NON-EXTENDABILITY OF SEMILATTICE-VALUED MEASURES ON PARTIALLY ORDERED SETS

FRIEDRICH WEHRUNG

Abstract. For a poset P and a distributive $\langle\vee,0\rangle$-semilattice S, a S-*valued poset measure* on P is a map $\mu\colon P\times P\to S$ such that $\mu(x,z)\leq\mu(x,y)\vee\mu(y,z)$, and $x\leq y$ implies that $\mu(x,y)=0$, for all $x,y,z\in P$. In relation with congruence lattice representation problems, we consider the problem whether such a measure can be extended to a poset measure $\overline{\mu}\colon\overline{P}\times\overline{P}\to S$, for a larger poset $\overline{P}$, such that for all $\boldsymbol{a},\boldsymbol{b}\in S$ and all $x\leq y$ in $\overline{P}$, $\overline{\mu}(y,x)=\boldsymbol{a}\vee\boldsymbol{b}$ implies that there are a positive integer n and a decomposition $x=z_0\leq z_1\leq\cdots\leq z_n=y$ in $\overline{P}$ such that either $\overline{\mu}(z_{i+1},z_i)\leq\boldsymbol{a}$ or $\overline{\mu}(z_{i+1},z_i)\leq\boldsymbol{b}$, for all $i<n$.

In this note we prove that this is not possible as a rule, even in case the poset P we start with is a *chain* and S has size $\aleph_1$. The proof uses a "monotone refinement property" that holds in S provided S is either a lattice, or countable, or strongly distributive, but fails for our counterexample. This strongly contrasts with the analogue problem for *distances* on (discrete) sets, which is known to have a positive (and even *functorial*) solution.

1. Introduction

In the paper [5], the author proved that for any lattice K, any distributive *lattice* S with zero, and any $\langle\vee,0\rangle$-homomorphism φ from the $\langle\vee,0\rangle$-semilattice $\mathrm{Con_c}\,K$ of all finitely generated congruences of K to S, there are a lattice L, a lattice homomorphism $f\colon K\to L$, and an isomorphism $\alpha\colon\mathrm{Con_c}\,L\to S$ such that $\varphi=\alpha\circ\mathrm{Con_c}\,f$. In the paper [4], J. Tůma and the author proved that for a $\langle\vee,0\rangle$-semilattice S, this statement characterizes S being a lattice. The proof of this negative result strongly uses the lattice structure of the hypothetical lattice L, see the proof of [4, Corollary 1.3].

In the present paper, we show that for a certain semilattice S of cardinality $\aleph_1$, the poset structure alone is sufficient to get a related counterexample. More precisely, for a $\langle\vee,0\rangle$-semilattice S, a S-*valued poset measure* on a poset P is a map $\mu\colon P\times P\to S$ such that $\mu(x,z)\leq\mu(x,y)\vee\mu(y,z)$ (*triangular inequality*) and $x\leq y$ implies that $\mu(x,y)=0$, for all $x,y,z\in P$. We say that μ is a *V-measure*, if for all $x\leq y$ in P and all $\boldsymbol{a},\boldsymbol{b}\in S$, if $\mu(y,x)\leq\boldsymbol{a}\vee\boldsymbol{b}$, then there are a positive integer n and a decomposition $x=z_0\leq z_1\leq\cdots\leq z_n=y$ in P such that either $\mu(z_{i+1},z_i)\leq\boldsymbol{a}$ or $\mu(z_{i+1},z_i)\leq\boldsymbol{b}$ for all $i<n$. In particular, if P is a lattice and $S=\mathrm{Con_c}\,P$, then the map μ defined by $\mu(x,y)=\Theta^+(x,y)=\Theta(y,x\vee y)$ is a $\mathrm{Con_c}\,P$ -valued V-measure on P.

This yields the following poset analogue of the abovementioned lattice-theoretical problem.

Date: February 28, 2006.

2000 *Mathematics Subject Classification.* 06A12, 06A06, 06A05.

Key words and phrases. Semilattice, poset, distributive, isotone, measure, Δ-Lemma, closed unbounded.

Problem. *Let S be a distributive $\langle\vee, 0\rangle$-semilattice. Does any S-valued poset measure on a given poset extend to some S-valued poset V-measure on a larger poset?*

A version of this problem for so-called *distances* (instead of measures) on discrete sets (instead of posets) is stated in [3]. The answer to this related question turns out to be positive (and easy). More surprisingly, this positive solution can be made *functorial*.

Nevertheless, we prove in the present paper that the problem above has a *negative* solution. Unlike what is done in [4], we do not reach here a characterization of all lattices among distributive $\langle\vee, 0\rangle$-semilattices. Our counterexample, denoted by $\mathcal{F}(\omega_1)$ (see Corollary 4.9) is obtained as an application of a certain "free construction" used by M. Ploščica and J. Tůma in [2]. The semilattice D of [4, Section 2], which is the simplest example of a $\langle\vee, 0\rangle$-semilattice which is not a lattice, does not satisfy the negative property used here. This is because D is *countable*, while we prove in Proposition 4.10 that no countable distributive $\langle\vee, 0\rangle$-semilattice can have the required negative property. On the other hand, in relation to [4, Problem 4], the proof of our counterexample uses very little of the Axiom of Choice (namely, only the Axiom of *countable* choices), while the proof of the negative property of the abovementioned semilattice D established in [4, Corollary 2.4] uses the existence of an embedding from ω_1 into the reals.

2. Basic concepts

For posets (i.e., partially ordered sets) P and Q, a map $f\colon P \to Q$ is *isotone*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in P$. In addition, we say that f is *join-preserving*, if for any subset X of P, whenever the join $\bigvee X$ of X exists in P, $\bigvee f[X]$ exists in Q, and $\bigvee f[X] = f\bigl(\bigvee X\bigr)$. For a subset X of a poset P, we shall put $\downarrow X = \{p \in P \mid \exists x \in X \text{ such that } p \leq x\}$, and then $\downarrow a = \downarrow\{a\}$, for all $a \in P$. We say that X is a *lower subset of* P, if $X = \downarrow X$.

