

FRAGMENTATION AT HEIGHT ASSOCIATED TO LÉVY PROCESSES

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ABSTRACT. We consider the height process of a Lévy process with no negative jumps, and its associated continuous tree representation. Using tools developed by Duquesne and Le Gall, we construct a fragmentation process at height, which generalizes the fragmentation at height of stable trees given by Miermont. In this more general framework, we recover that the dislocation measures are the same as the dislocation measures of the fragmentation at node introduced by Abraham and Delmas, up to a factor equal to the fragment size. We also compute the asymptotic for the number of small fragments.

1. INTRODUCTION

In [13] and [12], Le Gall and Le Jan associated to a Lévy process with no negative jumps that does not drift to infinity, $X = (X_s, s \geq 0)$ with Laplace exponent ψ , a continuous state branching process (CSBP) and a Lévy continuous random tree (CRT) which keeps track of the genealogy of the CSBP. The Lévy CRT can be coded by the so called height process, $H = (H_s, s \geq 0)$. Informally H_s gives the distance (which can be understood as the number of generations) between the individual labeled s and the root, 0, of the CRT. We can consider the excursion of the process H above level $t > 0$. Even if H is non Markov, there is a natural way, we shall recall later, to define the distribution \mathbb{N} of the excursion of H above 0 and the local time of H at level t under \mathbb{N} . Let (α_i, β_i) , $i \in I$, be the connected component of the open set $\{s \in [0, \sigma]; H_s > t\}$ (H is lower semi-continuous), where σ is the length of the excursion of H under \mathbb{N} . We denote by $\Lambda(t)$ the non-increasing reordering of the sequence $(\beta_i - \alpha_i, i \in I)$, and define the fragmentation at height: $(\Lambda(t), t \geq 0)$.

A fragmentation process is a Markov process which describes how an object with given total mass evolves as it breaks into several fragments randomly as time passes. Notice there may be loss of mass but no creation. Those processes have been widely studied in the recent years, see Bertoin [7] and references therein. To be more precise, the state space of a fragmentation process is the set of the non-increasing sequences of masses with finite total mass

$$\mathcal{S}^\downarrow = \left\{ s = (s_1, s_2, \dots); s_1 \geq s_2 \geq \dots \geq 0 \quad \text{and} \quad \Sigma(s) = \sum_{k=1}^{+\infty} s_k < +\infty \right\}.$$

If we denote by P_s the law of a \mathcal{S}^\downarrow -valued process $\Lambda = (\Lambda(t), t \geq 0)$ starting at $s = (s_1, s_2, \dots) \in \mathcal{S}^\downarrow$, we say that Λ is a fragmentation process if it is a Markov process such that $t \mapsto \Sigma(\Lambda(t))$ is non-increasing and if it fulfills the fragmentation property: the law of $(\Lambda(t), t \geq 0)$ under P_s is the non-increasing reordering of the fragments of independent

Date: March 8, 2006.

2000 Mathematics Subject Classification. 60J25, 60G57.

Key words and phrases. Fragmentation, Lévy snake, dislocation measure, local time, continuous random tree.

This research was partially supported by NSERC Discovery Grants of the Probability group at Univ. of British Columbia.

processes of respective laws $P_{(s_1,0,\dots)}, P_{(s_2,0,\dots)}, \dots$. In other words, each fragment after dislocation behaves independently of the others, and its evolution depends only on its initial mass. As a consequence, to describe the law of the fragmentation process with any initial condition, it suffices to study the laws $P_r := P_{(r,0,\dots)}$ for any $r \in (0, +\infty)$, i.e. the law of the fragmentation process starting with a single mass r .

Theorem 3.2 states that the fragmentation at height is indeed a fragmentation process. This was already observed for the stable case $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2]$, by Bertoin [5] ($\alpha = 2$) and Miermont [14] ($\alpha \in (1, 2)$).

A fragmentation process is said to be self-similar of index α if, for any $r > 0$, the process Λ under P_r is distributed as the process $(r\Lambda(r^\alpha t), t \geq 0)$ under P_1 . Bertoin [5] proved that the law of a self-similar fragmentation is characterized by: the index of self-similarity α , an erosion coefficient which corresponds to a rate of mass loss, and a dislocation measure ν on \mathcal{S}^\downarrow which describes sudden dislocations of a fragment of mass 1. The dislocation measure of a fragment of size r , ν_r is given by $\int F(s)\nu_r(ds) = r^\alpha \int F(rs)\nu(ds)$.

In the stable cases $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2]$, the fragmentation is self-similar with index $-1 + 1/\alpha$ and with a zero erosion coefficient. The authors computed in both cases the dislocation measure and observed it is the same as the dislocation measure associated to the fragmentation at “nodes” of the corresponding CRT see [2] and [4] for $\alpha = 2$, and [15] for $\alpha \in (1, 2)$.

For a general sub-critical or critical CRT, there is no more scaling property, and the dislocation measure, which describes how a fragment of size $r > 0$ is cut in smaller pieces, can't be expressed as a nice function of the dislocation measure of a fragment of size 1. In [1], the authors give the family of dislocation measures $(\nu_r, r > 0)$ for the fragmentation at node of a general sub-critical or critical CRT. We set $\nu_r^* = r^{-1}\nu_r$. Theorem 5.1 states that $(\nu_r^*, r > 0)$ is the family of dislocation measures for the fragmentation at height, see also Remark 5.2. Intuitively ν_r^* describes the way a mass r breaks in smaller pieces.

We also compute the asymptotic of the number of small fragments at time t (see [6] and [10] for results in the self-similar case). With a suitable renormalization, it converges to the local time of the height process at level t , see Proposition 4.2 and Corollary 4.4 for the stable case $\alpha \in (1, 2)$. We also characterize the law of this local time at level t under P_r in Lemma 4.5.

The paper is organized as follows. In Section 2, we recall the definition and properties of the height and exploration processes. In the very short Section 3 we state and prove the fragmentation at height is indeed a fragmentation process. The number of small fragments is studied in Section 4. And the dislocation measures are computed in Section 5.

2. NOTATIONS

We denote by $\mathcal{B}_+(\mathbb{R}_+)$ the set of measurable non-negative functions defined on \mathbb{R}_+ . Let \mathcal{M}_f be the set of finite measures on \mathbb{R}_+ , endowed with the topology of weak convergence. For $\mu \in \mathcal{M}_f$, we set $H^\mu = \sup \text{Supp}(\mu) \in [0, \infty]$ the supremum of its closed support. For $f \in \mathcal{B}_+(\mathbb{R}_+)$, we write $\langle \mu, f \rangle$ for $\int_{\mathbb{R}_+} f(x) \mu(dx)$.

Let ψ denote the Laplace exponent of X : $\mathbb{E}[e^{-\lambda X_t}] = e^{t\psi(\lambda)}$, $\lambda > 0$. We shall assume there is no Brownian part, so that

$$\psi(\lambda) = \alpha_0 \lambda + \int_{(0, +\infty)} \pi(d\ell) \left[e^{-\lambda \ell} - 1 + \lambda \ell \right],$$

with $\alpha_0 \geq 0$ and the Lévy measure π is a positive σ -finite measure on $(0, +\infty)$ such that $\int_{(0, +\infty)} (\ell \wedge \ell^2) \pi(d\ell) < \infty$. Following [9], we shall also assume that X is of infinite variation a.s. which implies that $\int_{(0,1)} \ell \pi(d\ell) = \infty$. Notice those hypothesis are fulfilled in the stable case: $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2)$. For $\lambda \geq 1/\varepsilon > 0$, we have $e^{-\lambda\ell} - 1 + \lambda\ell \geq \frac{1}{2}\lambda\ell \mathbf{1}_{\{\ell \geq 2\varepsilon\}}$, which implies that $\lambda^{-1}\psi(\lambda) \geq \alpha_0 + \int_{(2\varepsilon, \infty)} \ell \pi(d\ell)$. We deduce that

$$(1) \quad \lim_{\lambda \rightarrow \infty} \frac{\lambda}{\psi(\lambda)} = 0.$$

The so-called exploration process $\rho = (\rho_t, t \geq 0)$ is an \mathcal{M}_f -valued càd-làg Markov process. The height process at time t is defined as the supremum of the closed support of ρ_t (with the convention that $H_t = 0$ if $\rho_t = 0$). Informally, H_t gives the distance (which can be understood as the number of generations) between the individual labeled t and the root, 0, of the CRT. In some sense $\rho_t(dv)$ records the “number” of brothers, with labels larger than t , of the ancestor of t at generation v .

