{m!} \int |v|^{m+1} (1+|v|)^{-m-3} \left(\int_0^1 \underline{(1+|y-\varepsilon uv|)^{-q} (1+|v|)^{-q}}\,\mathrm{d}u \right) \mathrm{d}v.$$

Using the same technic as in the proof of theorem 7, we obtain that the underlined term is less than $(1+|y|)^{-q}$, for all $\varepsilon \in (0,1]$, $u \in [0,1]$ and $v \in \mathbb{R}$. This implies the existence of a constant $C_4 > 0$ such that

$$\forall y \in \mathbb{R} \qquad |\Delta_\varepsilon(y)| \leq C_4\, \varepsilon^{m+1} (1+|y|)^{-q}.$$

As $(\Delta_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}(\mathbb{R})$ and $\sup_{y\in\mathbb{R}} (1+|y|)^q |\Delta_\varepsilon(y)| = \mathrm{O}\left(\varepsilon^{m+1}\right)$ for all $m > 0$, we can conclude directly (without estimating the derivatives) that $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{S}}(\mathbb{R})$, by using lemma 6. ■

Consider

$$\mathcal{R}_1' = \left\{ N' \in \mathbb{R}_+^{\mathbb{N}^2} \mid \exists N \in \mathcal{R}_1 \;\; N' = 1 \otimes N \right\}, \tag{17}$$

where $\mathcal{R}_1$ is defined by (6). (This amounts to: $N \in \mathcal{R}_1'$ iff there exists $b \in \mathbb{R}_+$ such that $N(q,l) \leq l + b$.) The set $\mathcal{R}_1'$ is clearly regular and we note

$$\mathcal{G}_{\mathcal{S}}^{(1)}\left(\mathbb{R}^d\right) = \mathcal{X}_{\mathcal{S}}^{\mathcal{R}_1'}\left(\mathbb{R}^d\right) / \mathcal{N}_{\mathcal{S}}\left(\mathbb{R}^d\right). \tag{18}$$

Proposition 9 *The image of $\mathcal{O}'_M\left(\mathbb{R}^d\right)$ by $\iota_{\mathcal{S}}$ is included in $\mathcal{G}^{(1)}_{\mathcal{S}}\left(\mathbb{R}^d\right)$.*

Proof. Let u be in $\mathcal{O}'_M\left(\mathbb{R}^d\right)$. According to the characterization of elements of $\mathcal{O}'_M\left(\mathbb{R}^d\right)$ [11], there exists a finite family $(f_j)_{1\leq j\leq l}$ of rapidly decreasing continuous functions and $(\alpha_j)_{1\leq j\leq l}\in\left(\mathbb{N}^d\right)^l$ such that $u=\sum_{j=1}^{l}\partial^{\alpha_j}f_j$. In order to simplify, we shall suppose that this family is reduced to one element f, that is $u=\partial^\alpha f$. For $\beta\in\mathbb{N}^d$, the same estimates as in proof of theorem 7 lead to the following property

$$\forall q\in\mathbb{N},\ \ \exists C_q>0,\ \ \forall x\in\mathbb{R}^d\quad (1+|x|)^q\left|\partial^\beta\left(u*\rho_\varepsilon\right)(x)\right|\leq C_q\,\varepsilon^{-|\alpha|-|\beta|},$$

since, in the present case, f is rapidly decreasing. (The only difference is here that f and α do not depend on the chosen integer q.) Then

$$\mu_{q,l}\left(u*\rho_\varepsilon\right)\leq C_q\,\varepsilon^{-l-|\alpha|}.$$

Our claim follows, with $N'(q,l)=l+|\alpha|$, where $|\alpha|$ only depends on u. ■

We can summarize theorems 7 and 8 and proposition 9 in the following commutative diagram in which all arrows are embeddings (compare with diagram (7)):

$$\begin{array}{ccccc}
\mathcal{S}\left(\mathbb{R}^d\right) & \longrightarrow & \mathcal{O}'_M\left(\mathbb{R}^d\right) & \longrightarrow & \mathcal{O}'_C\left(\mathbb{R}^d\right)\\
 & \searrow\sigma_{\mathcal{S}} & \downarrow\iota_{\mathcal{S}} & & \downarrow\iota_{\mathcal{S}}\\
 & & \mathcal{G}^{(1)}_{\mathcal{S}}\left(\mathbb{R}^d\right) & \longrightarrow & \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)\ .
\end{array} \tag{19}$$

3.2.2 Embedding of $\mathcal{S}'\left(\mathbb{R}^d\right)$ into $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$

In order to embed $\mathcal{S}'\left(\mathbb{R}^d\right)$ into an algebra playing the role of $\mathcal{G}\left(\mathbb{R}^d\right)$ for $\mathcal{D}'\left(\mathbb{R}^d\right)$, a space $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ of tempered generalized functions is often introduced (see [2], [10]). This space $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ does not fit in the general scheme of construction of Colombeau type algebras, since the growth estimates for $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ are not based on the natural topology of the space $\mathcal{O}_M\left(\mathbb{R}^d\right)$, which replaces $\mathrm{C}^\infty\left(\mathbb{R}^d\right)$ in this case. Although it is possible to construct a space $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ based on the topology of $\mathcal{O}_M\left(\mathbb{R}^d\right)$, we don't need it in the sequel and we only show that $\mathcal{S}'\left(\mathbb{R}^d\right)$ can be embedded into $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ by means of a cutoff of the embedding $\iota_{\mathcal{S}}:\mathcal{O}'_C\left(\mathbb{R}^d\right)\to\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$.

Proposition 10 *With the notations (14) and (15), the map*

$$\iota_{\mathcal{S}'}:\mathcal{S}'\left(\mathbb{R}^d\right)\to\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right),\ u\mapsto\left[\left(\left(u*\rho_\varepsilon\right)\widehat{\rho}_\varepsilon\right)_\varepsilon\right]_{\mathcal{S}}$$

is a linear embedding.

Proof. For $u\in\mathcal{S}'\left(\mathbb{R}^d\right)$, the net $\left(\left(u*\rho_\varepsilon\right)\right)_\varepsilon$ belongs to the space

$$\mathcal{X}_\tau\left(\mathbb{R}^d\right)=\left\{\left(f_\varepsilon\right)_\varepsilon\in\mathrm{C}^\infty(\Omega)^{(0,1]}\ \middle|\ \forall\alpha\in\mathbb{N}^d,\ \exists q\in\mathbb{N},\ \exists N\in\mathbb{N}\ \ \mu_{-q,\alpha}\left(f_\varepsilon\right)=\mathrm{O}\left(\varepsilon^{-N}\right)\ \text{ as }\varepsilon\to0\right\}.$$

(See, for example, theorem 1.2.27 in [10]: The proof is similar to the one of theorem 7.) A straightforward calculus shows that $\mathcal{X}_{\mathcal{S}}$ is an ideal of $\mathcal{X}_\tau$. It follows that $\left(\left(u*\rho_\varepsilon\right)\widehat{\rho}_\varepsilon\right)_\varepsilon\in\mathcal{X}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ since $\left(\widehat{\rho}_\varepsilon\right)_\varepsilon\in\mathcal{X}_{\mathcal{S}}\left(\mathbb{R}^d\right)$. Then, the map $\iota_{\mathcal{S}'}$ is well defined.

Note that, for $u\in\mathcal{S}'\left(\mathbb{R}^d\right)$, we have $u*\rho_\varepsilon\overset{\mathcal{S}'}{\longrightarrow}u$ as $\varepsilon\to0$ and, therefore, $u*\rho_\varepsilon\overset{\mathcal{D}'}{\longrightarrow}u$ as $\varepsilon\to0$. As $\widehat{\rho}_\varepsilon=1$ on a compact set K_ε such that $K_\varepsilon\to\mathbb{R}^d$ as $\varepsilon\to0$, we get that

$$\left(u*\rho_\varepsilon\right)\widehat{\rho}_\varepsilon\overset{\mathcal{D}'}{\longrightarrow}u\text{ as }\varepsilon\to0. \tag{20}$$

Finally, take $u\in\mathcal{S}'\left(\mathbb{R}^d\right)$ with $\iota_{\mathcal{S}'}(u)=0$, that is $\left(\left(u*\rho_\varepsilon\right)\widehat{\rho}_\varepsilon\right)_\varepsilon\in\mathcal{N}_{\mathcal{S}}\left(\mathbb{R}^d\right)$. We have $\left(u*\rho_\varepsilon\right)\widehat{\rho}_\varepsilon\overset{\mathcal{S}'}{\longrightarrow}0$ and, consequently, $\left(u*\rho_\varepsilon\right)\widehat{\rho}_\varepsilon\overset{\mathcal{D}'}{\longrightarrow}0$. Thus, $u=0$ according to (20). ■

4 Fourier transform and exchange theorem

4.1 Fourier transform in $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$

The Fourier transform $\mathcal{F}$ is a continuous linear map from $\mathcal{S}\left(\mathbb{R}^d\right)$ to $\mathcal{S}\left(\mathbb{R}^d\right)$. According to proposition 3.2. of [7], $\mathcal{F}$ has a canonical extension $\mathcal{F}_{\mathcal{S}}$ from $\mathcal{G}_{\mathcal{S}}$ to $\mathcal{G}_{\mathcal{S}}$ defined by

$$\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \to \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \quad u \mapsto \widehat{u} = \left[\left(x \mapsto \int \mathrm{e}^{-ix\xi} u_\varepsilon(\xi)\, \mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{S}}, \tag{21}$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ is any representative of u.

