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A posteriori error estimates for lowest-order mixed finite element discretizations of convection–diffusion–reaction equations

Martin Vohralík[1,2]

[1] Laboratoire de Mathématiques, Analyse Numérique et EDP, Université de Paris-Sud et CNRS, Bât. 425, 91 405 Orsay, France. martin.vohralik@math.u-psud.fr

[2] Department of Process Modeling, Faculty of Mechatronics and Interdisciplinary Engineering Studies, Technical University of Liberec, Hálkova 6, 461 17 Liberec 1, Czech Republic. martin.vohralik@tul.cz

Abstract

We establish residual a posteriori error estimates for lowest-order Raviart–Thomas mixed finite element discretizations of convection–diffusion–reaction equations on simplicial meshes in two or three space dimensions. The upwind-mixed scheme is considered as well and the emphasis is put on the presence of an inhomogeneous and anisotropic diffusion–dispersion tensor and on a possible convection dominance. Global upper bounds in the energy norm for the approximation error are derived, where in particular all constants are evaluated explicitly, so that the estimators are fully computable. Our estimators give local lower bounds for the error as well, hold from the cases where convection or reaction are not present to convection-dominated problems, and their efficiency only depends on local variations in the coefficients and is shown to be optimal as the local Péclet number gets small. The main idea of the proof is a construction of a new scalar variable based on a simple local postprocessing in each element and a subsequent use of the abstract framework arising from the primal weak formulation of the continuous problem. An interesting particular consequence is that the postprocessed variable coincides with the exact solution for one-dimensional pure diffusion problems with piecewise constant coefficients. Numerical experiments confirm the efficiency and robustness of the derived estimators.

Key words: convection–diffusion–reaction equation, mixed finite element method, upwind weighting, a posteriori error estimates, convection dominance, inhomogeneous and anisotropic diffusion

AMS subject classifications: 65N15, 65N30, 76S05

1 Introduction

We consider in this paper the convection–diffusion–reaction problem

$$-\nabla\cdot(\mathbf{S}\nabla p)+\nabla\cdot(p\mathbf{w})+rp = f \quad \text{in } \Omega\,, \tag{1.1a}$$

$$p = 0 \quad \text{on } \partial\Omega\,, \tag{1.1b}$$

where $\mathbf{S}$ is in general an inhomogeneous and anisotropic (nonconstant full-matrix) diffusion–dispersion tensor, $\mathbf{w}$ is a (dominating) velocity field, r a reaction function, f a source term, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal (polyhedral) domain (open, bounded, and connected set). Our purpose is to derive a posteriori error estimates for the lowest-order Raviart–Thomas mixed finite element discretization of the problem (1.1a)–(1.1b) on simplicial meshes (consisting of triangles if

$d = 2$ and tetrahedra if $d = 3$), as well as for its upwind variant, cf. Douglas and Roberts [16, 17] and Dawson [14], Dawson and Aizinger [15], and Jaffré [26].

A posteriori error estimates, pioneered by Babuška and Rheinboldt [5], are nowadays well established for primal discretizations of second-order elliptic problems only involving a diffusion term, cf. for example the survey by Verfürth [36] for the conforming finite element method, Hoppe and Wohlmuth [24] for the nonconforming finite element method, Nicaise [30] for the finite volume method, and Karakashian and Pascal [27] for the discontinuous Galerkin method. In the majority of the cases the analysis is only given for $\mathbf{S}$ being an identity matrix; an in-depth analysis for the general inhomogeneous and anisotropic diffusion tensor in the frame of the finite element method was presented by Bernardi and Verfürth [7]. Similar results for the finite volume box scheme (in the given case actually equivalent to the lowest-order Raviart–Thomas mixed finite element one) have been obtained by El Alaoui and Ern in [19]. In recent years a posteriori error estimates have been extended to convection–diffusion problems as well. We cite in particular Verfürth [37], who derived estimates in the appropriate energy norm for the conforming Galerkin method and its stabilized version (the SUPG method of Franca *et al* [23]). His estimates are both reliable (yielding a global upper bound on the error between the exact and approximate solutions) and efficient (giving a local lower bound) and the lower and upper bounds differ by constants whose dependence on the local mesh discretization parameter vanishes as this approaches the ratio of the smallest eigenvalue of $\mathbf{S}$ and the local size of the velocity field (i.e. when the local Péclet number gets sufficiently small). Similar estimates are presented by El Alaoui and Ern in [20] for the nonconforming finite element method or (this time however without a careful analysis of the convection-dominated case) by Lazarov and Tomov [29] for the finite volume element method. Finally, a different approach, yielding an estimate in the L^1-norm, independent of the size of the diffusion tensor, is given by Ohlberger [31] in the frame of the vertex-centered finite volume method.

In comparison with primal methods, the literature on a posteriori error estimates in the mixed finite element method is much less extensive. Most of the results have been obtained for the Poisson equation (i.e. $r = \mathbf{w} = 0$ in (1.1a)–(1.1b)) in two space dimensions: Alonso [3] derived estimates for the error in the flux $\mathbf{u} := -\mathbf{S}\nabla p$ of the scalar variable p and either Raviart–Thomas [33] or Brezzi–Douglas–Marini [9] mixed finite elements. Braess and Verfürth [8] proved estimates for both $\mathbf{u}$ and p for Raviart–Thomas elements, based on mesh-dependent norms and a saturation assumption. Carstensen [11] derived optimal estimates for various mixed finite element schemes and for both $\mathbf{u}$ and p. Achchab *et al* [1] can imbed Raviart–Thomas elements in their hierarchical a posteriori error estimates, whereas Carstensen and Bartels [12] give an upper bound using averaging techniques. Kirby [28] proposed simple residual-based estimates for Raviart–Thomas elements, where however the flux estimator is not proved to yield a lower bound and is moreover obtained under a saturation assumption. Recently, Wheeler and Yotov [41] were able to obtain a posteriori error estimates for the mortar version of all families of mixed finite elements, also including the three-dimensional case; saturation assumption was however necessary for the velocity estimate. Finally, Hoppe and Wohlmuth [25] treat a diffusion–reaction problem in two space dimensions and use the relation of lowest-order Raviart–Thomas mixed finite elements to certain nonconforming finite elements derived by Arnold and Brezzi in [4] in order to control, under a saturation assumption, the L^2-norm error in the primal variable p. A comparison of various estimators is given by Wohlmuth and Hoppe in [42].

To the author's knowledge, no estimates for mixed finite element discretizations of convection–diffusion(–reaction) problems have been presented in the literature so far. We do this in Section 4 of this paper, after stating the assumptions on the data and formulating the continuous problem in Section 2 and after defining the lowest-order Raviart–Thomas mixed and upwind-mixed schemes in Section 3. The estimates are derived in the energy norm for a new, locally (on each element) postprocessed scalar variable $\tilde{p}_h$ such that its flux $-\mathbf{S}\nabla\tilde{p}_h$ is equal to $\mathbf{u}_h$ and such that its mean on each element is equal to p_h. By this construction, we actually have the $L^2(\Omega)$ control over both $\mathbf{u}_h - \mathbf{u}$ and $\tilde{p}_h - p$. Our estimates in particular do not include any undetermined multiplicative constants, so that the upper bound for the error between the exact and approximate solutions is fully (and easily) computable. They represent local lower bounds for the error as well with the efficiency constants of the form $c_1 + c_2 \min\{\mathrm{Pe}, \varrho\}$, where Pe (the local Péclet number) and ϱ are given below by (4.3) and where c_1, c_2 only depend on local variations in $\mathbf{S}$ (i.e. on local inhomogeneities and anisotropies), on local variations in $\nabla \cdot \mathbf{w}$ and r, and on the shape-regularity parameter of the mesh. Our estimates are thus in particular optimally efficient as the local Péclet number gets sufficiently small. They are finally robust from the cases where convection or reaction are not present to convection-dominated problems. The above-cited results seem to in addition have very interesting consequences for mixed finite element discretizations of pure diffusion problems with piecewise constant coefficients. The lowest-order mixed finite elements namely reveal as an exact three-point scheme in one space dimension and in two or three space dimensions, the approximation is shown to be exact with respect to some generalized continuous solution. All these issues are in detail discussed in Section 5. Next, Section 6 presents some discrete properties of the schemes and of the postprocessed scalar variable $\tilde{p}_h$. We namely show that $\tilde{p}_h$ is nonconforming in the sense that $\tilde{p}_h \notin H^1_0(\Omega)$, but we prove that the means of its traces are continuous across interior sides (edges if $d = 2$, faces if $d = 3$) and equal to zero on exterior sides of the mesh; they are in fact shown to equal to the Lagrange multipliers from the hybridized forms of the schemes. The actual proofs of our a posteriori error estimates and of their efficiency are then given in Section 7. They rely on the use of the abstract framework (cf. [37]) arising from the primal weak formulation of the continuous problem for $\tilde{p}_h$. The nonconformity of $\tilde{p}_h$ is then treated by the techniques developed in [2, 19]. No additional regularity of the weak solution is needed. Finally, we illustrate the robustness of the derived estimates in Section 8 on several numerical examples and conclude by some technical lemmas in Section 9.

We only focus in this paper on lowest-order methods since they are almost exclusively used in practice and hence we believe they deserve a special treatment; we mention that we on the other hand do cover the three-dimensional case. Moreover, we have shown in previous works (cf. [38, 40]) that lowest-order mixed finite element schemes are equivalent to particular finite volume schemes and that they can namely be implemented with only one unknown per element, which enables to significantly decrease their traditional increased computational cost. The extension to higher-order schemes is an ongoing work.

2 Notation, assumptions, and the continuous problem

We first introduce here the notation, then define admissible triangulations to which the space $W_0(\mathcal{T}_h)$ and data will be related, and finally give details on the continuous problem (1.1a)–(1.1b).

2.1 Notation

For a domain $S \subset \mathbb{R}^d$, we denote by $L^p(S)$ and $\mathbf{L}^p(S) = [L^p(S)]^d$ the Lebesgue spaces, by $(\cdot,\cdot)_S$ the $L^2(S)$ or $\mathbf{L}^2(S)$ inner product, and by $\|\cdot\|_S$ the associated norm; $|S|$ stands for the d-dimensional Lebesgue measure of S. Next, $H^1(S)$ and $H^1_0(S)$ are the Sobolev spaces of functions with square-integrable weak derivatives and $\mathbf{H}(\mathrm{div}, S)$ is the space of vector functions with square-integrable weak divergences, $\mathbf{H}(\mathrm{div}, S) = \{\mathbf{v} \in \mathbf{L}^2(S); \nabla\cdot\mathbf{v} \in L^2(S)\}$, and $\langle\cdot,\cdot\rangle_{\partial S}$ stands for $(d-1)$-dimensional inner product on ∂S or the duality pairing between $H^{-\frac{1}{2}}(\partial S)$ and $H^{\frac{1}{2}}(\partial S)$. In the subsequent text we conceptually denote by C_A, c_A constants only dependent on a quantity A.

2.2 Triangulation, Poincaré and Friedrichs inequalities, and the space $W_0(\mathcal{T}_h)$

We now define admissible partitions $\mathcal{T}_h$ of the domain Ω; the data of the problem (1.1a)–(1.1b) will be directly related to a basic partition $\widetilde{\mathcal{T}}_h$ (whose all $\mathcal{T}_h$ will be refinements) and we will in the sequel work with approximate solutions and the function space $W_0(\mathcal{T}_h)$ likewise related to $\mathcal{T}_h$.

We suppose that $\mathcal{T}_h$ for all $h>0$ consists of closed simplices such that $\overline{\Omega}=\bigcup_{K\in\mathcal{T}_h} K$ and such that if $K,L\in\mathcal{T}_h$, $K\neq L$, then $K\cap L$ is either an empty set or a common face, edge, or vertex of K and L. Let h_K denote the diameter of K and let $h:=\max_{K\in\mathcal{T}_h} h_K$. We make the following shape regularity assumption on the family of triangulations $\{\mathcal{T}_h\}_h$, denoting $\kappa_K:=|K|/h_K^d$:

Assumption (A) (Shape regularity of the meshes)

There exists a constant $\kappa_\mathcal{T}>0$ such that

$$\min_{K\in\mathcal{T}_h}\kappa_K\geq\kappa_\mathcal{T}\qquad\forall h>0\,.$$

Let ρ_K denote the diameter of the largest ball inscribed in K. Then using the inequalities $|K|\geq h_K^{d-1}\rho_K/(d-1)/d$, $|K|\leq(d+1)h_K^{d-1}\rho_K/(d-1)/d$ following from geometrical properties of a triangle (tetrahedron) K, Assumption (A) is equivalent to the more common requirement of the existence of a constant $\theta_\mathcal{T}>0$ such that

$$\max_{K\in\mathcal{T}_h}\frac{h_K}{\rho_K}\leq\theta_\mathcal{T}\qquad\forall h>0\,.$$

We next denote by $\mathcal{E}_h$ the set of all sides of $\mathcal{T}_h$, by $\mathcal{E}_h^{\rm int}$ the set of interior, by $\mathcal{E}_h^{\rm ext}$ the set of exterior, and by $\mathcal{E}_K$ the set of all the sides of an element $K\in\mathcal{T}_h$. Let h_σ stand for the diameter of $\sigma\in\mathcal{E}_h$.