A $\langle\vee, 0\rangle$-semilattice S is *distributive*, if $\boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}$ in S implies that there are $\boldsymbol{x} \leq \boldsymbol{a}$ and $\boldsymbol{y} \leq \boldsymbol{b}$ in S such that $\boldsymbol{c} = \boldsymbol{x} \vee \boldsymbol{y}$. A distributive $\langle\vee, 0\rangle$-semilattice S is *strongly distributive*, if every element of S is the join of a finite set of join-irreducible elements of S; equivalently, S is isomorphic to the semilattice of all finitely generated lower subsets of some poset.

We shall denote by $\operatorname{otp} P$ the order-type of a well-ordered set P. Hence $\operatorname{otp} P$ is an ordinal. We shall also use standard set-theoretical notation and terminology, referring the reader to [1] for further information. In particular, we shall denote by ω_1 the first uncountable ordinal. A subset C of ω_1 is *closed unbounded*, if C is unbounded in ω_1 and the join of any nonempty bounded subset of C belongs to C. It is well-known that the closed unbounded subsets form a countably complete filterbasis on ω_1, see [1, Lemma 7.4]. Hence containing a closed unbounded set is a notion of "largeness" for subsets of ω_1.

3. Free distributive extension of a $\langle\vee, 0\rangle$-semilattice

There are several non-equivalent definitions of what should be the "free distributive extension" of a given $\langle\vee, 0\rangle$-semilattice. The one that we shall use is introduced in [2, Section 2]. Let us first recall the construction.

For a $\langle\vee, 0\rangle$-semilattice S, we shall put $\mathcal{C}(S) = \{\langle\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\rangle \in S^3 \mid \boldsymbol{w} \leq \boldsymbol{u} \vee \boldsymbol{v}\}$. A *finite* subset $\boldsymbol{x}$ of $\mathcal{C}(S)$ is *reduced*, if it satisfies the following conditions:

(1) $\boldsymbol{x}$ contains exactly one diagonal triple, that is, a triple of the form $\langle \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u} \rangle$; we put $\boldsymbol{u} = \pi(\boldsymbol{x})$.
(2) $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \boldsymbol{x}$ and $\langle \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w} \rangle \in \boldsymbol{x}$ implies that $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in S$.
(3) $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \boldsymbol{x} \setminus \{\langle \pi(\boldsymbol{x}), \pi(\boldsymbol{x}), \pi(\boldsymbol{x}) \rangle\}$ implies that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \not\leq \pi(\boldsymbol{x})$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in S$.

We denote by $\mathcal{R}(S)$ the set of all reduced subsets of $\mathcal{C}(S)$, endowed with the partial ordering $\leq$ defined by

$$\boldsymbol{x} \leq \boldsymbol{y} \iff \forall \langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \boldsymbol{x} \setminus \boldsymbol{y}, \text{ either } \boldsymbol{u} \leq \pi(\boldsymbol{y}) \text{ or } \boldsymbol{w} \leq \pi(\boldsymbol{y}). \tag{3.1}$$

Furthermore, we shall identify $\boldsymbol{x}$ with the element $\{\langle \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x} \rangle\}$ of $\mathcal{R}(S)$, for all $\boldsymbol{x} \in S$. For set-theoretical purists, this can for example be done by replacing $\mathcal{R}(S)$ by the disjoint union of S with the set of non-singletons in $\mathcal{R}(S)$. The disjointness can easily be achieved by a suitable modification of the standard definition of a triple. We shall use the symbol $\bowtie$ to denote the canonical generators of $\mathcal{R}(S)$, so that

$$\bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \begin{cases} \boldsymbol{w}, & \text{if either } \boldsymbol{u} = \boldsymbol{v} \text{ or } \boldsymbol{v} = 0 \text{ or } \boldsymbol{w} = 0, \\ 0, & \text{if } \boldsymbol{u} = 0, \\ \{\langle 0, 0, 0 \rangle, \langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle\}, & \text{otherwise,} \end{cases}$$

Observe that the canonical map $\pi \colon \mathcal{R}(S) \twoheadrightarrow S$ is *isotone* and that the restriction of π to S is the identity. Furthermore, $\pi(\bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})) = 0$, for any non-diagonal $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \mathcal{C}(S)$. The following is an easy consequence of (3.1).

$$\boldsymbol{x} \leq \boldsymbol{y} \iff \boldsymbol{x} \leq \pi(\boldsymbol{y}), \qquad \text{for all } \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in S \times \mathcal{R}(S). \tag{3.2}$$

We recall the standard facts established in [2] about this construction.

Proposition 3.1.
(1) *For any $\langle \vee, 0 \rangle$-semilattice S, $\mathcal{R}(S)$ is a $\langle \vee, 0 \rangle$-semilattice, and the inclusion map from S into $\mathcal{R}(S)$ is a $\langle \vee, 0 \rangle$-embedding.*
(2) *For $\langle \vee, 0 \rangle$-semilattices S and T, every $\langle \vee, 0 \rangle$-homomorphism $f \colon S \to T$ extends to a unique $\langle \vee, 0 \rangle$-homomorphism $\mathcal{R}(f) \colon \mathcal{R}(S) \to \mathcal{R}(T)$ such that $\mathcal{R}(f)(\bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})) = \bowtie(f(\boldsymbol{u}), f(\boldsymbol{v}), f(\boldsymbol{w}))$, for all $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \mathcal{C}(S)$.*
(3) *The assignment $S \mapsto \mathcal{R}(S)$, $f \mapsto \mathcal{R}(f)$ is a functor.*

The extension $\mathcal{R}(S)$ is defined in such a way that $\bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \leq \boldsymbol{u}$ and $\boldsymbol{w} = \bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \vee \bowtie(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w})$, for all $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \mathcal{C}(S)$. Hence, putting $\mathcal{R}^0(S) = S$ and $\mathcal{R}^{n+1}(S) = \mathcal{R}(\mathcal{R}^n(S))$ for each n, we obtain that the increasing union $\mathcal{D}(S) = \bigcup(\mathcal{R}^n(S) \mid n < \omega)$ is a distributive $\langle \vee, 0 \rangle$-semilattice, extending S. Furthermore, putting $\mathcal{D}(f) = \bigcup(\mathcal{D}^n(f) \mid n < \omega)$ for each $\langle \vee, 0 \rangle$-homomorphism f, we obtain that $\mathcal{D}$ is a functor. The proof of the following lemma is straightforward.