We recall the definition and properties of the exploration process which are given in [13], [12] and [9]. The results of this section are mainly extracted from [9].

Let $I = (I_t, t \geq 0)$ be the infimum process of X , $I_t = \inf_{0 \leq s \leq t} X_s$. We will also consider for every $0 \leq s \leq t$ the infimum of X over $[s, t]$:

$$I_t^s = \inf_{s \leq r \leq t} X_r.$$

There exists a sequence $(\varepsilon_n, n \in \mathbb{N}^*)$ of positive real numbers decreasing to 0 s.t.

$$\tilde{H}_t = \lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k} \int_0^t \mathbf{1}_{\{X_s < I_t^s + \varepsilon_k\}} ds$$

exists and is finite a.s. for all $t \geq 0$.

The point 0 is regular for the Markov process $X - I$, $-I$ is the local time of $X - I$ at 0 and the right continuous inverse of $-I$ is a subordinator with Laplace exponent ϕ , the inverse of ψ : $\psi(\phi(x)) = \phi(\psi(x)) = x$ (see [3], chap. VII). Notice this subordinator has no drift thanks to (1). Let π_* denote the corresponding Lévy measure.

Let \mathbb{N} be the associated excursion measure of the process $X - I$ out of 0, and $\sigma = \inf\{t > 0; X_t - I_t = 0\}$ be the length of the excursion of $X - I$ under \mathbb{N} . Under \mathbb{N} , $X_0 = I_0 = 0$. We shall use (see Section 3.2.2. in [9]) that

$$(2) \quad \mathbb{N}[1 - e^{-\lambda\sigma}] = \phi(\lambda).$$

In particular σ is distributed under \mathbb{N} according to the Lévy measure π_* .

From section 1.2 in [9], there exists a \mathcal{M}_f valued process, $\rho^0 = (\rho_t^0, t \geq 0)$, called the exploration process, such that :

- A.s., for every $t \geq 0$, we have $\langle \rho_t^0, 1 \rangle = X_t - I_t$, and the process ρ^0 is càd-làg.
- The process $(H_s^0 = H^{\rho_s^0}, s \geq 0)$ taking values in $[0, \infty]$ is lower semi-continuous.
- For each $t \geq 0$, a.s. $H_t^0 = \tilde{H}_t$.
- For every $f \in \mathcal{B}_+(\mathbb{R}_+)$,

$$\langle \rho_t^0, f \rangle = \int_{[0,t]} f(H_s^0) d_s I_t^s,$$

or equivalently, with δ_x being the Dirac mass at x ,

$$\rho_t^0(dr) = \sum_{\substack{0 < s \leq t \\ X_s - < I_t^s}} (I_t^s - X_{s-}) \delta_{H_s^0}(dr).$$

In the definition of the exploration process, as X starts from 0, we have $\rho_0 = 0$ a.s. To get the Markov property of ρ , we must define the process ρ started at any initial measure $\mu \in \mathcal{M}_f$. For $a \in [0, \langle \mu, 1 \rangle]$, we define the erased measure $k_a \mu$ by

$$k_a \mu([0, r]) = \mu([0, r]) \wedge (\langle \mu, 1 \rangle - a), \quad \text{for } r \geq 0.$$

If $a > \langle \mu, 1 \rangle$, we set $k_a \mu = 0$. In other words, the measure $k_a \mu$ is the measure μ erased by a mass a backward from H^μ .

For $\nu, \mu \in \mathcal{M}_f$, and μ with compact support, we define the concatenation $[\mu, \nu] \in \mathcal{M}_f$ of the two measures by:

$$\langle [\mu, \nu], f \rangle = \langle \mu, f \rangle + \langle \nu, f(H^\mu + \cdot) \rangle, \quad f \in \mathcal{B}_+(\mathbb{R}_+).$$

Eventually, we set for every $\mu \in \mathcal{M}_f$ and every $t > 0$,

$$\rho_t = [k_{-I_t} \mu, \rho_t^0].$$

We say that $\rho = (\rho_t, t \geq 0)$ is the process ρ started at $\rho_0 = \mu$, and write \mathbb{P}_μ for its law. We set $H_t = H^{\rho_t}$. The process ρ is càd-làg and strong Markov.

2.1. Poisson decomposition. Let \mathbb{P}_μ^* denote the law of ρ started at μ and killed when it reaches 0. We decompose the path of ρ under \mathbb{P}_μ^* according to excursions of the total mass of ρ above its minimum, see Section 4.2.3 in [9]. More precisely let $(\alpha_i, \beta_i), i \in I$ be the excursion intervals of the process $X - I$ away from 0 under \mathbb{P}_μ^* . For every $i \in I$, we define $h_i = H_{\alpha_i}$ and ρ^i by the formula $\rho_t^i = \rho_{(\alpha_i+t) \wedge \beta_i}^0$ or equivalently $[k_{-I_{\alpha_i}} \mu, \rho_t^i] = \rho_{(\alpha_i+t) \wedge \beta_i}$. We recall Lemma 4.2.4. of [9].

Lemma 2.1. *Let $\mu \in \mathcal{M}_f$. The point measure $\sum_{i \in I} \delta_{(h_i, \rho^i)}$ is under \mathbb{P}_μ^* a Poisson point measure with intensity $\mu(dr) \mathbb{N}[d\rho]$.*

Let $(L_s^t, s \geq 0, t > 0)$ be the local time of the height process under \mathbb{N} at level $t > 0$ at time $s \geq 0$: $L_s^t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t < H_r \leq t + \varepsilon\}} dr$ in L^1 -norm. The local time is jointly measurable in (s, t) , non-decreasing and continuous in s (see Proposition 1.3.3 in [9]).

Consider the right-continuous inverse of the time spent by the height process under level t : $\tilde{\tau}_s^t = \inf\{r \geq 0; \int_0^r \mathbf{1}_{\{H_v \leq t\}} dv > s\}$, and set $\tilde{\rho} = (\tilde{\rho}_s, s \geq 0)$ where $\tilde{\rho}_s = \rho_{\tilde{\tau}_s^t}$. Recall L_σ^t is measurable with respect to $\tilde{\rho}$ thanks to Proposition 1.3.3 in [9].

Notice the set $\{s \in [0, \sigma]; H_s > t\}$ is open since H is lower semi-continuous. Let $(\alpha_i, \beta_i), i \in I_t$ be the excursion of H (or ρ) above level t . Notice that $I_t = \emptyset$ if $\sup\{H_s, s \in [0, \sigma]\} < t$. If $I_t \neq \emptyset$, we define ρ^i such that $[\rho_{\alpha_i}, \rho_t^i] = \rho_{(\alpha_i+t) \wedge \beta_i}$, and $\sigma^i = \beta_i - \alpha_i$ the duration of the excursion ρ^i . By Proposition 1.3.1 in [9] and standard excursion theory, we have the next Lemma.