Let $K\in\mathcal{T}_h$ and let $\varphi\in H^1(K)$. Two inequalities will play an essential role in the derivation of our a posteriori error estimates. First, the Poincaré inequality states that

$$\|\varphi-\varphi_K\|_K^2\leq C_{\rm P}h_K^2\|\nabla\varphi\|_K^2\,,\tag{2.1}$$

where φ_K is the mean of φ over K given by $\varphi_K:=(\varphi,1)_K/|K|$ and where the constant $C_{\rm P}$ can for a simplex (using its convexity) be evaluated as d/π, cf. [32, 6]. Next, the following generalized Friedrichs inequalities have been proved in [39, Lemma 4.1]:

$$(\varphi_K-\varphi_\sigma)^2\leq C_{{\rm F},d}\frac{h_K^2}{|K|}\|\nabla\varphi\|_K^2\,,\quad\|\varphi-\varphi_\sigma\|_K^2\leq C_{{\rm F},d}h_K^2\|\nabla\varphi\|_K^2\,,\tag{2.2}$$

where φ_σ is the mean of φ over $\sigma\in\mathcal{E}_K$ given by $\varphi_\sigma:=\langle\varphi,1\rangle_\sigma/|\sigma|$ and where $C_{{\rm F},d}=3d$. Similarly,

$$\|\varphi-\varphi_\sigma\|_\sigma^2\leq\widetilde{C}_{{\rm F},d}\frac{h_K}{h_\sigma}h_K\|\nabla\varphi\|_K^2\tag{2.3}$$

has been shown in [30, Lemma 3.5] with $\widetilde{C}_{{\rm F},d}\approx1.55416$ for $d=2$ and $\widetilde{C}_{{\rm F},d}\approx11.53557$ for $d=3$.

We finally define the space $W_0(\mathcal{T}_h)$ of functions locally in $H^1(K)$ on each $K\in\mathcal{T}_h$ such that the mean values of their traces on interior sides coincide and that the mean values of their traces on exterior sides are equal to zero,

$$\begin{aligned}W_0(\mathcal{T}_h):=\big\{\varphi\in L^2(\Omega)\,;\varphi|_K&\in H^1(K)&&\forall K\in\mathcal{T}_h\,,\\ \langle\varphi|_K-\varphi|_L,1\rangle_{\sigma_{K,L}}&=0&&\forall\sigma_{K,L}\in\mathcal{E}_h^{\rm int}\,,\\ \langle\varphi,1\rangle_\sigma&=0&&\forall\sigma\in\mathcal{E}_h^{\rm ext}\big\}\,,\end{aligned}\tag{2.4}$$

and recall the discrete Friedrichs inequality

$$\|\varphi\|_\Omega^2\leq C_{\rm DF}\sum_{K\in\mathcal{T}_h}\|\nabla\varphi\|_K^2\qquad\forall\varphi\in W_0(\mathcal{T}_h)\,,\,\forall h>0\,,\tag{2.5}$$

where the constant $C_{\rm DF}$ only depends on $\kappa_\mathcal{T}$ and $\inf_{\mathbf{b}\in\mathbb{R}^d}\{{\rm thick}_\mathbf{b}(\Omega)\}$, cf. [39, Theorem 5.4].

2.3 Data

We suppose that there exists a basic triangulation $\widetilde{\mathcal{T}}_h$ of Ω such that the data of the problem (1.1a)–(1.1b) are related to $\widetilde{\mathcal{T}}_h$ in the following way:

Assumption (B) (Data)

(B1) $\mathbf{S}_K := \mathbf{S}|_K$ *is a constant, symmetric, bounded, and uniformly positive definite tensor for all* $K \in \widetilde{\mathcal{T}}_h$*, i.e.*

$$\mathbf{S}_K\mathbf{v}\cdot\mathbf{v} \geq c_{\mathbf{S},K}\,\mathbf{v}\cdot\mathbf{v}\,,\ c_{\mathbf{S},K} > 0 \qquad \forall \mathbf{v} \in \mathbb{R}^d\,,\ \forall K \in \widetilde{\mathcal{T}}_h\,,$$
$$\|\mathbf{S}_K\| = C_{\mathbf{S},K}\,,\ C_{\mathbf{S},K} > 0 \qquad \forall K \in \widetilde{\mathcal{T}}_h\,;$$

(B2) $\mathbf{w} \in \mathbf{RT}^0(\widetilde{\mathcal{T}}_h)$ *satisfies* $\left|\mathbf{w}|_K\right| \leq C_{\mathbf{w},K}$*,* $C_{\mathbf{w},K} \geq 0$*, for all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B3) $r_K := r|_K$ *is a constant for all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B4) $\left(\frac{1}{2}\nabla\cdot\mathbf{w}+r\right)|_K = c_{\mathbf{w},r,K}$ *and* $\left|(\nabla\cdot\mathbf{w}+r)|_K\right| = C_{\mathbf{w},r,K}$*,* $c_{\mathbf{w},r,K} \geq 0$*,* $C_{\mathbf{w},r,K} \geq 0$*, for all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B5) $f|_K$ *is a polynomial of degree at most* k *on each* $K \in \widetilde{\mathcal{T}}_h$*;*

(B6) if $c_{\mathbf{w},r,K} = 0$*, then* $C_{\mathbf{w},r,K} = r_K = 0$*.*

The assumptions that $\mathbf{S}$ and r are piecewise constant on $\widetilde{\mathcal{T}}_h$, that $\mathbf{w} \in \mathbf{RT}^0(\widetilde{\mathcal{T}}_h)$ (cf. Section 3.1 below for the definition of this space), and that f is a piecewise polynomial are made for the sake of simplicity and are usually satisfied in practice. Likewise, the homogeneous Dirichlet boundary condition (1.1b) is only considered for the sake of clarity of the exposition. Finally, note that Assumption (B6) allows $c_{\mathbf{w},r,K} = 0$ (but $\mathbf{w}|_K \neq 0$), in contrast to the assumptions made in [20, 37].

2.4 Continuous problem

Let $\mathcal{T}_h$ be, as throughout the whole paper, a refinement of $\widetilde{\mathcal{T}}_h$. We define a bilinear form $\mathcal{B}$ by

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h}\left\{(\mathbf{S}\nabla p,\nabla\varphi)_K + (\nabla\cdot(p\mathbf{w}),\varphi)_K + (rp,\varphi)_K\right\} \qquad p,\varphi \in W_0(\mathcal{T}_h) \tag{2.6}$$

and the corresponding energy norm by

$$|\!|\!|\,\varphi|\!|\!|_\Omega^2 := \sum_{K\in\mathcal{T}_h} |\!|\!|\varphi|\!|\!|_K^2\,,\quad |\!|\!|\varphi|\!|\!|_K^2 := c_{\mathbf{S},K}\|\nabla\varphi\|_K^2 + c_{\mathbf{w},r,K}\|\varphi\|_K^2 \qquad \varphi \in W_0(\mathcal{T}_h)\,. \tag{2.7}$$

In this way $\mathcal{B}(\cdot,\cdot)$ and $|\!|\!|\cdot|\!|\!|_\Omega$ are well-defined for $p,\varphi \in H^1(\Omega)$ as well as for p,φ that are only piecewise regular. Remark also that $|\!|\!|\cdot|\!|\!|_\Omega$ is a norm on $W_0(\mathcal{T}_h)$ even if there exists $K \in \mathcal{T}_h$ such that $c_{\mathbf{w},r,K} = 0$. The weak formulation of the problem (1.1a)–(1.1b) is then to find $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi)_\Omega \qquad \forall \varphi \in H_0^1(\Omega). \tag{2.8}$$

Assumptions (B1)–(B5), the Green theorem, and the Cauchy–Schwarz inequality imply that

$$\mathcal{B}(\varphi,\varphi) \geq |\!|\!|\varphi\,|\!|\!|_\Omega^2 \qquad \forall \varphi \in H_0^1(\Omega) \tag{2.9}$$

and

$$\begin{aligned}\mathcal{B}(p,\varphi) \;\leq\; & \left(\max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}\right\} + \max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\}\right)|\!|\!|\,p\,|\!|\!|_\Omega\,|\!|\!|\varphi\,|\!|\!|_\Omega \\ & + \max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}\right\}|\!|\!|\,p\,|\!|\!|_\Omega\,\|\varphi\|_\Omega \qquad \forall p,\varphi \in W_0(\mathcal{T}_h)\,. \end{aligned} \tag{2.10}$$

Problem (2.8) under Assumption (B) in particular admits a unique solution.

Remark 2.1 (Notation)**.** *In estimate* (2.10)*, if* $c_{\mathbf{w},r,K} = 0$*, the term* $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K}$ *should be evaluated as zero; since Assumption (B6) for this case gives* $C_{\mathbf{w},r,K} = 0$*, the term with* $C_{\mathbf{w},r,K}$ *in fact does not even enter the estimate. To simplify notation, we will systematically use the convention* $0/0 = 0$ *throughout the text.*

3 Mixed finite element schemes

We define in this section the centered and upwind-weighted mixed finite element schemes.

3.1 Function spaces

In the sequel we will use the spaces $\mathbf{RT}^0_{-1}(\mathcal{T}_h)$ and $\mathbf{RT}^0(\mathcal{T}_h)$ for the approximation of the vector variable $\mathbf{u}$ and $\Phi(\mathcal{T}_h)$ for the approximation of the scalar variable p. The space $\mathbf{RT}^0_{-1}(\mathcal{T}_h)$ is the space of elementwise linear vector functions $\mathbf{u}_h$ such that

$$\mathbf{u}_h|_K = \begin{pmatrix} a_K + d_K x \\ b_K + d_K y \end{pmatrix} \qquad \text{for all } K \in \mathcal{T}_h \text{ if } d = 2\,, \tag{3.1a}$$

$$\mathbf{u}_h|_K = \begin{pmatrix} a_K + d_K x \\ b_K + d_K y \\ c_K + d_K z \end{pmatrix} \qquad \text{for all } K \in \mathcal{T}_h \text{ if } d = 3\,. \tag{3.1b}$$

The Raviart–Thomas–Nédélec space $\mathbf{RT}^0(\mathcal{T}_h)$ then imposes the continuity of the normal trace across all $\sigma \in \mathcal{E}_h$ on the functions from $\mathbf{RT}^0_{-1}(\mathcal{T}_h)$ and is given by $\mathbf{RT}^0(\mathcal{T}_h) := \mathbf{RT}^0_{-1}(\mathcal{T}_h) \cap \mathbf{H}(\mathrm{div}, \Omega)$. There is one basis function $\mathbf{v}_\sigma$ associated with each $\sigma \in \mathcal{E}_h$. For an interior side $\sigma_{K,L}$ shared by simplices K and L, $\mathbf{v}_{\sigma_{K,L}}(\mathbf{x}) = \frac{1}{d|K|}(\mathbf{x} - V_K)$, $\mathbf{x} \in K$, $\mathbf{v}_{\sigma_{K,L}}(\mathbf{x}) = \frac{1}{d|L|}(V_L - \mathbf{x})$, $\mathbf{x} \in L$, $\mathbf{v}_{\sigma_{K,L}}(\mathbf{x}) = 0$ otherwise, where V_K is the vertex of K opposite to σ and V_L the vertex of L opposite to σ. We suppose that the orientation of $\mathbf{v}_{\sigma_{K,L}}$, i.e. the order of K and L, is fixed. For a boundary side σ, the support of $\mathbf{v}_\sigma$ only consists of $K \in \mathcal{T}_h$ such that $\sigma \in \mathcal{E}_K$. The space $\Phi(\mathcal{T}_h)$ finally consists of elementwise constant scalar functions; we denote $p_h|_K = p_K$ for $p_h \in \Phi(\mathcal{T}_h)$. Recall that $\nabla \cdot \mathbf{u}_h \in \Phi(\mathcal{T}_h)$ for each $\mathbf{u}_h \in \mathbf{RT}^0_{-1}(\mathcal{T}_h)$.