Lemma 3.2. *Let S be a $\langle \vee, 0 \rangle$-semilattice and let $\langle S_i \mid i \in I \rangle$ be a family of $\langle \vee, 0 \rangle$-subsemilattices of S. The following statements hold:*
(1) $\mathcal{R}\left(\bigcap_{i \in I} S_i\right) = \bigcap_{i \in I} \mathcal{R}(S_i)$ *and* $\mathcal{D}\left(\bigcap_{i \in I} S_i\right) = \bigcap_{i \in I} \mathcal{D}(S_i)$.
(2) *If I is a nonempty upward directed poset and $\langle S_i \mid i \in I \rangle$ is isotone, then* $\mathcal{R}\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} \mathcal{R}(S_i)$ *and* $\mathcal{D}\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} \mathcal{D}(S_i)$.

Definition 3.3. For a $\langle \vee, 0 \rangle$-semilattice S and an element $\boldsymbol{x} \in \mathcal{D}(S)$, we define the *rank* of $\boldsymbol{x}$, denoted by $\operatorname{rk} \boldsymbol{x}$, as the least natural number n such that $\boldsymbol{x} \in \mathcal{R}^n(S)$, and the *complexity* of $\boldsymbol{x}$, denoted by $\|\boldsymbol{x}\|$, by $\|\boldsymbol{x}\| = 0$ if $\boldsymbol{x} \in S$, and

$$\|\boldsymbol{x}\| = \sum \left(\|\boldsymbol{u}\| + \|\boldsymbol{v}\| + \|\boldsymbol{w}\| + 1 \mid \langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle \in \boldsymbol{x}\right), \qquad \text{for all } \boldsymbol{x} \in \mathcal{D}(S) \setminus S.$$

4. The semilattices $\mathcal{S}(\Lambda)$ and $\mathcal{F}(\Lambda)$

For any *chain* Λ, we shall denote by $\mathcal{S}(\Lambda)$ the $\langle\vee,0\rangle$-semilattice defined by generators $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}_i$, for $i\in\Lambda$, and relations $\boldsymbol{c}_i\leq\boldsymbol{a}\vee\boldsymbol{b}$ and $\boldsymbol{c}_i\leq\boldsymbol{c}_j$, for all $i\leq j$ in Λ. Hence the elements of $\mathcal{S}(\Lambda)$ either belong to $\mathcal{S}(\varnothing)=\{0,\boldsymbol{a},\boldsymbol{b},\boldsymbol{a}\vee\boldsymbol{b}\}$ or have the form $\boldsymbol{c}_i$, $\boldsymbol{a}\vee\boldsymbol{c}_i$, or $\boldsymbol{b}\vee\boldsymbol{c}_i$, for some $i\in\Lambda$. We shall identify $\mathcal{S}(X)$ with the $\langle\vee,0\rangle$-subsemilattice of $\mathcal{S}(\Lambda)$ generated by $\mathcal{S}(\varnothing)\cup\{\boldsymbol{c}_i\mid i\in X\}$, for any $X\subseteq\Lambda$.

For chains X and Y, any isotone map $f\colon X\to Y$ gives raise to a unique $\langle\vee,0\rangle$-homomorphism $\mathcal{S}(f)\colon\mathcal{S}(X)\to\mathcal{S}(Y)$ fixing $\boldsymbol{a}$ and $\boldsymbol{b}$ and sending $\boldsymbol{c}_i$ to $\boldsymbol{c}_{f(i)}$, for all $i\in X$. Of course, the assignment $\Lambda\mapsto\mathcal{S}(\Lambda)$, $f\mapsto\mathcal{S}(f)$ is a functor.

We denote by $\mathcal{F}=\mathcal{D}\circ\mathcal{S}$ the composition of the two functors $\mathcal{D}$ and $\mathcal{S}$.

The proof of the following lemma is straightforward.

Lemma 4.1. *Let Λ be a chain and let $\langle X_i\mid i\in I\rangle$ be a family of subsets of Λ. The following statements hold:*

(1) $\mathcal{S}\left(\bigcap_{i\in I}X_i\right)=\bigcap_{i\in I}\mathcal{S}(X_i)$.
(2) *If I is a nonempty upward directed poset and $\langle X_i\mid i\in I\rangle$ is isotone, then* $\mathcal{S}\left(\bigcup_{i\in I}X_i\right)=\bigcup_{i\in I}\mathcal{S}(X_i)$.

As an easy consequence of Lemmas 3.2 and 4.1, we get the following.

Lemma 4.2. *Let Λ be a chain. Then for any $\boldsymbol{x}\in\mathcal{F}(\Lambda)$, there exists a least* (*with respect to the inclusion*) *subset X of Λ such that $\boldsymbol{x}\in\mathcal{F}(X)$; this subset is finite.*

We denote by $\operatorname{supp}(\boldsymbol{x})$ the subset given by Lemma 4.2, and we call it the *support* of $\boldsymbol{x}$.

Notation 4.3. For a chain Λ, well-ordered subsets X and Y of Λ such that $\operatorname{otp}X\leq\operatorname{otp}Y$, and $\boldsymbol{x}\in\mathcal{F}(X)$, we set $\boldsymbol{x}[Y/X]=\mathcal{F}(e_{X,Y})(\boldsymbol{x})$, where $e_{X,Y}$ denotes the unique embedding from X into Y whose range is a lower subset of Y.