Lemma 2.2. *Under \mathbb{N} , the point measure $\sum_{i \in I_t} \delta_{\rho^i}$ is, conditionally on L_σ^t (or on $\tilde{\rho}$) a Poisson point measure with intensity $L_\sigma^t \mathbb{N}[d\rho]$.*

2.2. The dual process and representation formula. We shall need the \mathcal{M}_f -valued process $\eta = (\eta_t, t \geq 0)$ defined by

$$\eta_t(dr) = \sum_{\substack{0 < s \leq t \\ X_s < I_t^s}} (X_s - I_t^s) \delta_{H_s}(dr).$$

The process η is the dual process of ρ under \mathbb{N} (see Corollary 3.1.6 in [9]). Let $\Delta_s = X_s - X_{s-}$, $s > 0$, be the jumps of X . We write (recall $\Delta_s = X_s - X_{s-}$)

$$\kappa_t(dr) = \rho_t(dr) + \eta_t(dr) = \sum_{\substack{0 < s \leq t \\ X_{s-} < I_s^t}} \Delta_s \delta_{H_s}(dr).$$

We recall the Poisson representation of (ρ, η) under \mathbb{N} . Let $\mathcal{N}(dx d\ell du)$ be a Poisson point measure on $[0, +\infty)^3$ with intensity

$$dx \ell \pi(d\ell) \mathbf{1}_{[0,1]}(u) du.$$

For every $a > 0$, let us denote by \mathbb{M}_a the law of the pair (μ_a, ν_a) of finite measures on \mathbb{R}_+ defined by: for $f \in \mathcal{B}_+(\mathbb{R}_+)$

$$\begin{aligned} \langle \mu_a, f \rangle &= \int \mathcal{N}(dx d\ell du) \mathbf{1}_{[0,a]}(x) u \ell f(x), \\ \langle \nu_a, f \rangle &= \int \mathcal{N}(dx d\ell du) \mathbf{1}_{[0,a]}(x) \ell (1-u) f(x). \end{aligned}$$

We eventually set $\mathbb{M} = \int_0^{+\infty} da e^{-\alpha_0 a} \mathbb{M}_a$. For every non-negative measurable function F on \mathcal{M}_f^2 , we have

$$(3) \quad \mathbb{N} \left[\int_0^\sigma F(\rho_t, \eta_t) dt \right] = \int \mathbb{M}(d\mu d\nu) F(\mu, \nu),$$

where $\sigma = \inf\{s > 0; \rho_s = 0\}$ denotes the length of the excursion. Let $t > 0$. For every non-negative measurable function F on \mathcal{M}_f , we have

$$(4) \quad \mathbb{N} \left[\int_0^\sigma F(\rho_s) dL_s^t \right] = e^{-\alpha_0 t} \int \mathbb{M}_t(d\mu d\nu) F(\mu).$$

3. THE FRAGMENTATION AT HEIGHT

We keep notations introduced for Lemma 2.2 in section 2.1. We define at time t the fragmentation process at height, $\Lambda(t) \in \mathcal{S}^\downarrow$, as the sequence $(\sigma^i, i \in I_t)$ ranked in non-increasing order (if I_t is empty or finite, this sequence is completed by zeroes). If needed, we write $\Lambda^\rho(t)$ for $\Lambda(t)$ to stress that the fragmentation process is built from ρ .

Let π_* be the distribution of σ under \mathbb{N} . By decomposing the measure \mathbb{N} w.r.t. the distribution of σ , we get that $\mathbb{N}[d\rho] = \int_{(0,\infty)} \pi_*(dr) \mathbb{N}_r[d\rho]$, where $(\mathbb{N}_r, r \in (0, \infty))$ is a measurable family of probability measures on the set of excursions of the exploration process such that $\mathbb{N}_r[\sigma = r] = 1$ for $\pi_*(dr)$ -a.e. $r > 0$. From standard excursion theory, we have the next Lemma.

Lemma 3.1. *Let $t > 0$. The random variables $(\rho_i, i \in I_t)$ are, conditionally on the σ -field generated by $(H_s \wedge t, s \in [0, \sigma])$ and $((\alpha_i, \beta_i), i \in I_t)$, independent and ρ^i is distributed according to $\mathbb{N}_{\sigma^i}[d\rho]$.*

For $\pi_*(dr)$ -a.e., let \mathbb{P}_r denote the law of Λ under \mathbb{N}_r , and let \mathbb{P}_0 be the law of the constant process equal to $(0, \dots) \in \mathcal{S}^\downarrow$.

Theorem 3.2. *For $\pi_*(dr)$ -a.e., under \mathbb{P}_r , the process Λ is a \mathcal{S}^\downarrow -valued fragmentation process. More precisely, Λ is Markov and satisfy the fragmentation property: the law under \mathbb{P}_r of the process $(\Lambda(t+t'), t' \geq 0)$ conditionally on $\Lambda(t) = (\Lambda_i, i \in \mathbb{N}^*)$ is given by the decreasing reordering of independent processes of respective law \mathbb{P}_{Λ_i} , $i \in \mathbb{N}^*$.*

Proof. Notice that $(\Lambda(s), s \in [0, t])$ is measurable w.r.t. the σ -field generated by $(H_u \wedge t, u \in [0, \sigma])$ and $((\alpha_i, \beta_i), i \in I_t)$. Using notations of Lemma 3.1, for $t' > 0$, $\Lambda^\rho(t + t')$ is the non-increasing reordering of $(\Lambda^{\rho^i}(t'), i \in I_t)$. The Markov property and the fragmentation property are consequences of Lemma 3.1. \square

4. NUMBER OF SMALL FRAGMENTS

For the fragmentation at height, it is easy to give the asymptotic of the number of small fragments. We keep notations introduced for Lemma 2.2. Let $N_\varepsilon(t)$ be the number of fragments at time t of size bigger or equal to $\varepsilon > 0$ and $M_\varepsilon(t)$ the total mass of the fragments less or equal than ε :

$$N_\varepsilon(t) = \sum_{i \in I_t} \mathbf{1}_{\{\sigma^i \geq \varepsilon\}} \quad \text{and} \quad M_\varepsilon(t) = \sum_{i \in I_t} \sigma^i \mathbf{1}_{\{\sigma^i \leq \varepsilon\}}.$$

For $t > 0$, we write $\bar{\pi}_*(t) = \pi_*((t, \infty))$ and $\varphi(t) = \int_{(0,t)} r \pi_*(dr)$.

Lemma 4.1. *We have $\lim_{\varepsilon \rightarrow 0} \bar{\pi}_*(\varepsilon) = \infty$ and $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \varphi(\varepsilon) = \infty$.*

Proof. We deduce from (1) that

$$(5) \quad \lim_{\lambda \rightarrow \infty} \phi(\lambda) = \infty.$$

Notice $\phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda r}) \pi_*(dr)$ to obtain the first part of the Lemma.