3.2 Centered scheme

The centered mixed finite element scheme reads (cf. [16, 17]): find $\mathbf{u}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,, \tag{3.2a}$$

$$(\nabla \cdot \mathbf{u}_h, \phi_h)_\Omega - (\mathbf{S}^{-1}\mathbf{u}_h\mathbf{w}, \phi_h)_\Omega + ((r + \nabla \cdot \mathbf{w})p_h, \phi_h)_\Omega = (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi(\mathcal{T}_h)\,. \tag{3.2b}$$

3.3 Upwind-weighted scheme

The upwind-weighted mixed finite element scheme reads: find $\mathbf{u}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,, \tag{3.3a}$$

$$(\nabla \cdot \mathbf{u}_h, \phi_h)_\Omega + \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \hat{p}_\sigma w_{K,\sigma} \phi_K + (r p_h, \phi_h)_\Omega = (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi_h\,, \tag{3.3b}$$

where $w_{K,\sigma} := \langle \mathbf{w} \cdot \mathbf{n}, 1\rangle_\sigma$, $\sigma \in \mathcal{E}_K$, with $\mathbf{n}$ being the unit normal vector of the side σ, outward to K, and $\hat{p}_\sigma$ is the weighted upwind value defined by

$$\hat{p}_\sigma := \begin{cases} (1-\nu_\sigma)p_K + \nu_\sigma p_L & \text{if } w_{K,\sigma} \geq 0 \\ (1-\nu_\sigma)p_L + \nu_\sigma p_K & \text{if } w_{K,\sigma} < 0 \end{cases} \tag{3.4}$$

if σ is an interior side between elements K and L and

$$\hat{p}_\sigma := \begin{cases} (1-\nu_\sigma)p_K & \text{if } w_{K,\sigma} \geq 0 \\ \nu_\sigma p_K & \text{if } w_{K,\sigma} < 0 \end{cases} \tag{3.5}$$

if σ is a boundary side. Here, $\nu_\sigma \in [0, 1/2]$ is the coefficient of the amount of upstream weighting. The full-upwind scheme (with $\nu_\sigma = 0$ for all $\sigma \in \mathcal{E}_h$) has been studied in [14, 15]. The introduction of the parameter ν_σ has been motivated by its successful use in finite volume or combined finite volume–finite element schemes, cf. [22]. In the present case a reasonable choice for ν_σ to still guarantee the stability of the scheme while reducing the excessive numerical diffusion added by the full upstream weighting may be (for $w_{K,\sigma} \neq 0$)

$$\nu_\sigma := \min\left\{ c_{\mathbf{S},\sigma}\frac{|\sigma|}{h_\sigma |w_{K,\sigma}|}, \frac{1}{2}\right\}, \tag{3.6}$$

where $c_{\mathbf{S},\sigma}$ is the harmonic average of $c_{\mathbf{S},K}$ and $c_{\mathbf{S},L}$ if $\sigma = \partial K \cap \partial L$ and $c_{\mathbf{S},K}$ otherwise.

4 A posteriori error estimates

We summarize in this section our a posteriori estimates on the error between the weak solution p and a postprocessed variable $\tilde{p}_h$ which we shall define first.

4.1 A postprocessed scalar variable $\tilde{p}_h$

We define in the section a new postprocessed scalar variable $\tilde{p}_h$, which will serve as the basis for our a posteriori error estimates.

In standard mixed finite element theory (see e.g. Brezzi and Fortin [10] or Roberts and Thomas [35]) the two variables p_h and $\mathbf{u}_h$ are considered as independent. The basis for our a posteriori error estimates is however a construction of a postprocessed scalar variable $\tilde{p}_h$ which links p_h and $\mathbf{u}_h$ of (3.2a)–(3.2b), (3.3a)–(3.3b) respectively, on each simplex in the following way:

$$-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K \quad \forall K \in \mathcal{T}_h\,, \tag{4.1a}$$

$$\frac{(\tilde{p}_h, 1)_K}{|K|} = p_K \quad \forall K \in \mathcal{T}_h\,. \tag{4.1b}$$

Note that in particular if $\mathbf{S}_K = Id$, one immediately has the existence of such $\tilde{p}_h$ and $\tilde{p}_h|_K = -d_K/2\,(x^2+y^2) - a_K x - b_K y - e_K$ if $d = 2$ and $\tilde{p}_h|_K = -d_K/2\,(x^2+y^2+z^2) - a_K x - b_K y - c_K z - e_K$ if $d = 3$. Here a_K–d_K are the coefficients from (3.1a)–(3.1b) and e_K is given so that (4.1b) was satisfied. If $\mathbf{S}_K \neq Id$, then $\tilde{p}_h$ verifying (4.1a)–(4.1b) still exists due to the symmetry of $\mathbf{S}$ and is this time a full second-order polynomial on each $K \in \mathcal{T}_h$. The new variable $\tilde{p}_h$ is nonconforming in the sense that it is in general not included in $H^1_0(\Omega)$, but, by Lemma 6.1 below, $\tilde{p}_h \in W_0(\mathcal{T}_h)$, i.e. its means on interior sides are continuous and its means on exterior sides are equal to zero. In fact, by Lemma 6.4 below, these means coincide with the Lagrange multipliers of hybridized schemes (see (6.2a)–(6.2c) and (6.3a)–(6.3c) below). Moreover, the centered scheme can equivalently be rewritten with the help of $\tilde{p}_h$, see Lemma 6.2 below, which corresponds to the employment of the Lagrange multipliers in the discretization of the convection term. Note that the proposed postprocessing is local on each element and hence its cost is negligible.

4.2 A modified Oswald interpolation operator

We will need below an interpolation operator associating to $\tilde{p}_h$ a continuous (conforming, included in $H_0^1(\Omega)$) function that preserves the means of $\tilde{p}_h$ over the sides. We now modify for this purpose the Oswald interpolation operator.

Let $\mathbb{P}_l(\mathcal{T}_h)$ denote the space of polynomials of degree at most l on each simplex, not necessary continuous. The Oswald interpolation operator $\mathcal{I}_{\rm O} : \mathbb{P}_l(\mathcal{T}_h) \to \mathbb{P}_l(\mathcal{T}_h) \cap H_0^1(\Omega)$ has been considered in [24, 27], as well as in [2, 19]. Given a function $\varphi_h \in \mathbb{P}_l(\mathcal{T}_h)$, $\mathcal{I}_{\rm O}(\varphi_h)$ is given at the Lagrangian nodes (degrees of freedom, cf. [13, Section 2.2]) of $\mathbb{P}_l(\mathcal{T}_h) \cap H_0^1(\Omega)$ by the average of the values of φ_h at this node.

The modified Oswald interpolation operator $\mathcal{I}_{\rm MO} : \mathbb{P}_2(\mathcal{T}_h) \cap W_0(\mathcal{T}_h) \to \mathbb{P}_d(\mathcal{T}_h) \cap H_0^1(\Omega)$ is defined as follows: at all Lagrangian nodes (degrees of freedom) of $\mathbb{P}_d(\mathcal{T}_h) \cap H_0^1(\Omega)$, except of those lying at the barycentres of the sides, the value of $\mathcal{I}_{\rm MO}(\varphi_h)$ is given by the average of the values of φ_h at this node (i.e. in the same way as in the standard Oswald interpolation operator). The values at the barycentres of the sides are then established so that the means of $\mathcal{I}_{\rm MO}(\varphi_h)$ over the sides were given by the means of φ_h (the space $\mathbb{P}_2(\mathcal{T}_h) \cap H_0^1(\Omega)$ in three space dimensions does not have Lagrangian nodes at sides barycentres; this is the reason to use $\mathbb{P}_3(\mathcal{T}_h) \cap H_0^1(\Omega)$ in this case). It is easily verified that as in the case of the Oswald interpolation operator, $\mathcal{I}_{\rm MO}(\varphi_h)$ is uniquely defined piecewise polynomial continuous function. Let $[\varphi_h]$ be the jump of a function φ_h across a side σ: if $\sigma = \partial K \cap \partial L$, then $[\varphi_h]$ is the difference of the value of φ_h in K and L, and if $\sigma \in \mathcal{E}_h^{\rm ext}$, then $[\varphi_h] = \varphi_h$. Then the following lemma is an easy modification of [27, Theorem 2.2] ($\sigma \cap K \neq \emptyset$ when σ contains a vertex of K):

Lemma 4.1 (Modified Oswald interpolation operator)**.** *Let $\varphi_h \in \mathbb{P}_2(\mathcal{T}_h) \cap W_0(\mathcal{T}_h)$ and let $\mathcal{I}_{\rm MO}(\varphi_h)$ be constructed as described above. Then*

$$\|\nabla(\varphi_h - \mathcal{I}_{\rm MO}(\varphi_h))\|_K^2 \leq C_1 \sum_{\sigma; \sigma\cap K \neq \emptyset} h_\sigma^{-1} \|[\varphi_h]\|_\sigma^2 \,,$$

where the constant C_1 only depends on the space dimension d and on the shape regularity parameter $\kappa_{\mathcal{T}}$.

4.3 A posteriori error estimates

We now finally state the a posteriori error estimates. Let us first put

$$m_K^2 := \min\left\{ C_{\rm P} \frac{h_K^2}{c_{\mathbf{S},K}}, \frac{2}{c_{\mathbf{w},r,K}} \right\}$$

for all $K \in \mathcal{T}_h$. We define the *residual estimator* η_K associated with an element K by

$$\eta_K := m_K \| f + \nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h \|_K \,. \tag{4.2}$$

Let the local Péclet number Pe_K and ϱ_K be given by

$$\mathrm{Pe}_K := h_K \frac{C_{\mathbf{w},K}}{c_{\mathbf{S},K}} \,, \quad \varrho_K := \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{w},r,K}}\sqrt{c_{\mathbf{S},K}}} \,. \tag{4.3}$$

We next denote

$$\alpha_{*,K} := 2c_{\mathbf{S},K} + 4c_{\mathbf{S},K}\left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \varrho_K\right)^2 , \quad \beta_{*,K} := 2c_{\mathbf{w},r,K} + 4\frac{C_{\mathbf{w},r,K}^2}{c_{\mathbf{w},r,K}} \,,$$

$$\alpha_{\#,K} := 2c_{\mathbf{S},K} + 4c_{\mathbf{S},K}\left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \mathrm{Pe}_K C_{d,K}\right)^2 , \quad \beta_{\#,K} := 2c_{\mathbf{w},r,K} + 4\frac{r_K^2}{c_{\mathbf{w},r,K}} \,,$$

where

$$C_{d,K} := \sqrt{C_{\mathrm{F},d}} + \sum_{\sigma \in \mathcal{E}_K} \widetilde{C}_{\mathrm{F},d} \frac{h_K}{h_\sigma}\,, \tag{4.4}$$

and define, for $\varphi \in H^1(K)$,

$$|||\varphi|||_{*,K}^2 := \alpha_{*,K} \|\nabla \varphi\|_K^2 + \beta_{*,K} \|\varphi\|_K^2\,, \quad |||\varphi|||_{\#,K}^2 := \alpha_{\#,K} \|\nabla \varphi\|_K^2 + \beta_{\#,K} \|\varphi\|_K^2\,.$$

The *nonconformity estimator* ζ_K associated with an element K is then given by

$$\zeta_K := \min\big\{|||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{*,K}, |||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{\#,K}\big\}\,. \tag{4.5}$$

Finally, let

$$m_\sigma^2 := 2(d+1) \min\left\{ \max_{K;\,\sigma \in \mathcal{E}_K} \left\{ \frac{C_{\mathrm{F},d} h_K}{(d-1)\kappa_K c_{\mathbf{S},K}} \right\}, \max_{K;\,\sigma \in \mathcal{E}_K} \left\{ \frac{1}{(d-1)\kappa_K h_K c_{\mathbf{w},r,K}} \right\} \right\} \tag{4.6}$$

for all $\sigma \in \mathcal{E}_h$. We put $\tilde{p}_\sigma := \langle \tilde{p}_h, 1 \rangle_\sigma / |\sigma|$, the mean of the postprocessed scalar variable $\tilde{p}_h$ over a side $\sigma \in \mathcal{E}_h$, recall that $\hat{p}_\sigma$ is the upwind value given by (3.4) or (3.5), and define the *upwinding estimator* η_σ associated with a side σ by

$$\eta_\sigma := m_\sigma \|(\hat{p}_\sigma - \tilde{p}_\sigma)\mathbf{w} \cdot \mathbf{n}\|_\sigma\,. \tag{4.7}$$

We have the following a posteriori error estimates:

Theorem 4.2 (A posteriori error estimate for the centered mixed finite element scheme)**.** *Let p be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.8) *and let $\tilde{p}_h$ be the postprocessed solution of the mixed finite element scheme* (3.2a)–(3.2b) *given by* (4.1a)–(4.1b)*. Then*

$$|||p - \tilde{p}_h|||_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \zeta_K^2 \right\}^{\frac{1}{2}}.$$

Theorem 4.3 (A posteriori error estimate for the upwind-weighted mixed finite element scheme)**.** *Let p be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.8) *and let $\tilde{p}_h$ be the postprocessed solution of the mixed finite element scheme* (3.3a)–(3.3b) *given by* (4.1a)–(4.1b)*. Then*

$$|||p - \tilde{p}_h|||_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \zeta_K^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{\sigma \in \mathcal{E}_h} \eta_\sigma^2 \right\}^{\frac{1}{2}}.$$

4.4 Efficiency of the estimates

The theorem below discusses the efficiency of the a posteriori error estimators of Section 4.3.