Hence $\boldsymbol{x}[Y/X]$ belongs to $\mathcal{F}(Y)$, for all $\boldsymbol{x}\in\mathcal{F}(X)$.

Lemma 4.4. *Let Λ be a chain and let X, Y be well-ordered subsets of Λ such that $\operatorname{otp}X\leq\operatorname{otp}Y$ and $X\cap Y$ is a lower subset of both X and Y. Then $\boldsymbol{x}[Y/X]=\boldsymbol{x}$, for all $\boldsymbol{x}\in\mathcal{F}(X\cap Y)$.*

Proof. As the set $Z=X\cap Y$ is a lower subset of both X and Y, the homomorphism $\mathcal{F}(e_{Z,X})$ (resp., $\mathcal{F}(e_{Z,Y})$) is the inclusion map from $\mathcal{F}(Z)$ into $\mathcal{F}(X)$ (resp., $\mathcal{F}(Y)$). In particular, $\boldsymbol{x}=\mathcal{F}(e_{Z,X})(\boldsymbol{x})=\mathcal{F}(e_{Z,Y})(\boldsymbol{x})$. Therefore,

$$\boldsymbol{x}[Y/X]=\mathcal{F}(e_{X,Y})(\boldsymbol{x})=\mathcal{F}(e_{X,Y})\circ\mathcal{F}(e_{Z,X})(\boldsymbol{x})=\mathcal{F}(e_{Z,Y})(\boldsymbol{x})=\boldsymbol{x}.\qquad\square$$

We are now reaching a crucial lemma.

Lemma 4.5 (Interpolation Lemma)**.** *Let Λ be a chain, let X, Y be finite subsets of Λ, and let $\langle\boldsymbol{x},\boldsymbol{y}\rangle\in\mathcal{F}(X)\times\mathcal{F}(Y)$. If $\boldsymbol{x}\leq\boldsymbol{y}$, then either there exists $\boldsymbol{z}\in\mathcal{F}(X\cap Y)$ such that $\boldsymbol{x}\leq\boldsymbol{z}\leq\boldsymbol{y}$ or* ($Y\nsubseteq X$ *and* $\boldsymbol{c}_{\min(Y\setminus X)}\leq\boldsymbol{y}$).

Proof. We shall denote by π_k^l the canonical map from $\mathcal{R}^l\mathcal{S}(\Lambda)$ onto $\mathcal{R}^k\mathcal{S}(\Lambda)$, for all natural numbers $k\leq l$. Put $m=\operatorname{rk}\boldsymbol{x}$ and $n=\operatorname{rk}\boldsymbol{y}$. Observe that $\operatorname{supp}(\boldsymbol{x})\subseteq X$ and $\operatorname{supp}(\boldsymbol{y})\subseteq Y$. We argue by induction on $\|\boldsymbol{x}\|+\|\boldsymbol{y}\|$. If either $\operatorname{supp}(\boldsymbol{x})\subseteq Y$ or $\operatorname{supp}(\boldsymbol{y})\subseteq X$ then either $\boldsymbol{z}=\boldsymbol{x}$ or $\boldsymbol{z}=\boldsymbol{y}$ belongs to $\mathcal{F}(X\cap Y)$ and satisfies the inequalities $\boldsymbol{x}\leq\boldsymbol{z}\leq\boldsymbol{y}$, so we are done. So suppose that $\operatorname{supp}(\boldsymbol{x})\nsubseteq Y$ and $\operatorname{supp}(\boldsymbol{y})\nsubseteq X$. In particular, $X\nsubseteq Y$ and $Y\nsubseteq X$, and both $\operatorname{supp}(\boldsymbol{x})$ and $\operatorname{supp}(\boldsymbol{y})$ are nonempty. We put $\xi=\min(Y\setminus X)$.

Suppose that $m = n = 0$, that is, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}(\Lambda)$. Pick $i \in \operatorname{supp}(\boldsymbol{x})$. As $\boldsymbol{c}_i \leq \boldsymbol{x} \leq \boldsymbol{y}$, we obtain that either $\boldsymbol{y} = \boldsymbol{a} \vee \boldsymbol{b}$ (a contradiction, as then $\operatorname{supp}(\boldsymbol{y}) = \varnothing$) or $\boldsymbol{y} \in \{\boldsymbol{c}_j, \boldsymbol{a} \vee \boldsymbol{c}_j, \boldsymbol{b} \vee \boldsymbol{c}_j\}$ for some $j \geq i$. If $i = j$, then $\operatorname{supp}(\boldsymbol{x}) = \operatorname{supp}(\boldsymbol{y}) = \{i\}$, a contradiction. If $i < j$, then $\xi = j$ and so $\boldsymbol{c}_\xi \leq \boldsymbol{y}$.

Suppose now that $m < n$. Then $\boldsymbol{x} \leq \boldsymbol{y}$ means that $\boldsymbol{x} \leq \pi_m^n(\boldsymbol{y})$ (use (3.2)). As $\pi_m^n(\boldsymbol{y})$ has support contained in Y and rank at most m, it follows from the induction hypothesis that either $\boldsymbol{c}_\xi \leq \pi_m^n(\boldsymbol{y})$ (thus, *a fortiori*, $\boldsymbol{c}_\xi \leq \boldsymbol{y}$) or there exists $\boldsymbol{z} \in \mathcal{F}(X \cap Y)$ such that $\boldsymbol{x} \leq \boldsymbol{z} \leq \pi_m^n(\boldsymbol{y})$ (thus, *a fortiori*, $\boldsymbol{z} \leq \boldsymbol{y}$).