The proof of this result uses mainly the continuity of $\mathcal{F}$. More precisely, any linear continuous map is continuously moderate in the sense of [7] and, therefore, admits such a canonical extension. (The proof of lemma 15 below, with a slight adaptation, shows directly this result for the case of $\mathcal{F}$.)

Definition 6 *The map $\mathcal{F}_{\mathcal{S}}$ defined by (21) is called the Fourier transform in $\mathcal{G}_{\mathcal{S}}$.*

In the same way, we can define $\mathcal{F}_{\mathcal{S}}^{-1}$ by

$$\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \to \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \quad u \mapsto \left[\left(x \mapsto (2\pi)^{-d} \int \mathrm{e}^{ix\xi} u_\varepsilon(\xi)\, \mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{S}}, \tag{22}$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ is any representative of u.

Theorem 11 *$\mathcal{F}_{\mathcal{S}} : \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \to \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ is a one to one linear map, whose inverse is $\mathcal{F}_{\mathcal{S}}^{-1}$.*

Proof. Let u be in $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ be one of its representative. As representative of $\mathcal{F}\left(\mathcal{F}^{-1}(u)\right)$, we can choose $(\widetilde{u}_\varepsilon)_\varepsilon$ defined by

$$\forall \varepsilon \in (0,1],\ \ \forall x \in \mathbb{R}^d \quad \widetilde{u}_\varepsilon(x) = (2\pi)^{-d} \int \mathrm{e}^{ix\xi} \widehat{u}_\varepsilon(\xi)\, \mathrm{d}\xi.$$

Since the Fourier transform is an isomorphism in $\mathcal{S}\left(\mathbb{R}^d\right)$, we get $\widetilde{u}_\varepsilon = u_\varepsilon$, for all $\varepsilon \in (0,1]$, and $\mathcal{F}_{\mathcal{S}}\left(\mathcal{F}_{\mathcal{S}}^{-1}(u)\right) = [(u_\varepsilon(x))_\varepsilon]_{\mathcal{S}} = u$. ∎

4.2 Regular sub presheaves of $\mathcal{G}_{\mathcal{S}}(\cdot)$

We introduce here some regular sub presheaves of $\mathcal{G}_{\mathcal{S}}(\cdot)$ needed for our further microlocal analysis.

Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$ and set

$$\mathcal{R}_u = \left\{N' \in \mathbb{R}_+^{\mathbb{N}^2} \mid \exists N \in \mathcal{R} \quad N' = 1 \otimes N\right\}; \quad \mathcal{R}_\partial = \left\{N' \in \mathbb{R}_+^{\mathbb{N}^2} \mid \exists N \in \mathcal{R} \quad N = N \otimes 1\right\}.$$

In other words, $N' \in \mathcal{R}_u$ (*resp.* $\mathcal{R}_\partial$) iff there exists $N \in \mathcal{R}$ such that $N'(q,l) = N(l)$ (*resp.* $N'(q,l) = N(q)$) or, equivalently, iff N only depends (in a $\mathcal{R}$-regular way) of l (*resp.* q).

Notation 5 *We shall write, with a slight abuse, $\mathcal{R}_u = \{1\} \otimes \mathcal{R}$, $\mathcal{R}_\partial = \mathcal{R} \otimes \{1\}$.*

Obviously, $\mathcal{R}_u$ (*resp.* $\mathcal{R}_\partial$) is a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$.

Example 8 *Take $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$. We set: $\mathcal{G}_{\mathcal{S}}^u(\cdot) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\cdot)$ (resp. $\mathcal{G}_{\mathcal{S}}^\partial(\cdot) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\cdot)$). In this case, we have $\mathcal{R}_u = \{1\} \otimes \mathbb{R}_+^{\mathbb{N}}$ (resp. $\mathcal{R}_\partial = \mathbb{R}_+^{\mathbb{N}} \otimes \{1\}$).*

The elements of $\mathcal{G}_{\mathcal{S}}^{u}(\Omega)$ (Ω open subset of $\mathbb{R}^d$) have uniform growth bounds with respect to the regularization parameter ε for all factors $(1+|x|)^q$. For $\mathcal{G}_{\mathcal{S}}^{\partial}(\Omega)$, those bounds are uniform for all derivatives. For $\mathcal{G}_{\mathcal{S}}^{\infty}(\Omega)$, introduced in example 7, the uniformity is global, in some sense stronger than the $\mathcal{G}^{\infty}$-regularity considered for the algebra $\mathcal{G}$. (In this last case, the uniformity is not required with respect to the compact sets.)

We have the obvious inclusions (for $\mathcal{R}$ regular subset of $\mathbb{R}_+^{\mathbb{N}}$):

$$\begin{array}{ccccccc} & & \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\Omega) & \longrightarrow & \mathcal{G}_{\mathcal{S}}^{\partial}(\Omega) & & \\ & \nearrow & & & & \searrow & \\ \mathcal{G}_{\mathcal{S}}^{\infty}(\Omega) & & & & & & \mathcal{G}_{\mathcal{S}}(\Omega)\ . \\ & \searrow & & & & \nearrow & \\ & & \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\Omega) & \longrightarrow & \mathcal{G}_{\mathcal{S}}^{u}(\Omega) & & \end{array} \tag{23}$$

Example 9 *The algebra* $\mathcal{G}_{\mathcal{S}}^{(1)}\left(\mathbb{R}^d\right)=\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_1'}\left(\mathbb{R}^d\right)/\mathcal{N}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ *introduced in (18) for the embedding of* $\mathcal{O}_M'\left(\mathbb{R}^d\right)$ *into* $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$ *(proposition 9) can be written as* $\mathcal{G}_{\mathcal{S}}^{(\mathcal{R}_1)_u}$*, with*

$$\mathcal{R}_1=\left\{N\in\mathbb{R}_+^{\mathbb{N}}\mid\exists b\in\mathbb{R}_+,\quad\forall l\in\mathbb{R}\quad N(l)\leq l+b\right\}.$$

As a first illustration of the properties of these spaces, we can show the existence of a canonical embedding of algebras of compactly supported generalized functions into particular spaces of rapidly decreasing generalized functions.

Lemma 12 *Let* $\mathcal{R}$ *be a regular subset of* $\mathbb{R}_+^{\mathbb{N}}$ *and* u *be in* $\mathcal{G}_C^{\mathcal{R}}(\Omega)$ *(*Ω *open subset of* $\mathbb{R}^d$*), with* $(u_\varepsilon)_\varepsilon$ *a representative of* u*. Let* κ *be in* $\mathcal{D}(\Omega)$*, with* $0\leq\kappa\leq 1$ *and* $\kappa\equiv 1$ *on a neighborhood of* $\operatorname{supp} u$*. Then* $(\kappa u_\varepsilon)_\varepsilon$ *belongs to* $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)$ *and* $[(\kappa u_\varepsilon)_\varepsilon]_{\mathcal{S}}$ *only depends on* u*, but not on* $(u_\varepsilon)_\varepsilon$ *and* κ*.*

Proof. We first show that $(\kappa u_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)$ and then the independence with respect to the representation.
(a) There exists a compact set $K\subset\Omega$ such that, for all $\varepsilon\in(0,1]$, $\operatorname{supp}\kappa u_\varepsilon\subset K$. It follows that κu_ε is compactly supported and therefore rapidly decreasing. Moreover

$$\forall(q,l)\in\mathbb{N}^2,\ \forall\varepsilon\in(0,1]\quad \mu_{q,l}(\kappa u_\varepsilon)\leq\sup\nolimits_{x\in K}(1+|x|)^q\,p_{K,l}(\kappa u_\varepsilon)\leq C_{K,q}\,p_{K,l}(\kappa u_\varepsilon)\,,\quad C_{K,q}>0.$$