Theorem 4.4 (Efficiency of the a posteriori error estimators)**.** *For the residual estimator η_K, there holds*

$$\eta_K \leq C_2\, |||\, p - \tilde{p}_h\, |||_K \left\{ \left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \right) + \min\big\{\mathrm{Pe}_K, \varrho_K\big\} \right\},$$

where the constant C_2 is independent of h_K, $\mathbf{S}$, $\mathbf{w}$, and r (see Lemma 7.5 below). Next, for the nonconformity estimator ζ_K, we have

$$\begin{aligned}\zeta_K^2 \quad \le \quad & C_3 \min\left\{\frac{\alpha_{*,K}}{\min_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}} + \min\left\{\frac{\beta_{*,K}}{\min_{L;L\cap K\neq\emptyset} c_{\mathbf{w},r,L}}, \frac{\beta_{*,K}h_K^2}{\min_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}}\right\},\right.\\ & \left.\frac{\alpha_{\#,K}}{\min_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}} + \min\left\{\frac{\beta_{\#,K}}{\min_{L;L\cap K\neq\emptyset} c_{\mathbf{w},r,L}}, \frac{\beta_{\#,K}h_K^2}{\min_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}}\right\}\right\} \sum_{L;L\cap K\neq\emptyset} |||p-\tilde{p}_h|||_L^2\\ & + C_3 \max\{\beta_{*,K},\beta_{\#,K}\} \inf_{s_h\in\mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)} \sum_{L;L\cap K\neq\emptyset} \|p-s_h\|_L^2\,,\end{aligned}$$

where the constant C_3 only depends on the space dimension d and on the shape regularity parameter $\kappa_{\mathcal{T}}$ (see Lemma 7.6 below). Finally, we only have for the upwinding estimators η_σ

$$\sum_{\sigma\in\mathcal{E}_h} \eta_\sigma^2 \le C_4 \max_{\sigma\in\mathcal{E}_h} \varrho_\sigma \max_{K\in\mathcal{T}_h} \tilde{\varrho}_K \min\left\{\frac{1}{2}\sum_{K\in\mathcal{T}_h} \frac{\|f\|_K^2}{c_{\mathbf{w},r,K}}, \|f\|_\Omega^2 \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}}\right\},$$

where

$$c_{\mathbf{S},\Omega} := \min_{K\in\mathcal{T}_h} c_{\mathbf{S},K}\,, \tag{4.8}$$

$$\varrho_\sigma := \left(\frac{\max_{K;\sigma\in\mathcal{E}_K} c_{\mathbf{S},K}}{\min_{K;\sigma\in\mathcal{E}_K} c_{\mathbf{S},K}}\right)^2, \quad \tilde{\varrho}_K := \min\left\{(\mathrm{Pe}_K)^2, (\varrho_K)^2 \frac{\max_{L;L\cap K\in\mathcal{E}_h} c_{\mathbf{w},r,L}}{\min_{L;L\cap K\in\mathcal{E}_h} c_{\mathbf{w},r,L}}\right\},$$

C_{DF} is the constant from the discrete Friedrichs inequality (2.5), and where the constant C_4 only depends on d and $\kappa_{\mathcal{T}}$ (see Lemma 7.7 below).

5 Various remarks

We give several remarks in this section.

5.1 Nature of the estimates

The basis for the a posteriori error estimates derived in this paper is the construction of the postprocessed scalar variable $\tilde{p}_h$ and the consequent application of the abstract framework arising from the primal weak formulation (2.8) of the continuous problem. The variable $\tilde{p}_h$, an elementwise quadratic polynomial, has the crucial advantage over Galerkin finite element approximations that the normal traces of its flux $-\mathbf{S}\nabla\tilde{p}_h$ (which is by (4.1a) nothing else than the mixed finite element vector variable $\mathbf{u}_h$) are continuous across interior sides. Hence the edge error estimators penalizing the mass balance common in Galerkin finite element methods (cf. [37]) do not appear here at all. This advantage is however compensated by the fact that $\tilde{p}_h$ is nonconforming in the sense that it is not included in $H_0^1(\Omega)$, so that the estimators known from nonconforming and discontinuous Galerkin finite elements (cf. [24, 27]) come in place. However, the means of $\tilde{p}_h$ do are continuous on interior sides and equal to zero on exterior sides, which in particular enabled us to derive lower and upper bounds for the discretization error in the convection-dominated case whose ratio is well bounded provided that the local Péclet number is sufficiently small. Next, whereas in the lowest-order Galerkin finite element method, $\nabla\cdot\mathbf{S}_K\nabla p_h|_K$ is always equal to zero on all $K\in\mathcal{T}_h$, the element residuals (4.2) give a good sense even for the lowest-order mixed finite element method. Finally, the upwind residuals for the upwind-mixed scheme are similar to the ones arising in e.g. the finite volume element method, cf. [29]. We also notice that using (2.7), (4.1a), and (2.5),

$$\begin{aligned} |\!|\!|\, p - \tilde{p}_h |\!|\!|_\Omega^2 &= \sum_{K \in \mathcal{T}_h} \left\{ c_{\mathbf{S},K} \|\mathbf{S}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_K^2 + c_{\mathbf{w},r,K} \|p - \tilde{p}_h\|_K^2 \right\} \qquad (5.1) \\ &\geq \sum_{K \in \mathcal{T}_h} \left\{ \frac{c_{\mathbf{S},K}}{2} \|\mathbf{S}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_K^2 + c_{\mathbf{w},r,K} \|p - \tilde{p}_h\|_K^2 \right\} + \frac{c_{\mathbf{S},\Omega}}{2C_{\mathrm{DF}}} \|p - \tilde{p}_h\|_\Omega^2 \,, \end{aligned}$$

so that we have the usual mixed finite element control over both $\mathbf{u} - \mathbf{u}_h$ and $p - \tilde{p}_h$. This holds true even if $c_{\mathbf{w},r,K} = 0$ for some $K \in \mathcal{T}_h$.

5.2 The estimates and their efficiency with respect to S and w

We discuss in this remark our a posteriori error estimates and their efficiency with respect to inhomogeneities, anisotropies, and the convection dominance.

The residual estimator η_K (4.2) does not possess any direct dependence on inhomogeneities and anisotropies. When $c_{\mathbf{w},r,K} > 0$, the minimum in its definition prevents it from growing to extreme values on coarse elements with a small value $c_{\mathbf{S},K}$. Its efficiency only depends on anisotropy in its element expressed by the ratio $C_{\mathbf{S},K}/c_{\mathbf{S},K}$ and there is no dependency on inhomogeneities. Next, under the given assumptions, $\nabla \cdot \mathbf{w}$ is constant on each $K \in \mathcal{T}_h$ and hence $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K} \leq 2$ whenever r_K is nonnegative, so this term may usually not be very important. Finally, the minimum of the local Péclet number Pe_K and ϱ_K may well be ϱ_K if $c_{\mathbf{w},r,K} \neq 0$ and if h_K is large. However, refining so as $\mathrm{Pe}_K \approx 2$ brings the overall efficiency of η_K to optimal values.

The nonconformity estimator ζ_K (4.5) gives the efficiency up to higher-order terms if $c_{\mathbf{w},r,K} \neq 0$. It depends on anisotropy in the given K by the ratio $C_{\mathbf{S},K}/c_{\mathbf{S},K}$, but there is no direct dependence on inhomogeneities. The minimum in its definition prevents it from exploding when $c_{\mathbf{w},r,K} = 0$ but $C_{\mathbf{w},K} \neq 0$. Its efficiency is shown to be a function of a local (meaning all elements sharing a vertex with the given one) maximal ratio of inhomogeneities. El Alaoui and Ern were able to show the dependency on the ratios only across adjacent sides, however for the price of a hypothesis of "monotonicity around vertices" on the distribution of the inhomogeneities (see [19, Hypotheses 3.1 and 3.7]). An interesting comparison is with the results of Bernardi and Verfürth for the Galerkin finite element method as well. It seems that for this method, the dependency on the inhomogeneity ratios across adjacent sides is rather already in the estimator itself than in its efficiency, see [7, Theorem 2.9], derived however again under a "monotonicity" hypothesis. Finally, the efficiency with respect to anisotropy stays controlled in each element by the ratio $C_{\mathbf{S},K}/c_{\mathbf{S},K}$ and it gets into optimal values with respect to convection dominance as Pe_K gets sufficiently small. For additional comments on possible nonconformity estimates, see Section 5.3 below.

The fact that the upwinding estimator η_σ (4.7) cannot in general give a lower bound for the error is quite obvious: it is not hard to imagine a situation where $p = \tilde{p}_h$, whereas $(\hat{p}_\sigma - \tilde{p}_\sigma)$, the difference of the mean value of $\tilde{p}_h$ on a side σ and of the combination of the mean values of $\tilde{p}_h$ on the elements sharing σ, is generally nonzero. We however at least show that there is an upper bound for the contributions of this estimator, which moreover decreases with the local Péclet numbers as $O(h)$. Whereas it is a quadratic function of the largest inhomogeneity over a side in the mesh, it does not depend on anisotropy. It should however be noted that this estimator does not change the limit optimality of the schemes and estimates—as the local Péclet number gets sufficiently small, we can switch from the upwind-weighted to the centered scheme and hence the upwinding estimator disappears. A proposition for a smooth transition from the one scheme to the other is given below in Section 5.5.

5.3 The nonconformity estimate: an alternative form

Using estimate (2.10) instead of estimate (7.2) in Lemma 7.3 below, the nonconformity term can alternatively be bounded by

$$C_{\mathrm{NC}} \inf_{s\in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega\,,$$

where

$$C_{\mathrm{NC}} := \left(1 + \max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}\right\} + \max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\} + \min\left\{\max_{K\in\mathcal{T}_h}\varrho_K\,,\ \frac{\sqrt{C_F}}{\sqrt{c_{\mathbf{S},\Omega}}}\max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}\right\}\right\}\right)\,,$$

using that $s\in H_0^1(\Omega)$ in Lemma 7.1 was chosen arbitrarily. Here C_{F} is the constant in the Friedrichs inequality $\|\varphi\|_\Omega^2 \le C_{\mathrm{F}}\|\nabla\varphi\|_\Omega^2$, $\varphi\in H_0^1(\Omega)$. This immediately leads to the efficiency of this estimate with the constant C_{NC}, since

$$C_{\mathrm{NC}} \inf_{s\in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega \le C_{\mathrm{NC}}\, |||\, \tilde{p}_h - p\, |||_\Omega\ .$$

This estimate is of course hardly computable, so we can change it into (cf. the averaging a posteriori error estimates e.g. in [12])

$$C_{\mathrm{NC}} \inf_{s_h\in V(\mathcal{T}_h)} |||\tilde{p}_h - s_h|||_\Omega\,,$$

where $V(\mathcal{T}_h)$ is some finite-dimensional subspace of $H_0^1(\Omega)$, which then may be efficient up to higher-order terms,

$$C_{\mathrm{NC}} \inf_{s_h\in V(\mathcal{T}_h)} |||\tilde{p}_h - s_h|||_\Omega \le C_{\mathrm{NC}}\, |||\, \tilde{p}_h - p\, |||_\Omega + C_{\mathrm{NC}} \inf_{s_h\in V(\mathcal{T}_h)} |||p - s_h\, |||_\Omega\ . \tag{5.2}$$

To evaluate this estimate however still requires a computational effort comparable to that of solving the original discretized problem, it explodes in the convection-dominated case, and finally, we only have a global (not local) lower bound for the error between the exact and approximate solutions. Two aspects however make this estimate interesting. First, provided that $\max_{K\in\mathcal{T}_h}\varrho_K$ represents the minimum in the definition of C_{NC}, the estimate and its efficiency only depend on anisotropy and local ratios of convection, reaction, and diffusion in each single element, there is no dependence on the ratios between different elements. And secondly, shall C_{NC} be small, which will in particular be the case for pure diffusion problems with small anisotropy (but however large inhomogeneity), it tends to be almost asymptotically exact.