So suppose from now on that $m > 0$ (i.e., $\boldsymbol{x} \notin \mathcal{S}(\Lambda)$) and $m \geq n$. If $\boldsymbol{x} = \bigvee_{i<k} \boldsymbol{x}_i$ where $k \geq 2$ and each $\boldsymbol{x}_i$ has support contained in X and complexity less than $\|\boldsymbol{x}\|$, then we apply the induction hypothesis to each inequality $\boldsymbol{x}_i \leq \boldsymbol{y}$, for $i < k$. If $\boldsymbol{c}_\xi \nleq \boldsymbol{y}$, then for all $i < k$, there exists $\boldsymbol{z}_i \in \mathcal{F}(X \cap Y)$ such that $\boldsymbol{x}_i \leq \boldsymbol{z}_i \leq \boldsymbol{y}$. Hence $\boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{y}$, where $\boldsymbol{z} = \bigvee_{i<k} \boldsymbol{z}_i$ belongs to $\mathcal{F}(X \cap Y)$. This reduces the problem to the case where $\boldsymbol{x} = \bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$, where $\langle \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \rangle$ is a non-diagonal triple of elements of $\mathcal{R}^{m-1}\mathcal{S}(X)$ of complexity less than $\|\boldsymbol{x}\|$.

If $m > n$, then, as $\bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{x} \leq \boldsymbol{y}$ with $\operatorname{supp}(\boldsymbol{x}) \nsubseteq Y$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{R}^{m-1}\mathcal{S}(X)$, and $\boldsymbol{y} \in \mathcal{R}^{m-1}\mathcal{S}(Y)$, it follows from (3.1) that either $\boldsymbol{u} \leq \boldsymbol{y}$ or $\boldsymbol{w} \leq \boldsymbol{y}$. If, for example, $\boldsymbol{u} \leq \boldsymbol{y}$, then, by the induction hypothesis, either $\boldsymbol{c}_\xi \leq \boldsymbol{y}$ (in which case we are done) or there exists $\boldsymbol{z} \in \mathcal{F}(X \cap Y)$ such that $\boldsymbol{u} \leq \boldsymbol{z} \leq \boldsymbol{y}$. In the second case, $\boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{y}$. The argument is similar in case $\boldsymbol{w} \leq \boldsymbol{y}$.

The remaining case is $m = n > 0$. As $\bowtie(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{x} \leq \boldsymbol{y}$ with $\operatorname{supp}(\boldsymbol{x}) \nsubseteq Y$, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{R}^{m-1}\mathcal{S}(X)$, and $\boldsymbol{y} \in \mathcal{R}^{m}\mathcal{S}(Y)$, it follows from (3.1) that either $\boldsymbol{u} \leq \pi_{m-1}^m(\boldsymbol{y})$ or $\boldsymbol{w} \leq \pi_{m-1}^m(\boldsymbol{y})$. If $\boldsymbol{u} \leq \pi_{m-1}^m(\boldsymbol{y})$, then, by the induction hypothesis, either $\boldsymbol{c}_\xi \leq \pi_{m-1}^m(\boldsymbol{y})$ (thus, *a fortiori*, $\boldsymbol{c}_\xi \leq \boldsymbol{y}$) or there exists $\boldsymbol{z} \in \mathcal{F}(X \cap Y)$ such that $\boldsymbol{u} \leq \boldsymbol{z} \leq \pi_{m-1}^m(\boldsymbol{y})$ (in which case $\boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{y}$). The case where $\boldsymbol{w} \leq \pi_{m-1}^m(\boldsymbol{y})$ is similar. □

Lemma 4.6. *Let Λ be a chain and let X be a nonempty subset of Λ admitting a supremum, say, ξ, in Λ. Then $\boldsymbol{c}_\xi$ is the supremum of $\{\boldsymbol{c}_i \mid i \in X\}$ in $\mathcal{F}(\Lambda)$.*

Proof. Let $\boldsymbol{x} \in \mathcal{F}(\Lambda)$ such that $\boldsymbol{c}_i \leq \boldsymbol{x}$ for all $i \in X$, we prove that $\boldsymbol{c}_\xi \leq \boldsymbol{x}$. Put $n = \operatorname{rk} \boldsymbol{x}$ and $\boldsymbol{y} = \pi_0^n(\boldsymbol{x})$. Let $i \in X$. From $\boldsymbol{c}_i \leq \boldsymbol{x}$ and $\boldsymbol{c}_i \in \mathcal{S}(\Lambda)$ it follows that $\boldsymbol{c}_i \leq \boldsymbol{y}$. This holds for all $i \in X$, hence, as $\boldsymbol{c}_\xi$ is clearly the supremum of $\{\boldsymbol{c}_i \mid i \in X\}$ in $\mathcal{S}(\Lambda)$, we obtain that $\boldsymbol{c}_\xi \leq \boldsymbol{y}$. Therefore, $\boldsymbol{c}_\xi \leq \boldsymbol{x}$. □

Now we can state the main technical result of the paper. It says that $\langle \boldsymbol{c}_\xi \mid \xi < \omega_1 \rangle$ is the least non-eventually constant isotone ω_1-sequence in $\mathcal{F}(\omega_1)$ modulo the closed unbounded filter on ω_1.