The second limit is more involved. We have $\lambda \psi'(\lambda) = \alpha_0 \lambda + \int_{(0,\infty)} \lambda r (1 - e^{-\lambda r}) \pi(dr)$. Since for all $x \geq 0$,

$$e^{-x} - 1 + x \leq x(1 - e^{-x}) \leq 2(e^{-x} - 1 + x),$$

we deduce that $\psi(\lambda) \leq \lambda \psi'(\lambda) \leq 2\psi(\lambda)$. And we get that for $x > 0$,

$$(6) \quad \frac{1}{2} \phi(x) \leq x \phi'(x) \leq \phi(x).$$

The next part of the proof is inspired by a Theorem of Haan and Stadtmüller (see [8] p.118). We have

$$\phi'(\lambda) = \int_{(0,\infty)} e^{-\lambda r} r \pi_*(dr) = \int_{(0,\infty)} e^{-u} \varphi(u/\lambda) du.$$

Since the function φ is non-decreasing, we have for $z > 0$

$$(7) \quad \phi'(\lambda) \geq \int_{(z,\infty)} e^{-u} \varphi(u/\lambda) du \geq e^{-z} \varphi(z/\lambda).$$

We have for any $x > 0$,

$$\begin{aligned} \frac{1}{2\lambda} \phi(\lambda) &\leq \phi'(\lambda) \\ &\leq \int_{(0,x)} e^{-u} \varphi(u/\lambda) du + \int_{(x,\infty)} e^{-u} \varphi(u/\lambda) du \\ &\leq \varphi(x/\lambda)(1 - e^{-x}) + \int_{(x,\infty)} e^{-u} e^{u/2} \phi'(\lambda/2) du \\ &\leq \varphi(x/\lambda) + \frac{4}{\lambda} \phi(\lambda/2) e^{-x/2}, \end{aligned}$$

where we used (6) for the first inequality, the fact that φ is non-decreasing for the first term and (7) for the second term of the third inequality, and (6) again for the last. With $x = 6 \ln(2)$, we get

$$\varphi(x/\lambda) \geq \frac{1}{2\lambda}(\phi(\lambda) - \phi(\lambda/2)) = \frac{1}{2\lambda} \int_{[\lambda/2, \lambda]} \phi'(u) du \geq \frac{\phi'(\lambda)}{4},$$

where we used that ϕ' is non-increasing for the last inequality. Using (6), we deduce that

$$\lambda\varphi(1/\lambda) \geq \frac{\lambda}{4}\phi'(\lambda/x) \geq \frac{x}{8}\phi(\lambda/x).$$

The last part of the Lemma is then a consequence of (5). \square

Proposition 4.2. *Let $t > 0$. We have that, conditionally on L_σ^t , $N_\varepsilon(t)$ is a Poisson random variable with mean $\bar{\pi}_*(\varepsilon)L_\sigma^t$. Furthermore, there exists a sequence of positive numbers, $(\varepsilon_n, n \geq 1)$, decreasing towards 0, such that, for all $t > 0$, we have \mathbb{N} -a.e.*

$$\lim_{n \rightarrow \infty} \frac{N_{\varepsilon_n}(t)}{\bar{\pi}_*(\varepsilon_n)} = \lim_{n \rightarrow \infty} \frac{M_{\varepsilon_n}(t)}{\varphi(\varepsilon_n)} = L_\sigma^t.$$

We can replace the sequence $(\varepsilon_n, n \geq 1)$ by any sequence in some particular case (see for example Corollary 4.4). Those results extend [10] for the fragmentation at height.

Proof. Recall $\mathbb{N}[\sigma \geq \varepsilon] = \bar{\pi}_*(\varepsilon)$. Since conditionally on L_σ^t , $\sum_{i \in I_t} \delta_{\rho^i}$ is under \mathbb{N} a Poisson point measure with intensity $\bar{\pi}_*(\varepsilon)L_\sigma^t$ (see Lemma 2.2), we deduce that conditionally on L_σ^t , $N_\varepsilon(t)$ is a Poisson random variable with mean $\bar{\pi}_*(\varepsilon)L_\sigma^t$.

Since $\lim_{\varepsilon \downarrow 0} \bar{\pi}_*(\varepsilon) = \infty$, there exists a sequence $(\varepsilon_n, n \geq 1)$ decreasing towards 0 such that $\sum_{n \geq 1} 1/\bar{\pi}_*(\varepsilon_n) < \infty$. Since $\mathbb{N} \left[\left(\frac{N_{\varepsilon_n}(t)}{\bar{\pi}_*(\varepsilon_n)} - L_\sigma^t \right)^2 \middle| L_\sigma^t \right] = \frac{L_\sigma^t}{\bar{\pi}_*(\varepsilon_n)}$, the series $\sum_{n \geq 1} \left(\frac{N_{\varepsilon_n}(t)}{\bar{\pi}_*(\varepsilon_n)} - L_\sigma^t \right)^2$ is \mathbb{N} -a.e. finite. This implies that \mathbb{N} -a.e. $\lim_{n \rightarrow \infty} \frac{N_{\varepsilon_n}(t)}{\bar{\pi}_*(\varepsilon_n)} = L_\sigma^t$.

Since conditionally on L_σ^t , $\sum_{i \in I_t} \delta_{\rho^i}$ is under \mathbb{N} a Poisson point measure with intensity $\bar{\pi}_*(\varepsilon)L_\sigma^t$ (see Lemma 2.2), we deduce that for $\lambda > 0$, $\varepsilon > 0$,

$$\mathbb{N} \left[e^{-\lambda M_\varepsilon(t)} \middle| L_\sigma^t \right] = e^{-L_\sigma^t \int_{(0, \varepsilon)} \pi_*(dr) (1 - e^{-\lambda r})}.$$

Poisson point measure properties yield

$$\begin{aligned} \mathbb{N} [M_\varepsilon(t) | L_\sigma^t] &= L_\sigma^t \int_{(0, \varepsilon)} r \pi_*(dr) = \varphi(\varepsilon) L_\sigma^t \\ \mathbb{N} [M_\varepsilon(t)^2 | L_\sigma^t] &= \mathbb{N} [M_\varepsilon(t) | L_\sigma^t]^2 + L_\sigma^t \int_{(0, \varepsilon)} r^2 \pi_*(dr). \end{aligned}$$

We deduce that

$$\mathbb{N} \left[\left(\frac{M_\varepsilon(t)}{\varphi(\varepsilon)} - L_\sigma^t \right)^2 \middle| L_\sigma^t \right] = L_\sigma^t \frac{\int_{(0, \varepsilon)} r^2 \pi_*(dr)}{\varphi(\varepsilon)^2} \leq L_\sigma^t \frac{\varepsilon \int_{(0, \varepsilon)} r \pi_*(dr)}{\varphi(\varepsilon)^2} = \frac{\varepsilon}{\varphi(\varepsilon)} L_\sigma^t.$$

As $\lim_{\varepsilon \rightarrow 0} \varepsilon/\varphi(\varepsilon) = 0$ (see Lemma 4.1), we can use similar arguments as those used for $N_\varepsilon(t)$ to end the proof. Eventually, notice one can choose the sequence $(\varepsilon_n, n \geq 1)$ such that the two limits in the Proposition hold simultaneously. \square

Remark 4.3. As $\lim_{\varepsilon \rightarrow 0} \varepsilon/\varphi(\varepsilon) = 0$ (see Lemma 4.1), we have

$$\lim_{\varepsilon \rightarrow 0} \int_{(0, \infty)} \pi_*(dr) (1 - e^{-\lambda r/\varphi(\varepsilon)}) = \lambda.$$

This implies that from the above proof

$$\lim_{\varepsilon \rightarrow 0} \mathbb{N} \left[e^{-\lambda M_\varepsilon(t)/\varphi(\varepsilon)} \middle| L_\sigma^t \right] = e^{-L_\sigma^t \lim_{\varepsilon \rightarrow 0} \int_{(0, \varepsilon)} \pi_*(dr) (1 - e^{-\lambda r/\varphi(\varepsilon)})} = e^{-\lambda L_\sigma^t}.$$

This implies that conditionally on L_σ^t , $M_\varepsilon(t)/\varphi(\varepsilon)$ converges in probability to L_σ^t as ε goes down to 0. Notice also that conditionally on L_σ^t , $N_\varepsilon(t)/\bar{\pi}_*(\varepsilon)$ converges in probability to L_σ^t as ε goes down to 0.