Thus, $(\kappa u_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)$. Indeed, by using Leibniz' formula for estimating $p_{K,l}(\kappa u_\varepsilon)$, we can find a constant $C_{K,q,\kappa}>0$ such that

$$\forall(q,l)\in\mathbb{N}^2,\quad\forall\varepsilon\in(0,1]\qquad\mu_{q,l}(\kappa u_\varepsilon)\leq C_{K,q,\kappa}p_{K,l}(u_\varepsilon)\,. \tag{24}$$

(b) Let $(\widetilde{u}_\varepsilon)_\varepsilon$ be another representative of u and $\widetilde{\kappa}$ be in $\mathcal{D}(\Omega)$, with $0\leq\widetilde{\kappa}\leq 1$ and $\widetilde{\kappa}=1$ on a neighborhood of $\operatorname{supp}\widetilde{u}$. Let L be a compact set such that $\operatorname{supp}\kappa u_\varepsilon\cup\operatorname{supp}\widetilde{\kappa}\widetilde{u}_\varepsilon\subset L\subset\Omega$. According to the previous estimate, we have

$$\begin{aligned}\forall(q,l)\in\mathbb{N}^2,\quad\forall\varepsilon\in(0,1]\qquad\mu_{q,l}(\kappa u_\varepsilon-\widetilde{\kappa}\widetilde{u}_\varepsilon)&\leq\mu_{q,l}((\kappa-\widetilde{\kappa})\,u_\varepsilon)+\mu_{q,l}(\widetilde{\kappa}\,(u_\varepsilon-\widetilde{u}_\varepsilon))\\&\leq C_{L,q}\,p_{L,l}((\kappa-\widetilde{\kappa})\,u_\varepsilon)+C_{L,q}\,p_{L,l}(\widetilde{\kappa}\,(u_\varepsilon-\widetilde{u}_\varepsilon))\,.\end{aligned}$$

As $\kappa=\widetilde{\kappa}$ on a closed neighborhood V of $\operatorname{supp} u$, it follows that $p_{V,l}((\kappa-\widetilde{\kappa})\,u_\varepsilon)=0$. Moreover, for all $m\in\mathbb{N}$, $p_{L\backslash V,l}((\kappa-\widetilde{\kappa})\,u_\varepsilon)=\mathrm{O}(\varepsilon^m)$ as $\varepsilon\to 0$, since $(L\backslash V)\cap\operatorname{supp} u=\varnothing$. Then $p_{L,l}((\kappa-\widetilde{\kappa})\,u_\varepsilon)=\mathrm{O}(\varepsilon^m)$ as $\varepsilon\to 0$. As $[(u_\varepsilon)_\varepsilon]=[(\widetilde{u}_\varepsilon)_\varepsilon]$, we have $p_{L,l}(\widetilde{\kappa}\,(u_\varepsilon-\widetilde{u}_\varepsilon))=\mathrm{O}(\varepsilon^m)$ as $\varepsilon\to 0$. Then $\mu_{q,l}(\kappa u_\varepsilon-\widetilde{\kappa}\widetilde{u}_\varepsilon)=\mathrm{O}(\varepsilon^m)$ and $[(\kappa u_\varepsilon)_\varepsilon]_{\mathcal{S}}=[(\widetilde{\kappa}\widetilde{u})_\varepsilon]_{\mathcal{S}}$. ■

From lemma 12, we deduce easily the following proposition:

Proposition 13 *With the notations of lemma 12, the map*

$$\iota_{C,S} : \mathcal{G}_C^{\mathcal{R}}(\Omega) \to \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right) \qquad u \mapsto [(\kappa u_\varepsilon)_\varepsilon]_{\mathcal{S}}$$

is a linear embedding.

From the embedding $\iota_{C,S}$, one can then verify that the Fourier transform of a compactly supported generalized functions $u \in \mathcal{G}^{\mathcal{R}}(\Omega)$, which can be straightforwardly considered as an element of $\mathcal{G}_C\left(\mathbb{R}^d\right)$, is defined by one of the following equalities

$$\mathcal{F}(u) = \mathcal{F}(\iota_{C,S}(u)) = \left[\left(x \mapsto (2\pi)^{-d} \int_W \mathrm{e}^{ix\xi} u_\varepsilon(\xi)\, \mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{S}},$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}\left(\mathbb{R}^d\right)$ is any representative of u and W any relatively compact neighborhood of $\operatorname{supp} u$.

4.3 Exchange and regularity theorems

Theorem 14 ***(Exchange theorem)*** *For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, we have:*

$$\mathcal{F}\left(\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)\right) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right) \qquad\qquad \mathcal{F}\left(\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right)\right) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right). \tag{25}$$

Proof. The proof of this theorem is based on the following classical:

Lemma 15 *For all $u \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $(q,l) \in \mathbb{N}^2$, there exists a constant $C_{q,l} > 0$ such that*

$$\mu_{q,l}(\widehat{u}) \leq C_{q,l}\, \mu_{l+d+1,q}(u). \tag{26}$$

Let us prove this result first. Take $u \in \mathcal{S}\left(\mathbb{R}^d\right)$, $(q,l) \in \mathbb{N}^2$ and $(\alpha,\beta) \in \left(\mathbb{N}^d\right)^2$ with $|\alpha| = l$ and $|\beta| \leq q$. We have

$$\forall \xi \in \mathbb{R}^d \quad \partial^\alpha \widehat{u}(\xi) = \int \mathrm{e}^{-ix\xi} (-ix)^\alpha u(x)\, \mathrm{d}x.$$

By integration by parts, we obtain

$$\forall \xi \in \mathbb{R}^d \quad \xi^\beta \partial^\alpha \widehat{u}(\xi) = \int \xi^\beta \mathrm{e}^{-ix\xi} (-ix)^\alpha u(x)\, \mathrm{d}x = (-i)^{|\beta|} \int \mathrm{e}^{ix\xi} \partial^\beta \left[(-ix)^\alpha u(x)\right] \mathrm{d}x. \tag{27}$$

By the Leibniz formula, we have

$$\forall x \in \mathbb{R}^d \quad \partial^\beta \left[(-ix)^\alpha u(x)\right] = \sum_{\gamma \leq \beta} C_\beta^\gamma \partial^{\beta-\gamma} \left[(-ix)^\alpha\right] \partial^\gamma u(x), \tag{28}$$

where C_β^γ are the d dimensional binomial coefficients.

There exists a constant $C'_{\alpha,\beta} > 0$ such that, for all $\gamma \in \mathbb{N}^d$ with $\gamma \leq \beta$,

$$\begin{aligned}\forall x \in \mathbb{R}^d \quad (1+|x|)^{d+1} \left|\partial^{\beta-\gamma}\left[(-ix)^\alpha\right] \partial^\gamma u(x)\right| &\leq C'_{\alpha,\beta} (1+|x|)^{|\alpha|+d+1} \left|\partial^\gamma u(x)\right| \\ &\leq C'_{\alpha,\beta}\, \mu_{|\alpha|+d+1,\gamma}(u) \leq C'_{\alpha,\beta} \sup_{\gamma \leq \beta} \mu_{|\alpha|+d+1,\gamma}(u).\end{aligned}$$

Summing up these results in (28), we find a constant $C''_{\alpha,\beta} > 0$ such that

$$\forall x \in \mathbb{R}^d \quad (1+|x|)^{d+1} \left|\partial^\beta \left[(-ix)^\alpha u(x)\right]\right| \leq C''_{\alpha,\beta} \sup_{\gamma \leq \beta} \mu_{|\alpha|+d+1,\gamma}(u).$$

Going back to relation (27), we have

$$\forall \xi \in \mathbb{R}^d \quad \left| \xi^\beta \partial^\alpha \widehat{u}\left(\xi\right) \right| \leq C''_{\alpha,\beta} \sup_{\gamma \leq \beta} \mu_{|\alpha|+d+1,\gamma}\left(u\right) \int \left(1+|x|\right)^{-d-1} \mathrm{d}x.$$

Finally, we get the existence of a constant $C_{\alpha,\beta} > 0$

$$\forall \xi \in \mathbb{R}^d \quad \left| \xi^\beta \partial^\alpha \widehat{u}\left(\xi\right) \right| \leq C_{\alpha,\beta}\, \mu_{|\alpha|+d+1,|\beta|}\left(u\right) \leq C_{q,l}\, \mu_{l+d+1,q}\left(u\right),$$

where $C_{q,l}$ is a constant greater than all $C_{\alpha,\beta}$ for $|\alpha| = l$ and $|\beta| \leq q$. In the classical manner, we can deduce inequality (26) from this last estimate.