Theorem 4.7. *Let $\sigma = \langle \boldsymbol{x}_\xi \mid \xi < \omega_1 \rangle$ be an isotone ω_1-sequence of elements of $\mathcal{F}(\omega_1)$. Then either σ is eventually constant or there exists a closed unbounded subset C of ω_1 such that $\boldsymbol{c}_\xi \leq \boldsymbol{x}_\xi$ for all $\xi \in C$.*

Proof. Assume that σ is not eventually constant. We put $X_\xi = \operatorname{supp}(\boldsymbol{x}_\xi)$ and $n_\xi = |X_\xi|$, for all $\xi < \omega_1$. So $\boldsymbol{x}_\xi = \boldsymbol{x}'_\xi[X_\xi/n_\xi]$, for some $\boldsymbol{x}'_\xi \in \mathcal{F}(n_\xi)$. As all sets X_ξ are finite, it follows from the Δ-Lemma (see [1, Lemma 22.6]) that there are an uncountable subset I of ω_1 and a finite subset X of ω_1 such that $X_\xi \cap X_\eta = X$ for all distinct $\xi, \eta \in I$. We may further assume without loss of generality that there are $n < \omega$ and $\boldsymbol{x} \in \mathcal{F}(n)$ such that $n_\xi = n$ and $\boldsymbol{x}'_\xi = \boldsymbol{x}$, for all $\xi \in I$. Hence $\boldsymbol{x}_\xi = \boldsymbol{x}[X_\xi/n]$, for all $\xi \in I$. As σ is isotone but not eventually constant, it follows that X is a proper subset of X_ξ, for all $\xi \in I$. Put $Y_\xi = X_\xi \setminus X$. Define $\rho(\xi)$ as the least element of Y_ξ.

For subsets U and V of ω_1, let $U < V$ hold, if $u < v$ for all $\langle u, v\rangle \in U \times V$. By further shrinking I, we might assume that $X < Y_\xi$, for all $\xi \in I$. In particular, observe that $X = X_\xi \cap X_\eta$ is a lower subset of both X_ξ and X_η, for all $\xi \neq \eta$ in I.

Let $\xi < \eta$ in I and suppose that there exists $\boldsymbol{z} \in \mathcal{F}(X)$ such that $\boldsymbol{x}_\xi \leq \boldsymbol{z} \leq \boldsymbol{x}_\eta$. Applying the embedding $\mathcal{F}(e_{X_\xi, X_\eta})$ to the inequality $\boldsymbol{x}[X_\xi/n] \leq \boldsymbol{z}$ and using Lemma 4.4, we obtain the inequality $\boldsymbol{x}[X_\eta/n] \leq \boldsymbol{z}$, so $\boldsymbol{x}_\eta = \boldsymbol{x}[X_\eta/n] = \boldsymbol{z}$, a contradiction since the left hand side has support X_η while the right hand side has the smaller support X. Therefore, as $\boldsymbol{x}_\xi \leq \boldsymbol{x}_\eta$ and by Lemma 4.5, we obtain the inequality $\boldsymbol{c}_{\rho(\eta)} \leq \boldsymbol{x}_\eta$.

Hence, we may assume that $\boldsymbol{c}_{\rho(\xi)} \leq \boldsymbol{x}_\xi$ for all $\xi \in I$. It follows that

$$\boldsymbol{c}_{\overline{\rho}(\xi)} \leq \boldsymbol{x}_\xi, \qquad \text{for all } \xi < \omega_1, \tag{4.1}$$

where we put

$$\overline{\rho}(\xi) = \bigvee \left(\rho(\eta) \mid \eta \in I,\ \eta < \xi\right), \qquad \text{for all } \xi < \omega_1.$$

As the range of ρ is unbounded, so is the range of $\overline{\rho}$. Hence, as $\overline{\rho}$ is a complete join-homomorphism from ω_1 to ω_1, the set $C = \{\xi < \omega_1 \mid \overline{\rho}(\xi) = \xi\}$ is a closed unbounded subset of ω_1. It follows from (4.1) that the inequality $\boldsymbol{c}_\xi \leq \boldsymbol{x}_\xi$ holds for all $\xi \in C$. □

The following corollary expresses that $\mathcal{F}(\omega_1)$ fails a certain "monotone refinement property".

Corollary 4.8. *There are no positive integer n and no finite collection of isotone ω_1-sequences $\langle \boldsymbol{x}_{i,\xi} \mid \xi < \omega_1\rangle$ of elements of $\mathcal{F}(\omega_1)$, for $0 \leq i \leq n$, such that*

(1) *$\boldsymbol{x}_{0,\xi} = 0$ and $\boldsymbol{x}_{n,\xi} = \boldsymbol{c}_\xi$, for all large enough $\xi < \omega_1$.*
(2) *$\boldsymbol{x}_{i,\xi} \leq \boldsymbol{c}_\xi$, for all $i \leq n$ and all large enough $\xi < \omega_1$.*
(3) *Either $\boldsymbol{x}_{i+1,\xi} \leq \boldsymbol{a} \vee \boldsymbol{x}_{i,\xi}$ or $\boldsymbol{x}_{i+1,\xi} \leq \boldsymbol{b} \vee \boldsymbol{x}_{i,\xi}$, for all $i < n$ and all $\xi < \omega_1$.*

Proof. We prove that for all $i \leq n$, there exists $\eta_i < \omega_1$ such that $\boldsymbol{x}_{i,\xi} \leq \boldsymbol{c}_{\eta_i}$ for all $\xi < \omega_1$. We argue by induction on i. For $i = 0$ it holds by assumption, with $\eta_0 = 0$. Suppose that $\boldsymbol{x}_{i,\xi} \leq \boldsymbol{c}_{\eta_i}$, for all $\xi < \omega_1$. Let $\xi > \eta_i$. Assume, for example, that $\boldsymbol{x}_{i+1,\xi} \leq \boldsymbol{a} \vee \boldsymbol{x}_{i,\xi}$; so $\boldsymbol{x}_{i+1,\xi} \leq \boldsymbol{a} \vee \boldsymbol{c}_{\eta_i}$. Observing that $\boldsymbol{c}_\xi \nleq \boldsymbol{a} \vee \boldsymbol{c}_{\eta_i}$, we get that $\boldsymbol{c}_\xi \nleq \boldsymbol{x}_{i+1,\xi}$. As this holds for all $\xi > \eta_i$ and by Theorem 4.7, we obtain that $\langle \boldsymbol{x}_{i+1,\xi} \mid \xi < \omega_1\rangle$ is eventually constant, and hence, by (2), below some $\boldsymbol{c}_{\eta_{i+1}}$, therefore completing the induction step.