We consider the stable case $\psi(\lambda) = \lambda^\alpha$. We have $\pi_*(dr) = (\alpha\Gamma(1 - \alpha^{-1}))^{-1} r^{-1-1/\alpha} dr$, $\bar{\pi}_*(\varepsilon) = \Gamma(1 - \alpha^{-1})^{-1} \varepsilon^{-1/\alpha}$, and $\varphi(\varepsilon) = \left((\alpha - 1)\Gamma(1 - \alpha^{-1}) \right)^{-1} \varepsilon^{1-\alpha^{-1}}$. There is a version of $(\mathbb{N}_r, r > 0)$ such that for all $r > 0$ we have $\mathbb{N}_r[F((X_s, s \in [0, r]))] = \mathbb{N}_1[F((r^{1/\alpha} X_{s/r}, s \in [0, r]))]$ for any non-negative measurable function F defined on the set of càd-làg paths. For the next Corollary, see also general results from [10].

Corollary 4.4. *Let $\psi(\lambda) = \lambda^\alpha$, for $\alpha \in (1, 2)$. Let $t > 0$. We have, under \mathbb{N} or \mathbb{N}_1 , that conditionally on L_σ^t , $N_\varepsilon(t)$ is a Poisson random variable with mean $\Gamma(1 - \alpha^{-1})^{-1} \varepsilon^{-1/\alpha} L_\sigma^t$. Furthermore, for all $t > 0$, we have \mathbb{N} -a.e. or \mathbb{N}_1 -a.s.*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/\alpha} N_\varepsilon(t) = \lim_{\varepsilon \rightarrow 0} (\alpha - 1) \varepsilon^{\alpha^{-1}-1} M_\varepsilon(t) = \frac{L_\sigma^t}{\Gamma(1 - \alpha^{-1})}.$$

Proof. It is enough to check that we can replace the sequence $(\varepsilon_n, n \geq 1)$ in Proposition 4.2 by any sequences. Notice that $\varepsilon_n = n^{-2\alpha}$, $n \geq 1$, satisfies $\sum_{n \geq 1} 1/\bar{\pi}_*(\varepsilon_n) < \infty$. From the proof of Proposition 4.2, we get that \mathbb{N} -a.e. or \mathbb{N}_1 -a.s., $\lim_{n \rightarrow \infty} n^{-2} N_{n^{-2\alpha}}(t) = L_\sigma^t/\Gamma(1 - \alpha^{-1})$. Since $N_\varepsilon(t)$ is a non-increasing function of ε , we get that for any $\varepsilon \in [(n+1)^{-2\alpha}, n^{-2\alpha}]$, we have

$$\frac{n^2}{(n+1)^2} n^{-2} N_{n^{-2\alpha}}(t) \leq \varepsilon^{1/\alpha} N_\varepsilon(t) \leq \frac{(n+1)^2}{n^2} (n+1)^{-2} N_{(n+1)^{-2\alpha}}(t).$$

Hence we deduce that \mathbb{N} -a.e. or \mathbb{N}_1 -a.s., $\lim_{\varepsilon \rightarrow 0} \varepsilon^{1/\alpha} N_\varepsilon(t) = L_\sigma^t/\Gamma(1 - \alpha^{-1})$. The proof for $M_\varepsilon(t)$ is similar, as $M_\varepsilon(t)$ is a non-decreasing function of ε . \square

The next Lemma characterizes the law of L_σ^t under \mathbb{N}_r . (Recall $\int F(r, \rho) \pi_*(dr) \mathbb{N}_r[d\rho] = \mathbb{N}[F(s, \rho)]$.)

Lemma 4.5. *Let $\lambda \geq 0$ and $\gamma \geq 0$. Let $w(t) = \mathbb{N} \left[e^{-\lambda\sigma} \left(1 - e^{-\gamma L_\sigma^t} \right) \right]$, for $t > 0$ and $w(0) = \gamma$. Then w belongs to $\mathcal{C}^1(\mathbb{R}_+)$, is non-increasing, such that $\lim_{t \rightarrow \infty} w(t) = 0$ and solves*

$$w'(t) = \lambda - \psi(\phi(\lambda) + w(t)), \quad t > 0.$$

Remark 4.6. For the α -stable case, we can characterize the law of L_σ^t under \mathbb{N}_1 . Scaling properties yield the processes $r^{1/\alpha} X_{s/r}$, $r^{1-\alpha^{-1}} H_{s/r}$, $r^{1/\alpha} \rho_{s/r}(d(u r^{1-\alpha^{-1}}))$ and $(r^{1/\alpha} L_{s/r}^{t r^{-1+\alpha^{-1}}}, t \geq 0)$ are distributed as the processes X_s , H_s , $\rho_s(du)$ and $(L_s^t, t \geq 0)$. We deduce from the definition of w in Lemma 4.5 that

$$\frac{1}{\alpha\Gamma(1 - \alpha^{-1})} \int_{(0, \infty)} dr r^{-1-1/\alpha} e^{-\lambda r} \left[1 - \mathbb{N}_1 \left[e^{-\gamma r^{1/\alpha} L_1^{t r^{-(1+\alpha^{-1})}}} \right] \right] = w(t).$$

And this is enough to characterize the law of L_1^t under \mathbb{N}_1 , for all $t > 0$.

Proof of Lemma 4.5. We first consider for $\lambda > 0$, $\gamma > 0$ and $t > 0$

$$v(t) = \mathbb{N} \left[1 - e^{-\lambda\sigma - \gamma L_\sigma^t} \right].$$

Notice that $\sigma = \sum_{i \in I_t} \sigma^i + \tilde{\sigma}$, where $\tilde{\sigma} = \int_0^\sigma \mathbf{1}_{\{H_s \leq t\}} ds$ is the duration of the excursion of $\tilde{\rho}$ (defined in Section 2.1) under \mathbb{N} . From Lemma 2.2, we get that by conditioning with respect to $\tilde{\rho}$,

$$v(t) = \mathbb{N} \left[1 - e^{-\lambda \int_0^\sigma \mathbf{1}_{\{H_s \leq t\}} ds - L_\sigma^t (\gamma + \mathbb{N}[1 - e^{-\lambda\sigma}])} \right].$$

We consider the additive functional of ρ given by

$$dA_s = \lambda \mathbf{1}_{\{H_s \leq t\}} ds + (\gamma + \phi(\lambda)) dL_s^t.$$

We get

$$v(t) = \mathbb{N} \left[1 - e^{-A_\sigma} \right] = \mathbb{N} \left[\int_0^\sigma e^{-(A_\sigma - A_s)} dA_s \right].$$

Then we can replace $e^{-(A_\sigma - A_s)}$ by its optional projection $B = \mathbb{E}_{\rho_s}^* [e^{-A_\sigma}]$. Thanks to Lemma 2.2, we can replace A_σ in B by $\lambda\sigma + \gamma L_\sigma^t$. Using notations introduced for Lemma 2.1, we have under \mathbb{E}_μ^* that $\sigma = \sum_{i \in I} \sigma^i$ (where σ^i is the length of the excursion ρ^i) and $L_\sigma^t = \sum_{i \in I} L_{\sigma^i, i}^{t-h_i}$, where $L_{s,i}^a$ is the local time at level a at time s of the exploration process ρ^i . Notice that $H_s \leq t$ dA_s -a.e. From Lemma 2.1, we get

$$B = e^{-\int \rho_s(du) \mathbb{N} [1 - e^{-\lambda\sigma - \gamma L_\sigma^{t-u}}]} = e^{-\int \rho_s(du) v(t-u)}.$$