We return to the proof of theorem 14. Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$.
(a) Take $u \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)$ and $\left(u_\varepsilon\right)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)$ a representative of u. There exists a sequence $N \in \mathcal{R}$ such that $\mu_{r,q}\left(u_\varepsilon\right) = \mathrm{O}\left(\varepsilon^{-N(q)}\right)$ as $\varepsilon \to 0$, for all $r \in \mathbb{N}$. Lemma 15 implies that $\mu_{q,l}\left(\widehat{u}_\varepsilon\right) = \mathrm{O}\left(\varepsilon^{-N(q)}\right)$ as $\varepsilon \to 0$, for all $l \in \mathbb{N}$. Thus, $\mathcal{F}\left(u\right) \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right)$.
(b) Conversely, take $u \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right)$ and $\left(u_\varepsilon\right)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right)$ a representative of u. There exists a sequence $N \in \mathcal{R}$ such that $\mu_{r,m}\left(u_\varepsilon\right) = \mathrm{O}\left(\varepsilon^{-N(r)}\right)$ as $\varepsilon \to 0$, for all $r \in \mathbb{N}$. According to the stability of regular sets, there exists a sequence $N' \in \mathcal{R}$ such that

$$\forall l \in \mathbb{N} \quad N(l+d+1) \leq N'(l).$$

Lemma 15 implies that $\mu_{q,l}\left(\widehat{u}_\varepsilon\right) = \mathrm{O}\left(\varepsilon^{-N'(q)}\right)$ as $\varepsilon \to 0$, for all $l \in \mathbb{N}$. Thus, $\mathcal{F}\left(u\right) \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right)$.

So, we proved the inclusions of the sets in the left hand side of relations (25), into the sets of the right hand side. The equalities follow directly from a similar study with the inverse Fourier transform. ■

Example 10 *Take $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$. We get $\mathcal{F}\left(\mathcal{G}_{\mathcal{S}}^{\partial}\left(\mathbb{R}^d\right)\right) = \mathcal{G}_{\mathcal{S}}^{u}\left(\mathbb{R}^d\right)$ and $\mathcal{F}\left(\mathcal{G}_{\mathcal{S}}^{u}\left(\mathbb{R}^d\right)\right) = \mathcal{G}_{\mathcal{S}}^{\partial}\left(\mathbb{R}^d\right)$, result which is closely related to the classical exchange theorem between $\mathcal{O}_M\left(\mathbb{R}^d\right)$ and $\mathcal{O}'_C\left(\mathbb{R}^d\right)$.*

Indeed, take $u \in \mathcal{O}'_C\left(\mathbb{R}^d\right)$ and consider $\left(u_\varepsilon\right)_\varepsilon = \left(u * \rho_\varepsilon\right)_\varepsilon$ which is a representative of its image by the embedding $\iota_{\mathcal{S}}$. Its Fourier image $\mathcal{F}\left(\iota_{\mathcal{S}}\left(u\right)\right) = \left[\left(\widehat{u}\widehat{\rho}_\varepsilon\right)_\varepsilon\right]_{\mathcal{S}}$ belongs to $\mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right)$, with $\widehat{u} \in \mathcal{O}_M\left(\mathbb{R}^d\right)$ and $\widehat{\rho}_\varepsilon \in \mathcal{S}\left(\mathbb{R}^d\right)$. As $\lim_{\varepsilon\to 0} \widehat{\rho}_\varepsilon = 1$, we get $\lim_{\varepsilon \to 0}\left(\widehat{u}\widehat{\rho}_\varepsilon\right)_\varepsilon \in \mathcal{O}_M\left(\mathbb{R}^d\right)$. (For those limits, we consider $\mathcal{O}_M\left(\mathbb{R}^d\right)$ equipped with its usual topology: See [16], [24].) This shows the consistency of our result with the classical one. The generalized function $\mathcal{F}\left(\iota_{\mathcal{S}}\left(u\right)\right)$ belongs to a space of rapidly decreasing generalized functions, but the limit of its representatives when $\varepsilon \to 0$ is in a space of functions of moderate growth.

Corollary 16 *(**Regularity theorem**) We have: $\mathcal{F}\left(\mathcal{G}_{\mathcal{S}}^{\infty}\left(\mathbb{R}^d\right)\right) = \mathcal{G}_{\mathcal{S}}^{\infty}\left(\mathbb{R}^d\right)$.*

Proof. Apply theorem 14 with $\mathcal{R} = \mathcal{B}$, the set of bounded sequences, for which $\mathcal{B}_u = \mathcal{B}_\partial$. ■

We can now complete diagram 23 in the case of $\Omega = \mathbb{R}^d$:

$$\begin{array}{ccccccc} & & \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right) & \longrightarrow & \mathcal{G}_{\mathcal{S}}^{\partial}\left(\mathbb{R}^d\right) & & \\ & \nearrow & & & & \searrow & \\ \mathcal{G}_{\mathcal{S}}^{\infty}\left(\mathbb{R}^d\right) & & \updownarrow \mathcal{F} & & \updownarrow \mathcal{F} & & \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) \\ & \searrow & & & & \nearrow & \\ & & \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\left(\mathbb{R}^d\right) & \longrightarrow & \mathcal{G}_{\mathcal{S}}^{u}\left(\mathbb{R}^d\right) & & \end{array} . \quad (29)$$

An interesting consequence of corollary 16 is the following property, also proved in [8], which is the equivalent for rapidly decreasing generalized functions of the result mentioned in the introduction for the $\mathcal{G}^\infty$ regularity ($\mathcal{D}'\left(\Omega\right) \cap \mathcal{G}^\infty\left(\Omega\right) = \mathrm{C}^\infty\left(\Omega\right)$, [20]).

Proposition 17 *We have* $\mathcal{O}'_C\left(\mathbb{R}^d\right) \cap \mathcal{G}^\infty_{\mathcal{S}}\left(\mathbb{R}^d\right) = \mathcal{S}\left(\mathbb{R}^d\right)$.

Proof. We follow here the ideas of [19] for the proof of the above mentioned result about $\mathcal{G}^\infty\left(\mathbb{R}^d\right)$. Let u be in $\mathcal{O}'_C\left(\mathbb{R}^d\right)$ and set $(u_\varepsilon)_\varepsilon = (u * \rho_\varepsilon)_\varepsilon$. By assumption $[(u*\rho_\varepsilon)_\varepsilon]_{\mathcal{S}}$ is in $\mathcal{G}^\infty_{\mathcal{S}}\left(\mathbb{R}^d\right)$. According to corollary 16, $\mathcal{F}_S\left([(u*\rho_\varepsilon)_\varepsilon]_{\mathcal{S}}\right)$ is also in $\mathcal{G}^\infty_{\mathcal{S}}\left(\mathbb{R}^d\right)$. It follows that there exists $N \in \mathbb{N}$ such that

$$\forall q \in \mathbb{N},\ \ \exists C_q > 0 \quad \sup_{\xi\in\mathbb{R}^d} (1+|\xi|)^q \left|\widehat{u}(\xi)\,\widehat{\rho}_\varepsilon(\xi)\right| \leq C_q \varepsilon^{-N}, \text{ for } \varepsilon \text{ small enough.}$$

By choice of ρ, $\widehat{\rho}_\varepsilon$ is an element of $\mathcal{D}\left(\mathbb{R}^d\right)$. Moreover, a straightforward calculation shows that $\widehat{\rho}_\varepsilon(\xi) = \widehat{\rho}(\varepsilon\xi)$, for all $\xi \in \mathbb{R}^d$, with $\widehat{\rho}$ equal to 1 on a neighborhood of 0. It follows that, for all $q \in \mathbb{N}$, we have

$$\begin{aligned}\forall \xi \in \mathbb{R}^d \quad (1+|\xi|)^q \left|\widehat{u}(\xi)\right| &\leq (1+|\xi|)^q \left|\widehat{u}(\xi)\right| \left(\left|1-\widehat{\rho}(\varepsilon\xi)\right| + \left|\widehat{\rho}(\varepsilon\xi)\right|\right) \\ &\leq (1+|\xi|)^q \left|\widehat{u}(\xi)\right| \left|1-\widehat{\rho}(\varepsilon\xi)\right| + C_q\varepsilon^{-N}.\end{aligned}$$

Since $1-\widehat{\rho}(\varepsilon\xi) = \widehat{\rho}(0) - \widehat{\rho}(\varepsilon\xi) = -\varepsilon\xi \int_0^1 \widehat{\rho}'(\varepsilon\xi t)\,\mathrm{d}t$, with $\widehat{\rho}'$ bounded, there exists a constant $C > 0$ such that

$$\forall \xi \in \mathbb{R}^d \quad (1+|\xi|)^q \left|\widehat{u}(\xi)\right| \leq C\,(1+|\xi|)^q \left|\widehat{u}(\xi)\right| \varepsilon\,|\xi| + C_q\varepsilon^{-N}.$$

As $\widehat{u}$ is in $\mathcal{O}_M\left(\mathbb{R}^d\right)$, there exist $m \in \mathbb{N}$ and a constant $C_1 > 0$ such that

$$\sup\nolimits_{\xi\in\mathbb{R}^d} (1+|\xi|)^{-m+1} \left|\widehat{u}(\xi)\right| \leq C_1.$$