In particular, for $i = n$, we obtain that the ω_1-sequence $\langle \boldsymbol{c}_\xi \mid \xi < \omega_1\rangle$ is eventually dominated by the constant $\boldsymbol{c}_{\eta_n}$, a contradiction. □

Hence we get a negative extension property for posets.

Corollary 4.9. *There are a poset measure $\mu\colon (\omega_1 + 1) \times (\omega_1 + 1) \to \mathcal{F}(\omega_1)$ such that $\mu(\omega_1, 0) = \boldsymbol{a} \vee \boldsymbol{b}$ but there are no poset P containing $\omega_1 + 1$, no poset measure $\overline{\mu}\colon P \times P \to \mathcal{F}(\omega_1)$ extending μ, no positive integer n, and no decomposition $0 = z_0 \leq z_1 \leq \cdots \leq z_n = \omega_1$ in P such that either $\overline{\mu}(z_{i+1}, z_i) \leq \boldsymbol{a}$ or $\overline{\mu}(z_{i+1}, z_i) \leq \boldsymbol{b}$ for all $i < n$.*

Proof. Define $\mu\colon (\omega_1+1)\times(\omega_1+1)\to \mathcal{F}(\omega_1)$ by

$$\mu(\xi,\eta)=\begin{cases}0, & \text{if } \xi\leq\eta,\\ \boldsymbol{c}_\xi, & \text{if } \eta<\xi<\omega_1,\\ \boldsymbol{a}\vee\boldsymbol{b}, & \text{if } \eta<\xi=\omega_1.\end{cases}\qquad \text{for all } \xi,\eta\leq\omega_1.$$

It is straightforward to verify that μ is a poset measure on ω_1+1. Suppose that P, $\overline{\mu}$, n, z_0, ..., z_n satisfy the given conditions. We put $\boldsymbol{x}_{i,\xi}=\overline{\mu}(\xi,z_{n-i})$, for all $\xi<\omega_1$. It is not hard, using the triangular inequality, to verify that the elements $\boldsymbol{x}_{i,\xi}$ satisfy the assumptions (1)–(3) of Corollary 4.8, a contradiction. □

As the following result shows, more "amenable" semilattices do satisfy a certain "monotone refinement property".

Proposition 4.10. *Let S be a distributive $\langle\vee,0\rangle$-semilattice. If S is either a lattice, or strongly distributive, or countable, then for all $\boldsymbol{a},\boldsymbol{b}\in S$, every chain Λ, and every isotone Λ-sequence $\langle\boldsymbol{c}_i\mid i\in\Lambda\rangle$ of elements of S such that $\boldsymbol{c}_i\leq\boldsymbol{a}\vee\boldsymbol{b}$ for all $i\in\Lambda$, then there are isotone Λ-sequences $\langle\boldsymbol{a}_i\mid i\in\Lambda\rangle$ and $\langle\boldsymbol{b}_i\mid i\in\Lambda\rangle$ of elements of S such that $\boldsymbol{a}_i\leq\boldsymbol{a}$, $\boldsymbol{b}_i\leq\boldsymbol{b}$, and $\boldsymbol{c}_i=\boldsymbol{a}_i\vee\boldsymbol{b}_i$, for all $i\in\Lambda$.*

Proof. If S is a lattice the conclusion is trivial: put $\boldsymbol{a}_i=\boldsymbol{a}\wedge\boldsymbol{c}_i$ and $\boldsymbol{b}_i=\boldsymbol{b}\wedge\boldsymbol{c}_i$, for all $i\in\Lambda$.

Now assume that S is strongly distributive. Denote by C_i the (finite) set of all maximal join-irreducible elements of S below $\boldsymbol{c}_i$, for all $i\in\Lambda$. Observe that $C_i\subseteq{\downarrow}C_j$, for all $i\leq j$ in Λ. For every finite subset I of Λ, denote by X_I the set of all families $\langle\langle A_i,B_i\rangle\mid i\in I\rangle$ such that

(1) $A_i\subseteq{\downarrow}\boldsymbol{a}$, $B_i\subseteq{\downarrow}\boldsymbol{b}$, and $A_i\cup B_i=C_i$, for all $i\in I$.
(2) For all $i\leq j$ in I, $A_i\subseteq{\downarrow}A_j$ and $B_i\subseteq{\downarrow}B_j$.

We claim that X_I is nonempty, for every finite subset I of Λ. We argue by induction on $|I|$. The conclusion is obvious for $I=\varnothing$. For $I=\{i\}$, put $A_i={\downarrow}\boldsymbol{a}\cap C_i$ and $B_i={\downarrow}\boldsymbol{b}\cap C_i$. Now suppose that $I=\{i\}\cup J$, where $i<j$ for all $j\in J$ and J is nonempty. By induction hypothesis, there exists an element $\langle\langle A_k,B_k\rangle\mid k\in J\rangle$ in X_J. Put $j=\min J$, $A_i={\downarrow}A_j\cap C_i$, and $B_i={\downarrow}B_j\cap C_i$. It is straightforward to verify that $\langle\langle A_k,B_k\rangle\mid i\in I\rangle$ belongs to X_I. This completes the induction step.

It follows that the set Ω_I of all families $\langle\langle A_i,B_i\rangle\mid i\in\Lambda\rangle$ of elements of the Cartesian product $\Omega=\prod(\mathfrak{P}(C_i)\times\mathfrak{P}(C_i)\mid i\in\Lambda)$ (where $\mathfrak{P}(X)$ denotes the powerset of a set X) whose restriction to I belongs to X_I is nonempty, for every finite subset I of Λ. Endow Ω with the product topology of the discrete topologies on all (finite) sets $\mathfrak{P}(C_i)\times\mathfrak{P}(C_i)$. By Tychonoff's Theorem, Ω is compact. Hence the intersection of all Ω_I, for I a finite subset of Λ, is nonempty. Let $\langle\langle A_i,B_i\rangle\mid i\in\Lambda\rangle$ be an element of that intersection. Then the collection of all elements $\boldsymbol{a}_i=\bigvee A_i$ and $\boldsymbol{b}_i=\bigvee B_i$, for $i\in\Lambda$, satisfies the required conditions.