Eventually, we have

$$\begin{aligned} v(t) &= \mathbb{N} \left[\int_0^\sigma e^{-\int \rho_s(du) v(t-u)} dA_s \right] \\ &= \lambda \int_0^t da e^{-\alpha_0 a} e^{-\int_0^a dx \int_0^1 du \int_{(0,\infty)} \ell \pi(d\ell) [1 - e^{-v(t-x)u\ell}]} \\ &\quad + (\gamma + \phi(\lambda)) e^{-\alpha_0 t} e^{-\int_0^t dx \int_0^1 du \int_{(0,\infty)} \ell \pi(d\ell) [1 - e^{-v(t-x)u\ell}]} \\ &= \lambda \int_0^t da e^{-\int_0^a dx \int_0^1 du \psi'(v(t-x)u)} + (\gamma + \phi(\lambda)) e^{-\int_0^t dx \int_0^1 du \psi'(v(t-x)u)} \\ &= \lambda \int_0^t da e^{-\int_0^a dx \frac{\psi(v(t-x))}{v(t-x)}} + (\gamma + \phi(\lambda)) e^{-\int_0^t dx \frac{\psi(v(x))}{v(x)}} \\ (8) \quad &= \lambda \int_0^t da e^{-\int_a^t dx \frac{\psi(v(x))}{v(x)}} + (\gamma + \phi(\lambda)) e^{-\int_0^t dx \frac{\psi(v(x))}{v(x)}}, \end{aligned}$$

where we used formulas (3) and (4) for the second equality and the convention $\frac{\psi(\infty)}{\infty} = \psi'(\infty) = \infty$ for the fourth. Notice that for all $t > 0$, (8) implies

$$v(t) \leq \lambda t + \gamma + \phi(\lambda).$$

We set $v(0) = \gamma + \phi(\lambda)$. Since the function $t \mapsto v(t)$ is locally bounded and measurable, we deduce from (8) that v is continuous and even of class $\mathcal{C}^1(\mathbb{R}_+)$. By differentiation w.r.t. t , we get

$$v'(t) = \lambda - \psi(v(t)).$$

Since $\int_{\phi(\lambda)} dv/(\lambda - \psi(v)) = \infty$ and $v'(0) < 0$, it is easy to check that v is decreasing and $\lim_{t \rightarrow \infty} v(t) = \phi(\lambda)$. Then notice that $w(t) = v(t) - \mathbb{N}[1 - e^{-\lambda\sigma}] = v(t) - \phi(\lambda)$ to conclude. The case $\lambda = 0$ is similar. The case $\gamma = 0$ is immediate. \square

5. DISLOCATION MEASURES

For $s \in (0, \sigma)$, let $\sigma^{s,t}$ be the size (i.e. the Lebesgue measure) of the fragment at time t which contains s :

$$\sigma^{s,t} = \int_0^\sigma du \mathbf{1}_{\{H_{[u,s]} > t\}},$$

where $H_{[u,s]} = \min\{H_r; r \in [s \wedge u, s \vee u]\}$. We consider a tagged fragment. More precisely, let $s^* \in [0, \sigma]$ be, conditionally on σ , chosen uniformly and independently of ρ . The process $(\sigma^{s^*,t}, t \geq 0)$ is a non-increasing process. Let \mathcal{T}_{s^*} the set of its jumping times. Notice there is a jump at time t if and only if there is a node of the CRT at level t for s^* , which is equivalent to say that $\kappa_{s^*}(\{t\}) > 0$. Let $x(t) = (x_1(t), \dots) \in \mathcal{S}^\downarrow$ be the sequence of the Lebesgue measures for the different fragments coming from the fragmentation at time t of the tagged fragment. In particular, we have $\sum_{i=1}^\infty x_i(t) = \sigma^{s^*,t^-}$.

Let S be a subordinator with Laplace index ϕ . Denote by $(\Delta S_t, t \geq 0)$ its jumps. Let μ the measure on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$ such that for any non-negative measurable function, F , on $\mathbb{R}_+ \times \mathcal{S}^\downarrow$,

$$(9) \quad \int_{(0,+\infty) \times \mathcal{S}^\downarrow} F(r, x) \mu(dr, dx) = \int \pi(dv) \mathbb{E}[F(S_v, (\Delta S_t, t \leq v))],$$

where $(\Delta S_t, t \leq v)$ has to be understood as the family of jumps of the subordinator up to time v ranked in non-increasing order. Intuitively, μ is the law of S_T and the jumps of S up to time T , where T and S are independent, and T is distributed according to the infinite measure π . Recall π_* is the ‘‘distribution’’ of σ under \mathbb{N} (this is the Lévy measure associated to the Laplace exponent ϕ). From Theorem 9.1 in [1], we have that $\mu(dr, dx)$ is absolutely continuous with respect to $\pi_*(dr)$, and more precisely

$$r\mu(dr, dx) = \nu_r(dx)\pi_*(dr),$$

where $(\nu_r, r > 0)$ is the measurable family of dislocation measures of the fragmentation at nodes introduced in [1]. Let $(\nu_r^*, r > 0)$ defined by $r\nu_r^*(dx) = \nu_r(dx)$ for $r > 0$, so that

$$(10) \quad \mu(dr, dx) = \nu_r^*(dx)\pi_*(dr).$$

We refer to [11] for the definition of intensity of a random point measure. Recall $\sigma^{s,t}$ is the size of the fragment at time t which contains s , and s^* is uniform on $[0, \sigma]$.

Theorem 5.1. *The intensity of the random point measure $\sum_{t \in \mathcal{T}_{s^*}} \delta_{t, x(t)}(dt, dx)$ is given by $\mathbf{1}_{\{\sigma^{s^*,t^-} > 0\}} dt \nu_{\sigma^{s^*,t^-}}(dx)$.*

Remark 5.2. For the α -stable case, $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (1, 2)$, we deduce from Corollary 9.3 in [1], that the fragmentation at height is a self-similar fragmentation with index $\alpha^{-1} - 1$, and with the same dislocation measure ν_1^* as for the fragmentation at node (ν_1 in Corollary 9.3 [1]). This result was proved by Miermont [14], see also [15]. This result was previously observed by Bertoin [5] for the case $\alpha = 2$ (in this case the exploration process is a reflected Brownian motion).

The rest of this section is devoted to the proof of Theorem 5.1, which is based on the next three lemmas. For a function G defined on \mathcal{S}^\downarrow , $G((\Delta S_t, t \leq r))$ has to be understood as

the function G evaluated on the sequence $(\Delta S_t, t \leq r)$ ranked in non-increasing order and eventually completed by zeroes if this sequence is finite.

Lemma 5.3. *Let $g \in \mathcal{B}_+(\mathbb{R}_+)$ and G a measurable non-negative function defined on \mathcal{S}^\downarrow . We have*

$$\mathbb{N} \left[\sigma e^{-\lambda\sigma} \sum_{t \in \mathcal{T}_{s^*}} g(t) G(x(t)) \right] = \int_{\mathbb{R}_+} dt g(t) e^{-t\psi'(\phi(\lambda))} \int_{(0,\infty) \times \mathcal{S}^\downarrow} \mu(dr, dx) r e^{-\lambda r} G(x).$$

Proof. For $t \in \mathcal{T}_{s^*}$, the quantity σ^{s^*, t^-} is the Lebesgue measure of $\{u \in [0, \sigma]; H_{[u, s^*]} \geq t\}$. From the property of the height process, the set $\{u \in [0, \sigma]; H_{[u, s^*]} > t\}$ is open and can be written as the union of (α_i, β_i) , $i \in I_t$. We write $\sigma^i = \beta_i - \alpha_i$. Notice that $x(t)$ is the sequence $(\sigma^i, i \in I_t)$ ranked in non-increasing order. Let I_t^+ (resp. I_t^-) the subset of I_t of indexes such that $\alpha_i > s^*$ (resp. $\beta_i < s^*$). Notice the sequence $(\sigma^i, i \in I_t)$ is the union of $\sigma^{s^*, t}$ and $(\sigma^i, i \in I_t^+ \cup I_t^-)$.