5.4 The estimate for pure diffusion problems

Let us now consider the pure diffusion problem, i.e. $r = \mathbf{w} = 0$ in (1.1a)–(1.1b). Using that in this case $-\nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h|_K = \nabla\cdot\mathbf{u}_h|_K = f_K$ for all $K\in\mathcal{T}_h$, where f_K is the mean value of f over K, $f_K := (f,1)_K/|K|$, the analysis for the convection–diffusion–reaction case simplifies to the a posteriori error estimate

$$|||\, p - \tilde{p}_h|||_\Omega \le \left\{\sum_{K\in\mathcal{T}_h}\eta_K^2\right\}^{\frac12} + \left\{\sum_{K\in\mathcal{T}_h}\zeta_K^2\right\}^{\frac12}, \tag{5.3}$$

where

$$\eta_K^2 := C_{\mathrm{P}}\frac{h_K^2}{c_{\mathbf{S},K}}\|f - f_K\|_K^2\,, \tag{5.4}$$

$$\zeta_K^2 := \left(2c_{\mathbf{S},K} + 2\frac{(C_{\mathbf{S},K})^2}{c_{\mathbf{S},K}}\right)\|\nabla(\tilde{p}_h - \mathcal{I}(\tilde{p}_h))\|_K^2\,, \tag{5.5}$$

and where $\mathcal{I} : \mathbb{P}_2(\mathcal{T}_h) \to H_0^1(\Omega)$ is e.g. the Oswald or the modified Oswald interpolation operator (see Section 4.2). Note that since $\nabla \cdot (\mathbf{u} - \mathbf{u}_h)|_K = f - f_K$ is fully computable for all $K \in \mathcal{T}_h$, the control over $\|\mathbf{u} - \mathbf{u}_h\|_\Omega + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_\Omega$ immediately follows using (5.1).

Let us now remark that in the pure diffusion case, we actually have (compare with (2.9))

$$\mathcal{B}(\varphi, \varphi) \geq |||\varphi|||_\Omega^2 \qquad \forall \varphi \in W_0(\mathcal{T}_h)\,. \tag{5.6}$$

This leads us to the following considerations. Let us first generalize the classical weak solution to a function $\tilde{p} \in W_0(\mathcal{T}_h)$ (only satisfying the continuity of the means of the traces at interior sides and the equality of the means of the traces to zero at exterior sides) such that

$$\mathcal{B}(\tilde{p}, \varphi) = (f, \varphi)_\Omega \qquad \forall \varphi \in W_0(\mathcal{T}_h)\,. \tag{5.7}$$

Inequalities (5.6) and (2.10) together with the discrete Friedrichs inequality (2.5) assure the existence of a unique solution of (5.7). Next,

$$|||\tilde{p} - \tilde{p}_h|||_\Omega \leq \frac{\mathcal{B}(\tilde{p} - \tilde{p}_h, \tilde{p} - \tilde{p}_h)}{|||\tilde{p} - \tilde{p}_h|||_\Omega} \leq \sup_{\varphi \in W_0(\mathcal{T}_h),\, |||\varphi|||_\Omega = 1} \mathcal{B}(\tilde{p} - \tilde{p}_h, \varphi)\,,$$

using (5.6). We further estimate, similarly as in the proof of Lemma 7.1 below,

$$\begin{aligned}
\mathcal{B}(\tilde{p} - \tilde{p}_h, \varphi) &= (f, \varphi)_\Omega + \sum_{K \in \mathcal{T}_h} \left\{ (\nabla \cdot \mathbf{S} \nabla \tilde{p}_h, \varphi)_K - \langle \mathbf{S} \nabla \tilde{p}_h \cdot \mathbf{n}, \varphi \rangle_{\partial K} \right\} \\
&= \sum_{K \in \mathcal{T}_h} (f - \nabla \cdot \mathbf{u}_h, \varphi)_K + \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}} \langle \mathbf{u}_h \cdot \mathbf{n}_K, \varphi|_K - \varphi|_L \rangle_{\sigma_{K,L}} + \sum_{\sigma \in \mathcal{E}_h^{\mathrm{ext}}} \langle \mathbf{u}_h \cdot \mathbf{n}, \varphi \rangle_\sigma \\
&= \sum_{K \in \mathcal{T}_h} (f - \nabla \cdot \mathbf{u}_h, \varphi)_K = \sum_{K \in \mathcal{T}_h} (f - \nabla \cdot \mathbf{u}_h, \varphi - \varphi_K)_K\,,
\end{aligned}$$

using the bilinearity of $\mathcal{B}(\cdot, \cdot)$, the definition (5.7) of the generalized weak solution $\tilde{p}$, the Green theorem in each $K \in \mathcal{T}_h$, the relation (4.1a) between $\tilde{p}_h$ and $\mathbf{u}_h$, reordering the summation over the boundaries of elements to the summation over the sides, using the continuity of the normal trace of $\mathbf{u}_h$ expressed by $\mathbf{u}_h|_K \cdot \mathbf{n}_K = -\mathbf{u}_h|_L \cdot \mathbf{n}_L$ on $\sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}$, the fact that $\mathbf{u}_h \cdot \mathbf{n}$ is moreover constant on all sides $\sigma \in \mathcal{E}_h$ and the definition (2.4) of the space $W_0(\mathcal{T}_h)$, and finally the second equation (3.2b) of the definition of the mixed finite element scheme (φ_K is the mean of φ over K). Next, estimate (9.1) from Lemma 9.1 below holds true also in this case, so that finally the Cauchy–Schwarz inequality leads to

$$|||\tilde{p} - \tilde{p}_h|||_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K^2 \right\}^{\frac{1}{2}}$$

with η_K given by (5.4).

First, this is a completely data-dependent a posteriori error estimate and secondly, this is in fact an a priori error estimate as well: it shows that the mixed finite element solutions $\tilde{p}_h$ and $\mathbf{u}_h$ (cf. (5.1) which still holds true) converge both as $O(h^2)$ in the $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$ respectively, norms to the generalized weak solution $\tilde{p}$ given by (5.7) and its flux $\mathbf{u}$, $\mathbf{u}|_K := -\mathbf{S}\nabla \tilde{p}|_K$ (for e.g. $f \in H^1(K)$ on all $K \in \mathcal{T}_h$). Moreover, as soon as f is piecewise constant, $\tilde{p}_h$ is directly equal to the generalized solution! We emphasize that these results hold true for $\mathbf{S}$ piecewise constant but arbitrarily inhomogeneous and anisotropic; they apparently confirm the observations of a very good behavior of mixed methods in these circumstances. There are also very interesting consequences in one space dimension, cf. Section 5.7 below. Finally, it should be noted that the above results hold independently of the convexity of Ω and of the additional regularity of the weak solution.

5.5 A combination of the centered and upwind-weighted schemes

The upwind scheme (3.3a)–(3.3b) guarantees stability in the convection-dominated case, but the additional a posteriori error estimator η_σ given by (4.7) is unfortunately not efficient, even with the introduction of the local Péclet upstream weighting (3.6) (see the discussion in Section 5.2). On the other hand, the centered scheme, however precise if h is sufficiently small, may give completely wrong results for coarse meshes. Hence a good idea may be a smooth transition from the one scheme to the other under the form

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,,$$

$$(\nabla \cdot \mathbf{u}_h, \phi_K)_K + \sum_{\sigma \in \mathcal{E}_K} \left\{ (\mu_\sigma \hat{p}_\sigma + (1-\mu_\sigma)\tilde{p}_\sigma) w_{K,\sigma} \phi_K \right\} + (r p_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K \in \mathcal{T}_h\,,$$

where $\hat{p}_\sigma$ is the upstream value and where the parameter μ_σ is set to $1-2\nu_\sigma$ with ν_σ given by (3.6). Hence μ_σ will be close to 1 (and thus the scheme to the upwind one) in the convection-dominated regime, while when the local Péclet number (4.3) gets smaller than 2, μ_σ will equal to 0 (and thus the scheme to the centered one). Notice finally that such scheme is fully rewritable in terms of the original unknowns p_h, $\mathbf{u}_h$, using that $\sum_{\sigma \in \mathcal{E}_K} \tilde{p}_\sigma w_{K,\sigma} \phi_K = \langle \tilde{p}_h \mathbf{w} \cdot \mathbf{n}, \phi_K \rangle_{\partial K}$ and Lemma 6.2 below.

5.6 Implementation with one unknown per element

It is shown in [40] (cf. alternatively the abbreviated version [38]) that in the lowest-order mixed finite element scheme for pure diffusion problems, as well as in the upwind-weighted scheme (3.3a)–(3.3b), there exist local flux expressions. Hence these schemes are in fact equivalent to particular finite volume schemes and can namely be implemented with only one unknown (p_K) per element. The associated matrices are then in the majority of the cases (in dependence on the mesh $\mathcal{T}_h$ and the tensor $\mathbf{S}$) positive definite, although in general nonsymmetric. It is shown in these references that one can in this way considerably reduce the CPU time necessary to solve the liner systems arising from these mixed finite element schemes. The same results hold true as well for a centered scheme in the form: find $\mathbf{u}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega - (p_h \mathbf{w}, \mathbf{S}^{-1}\mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,,$$

$$(\nabla \cdot \mathbf{u}_h, \phi_h)_\Omega + (r p_h, \phi_h)_\Omega = (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi(\mathcal{T}_h)\,.$$

The centered scheme (3.2a)–(3.2b) is different from the above one (although the are actually very close namely in case that $\nabla \cdot \mathbf{w} = 0$; then their system matrices are only transposed). It may however be possible to use the same approach as well.

5.7 The estimates in one space dimension

It appears that the above results have interesting consequences in one space dimension, where the two schemes (3.2a)–(3.2b) and (3.3a)–(3.3b) can likewise be defined.

5.7.1 One dimension: no nonconformity

First of all, Lemma 6.1 below reduces in one space dimension to the assertion that the postprocessed variable $\tilde{p}_h$ given by (4.1a)–(4.1b) is continuous, i.e. that in this case $\tilde{p}_h \in H_0^1(\Omega)$. An immediate consequence is that the parts of the a posteriori error estimates of Theorems 4.2–4.3 because of nonconformity disappear.

5.7.2 Lowest-order mixed finite elements: an exact three-point scheme for one-dimensional diffusion problems with piecewise constant coefficients

Another quite interesting consequence is related to the remarks of Sections 5.4 and 5.6. As there is no nonconformity, the superconvergence $O(h^2)$ of both $\tilde{p}_h$ and $\mathbf{u}_h$ (this time towards the weak solution, coinciding with the generalized one) holds always true, and moreover, it appears that in one space dimension, one can always rewrite the schemes with only p_K, $K \in \mathcal{T}_h$, as unknowns. Hence the lowest-order mixed finite elements represent a scheme with a three-point stencil (there are at most three nonzero entries on each matrix row), which is exact for one-dimensional pure diffusion problems, where the diffusion tensor $\mathbf{S}$ (this time a scalar function) and the right-hand side f are piecewise constant (and hence possibly arbitrarily discontinuous). This should namely be compared to the known results for the finite volume/finite difference method. In particular, the (best known?) scheme proposed by Ewing *et al* in [21] is only exact when the right-hand side is constant (the diffusion tensor may be piecewise constant), cf. Remark 2.4 in the above reference. It would be interesting to investigate in detail the relation between these two schemes.

6 Discrete properties of the schemes

We prove in this section different properties of the schemes (3.2a)–(3.2b) and (3.3a)–(3.3b) and of the postprocessed scalar variable $\tilde{p}_h$ needed in the paper.