Therefore, by setting $C_2 = \max(CC_1, C_q)$, we get

$$\begin{aligned}\forall \xi \in \mathbb{R}^d \quad (1+|\xi|)^q \left|\widehat{u}(\xi)\right| &\leq C_2 \left((1+|\xi|)^{q+m-1} \varepsilon\,|\xi| + \varepsilon^{-N}\right) \\ &\leq C_2 \left((1+|\xi|)^{q+m} \varepsilon + \varepsilon^{-N}\right).\end{aligned}$$

By minimizing the function $f_\xi : \varepsilon \mapsto (1+|\xi|)^{q+m} \varepsilon + \varepsilon^{-N}$, we get the existence of a constant $C_3 > 0$ such that

$$\forall \xi \in \mathbb{R}^d \quad (1+|\xi|)^q \left|\widehat{u}(\xi)\right| \leq C_3 \left((1+|\xi|)^{N(q+m)/(N+1)}\right),$$

and

$$\forall \xi \in \mathbb{R}^d \quad \left|\widehat{u}(\xi)\right| \leq C_3 \left((1+|\xi|)^{-q/(N+1)+mN/(N+1)}\right),$$

for all $q \in \mathbb{N}$. (m only depends on u.) Treating the derivatives in the same way, we obtain the same type of estimates. Therefore $\widehat{u}$ and its derivatives are rapidly decreasing. This shows that $\mathcal{O}'_C(\Omega) \cap \mathcal{G}^\infty_{\mathcal{S}}(\Omega) \subset \mathcal{S}(\Omega)$. As the other inclusion is obvious, our claim is proved. ■

5 Global regularity of compactly supported generalized functions

5.1 C^∞-regularity for compactly supported distributions

In order to render easier the comparison between the distributional case and the generalized case, we are going to recall the classical theorem and complete it by some equivalent statements.

Theorem 18 *For u in $\mathcal{E}'\left(\mathbb{R}^d\right)$, the following equivalences hold:*

$$\begin{aligned} (i)\quad u\in \mathrm{C}^\infty\left(\mathbb{R}^d\right) \quad &\Leftrightarrow (ii) \quad \mathcal{F}(u)\in \mathcal{S}\left(\mathbb{R}^d\right) \\ &\Leftrightarrow (iii) \quad \mathcal{F}(u)\in \mathcal{O}'_C\left(\mathbb{R}^d\right) \\ &\Leftrightarrow (iv) \quad \mathcal{F}(u)\in \mathcal{O}'_M\left(\mathbb{R}^d\right) \\ &\Leftrightarrow (v) \quad \mathcal{F}(u)\in \mathcal{O}'_C\left(\mathbb{R}^d\right). \end{aligned}$$

Proof. Equivalence $(i)\Leftrightarrow(ii)$ is the classical result. The trivial inclusion $\mathcal{S}\left(\mathbb{R}^d\right)\subset\mathcal{S}_*\left(\mathbb{R}^d\right)$ shows $(ii)\Rightarrow(iii)$. Then, the structure of elements of $\mathcal{O}'_M\left(\mathbb{R}^d\right)$ [21] shows that $\mathcal{S}_*(\Omega)$ is canonically embedded in $\mathcal{O}'_M\left(\mathbb{R}^d\right)$: This shows $(iii)\Rightarrow(iv)$. As $\mathcal{O}'_M\left(\mathbb{R}^d\right)\subset\mathcal{O}'_C\left(\mathbb{R}^d\right)$, $(iv)\Rightarrow(v)$ is obvious. For $(v)\Rightarrow(i)$, note that $\mathcal{F}(u)$ belongs to $\mathcal{O}_M\left(\mathbb{R}^d\right)$ and better to $\mathcal{O}_C\left(\mathbb{R}^d\right)$ since u is in $\mathcal{E}'\left(\mathbb{R}^d\right)$. (This last assertion is a refinement of the classical previous one.) Then, if (v) holds, $\mathcal{F}(u)$ is in $\mathcal{O}_C\left(\mathbb{R}^d\right)\cap\mathcal{O}'_C\left(\mathbb{R}^d\right)$ which is equal to $\mathcal{S}\left(\mathbb{R}^d\right)$ [21]. Then (ii) holds. ■

Theorem 18 shows, at least, that there is no need to consider spaces of functions with all the derivatives rapidly decreasing to characterize elements of $\mathcal{E}'\left(\mathbb{R}^d\right)$ which are C^∞. In fact, we can only consider functions rapidly decreasing, with no other hypothesis on the derivatives. A similar situation holds for generalized functions, justifying the introduction of rough generalized functions in the following subsection.

5.2 Rough rapidly decreasing generalized functions

5.2.1 Definitions

Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$ and Ω an open subset or $\mathbb{R}^d$. Set

$$\begin{aligned} \mathcal{S}_*(\Omega) &= \left\{f\in \mathrm{C}^\infty(\Omega)\mid \forall q\in\mathbb{N}\ \ \mu_{q,0}(f)<+\infty\right\}, \\ \mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega) &= \left\{(f_\varepsilon)_\varepsilon\in \mathcal{S}_*(\Omega)^{(0,1]}\ \middle|\ \exists N\in\mathcal{R},\ \forall q\in\mathbb{N}\ \ \mu_{q,0}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{-N(q)}\right)\text{ as }\varepsilon\to 0\right\}, \qquad (30) \\ \mathcal{N}_{\mathcal{S}_*}(\Omega) &= \left\{(f_\varepsilon)_\varepsilon\in \mathcal{S}_*(\Omega)^{(0,1]}\ \middle|\ \forall N\in\mathbb{R}_+^{\mathbb{N}},\ \forall l\in\mathbb{N}\ \ \mu_{q,0}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{N(l)}\right)\text{ as }\varepsilon\to 0\right\}. \end{aligned}$$

One can show that $\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$ is a subalgebra of $\mathcal{S}_*(\Omega)^{(0,1]}$ and $\mathcal{N}_{\mathcal{S}_*}(\Omega)$ an ideal of $\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$. (In fact, these spaces fit in the general scheme of construction of Colombeau type algebra [1], [6], [17].)

Definition 7 *The space $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)=\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)/\mathcal{N}_{\mathcal{S}_*}(\Omega)$ is called the algebra of $\mathcal{R}$-*regular rough rapidly decreasing generalized functions.

Example 11 *Taking $\mathcal{R}=\mathbb{R}_+^{\mathbb{N}}$, we obtain the space $\mathcal{G}_{\mathcal{S}_*}(\Omega)$ of* rough rapidly decreasing generalized functions.

Example 12 *Taking $\mathcal{R}=\mathcal{B}$, the set of bounded sequences, we obtain the space $\mathcal{G}_{\mathcal{S}_*}^{\infty}(\Omega)$, of* regular rough rapidly decreasing generalized functions.

Lemma 6 implies immediately the following proposition:

Proposition 19 *If the open set Ω is a box and $\mathcal{R}'$ a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$, then $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$ is included in $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}'_0}(\Omega)$, where $\mathcal{R}'_0$ is the (regular) subset of $\mathbb{R}_+^{\mathbb{N}}$ equal to $\{N(\cdot,0),\ N\in\mathcal{R}'\}$.*

Example 13 *If Ω is a box, for all $\mathcal{R}\subset\mathbb{R}_+^{\mathbb{N}}$, $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\Omega)$ is included in $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$.*

Indeed, $\mathcal{R}_\partial=\mathcal{R}\otimes\{1\}$, which implies that $(\mathcal{R}_\partial)_0=\mathcal{R}$. Let us quote two other examples of application of proposition 19.

Corollary 20 *If the open set Ω is a box, then*
(i) $\mathcal{G}_{\mathcal{S}}(\Omega)$, obtained for $\mathcal{R}'=\mathbb{R}_+^{\mathbb{N}^2}$, is included in $\mathcal{G}_{\mathcal{S}_*}(\Omega)$.
(ii) $\mathcal{G}_{\mathcal{S}}^{\infty}(\Omega)$, obtained for $\mathcal{R}'=\mathcal{B}'$, is included in $\mathcal{G}_{\mathcal{S}_*}^{\infty}(\Omega)$.