Notice that a jump of $\sigma^{s^*, t}$ happens only if there is a node at height t in the ancestor line of s^* . This is equivalent to say that $\kappa_{s^*}(\{t\}) > 0$. In particular, we have

$$(11) \quad \sum_{t \in \mathcal{T}_{s^*}} g(t) G(x(t)) = \sum_{t; \kappa_{s^*}(\{t\}) > 0} g(t) G((\sigma^i, i \in I_t)) = \int_0^\infty \frac{\kappa_{s^*}(dt)}{\kappa_{s^*}(\{t\})} g(t) G((\sigma^i, i \in I_t)),$$

with the convention that $\kappa(dr)/\kappa(\{r\}) = 0$ if $\kappa(\{r\}) = 0$. We first consider

$$(12) \quad J = \mathbb{N} \left[e^{-\lambda\sigma} \int_0^\sigma ds \int_0^\infty \frac{\kappa_s(dt)}{\kappa_s(\{t\})} g(t) e^{-p\sigma^{s,t}} K \left(\sum_{i \in I_t^+ \cup I_t^-} \delta_{\sigma^i} \right) \right],$$

where K is a non-negative measurable function defined on the set of σ -measures on $(0, \infty)$. The next computations are similar to those in Section 9.3 of [1].

We set $\sigma_-^{0,t} = \int_0^s du \mathbf{1}_{\{H_{[u,s]} < t\}}$, $\sigma_-^{s,t} = \int_0^s du \mathbf{1}_{\{H_{[u,s]} > t\}}$, $\sigma_+^{0,t} = \int_s^\sigma du \mathbf{1}_{\{H_{[u,s]} < t\}}$, $\sigma_+^{s,t} = \int_s^\sigma du \mathbf{1}_{\{H_{[u,s]} > t\}}$. Notice that

$$\sigma = \sigma_-^{0,t} + \sum_{i \in I_t^-} \sigma^i + \sigma_-^{s,t} + \sigma_+^{s,t} + \sum_{i \in I_t^+} \sigma^i + \sigma_+^{0,t}.$$

We write $K_\lambda(\mu) = K(\mu) e^{-\lambda \langle \mu, I_d \rangle}$, where $I_d(x) = x$ for $x \in \mathbb{R}_+$. In the integral in (12), we can replace

$$e^{-\lambda\sigma - p\sigma^{s,t}} K \left(\sum_{i \in I_t^+ \cup I_t^-} \delta_{\sigma^i} \right) = e^{-\lambda\sigma_-^{0,t} - (\lambda+p)\sigma_-^{s,t} - \lambda\sigma_+^{0,t} - (\lambda+p)\sigma_+^{s,t}} K_\lambda \left(\sum_{i \in I_t^-} \delta_{\sigma^i} + \sum_{i \in I_t^+} \delta_{\sigma^i} \right)$$

by its optional projection

$$B = e^{-\lambda\sigma_-^{0,t} - (\lambda+p)\sigma_-^{s,t}} \mathbb{E}_{\rho_s}^* \left[e^{-\lambda \int_0^\sigma \mathbf{1}_{\{H_{[0,u]} < t\}} du - (\lambda+p) \int_0^\sigma \mathbf{1}_{\{H_{[0,u]} > t\}} du} K_\lambda \left(\sum_{i \in I_{t,+}} \delta_{\sigma^i} + \mu' \right) \right],$$

with $\mu' = \sum_{i \in I_{t,-}} \delta_{\sigma^i}$. Using notations introduced for Lemma 2.1, we have $\int_0^\sigma \mathbf{1}_{\{H_{[0,u]} < t\}} du = \sum_{k \in I} \sigma_k \mathbf{1}_{\{h_k < t\}}$, $\int_0^\sigma \mathbf{1}_{\{H_{[0,u]} > t\}} du = \sum_{k \in I} \sigma_k \mathbf{1}_{\{h_k > t\}}$ and $\sum_{i \in I_{t,+}} \delta_{\sigma^i} = \sum_{k \in I; h_k = t} \delta_{\sigma_k}$. Then we deduce from Lemma 2.1 and (2), that

$$B = e^{-\lambda\sigma_-^{0,t} - (\lambda+p)\sigma_-^{s,t} - \rho_s((0,t))\phi(\lambda) - \rho_s((t,\infty))\phi(\lambda+p)} \mathbb{E}[K_\lambda(\mathcal{P} + \mu')],$$

where \mathcal{P} is under \mathbb{P} a Poisson point measure with intensity $\rho_s(\{t\})\mathbb{N}[d\sigma] = \rho_s(\{t\})\pi_*(d\sigma)$. By time reversibility (see Corollary 3.1.6 in [9]), we get

$$\begin{aligned} J &= \mathbb{N} \left[\int_0^\sigma ds \int_0^\infty \frac{\kappa_s(dt)}{\kappa_s(\{t\})} g(t) e^{-\lambda\sigma_-^{0,t} - (\lambda+p)\sigma_-^{s,t} - \rho_s((0,t))\phi(\lambda) - \rho_s((t,\infty))\phi(\lambda+p)} \right. \\ &\quad \left. \mathbb{E}[K_\lambda(\mathcal{P} + \mu')] \Big|_{\mu' = \sum_{i \in I_{t,-}} \delta_{\sigma^i}} \right] \\ &= \mathbb{N} \left[\int_0^\sigma ds \int_0^\infty \frac{\kappa_s(dt)}{\kappa_s(\{t\})} g(t) e^{-\lambda\sigma_+^{0,t} - (\lambda+p)\sigma_+^{s,t} - \eta_s((0,t))\phi(\lambda) - \eta_s((t,\infty))\phi(\lambda+p)} \right. \\ &\quad \left. \mathbb{E}[K_\lambda(\mathcal{P}' + \mu')] \Big|_{\mu' = \sum_{i \in I_{t,+}} \delta_{\sigma^i}} \right], \end{aligned}$$

where \mathcal{P}' is under \mathbb{P} a Poisson point measure with intensity $\eta_s(\{t\})\pi_*(dr)$. Using similar computation as above, we get

$$J = \mathbb{N} \left[\int_0^\sigma ds \int_0^\infty \frac{\kappa_s(dt)}{\kappa_s(\{t\})} g(t) e^{-\kappa_s((0,t))\phi(\lambda) - \kappa_s((t,\infty))\phi(\lambda+p)} \mathbb{E}[K_\lambda(\mathcal{P}'')] \right],$$

where \mathcal{P}'' is under \mathbb{P} a Poisson point measure with intensity $\kappa_s(\{t\})\pi_*(d\sigma)$. We write $f(r)$ for $\mathbb{E}[K_\lambda(\mathcal{P}''')]$, where \mathcal{P}''' is under \mathbb{P} a Poisson point measure with intensity $r\pi_*(d\sigma)$. Thanks to the Poisson representation of (3), using notation $\mathcal{N}(dx, dl, du) = \sum_i \delta_{x_i, \ell_i, u_i}$, we get

$$\begin{aligned} J &= \mathbb{E} \left[\int_0^\infty da e^{-\alpha_0 a} \sum_{x_i \leq a} \ell_i \frac{1}{\ell_i} g(x_i) f(\ell_i) e^{-\sum_{x_k < x_i} \ell_k \phi(\lambda) - \sum_{a \geq x_j > x_i} \ell_j \phi(\lambda+p)} \right] \\ &= \int_0^\infty da e^{-\alpha_0 a} \mathbb{E} \left[\sum_{x_i \leq a} g(x_i) f(\ell_i) e^{-x_i \int \ell \pi(dl) [1 - e^{-\ell \phi(\lambda)}] - (a - x_i) \int \ell \pi(dl) [1 - e^{-\ell \phi(\lambda+p)}]} \right] \\ &= \int_0^\infty da \mathbb{E} \left[\sum_{x_i \leq a} g(x_i) f(\ell_i) e^{-x_i \psi'(\phi(\lambda)) - (a - x_i) \psi'(\phi(\lambda+p))} \right] \\ &= \int_0^\infty da \int_{(0, \infty)} \ell \pi(dl) \int dx \mathbf{1}_{[0, a]}(x) g(x) f(\ell) e^{-x \psi'(\phi(\lambda)) - (a - x) \psi'(\phi(\lambda+p))} \\ &= \frac{1}{\psi'(\phi(\lambda+p))} \int_0^\infty dt g(t) e^{-t \psi'(\phi(\lambda))} \int \ell \pi(dl) f(\ell). \end{aligned}$$