Lemma 6.1 (Continuity of the means of the traces of $\tilde{p}_h$)**.** *It holds that* $\tilde{p}_h \in W_0(\mathcal{T}_h)$*, i.e.*

$$\begin{aligned} \langle \tilde{p}_h|_K - \tilde{p}_h|_L, 1\rangle_{\sigma_{K,L}} &= 0 \qquad \forall \sigma_{K,L} \in \mathcal{E}_h^{\rm int}\,, \\ \langle \tilde{p}_h, 1\rangle_\sigma &= 0 \qquad \forall \sigma \in \mathcal{E}_h^{\rm ext}\,. \end{aligned}$$

PROOF:
Let us consider a side $\sigma_{K,L} \in \mathcal{E}_h^{\rm int}$. Then taking $\mathbf{v}_h$ equal to the basis function $\mathbf{v}_{\sigma_{K,L}}$ (cf. Section 3.1) in (3.2a) or (3.3a) yields

$$\begin{aligned} 0 &= -(\nabla \tilde{p}_h, \mathbf{v}_{\sigma_{K,L}})_{K\cup L} - (\tilde{p}_h, \nabla\cdot\mathbf{v}_{\sigma_{K,L}})_{K\cup L} = -\langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial K} - \langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial L} \\ &= \langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}_K, \tilde{p}_h|_L - \tilde{p}_h|_K\rangle_{\sigma_{K,L}}\,, \end{aligned}$$

using the definition (4.1a)–(4.1b) of $\tilde{p}_h$, the fact that $\nabla\cdot\mathbf{v}_h$ for $\mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ is constant in each simplex (which allows us to replace p_h by $\tilde{p}_h$), the Green theorem, and the fact that $\mathbf{v}_{\sigma_{K,L}}$ has a nonzero normal flux only through $\sigma_{K,L}$. The first assertion of the lemma follows by the fact that $\mathbf{v}_h\cdot\mathbf{n}$ for $\mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ is constant on each side $\sigma \in \mathcal{E}_h$. The proof for boundary sides is completely similar. □

Lemma 6.2 (Equivalent form of the centered scheme)**.** *The centered scheme* (3.2a)–(3.2b) *can be equivalently written: find* $\mathbf{u}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ *and* $p_h \in \Phi(\mathcal{T}_h)$ *such that*

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (\tilde{p}_h, \nabla\cdot\mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,, \tag{6.1a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_K)_K + \langle \tilde{p}_h \mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K} + (r\tilde{p}_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K \in \mathcal{T}_h\,, \tag{6.1b}$$

where $\tilde{p}_h$ *is defined by* (4.1a)–(4.1b)*.*

PROOF:
Since $\nabla\cdot\mathbf{v}_h$ for $\mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ is constant in each simplex and since r was supposed piecewise

constant as well, one can replace p_h by $\tilde{p}_h$ in the terms $(p_h, \nabla\cdot\mathbf{v}_h)_\Omega$ and $(rp_h, \phi_K)_K$ using (4.1b). Similarly, using in addition the Green theorem,

$$\begin{aligned}-(\mathbf{S}_K^{-1}\mathbf{u}_h\mathbf{w}, \phi_K)_K + (p_K\nabla\cdot\mathbf{w}, \phi_K)_K &= (\nabla\tilde{p}_h\mathbf{w}, \phi_K)_K + (\tilde{p}_h\nabla\cdot\mathbf{w}, \phi_K)_K \\ &= (\nabla\cdot(\tilde{p}_h\mathbf{w}), \phi_K)_K = \langle\tilde{p}_h\mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K}\,. \quad \square\end{aligned}$$

Remark 6.3 (Hybridization of the schemes)**.** *Mixed finite element schemes can equivalently be reformulated while relaxing the continuity of the normal trace of* $\mathbf{u}_h$ *required in the definition of the space* $\mathbf{RT}^0(\mathcal{T}_h)$ *and imposing it instead with the help of Lagrange multipliers* λ_σ, $\sigma\in\mathcal{E}_h^{\rm int}$, *cf. [10, Section V.1.2]. The centered scheme* (3.2a)–(3.2b)*, taking into account its equivalent form given by Lemma 6.2, then changes to: find* $\mathbf{u}_h\in\mathbf{RT}^0_{-1}(\mathcal{T}_h)$, $p_h\in\Phi(\mathcal{T}_h)$, *and* λ_σ, $\sigma\in\mathcal{E}_h^{\rm int}$, *with* $\tilde{p}_h$ *defined by* (4.1a)–(4.1b)*, such that*

$$\sum_{K\in\mathcal{T}_h}\Big\{(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K - (\tilde{p}_h, \nabla\cdot\mathbf{v}_h)_K + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\rm int}}\langle\mathbf{v}_h\cdot\mathbf{n}, \lambda_\sigma\rangle_\sigma\Big\} = 0 \qquad \forall\,\mathbf{v}_h\in\mathbf{RT}^0_{-1}(\mathcal{T}_h)\,, \tag{6.2a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_K)_K + \langle\tilde{p}_h\mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K} + (r\tilde{p}_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K\in\mathcal{T}_h\,, \tag{6.2b}$$

$$\langle(\mathbf{u}_h\cdot\mathbf{n})|_K + (\mathbf{u}_h\cdot\mathbf{n})|_L, \lambda_{\sigma_{K,L}}\rangle_{\sigma_{K,L}} = 0 \qquad \forall\sigma_{K,L}\in\mathcal{E}_h^{\rm int}\,, \tag{6.2c}$$

whereas the upwind-weighted scheme (3.3a)–(3.3b) *then writes: find* $\mathbf{u}_h\in\mathbf{RT}^0_{-1}(\mathcal{T}_h)$, $p_h\in\Phi(\mathcal{T}_h)$, *and* λ_σ, $\sigma\in\mathcal{E}_h^{\rm int}$, *such that*

$$\sum_{K\in\mathcal{T}_h}\Big\{(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K - (p_h, \nabla\cdot\mathbf{v}_h)_K + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\rm int}}\langle\mathbf{v}_h\cdot\mathbf{n}, \lambda_\sigma\rangle_\sigma\Big\} = 0 \qquad \forall\,\mathbf{v}_h\in\mathbf{RT}^0_{-1}(\mathcal{T}_h)\,, \tag{6.3a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_K)_K + \sum_{\sigma\in\mathcal{E}_K}\hat{p}_\sigma w_{K,\sigma}\phi_K + (rp_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K\in\mathcal{T}_h\,, \tag{6.3b}$$

$$\langle(\mathbf{u}_h\cdot\mathbf{n})|_K + (\mathbf{u}_h\cdot\mathbf{n})|_L, \lambda_{\sigma_{K,L}}\rangle_{\sigma_{K,L}} = 0 \qquad \forall\sigma_{K,L}\in\mathcal{E}_h^{\rm int}\,. \tag{6.3c}$$

Lemma 6.4 (Relation of $\tilde{p}_h$ to the Lagrange multipliers λ_σ)**.** *It holds that*

$$\lambda_\sigma = \tilde{p}_\sigma = \frac{\langle\tilde{p}_h, 1\rangle_\sigma}{|\sigma|} \qquad \forall\sigma\in\mathcal{E}_h^{\rm int}\,.$$

Proof:

The proof is similar to that of Lemma 6.1. Let $K\in\mathcal{T}_h$ and $\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\rm int}$. Then taking $\mathbf{v}_h = \mathbf{v}_\sigma$ in (6.2a) or (6.3a), we have

$$0 = -(\nabla\tilde{p}_h, \mathbf{v}_\sigma)_K - (\tilde{p}_h, \nabla\cdot\mathbf{v}_\sigma)_K + \langle\mathbf{v}_\sigma\cdot\mathbf{n}, \lambda_\sigma\rangle_\sigma = \langle\mathbf{v}_\sigma\cdot\mathbf{n}, \lambda_\sigma - \tilde{p}_h\rangle_\sigma\,,$$

using the definition (4.1a)–(4.1b) of $\tilde{p}_h$, the fact that $\nabla\cdot\mathbf{v}_\sigma$ is constant in each simplex, the fact that $\mathbf{v}_\sigma$ has a nonzero normal flux only through σ, and the Green theorem. The assertion of the lemma follows by the fact that $\mathbf{v}_\sigma\cdot\mathbf{n}$ is constant on σ. $\square$

Lemma 6.5 (A priori estimate for the upwind-weighted scheme)**.** *Let* $\mathbf{u}_h$, p_h *be the solutions of the upwind-weighted scheme* (3.3a)–(3.3b) *and let* $\tilde{p}_h$ *be the postprocessed scalar variable given by* (4.1a)–(4.1b)*. Then*

$$\sum_{K\in\mathcal{T}_h}\Big\{c_{\mathbf{S},K}\|\nabla\tilde{p}_h\|_K^2 + \frac{1}{2}c_{\mathbf{w},r,K}\,\|p_h\|_K^2\Big\} \le \frac{1}{2}\sum_{K\in\mathcal{T}_h}\frac{\|f\|_K^2}{c_{\mathbf{w},r,K}}$$

if $c_{\mathbf{w},r,K} > 0$ *for all* $K \in \mathcal{T}_h$ *and*

$$\sum_{K\in\mathcal{T}_h} \left\{ \frac{1}{2} c_{\mathbf{S},K} \|\nabla \tilde{p}_h\|_K^2 + c_{\mathbf{w},r,K} \|p_h\|_K^2 \right\} \leq \frac{\|f\|_\Omega^2}{2} \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}},$$

where $c_{\mathbf{S},\Omega}$ *is given by* (4.8) *and where* C_{DF} *is the constant from the discrete Friedrichs inequality* (2.5).

PROOF:
Let us put $\phi_h = p_h$ in (3.3b). We then can rewrite the first term of the left-hand side of (3.3b) as

$$\sum_{K\in\mathcal{T}_h} (\nabla\cdot\mathbf{u}_h, p_K)_K = \sum_{K\in\mathcal{T}_h} \{-(\mathbf{u}_h, \nabla\tilde{p}_h)_K + \langle \mathbf{u}_h\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial K}\} = \sum_{K\in\mathcal{T}_h} (\mathbf{S}_K\nabla\tilde{p}_h, \nabla\tilde{p}_h)_K$$
$$+ \sum_{\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}}} \langle \mathbf{u}_h\cdot\mathbf{n}_K, \tilde{p}_h|_K - \tilde{p}_h|_L\rangle_{\sigma_{K,L}} + \sum_{\sigma\in\mathcal{E}_h^{\mathrm{ext}}} \langle \mathbf{u}_h\cdot\mathbf{n}, \tilde{p}_h\rangle_\sigma = \sum_{K\in\mathcal{T}_h} c_{\mathbf{S},K}\|\nabla\tilde{p}_h\|_K^2,$$

using the fact that $\nabla\cdot\mathbf{u}_h$ is constant on each $K\in\mathcal{T}_h$ and we thus can replace p_h by $\tilde{p}_h$ employing (4.1b), the Green theorem, (4.1a), the fact that $\mathbf{u}_h\cdot\mathbf{n}$ is constant on each $\sigma\in\mathcal{E}_h$, and the continuity of the means of the traces of $\tilde{p}_h$ given by Lemma 6.1. Next,

$$\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K} \hat{p}_\sigma w_{K,\sigma} p_K = \sum_{\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}}} \{\hat{p}_\sigma w_{K,\sigma} p_K + \hat{p}_\sigma w_{L,\sigma} p_L\} + \sum_{\sigma_K\in\mathcal{E}_h^{\mathrm{ext}}} \hat{p}_\sigma w_{K,\sigma} p_K$$
$$= \sum_{\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}},\, w_{K,\sigma}\geq 0} w_{K,\sigma}\left(p_K(p_K-p_L) - \nu_\sigma (p_L-p_K)^2\right) + \sum_{\sigma_K\in\mathcal{E}_h^{\mathrm{ext}}} \hat{p}_\sigma w_{K,\sigma} p_K$$
$$= \frac{1}{2}\sum_{\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}},\, w_{K,\sigma}\geq 0} w_{K,\sigma}(p_K^2 - p_L^2) + \sum_{\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}}} |w_{K,\sigma}|(p_L-p_K)^2\left(\frac{1}{2}-\nu_\sigma\right)$$
$$+ \sum_{\sigma_K\in\mathcal{E}_h^{\mathrm{ext}}} \left\{\frac{1}{2}p_K^2 w_{K,\sigma} + |w_{K,\sigma}| p_K^2\left(\frac{1}{2}-\nu_\sigma\right)\right\} \geq \frac{1}{2}\sum_{K\in\mathcal{T}_h} p_K^2 (\nabla\cdot\mathbf{w}, 1)_K,$$