Assume, finally, that S is countable. Define an equivalence relation $\equiv$ on Λ by $i\equiv j$ iff $\boldsymbol{c}_i=\boldsymbol{c}_j$, for all $i,j\in\Lambda$, and denote by $[i]$ the $\equiv$-equivalence class of i, for any $i\in\Lambda$. Putting $\boldsymbol{c}_{[i]}=\boldsymbol{c}_i$ makes it possible to replace Λ by $\Lambda/{\equiv}$. In particular, as S is countable, Λ becomes countable as well. Now write $\Lambda=\bigcup(\Lambda_n\mid n<\omega)$, where $\langle\Lambda_n\mid n<\omega\rangle$ is an increasing sequence of finite subsets of Λ with $|\Lambda_n|=n$, for all $n<\omega$. Denote by Y_n the set of all families $\langle\langle\boldsymbol{a}_l,\boldsymbol{b}_l\rangle\mid l\in\Lambda_n\rangle$ such that $\boldsymbol{a}_i\leq\boldsymbol{a}$, $\boldsymbol{b}_i\leq\boldsymbol{b}$, $\boldsymbol{c}_i=\boldsymbol{a}_i\vee\boldsymbol{b}_i$, and $i\leq j$ implies that $\boldsymbol{a}_i\leq\boldsymbol{a}_j$ and $\boldsymbol{b}_i\leq\boldsymbol{b}_j$, for all $i\leq j$ in Λ_n. Suppose that we are given an element of Y_n as above, and denote by k the unique element of $Y_{n+1} \setminus Y_n$. Suppose, for example, that $\min \Lambda_n < k < \max \Lambda_n$, and denote by i (resp., j) the largest (resp., least) element of Λ_n below k (resp., above k). As $\boldsymbol{c}_k \leq \boldsymbol{c}_j = \boldsymbol{a}_j \vee \boldsymbol{b}_j$, there are $\boldsymbol{a}' \leq \boldsymbol{a}_j$ and $\boldsymbol{b}' \leq \boldsymbol{b}_j$ such that $\boldsymbol{c}_k = \boldsymbol{a}' \vee \boldsymbol{b}'$. Put $\boldsymbol{a}_k = \boldsymbol{a}_i \vee \boldsymbol{a}'$ and $\boldsymbol{b}_k = \boldsymbol{b}_i \vee \boldsymbol{b}'$. Then $\langle \langle \boldsymbol{a}_l, \boldsymbol{b}_l \rangle \mid l \in \Lambda_{n+1} \rangle$ is an element of Y_{n+1}. So every element of Y_n extends to an element of Y_{n+1}. The proof is even easier in case either $k < \min \Lambda_n$ or $k > \max \Lambda_n$. Hence we have constructed inductively a family $\langle \langle \boldsymbol{a}_i, \boldsymbol{b}_i \rangle \mid i \in \Lambda \rangle$ whose restriction to Λ_n belongs to Y_n, for all $n < \omega$. Therefore, the elements $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$, for $i \in \Lambda$, are as required. □

The "monotone refinement property" described above fails in $\mathcal{F}(\omega_1)$. Indeed, consider the isotone ω_1-sequence $\langle \boldsymbol{c}_\xi \mid \xi < \omega_1 \rangle$ together with the inequalities $\boldsymbol{c}_\xi \leq \boldsymbol{a} \vee \boldsymbol{b}$, for $\xi < \omega_1$. Suppose that $\langle \boldsymbol{a}_\xi \mid \xi < \omega_1 \rangle$ and $\langle \boldsymbol{b}_\xi \mid \xi < \omega_1 \rangle$ are isotone ω_1-sequences in $\mathcal{F}(\omega_1)$ such that $\boldsymbol{c}_\xi = \boldsymbol{a}_\xi \vee \boldsymbol{b}_\xi$ while $\boldsymbol{a}_\xi \leq \boldsymbol{a}$ and $\boldsymbol{b}_\xi \leq \boldsymbol{b}$, for all $\xi < \omega_1$. Set $\boldsymbol{x}_{0,\xi} = 0$, $\boldsymbol{x}_{1,\xi} = \boldsymbol{a}_\xi$, and $\boldsymbol{x}_{2,\xi} = \boldsymbol{c}_\xi$, for all $\xi < \omega_1$. Then the isotone ω_1-sequences $\langle \boldsymbol{x}_{i,\xi} \mid \xi < \omega_1 \rangle$, for $i \in \{0,1,2\}$, satisfy (1)–(3) of Corollary 4.8 (with $n = 2$), a contradiction.

Nevertheless we do not know whether the monotone refinement property of S either implies or is implied by the statement that every S-valued poset measure extends to a V-measure.

References

[1] T. Jech, "Set Theory", Academic Press, Harcourt Brace Jovanovich, New York - San Francisco - London, xi + 621 p, 1978.

[2] M. Ploščica and J. Tůma, *Uniform refinements in distributive semilattices*, Contributions to General Algebra **10**, Proceedings of the Klagenfurt Conference, May 29 – June 1, 1997. Verlag Johannes Heyn, Klagenfurt 1998.

[3] P. Růžička, J. Tůma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, preprint.

[4] J. Tůma and F. Wehrung, *Unsolvable one-dimensional lifting problems for congruence lattices of lattices*, Forum Math. **14**, no. 4 (2002), 483–493.

[5] F. Wehrung, *Forcing extensions of partial lattices*, J. Algebra **262**, no. 1 (2003), 127–193.

CNRS, UMR 6139, Département de Mathématiques, BP 5186, Université de Caen, Campus 2, 14032 Caen cedex, France

E-mail address: wehrung@math.unicaen.fr

URL: http://www.math.unicaen.fr/~wehrung