Indeed, (i) (*resp.* (ii)) holds, since $\left(\mathbb{R}_+^{\mathbb{N}^2}\right)_0=\mathbb{R}_+^{\mathbb{N}}$ (*resp.* $(\mathcal{B}')_0=\mathcal{B}'$). Note that the proof of proposition 17 shows that

$$\mathcal{G}_{\mathcal{S}_*}^{\infty}\left(\mathbb{R}^d\right)\cap\mathcal{O}_C'\left(\mathbb{R}^d\right)=\mathcal{S}_*\left(\mathbb{R}^d\right).$$

We turn to the question of embeddings. First, the structure of elements of $\mathcal{O}_C'\left(\mathbb{R}^d\right)$ ([16], [21], [24]) shows that $\mathcal{S}_*\left(\mathbb{R}^d\right)$ is canonically embedded in $\mathcal{O}_C'\left(\mathbb{R}^d\right)$. The embedding of $\mathcal{S}\left(\mathbb{R}^d\right)$ into $\mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)$ is done by the canonical injective map

$$\sigma_{\mathcal{S}_*}:\mathcal{S}_*\left(\mathbb{R}^d\right)\to\mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)\quad f\mapsto(f_\varepsilon)_\varepsilon+\mathcal{N}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)\quad\text{with } f_\varepsilon=f\text{ for }\varepsilon\in(0,1].$$

Finally, a simplification of the proofs of theorems 7, 8 and proposition 9 leads to the following theorem, where $(\rho_\varepsilon)_\varepsilon$ is defined by (14) and (15).

Theorem 21
(i) The map

$$\iota_{\mathcal{S}_*}:\mathcal{O}_C'\left(\mathbb{R}^d\right)\to\mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)\qquad u\mapsto(u*\rho_\varepsilon)_\varepsilon+\mathcal{N}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)$$

is a linear embedding which commutes with partial derivatives.
(ii) We have: $\iota_{\mathcal{S}_*|\mathcal{S}_*(\mathbb{R}^d)}=\sigma_{\mathcal{S}_*}$.
(iii) We have: $\iota_{\mathcal{S}_*}\left(\mathcal{O}_M'\left(\mathbb{R}^d\right)\right)\subset\mathcal{G}_{\mathcal{S}_*}^{\infty}\left(\mathbb{R}^d\right)$.

Remark 2 *Theorems 7, 8 and 21 combined together show that all the arrows are injective and all diagrams commutative in the following schemes:*

$$\begin{array}{ccccc} \mathcal{S}\left(\mathbb{R}^d\right) & & \longrightarrow & & \mathcal{S}_*\left(\mathbb{R}^d\right)\\ & \searrow & & \swarrow & \\ \downarrow & & \mathcal{O}_C'\left(\mathbb{R}^d\right) & & \downarrow\\ & \swarrow & & \searrow & \\ \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right) & & \longrightarrow & & \mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right) \end{array}\qquad \begin{array}{ccccc} \mathcal{S}\left(\mathbb{R}^d\right) & & \longrightarrow & & \mathcal{S}_*\left(\mathbb{R}^d\right)\\ & \searrow & & \swarrow & \\ \downarrow & & \mathcal{O}_M'\left(\mathbb{R}^d\right) & & \downarrow\\ & \swarrow & & \searrow & \\ \mathcal{G}_{\mathcal{S}}^{u}\left(\mathbb{R}^d\right) & & \longrightarrow & & \mathcal{G}_{\mathcal{S}_*}^{\infty}\left(\mathbb{R}^d\right) \end{array}.$$

5.2.2 Fourier transform in $\mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)$

We need in the sequel to define a Fourier transform (or an inverse Fourier transform) in $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}\left(\mathbb{R}^d\right)$. This is done in the following way. Set, for any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$,

$$\mathcal{X}_{\mathcal{B}}(\Omega)=\left\{(f_\varepsilon)_\varepsilon\in\mathrm{C}^\infty(\Omega)^{(0,1]}\ \middle|\ \exists N\in\mathbb{R}_+^{\mathbb{N}},\ \forall l\in\mathbb{N}\ \ \mu_{0,l}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{-N(l)}\right)\text{ as }\varepsilon\to0\right\},$$
$$\mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)=\left\{(f_\varepsilon)_\varepsilon\in\mathrm{C}^\infty(\Omega)^{(0,1]}\ \middle|\ \exists N\in\mathcal{R},\ \forall l\in\mathbb{N}\ \ \mu_{0,l}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{-N(l)}\right)\text{ as }\varepsilon\to0\right\},$$
$$\mathcal{N}_{\mathcal{B}}(\Omega)=\left\{(f_\varepsilon)_\varepsilon\in\mathrm{C}^\infty(\Omega)^{(0,1]}\ \middle|\ \forall N\in\mathbb{R}_+^{\mathbb{N}},\ \forall l\in\mathbb{N}\ \ \mu_{0,l}(f_\varepsilon)=\mathrm{O}\left(\varepsilon^{N(l)}\right)\text{ as }\varepsilon\to0\right\}.$$

According to the general scheme of construction of Colombeau type algebras, $\mathcal{G}_{\mathcal{B}}(\Omega)=\mathcal{X}_{\mathcal{B}}(\Omega)/\mathcal{N}_{\mathcal{B}}(\Omega)$ is an algebra, named the *algebra of bounded generalized functions*. Moreover, $\mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)$ is a subalgebra of $\mathcal{X}_{\mathcal{B}}(\Omega)$. (The proof is similar to the one of proposition 1.) The space $\mathcal{G}_{\mathcal{B}}^{\mathcal{R}}(\Omega)=\mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)/\mathcal{N}_{\mathcal{B}}(\Omega)$ is called the space of *$\mathcal{R}$-regular bounded generalized functions.*

Notation 6 *We shall note $[(f_\varepsilon)_\varepsilon]_{\mathcal{B}}$ the class of $(f_\varepsilon)_\varepsilon$ in $\mathcal{G}_{\mathcal{B}}^{\mathcal{R}}(\Omega)$.*

Remark 3 *One can verify that* $\mathcal{G}_C(\Omega)$ *(resp.* $\mathcal{G}_C^{\mathcal{R}}(\Omega)$*) is embedded in* $\mathcal{G}_{\mathcal{B}}(\Omega)$ *(resp.* $\mathcal{G}_{\mathcal{B}}^{\mathcal{R}}(\Omega)$*).*

Proposition 22
(i) For all $u \in \mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ *and* $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$ *a representative of* u*, the expression*

$$\widehat{u} : \left[\widehat{u}_\varepsilon = \left(\xi \mapsto \int \mathrm{e}^{-ix\xi} u_\varepsilon(x)\,\mathrm{d}x\right)_\varepsilon\right]_{\mathcal{B}} \tag{31}$$

defines an element of $\mathcal{G}_{\mathcal{B}}(\Omega)$ *depending only on* u*.*
(ii) For any regular subspace $\mathcal{R}$ *of* $\mathbb{R}_+^{\mathbb{N}}$ *and* $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}}(\Omega)$*, we have* $(\widehat{u}_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)$*.*

The **proof** of proposition 22 is mainly a consequence of lemma 15.
Assertion (i). Take $u \in \mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$ a representative of u. Then lemma 15 (applied with $q = 0$) implies that

$$\forall l \in \mathbb{N},\ \exists C_l > 0,\ \forall \varepsilon \in (0,1] \qquad \mu_{0,l}(\widehat{u}_\varepsilon) \leq C_\alpha\, \mu_{l+d+1,0}(u_\varepsilon). \tag{32}$$

This estimate shows that $(\widehat{u}_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{B}}(\mathbb{R}^d)$. Indeed, if $(u_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$, there exists a sequence $N \in \mathcal{R}$ such that $\mu_{l,0}(u_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(l)}\right)$ as $\varepsilon \to 0$ and setting $N' : l \mapsto N(l+d+1)$, we get that $\mu_{0,l}(\widehat{u}_\varepsilon) = \left(\varepsilon^{-N'(l)}\right)$ as $\varepsilon \to 0$. According to the overstability by translation of the subset $\mathcal{R}$, $(\widehat{u}_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_{\mathcal{B}}(\mathbb{R}^d)$. Similar arguments show that, if $(\eta_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{S}_*}(\mathbb{R}^d)$, then $(\widehat{\eta}_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{B}}(\Omega)$. Therefore, relation (31) defines an element of $\mathcal{G}_{\mathcal{B}}(\mathbb{R}^d)$, depending only on u.
Assertion (ii). The estimate (32) implies that the regularity of the sequences in the definition of moderate elements transfers by Fourier transform from the space index q in the $\mathcal{S}_*$-type spaces to the derivative index l in the Colombeau type space (here of bounded functions), showing our claim.

We define the **Fourier transform** $\mathcal{F}_*$ on $\mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ by the formula

$$\mathcal{F}_* : \mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right) \to \mathcal{G}_B\left(\mathbb{R}^d\right) \quad u \mapsto \left[\left(x \mapsto \int \mathrm{e}^{-ix\xi} u_\varepsilon(\xi)\,\mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{B}}$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$ is any representative of u. (The inverse Fourier on $\mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ is defined analogously.)