where we have rewritten the summation over the sides and fixed denotation of $K, L\in\mathcal{T}_h$ sharing a side $\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}}$ such that $w_{K,\sigma}\geq 0$, used that $w_{K,\sigma} = -w_{L,\sigma}$, the definition (3.4)–(3.5) of $\hat{p}_\sigma$, and the relation $2a(a-b) = (a-b)^2 + a^2 - b^2$, estimated using $0\leq\nu_\sigma\leq 1/2$, rewritten the summation back over the elements and their sides, and finally employed the Green theorem giving $\sum_{\sigma\in\mathcal{E}_K} w_{K,\sigma} = (\nabla\cdot\mathbf{w}, 1)_K$. Finally,

$$(rp_h, p_h)_\Omega = \sum_{K\in\mathcal{T}_h} p_K^2 (r, 1)_K.$$

The right-hand side of (3.3b) with $\phi_h = p_h$ can be estimated either by

$$(f, p_h)_\Omega \leq \sum_{K\in\mathcal{T}_h} \|f\|_K \frac{\sqrt{c_{\mathbf{w},r,K}}}{\sqrt{c_{\mathbf{w},r,K}}} \|p_h\|_K \leq \frac{1}{2}\sum_{K\in\mathcal{T}_h} \frac{\|f\|_K^2}{c_{\mathbf{w},r,K}} + \frac{1}{2}\sum_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K}\|p_h\|_K^2$$

or by

$$(f, p_h)_\Omega \leq \|f\|_\Omega \|p_h\|_\Omega \leq \frac{\|f\|_\Omega^2}{2}\frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} + \frac{c_{\mathbf{S},\Omega}}{C_{\mathrm{DF}}}\frac{\|\tilde{p}_h\|_\Omega^2}{2} \leq \frac{\|f\|_\Omega^2}{2}\frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} + \frac{c_{\mathbf{S},\Omega}}{2}\sum_{K\in\mathcal{T}_h} \|\nabla\tilde{p}_h\|_K^2,$$

using the Cauchy–Schwarz, $ab \leq \varepsilon a^2/2 + b^2/(2\varepsilon)$, $\varepsilon > 0$, $\|p_h\|_K \leq \|\tilde{p}_h\|_K$, and the discrete Friedrichs (2.5) inequalities. The assertion follows by combining the above estimates. □

Remark 6.6 (Existence and uniqueness for the upwind-weighted scheme)**.** *From Lemma 6.5, existence and uniqueness for the upwind-weighted scheme* (3.3a)–(3.3b) *easily follows. Indeed, let* $f = 0$*. Then* $p_h = 0$ *and* $\mathbf{u}_h = -\mathbf{S}_K \nabla \tilde{p}_h = 0$ *for all* $K \in \mathcal{T}_h$*.*

Remark 6.7 (Existence and uniqueness for the centered scheme)**.** *In contrast with the upwind-weighted scheme, existence and uniqueness for the centered scheme* (3.2a)–(3.2b) *is in [17] only guaranteed for "h sufficiently small". Alternatively, there exists a unique solution if* $C_{\mathbf{w},K} \leq 2(1-\mu)\sqrt{c_{\mathbf{S},K}}\sqrt{\tilde{c}_{\mathbf{w},r,K}}$ *for some* $\mu \in (0,1)$ *and all* $K \in \mathcal{T}_h$*, where* $(\nabla\cdot\mathbf{w} + r)|_K = \tilde{c}_{\mathbf{w},r,K} > 0$*, which corresponds to the case that is not convection-dominated.*

7 Proofs of the a posteriori error estimates and of their efficiency

We shall prove in this section the a posteriori error estimates stated by Theorems 4.2–4.3, as well as their efficiency discussed in Theorem 4.4.

7.1 Proofs of the a posteriori error estimates

To begin with, the following bound for the error $|||p - \tilde{p}_h|||_\Omega$ holds:

Lemma 7.1 (Abstract framework)**.** *Let* p *be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.8) *and let* $s \in H^1_0(\Omega)$ *be arbitrary. If* $\tilde{p}_h$ *is the postprocessed solution of the centered mixed finite element scheme* (3.2a)–(3.2b)*, given by* (4.1a)–(4.1b)*, then*

$$|||p - \tilde{p}_h|||_\Omega \leq |||s - \tilde{p}_h\,|||_\Omega + \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||_\Omega = 1} \left\{ T_{\mathrm{R}}(\varphi) + T_{\mathrm{NC}}(\varphi) \right\},$$

and if $\tilde{p}_h$ *is the postprocessed solution of the upwind-weighted mixed finite element scheme* (3.3a)–(3.3b)*, given by* (4.1a)–(4.1b)*, then*

$$|||p - \tilde{p}_h|||_\Omega \leq |||s - \tilde{p}_h\,|||_\Omega + \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||_\Omega = 1} \left\{ T_{\mathrm{R}}(\varphi) + T_{\mathrm{U}}(\varphi) + T_{\mathrm{NC}}(\varphi) \right\},$$

where

$$\begin{aligned} T_{\mathrm{R}}(\varphi) &:= \sum_{K \in \mathcal{T}_h} \left(f + \nabla\cdot\mathbf{S}_K \nabla \tilde{p}_h - \nabla\cdot(\tilde{p}_h \mathbf{w}) - r\tilde{p}_h, \varphi - \varphi_K \right)_K, \\ T_{\mathrm{NC}}(\varphi) &:= \mathcal{B}(\tilde{p}_h - s, \varphi), \\ T_{\mathrm{U}}(\varphi) &:= \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \langle (\hat{p}_\sigma - \tilde{p}_h)\mathbf{w}\cdot\mathbf{n}, \varphi_K \rangle_\sigma, \end{aligned}$$

and where φ_K *is the mean of* φ *over* K*,* $\varphi_K := (\varphi, 1)_K / |K|$*.*

PROOF:
The triangle inequality implies

$$|||p - \tilde{p}_h|||_\Omega \leq |||p - s\,|||_\Omega + |||\, s - \tilde{p}_h\,|||_\Omega\ .$$

Now since $(p - s) \in H^1_0(\Omega)$, we can use the coercivity of the form $\mathcal{B}(\cdot,\cdot)$ given by (2.9), so that

$$\begin{aligned} |||\, p - s|||_\Omega &\leq \frac{\mathcal{B}(p-s, p-s)}{|||p-s|||_\Omega} \leq \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||_\Omega = 1} \mathcal{B}(p - s, \varphi) \\ &= \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||_\Omega = 1} \left\{ \mathcal{B}(p - \tilde{p}_h, \varphi) + \mathcal{B}(\tilde{p}_h - s, \varphi) \right\}. \end{aligned} \tag{7.1}$$

Let us consider an arbitrary $\varphi \in H_0^1(\Omega)$. We have, using the bilinearity of $\mathcal{B}(\cdot,\cdot)$, the definition (2.8) of the weak solution p, and the Green theorem in each $K \in \mathcal{T}_h$,

$$\begin{aligned} \mathcal{B}(p-\tilde{p}_h,\varphi) &= (f,\varphi)_\Omega - \sum_{K\in\mathcal{T}_h} \left\{(\mathbf{S}\nabla\tilde{p}_h,\nabla\varphi)_K + \big(\nabla\cdot(\tilde{p}_h\mathbf{w}),\varphi\big)_K + (r\tilde{p}_h,\varphi)_K\right\} \\ &= \sum_{K\in\mathcal{T}_h} \big(f+\nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h,\varphi\big)_K - \sum_{K\in\mathcal{T}_h} \langle \mathbf{S}_K\nabla\tilde{p}_h\cdot\mathbf{n},\varphi\rangle_{\partial K} \\ &= \sum_{K\in\mathcal{T}_h} \big(f+\nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h,\varphi\big)_K . \end{aligned}$$

Note that we have in particular used the continuity of the normal trace of $\mathbf{S}\nabla\tilde{p}_h$ (i.e., by (4.1a), the typical mixed finite element continuity of the normal trace of $\mathbf{u}_h$) yielding

$$\langle(\mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n})|_K + (\mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n})|_L,\varphi\rangle_{\sigma_{K,L}} = \langle 0,\varphi\rangle_{\sigma_{K,L}} = 0 \qquad \forall\sigma_{K,L}\in\mathcal{E}_h^{\rm int}$$

(the fact that $\langle\mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n},\varphi\rangle_\sigma = 0$ for $\sigma\in\mathcal{E}_h^{\rm ext}$ follows by $\varphi\in H_0^1(\Omega)$).

Now the second equation (6.1b) of the equivalent form of the centered scheme by the definition of $\tilde{p}_h$ (4.1a)–(4.1b) and by the Green theorem implies that (recall that φ_K is the constant mean of φ over K)

$$\big(f+\nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h,\varphi_K\big)_K = 0 \qquad \forall K\in\mathcal{T}_h .$$

Hence in the case of the centered scheme,

$$\mathcal{B}(p-\tilde{p}_h,\varphi) = \sum_{K\in\mathcal{T}_h} \big(f+\nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h,\varphi-\varphi_K\big)_K = T_{\rm R}(\varphi) .$$

For the upwind-weighted scheme, there occurs an additional term

$$\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K} \langle(\hat{p}_\sigma-\tilde{p}_h)\mathbf{w}\cdot\mathbf{n},\varphi_K\rangle_\sigma = T_{\rm U}(\varphi) .$$

The nonconformity term $T_{\rm NC}(\varphi)$ appears directly as the second term of (7.1). □

We now estimate the terms $T_{\rm R}$, $T_{\rm NC}$, and $T_{\rm U}$ separately.

Lemma 7.2 (Residual estimate)**.** *There holds*

$$\sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||_\Omega=1} T_{\rm R}(\varphi) \le \left\{\sum_{K\in\mathcal{T}_h}\eta_K^2\right\}^{\frac12},$$

where η_K is given by (4.2)*.*

Proof:
Let $\varphi \in H_0^1(\Omega)$ be arbitrary. Then by the Cauchy–Schwarz inequality and by Lemma 9.1 from Section 9 below,

$$T_{\rm R}(\varphi) \le \sum_{K\in\mathcal{T}_h} \big\|f+\nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h\big\|_K \|\varphi-\varphi_K\|_K \le \left\{\sum_{K\in\mathcal{T}_h}\eta_K^2\right\}^{\frac12} |||\,\varphi\,|||_\Omega . \quad \square$$

Lemma 7.3 (Nonconformity estimate)**.** *Let us put $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$ in Lemma 7.1. Then*

$$|||s - \tilde{p}_h|||_\Omega + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||_\Omega = 1} T_{\mathrm{NC}}(\varphi) \leq \left\{ \sum_{K \in \mathcal{T}_h} \zeta_K^2 \right\}^{\frac{1}{2}},$$

where ζ_K is given by (4.5).

PROOF:
Let $\varphi \in H_0^1(\Omega)$ be arbitrary and let us denote $v := \tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$. Then, for each $K \in \mathcal{T}_h$,

$$\begin{aligned}
& (\mathbf{S}\nabla v, \nabla\varphi)_K + (\nabla\cdot(v\mathbf{w}), \varphi)_K + (rv, \varphi)_K \\
\leq\ & C_{\mathbf{S},K}\|\nabla v\|_K \|\nabla\varphi\|_K + C_{\mathbf{w},K}\|\nabla v\|_K\|\varphi\|_K + C_{\mathbf{w},r,K}\|v\|_K\|\varphi\|_K \\
\leq\ & \left(\left(\frac{C_{\mathbf{S},K}}{\sqrt{c_{\mathbf{S},K}}} + \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{w},r,K}}} \right) \|\nabla v\|_K + \frac{C_{\mathbf{w},r,K}}{\sqrt{c_{\mathbf{w},r,K}}}\|v\|_K \right) |||\,\varphi\,|||_K\ .
\end{aligned}$$

Remark that this estimate would stay valid for an arbitrary $s \in H_0^1(\Omega)$ instead of $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$.