On the other side, let $(\Delta S_t, t \geq 0)$ be the jumps of a subordinator $S = (S_t, t \geq 0)$ with Laplace exponent ϕ and Lévy measure π_* . Standard computations yield for $r > 0$,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda S_r} \sum_{t \leq r} \Delta S_t e^{-p \Delta S_t} K \left(\sum_{u \leq r, u \neq t} \delta_{\Delta S_u} \right) \right] &= \mathbb{E} \left[\sum_{t \leq r} \Delta S_t e^{-(\lambda+p) \Delta S_t} K_\lambda \left(\sum_{u \leq r, u \neq t} \delta_{\Delta S_u} \right) \right] \\ &= r \left[\int \pi_*(dl) \ell e^{-(\lambda+p)\ell} \right] \mathbb{E} \left[K_\lambda \left(\sum_{u \leq r} \delta_{\Delta S_u} \right) \right] \\ &= r \phi'(\lambda+p) f(r), \end{aligned}$$

as $\sum_{u \leq r} \delta_{\Delta S_u}$ is a Poisson measure with intensity $r\pi_*(dv)$. Notice that $\phi' = \phi^{-1'} = 1/\psi' \circ \phi$ to conclude from (12) that

$$(13) \quad \mathbb{N} \left[e^{-\lambda\sigma} \int_0^\sigma ds \int_0^\infty \frac{\kappa_s(dt)}{\kappa_s(\{t\})} g(t) e^{-p\sigma s, t} K \left(\sum_{i \in I_t^+ \cup I_t^-} \delta_{\sigma^i} \right) \right] \\ = \int_0^\infty dt g(t) e^{-t\psi'(\phi(\lambda))} \int \pi(dr) \mathbb{E} \left[e^{-\lambda S_r} \sum_{t \leq r} \Delta S_t e^{-p\Delta S_t} K \left(\sum_{u \leq r, u \neq t} \delta_{\Delta S_u} \right) \right].$$

Eventually from monotone class Theorem, we get

$$\mathbb{N} \left[e^{-\lambda\sigma} \int_0^\sigma ds \int_0^\infty \frac{\kappa_s(dt)}{\kappa_s(\{t\})} g(t) G((\sigma^i, i \in I_t)) \right] \\ = \int_0^\infty dt g(t) e^{-t\psi'(\phi(\lambda))} \int \pi(dr) \mathbb{E} \left[S_r e^{-\lambda S_r} G(\Delta S_u, u \leq r) \right].$$

Then we deduce from (11) that

$$\mathbb{N} \left[\sigma e^{-\lambda\sigma} \sum_{t \in \mathcal{T}_{s^*}} g(t) G(x(t)) \right] = \int_0^\infty dt g(t) e^{-t\psi'(\phi(\lambda))} \int \pi(dr) \mathbb{E} \left[S_r e^{-\lambda S_r} G(\Delta S_u, u \leq r) \right].$$

where we used that the tag s^* is chosen uniformly on $[0, \sigma]$. We conclude by using the definition (9) of the measure μ . \square

Lemma 5.4. *Let $H \in \mathcal{B}_+(\mathbb{R}_+)$, $t > 0$, $\lambda > 0$. We have*

$$(14) \quad \mathbb{N} \left[\sigma e^{-\lambda\sigma} H(\sigma^{s^*, t^-}) \mathbf{1}_{\{\sigma^{s^*, t^-} > 0\}} \right] = e^{-t\psi'(\phi(\lambda))} \int_{(0, \infty)} \pi_*(dr) r e^{-\lambda r} H(r).$$

Proof. Notice that $\sigma^{s^*, t^-} > 0$ if and only if $H_s > t$ \mathbb{N} -a.e. Let $p > 0$. We have

$$\mathbb{N} \left[\sigma e^{-\lambda\sigma - p\sigma^{s^*, t^-}} \mathbf{1}_{\{\sigma^{s^*, t^-} > 0\}} \right] = \mathbb{N} \left[e^{-\lambda\sigma} \int_0^\sigma ds e^{-p\sigma s, t^-} \mathbf{1}_{\{H_s > t\}} \right].$$

Similar computations as in the proof of Lemme 5.3 yield

$$\mathbb{N} \left[\sigma e^{-\lambda\sigma - p\sigma^{s^*, t^-}} \mathbf{1}_{\{H_s > t\}} \right] = \mathbb{N} \left[\int_0^\sigma ds e^{-\kappa_s((0, t)\phi(\lambda) - \kappa_s([t, \infty))\phi(\lambda + p))} \mathbf{1}_{\{H_s > t\}} \right] \\ = \mathbb{E} \left[\int_0^\infty da e^{-\alpha_0 a} e^{-\sum_{x_k < t} \ell_k \phi(\lambda) - \sum_{a \geq x_j \geq t} \ell_j \phi(\lambda + p)} \mathbf{1}_{\{a > t\}} \right] \\ = \int_t^\infty da e^{-t\psi'(\phi(\lambda)) - (a-t)\psi'(\phi(\lambda + p))} \\ = \frac{e^{-t\psi'(\phi(\lambda))}}{\psi'(\phi(\lambda + p))},$$

where we used the Poisson representation of (3) and notation $\mathcal{N}(dx, dl, du) = \sum_i \delta_{x_i, \ell_i, u_i}$ for the second equality. Since $\int_{(0, \infty)} \pi_*(dr) r e^{-(\lambda + p)r} = \phi'(\lambda + p)$ and $\phi' = 1/\psi' \circ \phi$, the Lemma is proved for $H(x) = e^{-px}$ and all $p \geq 0$. We use the monotone class Theorem to end the proof. \square

Lemma 5.5. *We have $\pi_*(dr)$ -a.e.*

$$(15) \quad \mathbb{N}_r \left[\sum_{t \in \mathcal{T}_{s^*}} g(t) G(x(t)) \right] = \mathbb{N}_r \left[\int_{\mathbb{R}_+} dt g(t) \mathbf{1}_{\{\sigma^{s^*}, t^- > 0\}} \int \nu_{\sigma^{s^*}, t^-}^*(dx) G(x) \right].$$

Proof. As a direct consequence of Lemma 5.3, Lemma 5.4 with $H(r) = \int_{\mathcal{S}^\downarrow} \nu_r^*(dx) G(x)$ and (10), we have for $\lambda > 0$,

$$\mathbb{N} \left[\sigma e^{-\lambda \sigma} \sum_{t \in \mathcal{T}_{s^*}} g(t) G(x(t)) \right] = \mathbb{N} \left[\sigma e^{-\lambda \sigma} \int_{\mathbb{R}_+} dt g(t) \mathbf{1}_{\{\sigma^{s^*}, t^- > 0\}} \int \nu_{\sigma^{s^*}, t^-}^*(dx) G(x) \right].$$

As Laplace transforms characterize measures, and since the distribution of σ under \mathbb{N} is given by π_* , we easily get the Lemma. \square

From the definition of intensity measure (see [11]), Lemma 5.5 readily implies Theorem 5.1.

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