The assertion (ii) of proposition 22 implies:

Proposition 23 *(Small exchange theorem) We have:* $\mathcal{F}\left(\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}\right) \subset \mathcal{G}_B^{\mathcal{R}}(\mathbb{R}^d)$*.*

5.3 $\mathcal{G}^{\mathcal{R}}$-regularity for compactly supported generalized functions

We have now all the elements to formulate and prove the following fundamental theorem:

Theorem 24 *Let* $\mathcal{R}$ *be regular subspace of* $\mathbb{R}_+^{\mathbb{N}}$*. For* u *in* $\mathcal{G}_C(\mathbb{R}^d)$*, the following equivalences hold:*

$$\begin{aligned}(i)\ u \in \mathcal{G}^{\mathcal{R}}\left(\mathbb{R}^d\right) \quad &\Leftrightarrow (ii) \quad \mathcal{F}(u) \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\left(\mathbb{R}^d\right)\\ &\Leftrightarrow (iii) \quad \mathcal{F}(u) \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}\left(\mathbb{R}^d\right).\end{aligned}$$

Proof.
$(i) \Rightarrow (ii)$ As u is in $\mathcal{G}_C(\mathbb{R}^d) \cap \mathcal{G}^{\mathcal{R}}(\mathbb{R}^d) = \mathcal{G}_C^{\mathcal{R}}(\mathbb{R}^d)$, u is in $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$ according to proposition 13. Then, applying theorem 14, $\mathcal{F}(u)$ is in $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\mathbb{R}^d)$.
$(ii) \Rightarrow (iii)$ We have $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\mathbb{R}^d) \subset \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\mathbb{R}^d)$, according to example 13.

$(iii) \Rightarrow (i)$ Let u be in $\mathcal{G}_C(\mathbb{R}^d)$, $(u_\varepsilon)_\varepsilon$ be a representative of u and K a compact set such that $\operatorname{supp} u_\varepsilon \subset K$, for all ε in $(0,1]$. We have $\mathcal{F}_{\mathcal{S}}(u) = [(\widehat{u}_\varepsilon)_\varepsilon]_{\mathcal{G}_{\mathcal{S}}}$ where $\widehat{}$ denotes the classical Fourier transform in $\mathcal{S}$. By assumption $\mathcal{F}_{\mathcal{S}}(u)$ is in $\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\mathbb{R}^d)$ and we can consider its inverse Fourier transform $\mathcal{F}_*^{-1}$, with $\mathcal{F}_*^{-1}(\mathcal{F}_{\mathcal{S}}(u))$ in $\mathcal{G}^{\mathcal{R}}_B(\mathbb{R}^d)$ and

$$\mathcal{F}_*^{-1}(\mathcal{F}_{\mathcal{S}}(u)) = \left[\left(\mathcal{F}^{-1}(\widehat{u}_\varepsilon)\right)_\varepsilon\right]_{\mathcal{B}}.$$

Using the classical isomorphism theorem in $\mathcal{S}$, we have $\mathcal{F}^{-1}(\widehat{u}_\varepsilon) = u_\varepsilon$ for all ε in $(0,1]$. Then

$$\mathcal{F}_*^{-1}(\mathcal{F}_{\mathcal{S}}(u)) = [(u_\varepsilon)_\varepsilon]_{\mathcal{B}}.$$

Since all the u_ε have their support included in the same compact set, we obviously have $[(u_\varepsilon)_\varepsilon]_{\mathcal{B}} = \iota_{C,B}(u)$ where $\iota_{C,B}$ is the canonical embedding of $\mathcal{G}_C(\mathbb{R}^d)$ in $\mathcal{G}_B(\mathbb{R}^d)$. Therefore, $u \in \mathcal{G}^{\mathcal{R}}_B(\mathbb{R}^d) \cap \mathcal{G}_C(\mathbb{R}^d) = \mathcal{G}^{\mathcal{R}}(\mathbb{R}^d) \cap \mathcal{G}_C(\mathbb{R}^d)$. ■

Example 14 *The case $\mathcal{R} = \mathcal{B}$ in theorem 24 gives a characterization of the global $\mathcal{G}^\infty$-regularity of compactly supported generalized functions.*

Moreover, we can refine theorem 24 in this particular case and prove:

Theorem 25 *For u in $\mathcal{G}_C(\mathbb{R}^d)$, the following statements are equivalent:*

$$\begin{aligned} (i) \quad u \in \mathcal{G}^\infty(\mathbb{R}^d) \quad &\Leftrightarrow (ii) \quad \mathcal{F}(u) \in \mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d) \\ &\Leftrightarrow (iii) \quad \mathcal{F}(u) \in \mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d) \\ &\Leftrightarrow (iv) \quad \mathcal{F}(u) \in \mathcal{G}^\infty_{\mathcal{S}_*}(\mathbb{R}^d). \end{aligned}$$

Indeed, $(i) \Rightarrow (ii)$ and $(iv) \Rightarrow (i)$ follow directly from theorem 24 applied with $\mathcal{R} = \mathcal{B}$, since $\mathcal{B}_\partial = \mathcal{B}'$, the set of bounded elements of $\mathbb{R}_+^{\mathbb{N}^2}$. For $(ii) \Rightarrow (iii)$, we have $\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d) \subset \mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d)$. For $(iii) \Rightarrow (iv)$, we remark that $\mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d)$ is obtained with $\mathcal{R}' = \{1\} \otimes \mathbb{R}_+^{\mathbb{N}}$ as regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$. This implies that $(\mathcal{R}')_0 = \mathcal{B}'$, with the notations of proposition 19.

6 Local and microlocal $\mathcal{R}$-regularity

We follow here the presentation of [15] and show that, with the previously introduced material, the $\mathcal{G}^{\mathcal{R}}$ wavefront of a generalized function is defined exactly like the C^∞ wavefront of a distribution. First, as $\mathcal{G}^{\mathcal{R}}$ is a subsheaf of $\mathcal{G}$, the following definition makes sense:

Definition 8 *Let u be in $\mathcal{G}(\Omega)$. The singular $\mathcal{G}^{\mathcal{R}}$-support of u is the set*

$$\operatorname{singsupp}_{\mathcal{R}} u = \Omega \setminus \left\{x \in \Omega \mid \exists V \in \mathcal{V}_x,\ \ u \in \mathcal{G}^{\mathcal{R}}(V)\right\}.$$

Proposition 26 *$\mathcal{G}_{\mathcal{S}_*} : \Omega \to \mathcal{G}_{\mathcal{S}_*}(\Omega)$ is a pre-sheaf: It allows restrictions.*

The **proof** is similar to the part (b) of the one of proposition 5.

Notation 7 *For $(x,\xi) \in \Omega \times \mathbb{R}^d \setminus \{0\}$ (Ω open subset of $\mathbb{R}^d$), we shall denote by:*
(i) $\mathcal{V}_x$ (resp. $\mathcal{V}^\Gamma_\xi$), the set of all open neighborhoods (resp. open convex conic neighborhoods) of x (resp. ξ),
(ii) $\mathcal{D}_x(\Omega)$, the set of elements $\mathcal{D}(\Omega)$ non vanishing at x.

For $\Gamma \in \mathcal{V}_\xi^\Gamma$, we say that $\widehat{u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$ if $u_{|\Gamma} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$. Let us fix a regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ and set, for $u \in \mathcal{G}_C(\mathbb{R}^d)$,

$$O^{\mathcal{R}}(u) = \left\{\xi \in \mathbb{R}^d \setminus \{0\} \mid \exists \Gamma \in \mathcal{V}_\xi^\Gamma \quad \widehat{u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)\right\} \qquad \Sigma^{\mathcal{R}}(u) = \left(\mathbb{R}^d \setminus \{0\}\right) \setminus O^{\mathcal{R}}(u).$$

Lemma 27 *For $u \in \mathcal{G}_C(\mathbb{R}^d)$ and $\varphi \in D(\mathbb{R}^d)$, $O^{\mathcal{R}}(u) \subset O^{\mathcal{R}}(\varphi u)$ (or, equivalently, $\Sigma^{\mathcal{R}}(\varphi u) \subset \Sigma^{\mathcal{R}}(u)$).*

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^d)$ be a representative of u with $\operatorname{supp} u_\varepsilon$ included in the same compact set, for all ε in $(0,1]$. We have

$$\widehat{\varphi u_\varepsilon}(y) = \widehat{\varphi} * \widehat{u}_\varepsilon(y) = \int \widehat{\varphi}(\eta)\, \widehat{u}_\varepsilon(y-\eta)\, \mathrm{d}\eta.$$

Let ξ be in $O^{\mathcal{R}}(u)$ and $\Gamma \in \mathcal{V}_\xi^\Gamma$ such that $\widehat{u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$. There exists an open conic neighborhood $\Gamma_1 \subset \Gamma$ of ξ and a real number $c \in (0,1)$ such that, for all (y,η) with $y \in \Gamma_1$ and $|\eta| \leq c|y|$, $y - \eta \in \Gamma$. Then

$$\begin{aligned}\widehat{\varphi u_\varepsilon}(y) &= \int_{|\eta| \leq c|y|} \widehat{\varphi}(\eta)\, \widehat{u}_\varepsilon(y-\eta)\, \mathrm{d}\eta + \int_{|\eta| > c|y|} \widehat{\varphi}(\eta)\, \widehat{u}_\varepsilon(y-\eta)\, \mathrm{d}\eta \\ &= \underbrace{\int_{|\eta| \leq c|y|} \widehat{\varphi}(\eta)\, \widehat{u}_\varepsilon(y-\eta)\, \mathrm{d}\eta}_{v_{1,\varepsilon}(y)} + \underbrace{\int_{|y-\eta| > c|y|} \widehat{\varphi}(y-\eta)\, \widehat{u}_\varepsilon(\eta)\, \mathrm{d}\eta}_{v_{2,\varepsilon}(y)}.\end{aligned}$$

In order to estimate $v_{1,\varepsilon}$, let us remark that $\widehat{u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$. There exists a sequence $N \in \mathcal{R}$ such that, for all $q \in \mathbb{N}$, there exists a constant $C_1 > 0$ with

$$\forall (y,\eta) \in \Gamma_1 \times \mathbb{R}^d \text{ with } |\eta| \leq c|y| \quad |\widehat{u}_\varepsilon(y-\eta)| \leq C_1\, \varepsilon^{-N(q)} (1 + |y-\eta|)^{-q},$$

for ε small enough.

As, for $|\eta| \leq c|y|$, we have $|y-\eta| \geq ||y| - |\eta|| \geq |y|(1-c)$, it follows that

$$\forall (y,\eta) \in \Gamma_1 \times \mathbb{R}^d \text{ with } |\eta| \leq c|y| \quad |\widehat{u}_\varepsilon(y-\eta)| \leq C_1\, \varepsilon^{-N(q)} (1 + |y|(1-c))^{-q}.$$

Since $\widehat{\varphi}$ is rapidly decreasing, we get the existence of a constant $C_2 > 0$ such that.

$$\forall \eta \in \mathbb{R}^d \qquad \widehat{\varphi}(\eta) \leq C_2 (1 + |\eta|)^{-d-1}.$$

Replacing in the definition of $|v_{1,\varepsilon}(y)|$, we get the existence of a constant $C_3 > 0$ such that

$$\forall y \in \Gamma_1 \qquad (1+|y|)^q\, |v_{1,\varepsilon}(y)| \leq C_3\, \varepsilon^{-N(q)} \int \left(\frac{1+|y|}{(1+|y|(1-c))}\right)^q \frac{1}{(1+|\eta|)^{d+1}}\, \mathrm{d}\eta.$$

The function $t \mapsto (1+t)/(1+t(1-c))$ is bounded on $\mathbb{R}_+$. It follows that the integral in the previous inequality converges, we finally get a constant $C_4 > 0$ such that

$$\forall y \in \Gamma_1 \qquad |v_{1,\varepsilon}(y)| \leq C_4\, \varepsilon^{-N(q)} (1+|y|)^{-q}. \tag{33}$$

For $v_{2,\varepsilon}$, note that $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$. Therefore, there exist $M > 0$ and $C_5 > 0$ such that $|\widehat{u}_\varepsilon(\eta)| \leq C_5\, \varepsilon^{-M} (1+|\eta|)^{-d-1}$ for ε small enough. As $\widehat{\varphi} \in S(\mathbb{R}^d)$, there exists $C_6 > 0$ such that

$$\forall (y,\eta) \in \Gamma_1 \times \mathbb{R}^d \text{ with } |y-\eta| \geq c|y| \quad |\widehat{\varphi}(y-\eta)| \leq C_6 (1+|y-\eta|)^{-q} \leq C_6 (1+c|y|)^{-q}.$$

Then $|\widehat{\varphi}(y-\eta)| = \mathrm{O}\left((1+|y|)^{-q}\right)$ as $y \to +\infty$. Thus, there exists a constant $C_7 > 0$ such that

$$\forall y \in \Gamma_1 \qquad |v_{2,\varepsilon}(y)| \leq C_7\, \varepsilon^{-M}\,(1+|y|)^{-q}\,, \quad \text{for } \varepsilon \text{ small enough.} \tag{34}$$

From (33) and (34), we gets that, for all $q \in \mathbb{N}$, there exists a constant $C > 0$ (depending on q) such that

$$\forall y \in \Gamma_1 \qquad |\widehat{\varphi u}_{\varepsilon}(y)| \leq C\, \varepsilon^{-(N(q)+M)}\,(1+|y|)^{-q}\,.$$

Since $\mathcal{R}$ is overstable by translation, there exists a sequence $N'(\cdot) \in \mathcal{R}$ such that $N(\cdot) + M \preceq N'(\cdot)$ and $\mu_{q,0}(\widehat{\varphi u}_{\varepsilon}) = \mathrm{O}\left(\varepsilon^{-N'(q)}\right)$ as $\varepsilon \to 0$. Finally $\widehat{\varphi u} = [(\widehat{\varphi u}_{\varepsilon})_{\varepsilon}]_{\mathcal{G}_{\mathcal{S}_*}} \in \mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\Gamma_1)$ and $\xi \in O^{\mathcal{R}}(\varphi u)$. ■

Definition 9 *An element $u \in \mathcal{G}(\Omega)$ is said to be $\mathcal{R}$ microregular on $(x,\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ if there exist $\varphi \in \mathcal{D}_x(\Omega)$ and $\Gamma \in \mathcal{V}^{\Gamma}_{\xi}$, such that $\widehat{\varphi u} \in \mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\Gamma)$.*

We set, for $u \in \mathcal{G}(\Omega)$ and $x \in \Omega$,

$$O^{\mathcal{R}}_x(u) = \cup_{\varphi \in \mathcal{D}_x} O^{\mathcal{R}}(\varphi u) = \left\{ \xi \in (\mathbb{R}^d \setminus \{0\}) \mid u \text{ is microregular on } (x,\xi) \right\},$$
$$\Sigma^{\mathcal{R}}_x(u) = \cap_{\varphi \in \mathcal{D}_x} \Sigma^{\mathcal{R}}(\varphi u) = \left(\mathbb{R}^d \setminus \{0\}\right) \setminus O^{\mathcal{R}}_x(u)\,.$$

Definition 10 *For $u \in \mathcal{G}(\Omega)$ the set*

$$WF_{\mathcal{R}}(u) = \left\{ (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \mid \xi \in \Sigma^{\mathcal{R}}_x(u) \right\}$$

is called the $\mathcal{R}$-wavefront of u.

Proposition 28 *For $u \in \mathcal{G}(\Omega)$, the projection on the first component of $WF_{\mathcal{R}}(u)$ is equal to* $\mathrm{singsupp}_{\mathcal{R}}\, u$.

The **proof** of this proposition follows the same lines as the one for the C^∞-wavefront of a distribution. First, for $u \in \mathcal{G}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, φu, which is *a priori* in $\mathcal{G}_C(\Omega)$, can be straightforwardly considered as an element of $\mathcal{G}_C(\mathbb{R}^d)$. As $\mathcal{G}_C(\mathbb{R}^d)$ is included in $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$ (see proposition 13), the Fourier transform of φu can be defined. (In the distributional case, that is $u \in \mathcal{D}'(\Omega)$, φu is identified to an element of $\mathcal{E}'(\mathbb{R}^d)$.) From this, we can follow the arguments of [15] (pages 253/254) for the C^∞-wavefront, which use mainly the compactness of the sphere S^{d-1} and lemma 27, which holds in both cases: See lemma 8.1.1. in [15] for the distributional case.

Example 15 *Taking $\mathcal{R} = \mathcal{B}$, the set of bounded sequences, we recover the $\mathcal{G}^\infty$-wavefront, which has here a definition independent of representatives.*

Example 16 *Taking $\mathcal{R} = \mathcal{R}_1$, we get a wavefront which "contains" the distributional microlocal singularities of a generalized function, since $\mathcal{D}'(\cdot)$ is embedded in $\mathcal{G}^{(1)}(\cdot)$.*

In [18], it is shown that the analogon of this lemma holds for the analytic singularities of a generalized function, giving rise to the corresponding wavefront set and the projection property of proposition 28. Our future aim is to apply this theory to the propagation of singularities through integral generalized operators [4]. We also refer the reader to [9], [12], [13], [14], [19] and the literature therein for other presentations of the $\mathcal{G}^\infty$-wavefront (which is a particular case of $\mathcal{R}$ wavefront).

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