We next estimate the term $(\nabla\cdot(v\mathbf{w}), \varphi)_K$, by the Green theorem equal to $-(v\mathbf{w}, \nabla\varphi)_K + \langle v\mathbf{w}\cdot\mathbf{n}, \varphi\rangle_{\partial K}$, in a different way. First,

$$\left|(v\mathbf{w}, \nabla\varphi)_K\right| \leq C_{\mathbf{w},K}\|v\|_K\|\nabla\varphi\|_K = C_{\mathbf{w},K}\|v - v_\sigma\|_K\|\nabla\varphi\|_K \leq \sqrt{C_{\mathrm{F},d}}C_{\mathbf{w},K}h_K\|\nabla v\|_K\|\nabla\varphi\|_K\,,$$

noticing that $v_\sigma := \langle v, 1\rangle_\sigma/|\sigma| = 0$ since the modified Oswald interpolation operator of Section 4.2 preserves the means of $\tilde{p}_h$ over the sides and applying the generalized Friedrichs inequality (2.2). Next, for each $\sigma \in \mathcal{E}_K$,

$$\begin{aligned}
\langle v\mathbf{w}\cdot\mathbf{n}, \varphi\rangle_\sigma &= \langle v\mathbf{w}\cdot\mathbf{n}, \varphi - \varphi_\sigma\rangle_\sigma = \langle (v - v_\sigma)\mathbf{w}\cdot\mathbf{n}, \varphi - \varphi_\sigma\rangle_\sigma \\
&\leq C_{\mathbf{w},K}\|v - v_\sigma\|_\sigma\|\varphi - \varphi_\sigma\|_\sigma \leq C_{\mathbf{w},K}\widetilde{C}_{\mathrm{F},d}\frac{h_K}{h_\sigma}h_K\|\nabla v\|_K\|\nabla\varphi\|_K\,,
\end{aligned}$$

where $\varphi_\sigma := \langle \varphi, 1\rangle_\sigma/|\sigma|$, using that $\langle v\mathbf{w}\cdot\mathbf{n}, \varphi_\sigma\rangle_\sigma = 0$ since $\mathbf{w}\cdot\mathbf{n}$ and φ_σ are constants and $v_\sigma = 0$, and finally applying the generalized Friedrichs inequality (2.3). Thus, applying the above estimate to each of the sides of K,

$$\left|(\nabla\cdot(v\mathbf{w}), \varphi)_K\right| \leq C_{\mathbf{w},K}h_K C_{d,K}\|\nabla v\|_K\|\nabla\varphi\|_K$$

with $C_{d,K}$ given by (4.4). This implies an alternative estimate

$$\begin{aligned}
& (\mathbf{S}\nabla v, \nabla\varphi)_K + (\nabla\cdot(v\mathbf{w}), \varphi)_K + (rv, \varphi)_K \\
\leq\ & (C_{\mathbf{S},K} + C_{\mathbf{w},K}h_K C_{d,K})\|\nabla v\|_K\|\nabla\varphi\|_K + |r_K|\|v\|_K\|\varphi\|_K \\
\leq\ & \left(\left(\frac{C_{\mathbf{S},K}}{\sqrt{c_{\mathbf{S},K}}} + \frac{C_{\mathbf{w},K}h_K C_{d,K}}{\sqrt{c_{\mathbf{S},K}}} \right) \|\nabla v\|_K + \frac{|r_K|}{\sqrt{c_{\mathbf{w},r,K}}}\|v\|_K \right) |||\,\varphi|||_K
\end{aligned}$$

for each $K \in \mathcal{T}_h$. Hence using the definition of $T_{\mathrm{NC}}(\varphi)$ in Lemma 7.1, that of $\mathcal{B}(\cdot,\cdot)$ by (2.6), the Cauchy–Schwarz inequality, and the inequality $(a+b)^2 \leq 2(a^2+b^2)$,

$$\begin{aligned}
T_{\mathrm{NC}}(\varphi) \leq\ & \left\{ \sum_{K \in \mathcal{T}_h} \min\left\{ 2\left(\frac{C_{\mathbf{S},K}}{\sqrt{c_{\mathbf{S},K}}} + \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{w},r,K}}} \right)^2 \|\nabla v\|_K^2 + 2\frac{C_{\mathbf{w},r,K}^2}{c_{\mathbf{w},r,K}}\|v\|_K^2\,, \right.\right. \\
& \left.\left. 2\left(\frac{C_{\mathbf{S},K}}{\sqrt{c_{\mathbf{S},K}}} + \frac{C_{\mathbf{w},K}h_K C_{d,K}}{\sqrt{c_{\mathbf{S},K}}} \right)^2 \|\nabla v\|_K^2 + 2\frac{r_K^2}{c_{\mathbf{w},r,K}}\|v\|_K^2 \right\} \right\}^{\frac{1}{2}} |||\,\varphi\,|||_\Omega\ .
\end{aligned} \tag{7.2}$$

Adding the term $|||v|||_\Omega$, using once more the inequality $(a+b)^2 \leq 2(a^2+b^2)$, and noticing the definition of ζ_K by (4.5) concludes the proof. □

Lemma 7.4 (Upwinding estimate)**.** *There holds*

$$\sup_{\varphi\in H_0^1(\Omega),\,|||\varphi|||_\Omega=1} T_{\rm U}(\varphi) \le \left\{\sum_{\sigma\in\mathcal{E}_h}\eta_\sigma^2\right\}^{\frac12},$$

where η_σ is given by (4.7).

PROOF:

We have, for an arbitrary $\varphi\in H_0^1(\Omega)$,

$$T_{\rm U}(\varphi) = \sum_{\sigma_{K,L}\in\mathcal{E}_h^{\rm int}} \langle(\widehat{p}_{\sigma_{K,L}}-\tilde{p}_{\sigma_{K,L}})\mathbf{w}\cdot\mathbf{n}_K,\varphi_K-\varphi_L\rangle_{\sigma_{K,L}} + \sum_{\sigma_K\in\mathcal{E}_h^{\rm ext}} \langle(\hat{p}_{\sigma_K}-\tilde{p}_{\sigma_K})\mathbf{w}\cdot\mathbf{n},\varphi_K\rangle_{\sigma_K},$$

using that $\mathbf{w}\cdot\mathbf{n}$ is constant on each $\sigma\in\mathcal{E}_h$ and Lemma 6.1. Let $\sigma_{K,L}\in\mathcal{E}_h^{\rm int}$ and let us put $\varphi_{\sigma_{K,L}} := \langle\varphi,1\rangle_{\sigma_{K,L}}/|\sigma_{K,L}|$. Then

$$\begin{aligned}\|\varphi_K-\varphi_L\|_{\sigma_{K,L}} &\le \|\varphi_K-\varphi_{\sigma_{K,L}}\|_{\sigma_{K,L}} + \|\varphi_L-\varphi_{\sigma_{K,L}}\|_{\sigma_{K,L}}\\ &\le \max_{M=\{K,L\}}\left\{\frac{C_{{\rm F},d}h_M}{(d-1)\kappa_M c_{\mathbf{S},M}}\right\}^{\frac12}(|||\varphi\,|||_K + |||\,\varphi|||_L)\end{aligned}$$

by the triangle inequality and the first estimate of Lemma 9.2 from Section 9 below. At the same time,

$$\|\varphi_K-\varphi_L\|_{\sigma_{K,L}} \le \|\varphi_K\|_{\sigma_{K,L}} + \|\varphi_L\|_{\sigma_{K,L}} \le \max_{M=\{K,L\}}\left\{\frac{1}{(d-1)\kappa_M h_M c_{\mathbf{w},r,M}}\right\}^{\frac12}(|||\varphi\,|||_K + |||\,\varphi|||_L),$$

using the triangle inequality and the second estimate of Lemma 9.2. Similar estimates on $\|\varphi_K\|_{\sigma_K}$ for $\sigma_K\in\mathcal{E}_h^{\rm ext}$ follow directly from Lemma 9.2 using that $\varphi_{\sigma_K}=0$ by $\varphi\in H_0^1(\Omega)$. Hence

$$\begin{aligned}T_{\rm U}(\varphi) &\le \sum_{\sigma\in\mathcal{E}_h}\left\{\|(\hat{p}_\sigma-\tilde{p}_\sigma)\mathbf{w}\cdot\mathbf{n}\|_\sigma\frac{m_\sigma}{\sqrt{2(d+1)}}\sum_{K;\,\sigma\in\mathcal{E}_K}|||\varphi|||_K\right\}\\ &\le \left\{\sum_{\sigma\in\mathcal{E}_h}\|(\hat{p}_\sigma-\tilde{p}_\sigma)\mathbf{w}\cdot\mathbf{n}\|_\sigma^2\frac{m_\sigma^2}{2(d+1)}\right\}^{\frac12}\left\{\sum_{\sigma\in\mathcal{E}_h}\sum_{K;\,\sigma\in\mathcal{E}_K}2\,|||\,\varphi|||_K^2\right\}^{\frac12} = \left\{\sum_{\sigma\in\mathcal{E}_h}\eta_\sigma^2\right\}^{\frac12}|||\,\varphi|||_\Omega\end{aligned}$$

with m_σ and η_σ given respectively by (4.6) and (4.7), using the Cauchy–Schwarz inequality, the inequality $(a+b)^2\le 2a^2+2b^2$ and the fact that each side is shared by at most two simplices, and the fact that each simplex has exactly $d+1$ sides. □

Lemmas 7.1–7.4 together prove Theorems 4.2–4.3.

7.2 Proofs of the efficiency of the estimates

Lemma 7.5 (Efficiency of the residual estimator)**.** *Let $K\in\mathcal{T}_h$ and let η_K be the residual estimator given by (4.2). There holds*

$$\eta_K \le C_2\,|||\,p-\tilde{p}_h\,|||_K\left\{\left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}+\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right)+\min\{{\rm Pe}_K,\varrho_K\}\right\},$$

where ${\rm Pe}_K$ and ϱ_K are given by (4.3) and where the constant C_2 is independent of h_K, $\mathbf{S}$, $\mathbf{w}$, and r.

Proof:

The proof follows the one given in [37]. Let ψ_K by the bubble function on K, given as the product of the $d+1$ linear functions that take the value 1 at one vertex of K and vanish at the other vertices, and let us denote $v := (f + \nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h)$ (note that v is a polynomial in K). Then an appropriate modification of [37, Lemma 3.3] gives

$$\begin{aligned} c\,\|v\|_K^2 &\le (v, \psi_K v)_K\,,\\ \|\psi_K v\|_K &\le \|v\|_K\,,\\ |||\psi_K v|||_K &\le C\min\left\{\frac{h_K}{\sqrt{c_{\mathbf{S},K}}}, \frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\right\}^{-1}\|v\|_K \end{aligned}$$

with the constants c and C depending on the polynomial degree k of f, d, and κ_K but independent of h_K, $\mathbf{S}$, $\mathbf{w}$, and r. Next, we immediately have (cf. the proof of Lemma 7.1)

$$\mathcal{B}(p - \tilde{p}_h, \psi_K v) = (v, \psi_K v)_K\,,$$

and, using (2.10),

$$\mathcal{B}(p - \tilde{p}_h, \psi_K v) \le \left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right) |||\, p - \tilde{p}_h\, |||_K\, |||\psi_K v\, |||_K + \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}\, |||\, p - \tilde{p}_h\, |||_K\, \|\psi_K v\|_K\,.$$

Combining the above estimates, one comes to

$$c\|v\|_K^2 \le |||p - \tilde{p}_h\,|||_K\,\|v\|_K\left\{\left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right) C\min\left\{\frac{h_K}{\sqrt{c_{\mathbf{S},K}}}, \frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\right\}^{-1} + \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}\right\}.$$

Considering the definition of η_K by (4.2) and of Pe_K and ϱ_K by (4.3) concludes the proof. □

Lemma 7.6 (Efficiency of the nonconformity estimator)**.** *Let $K \in \mathcal{T}_h$ and let ζ_K be the nonconformity estimator given by (4.5). There holds*

$$\begin{aligned} \zeta_K^2 \le\ & C_3\min\left\{\frac{\alpha_{*,K}}{\min\limits_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}} + \min\left\{\frac{\beta_{*,K}}{\min\limits_{L;L\cap K\neq\emptyset} c_{\mathbf{w},r,L}}, \frac{\beta_{*,K}h_K^2}{\min\limits_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}}\right\},\right.\\ & \left.\frac{\alpha_{\#,K}}{\min\limits_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}} + \min\left\{\frac{\beta_{\#,K}}{\min\limits_{L;L\cap K\neq\emptyset} c_{\mathbf{w},r,L}}, \frac{\beta_{\#,K}h_K^2}{\min\limits_{L;L\cap K\neq\emptyset} c_{\mathbf{S},L}}\right\}\right\}\sum_{L;L\cap K\neq\emptyset} |||p - \tilde{p}_h\,|||_L^2\\ & + C_3\max\{\beta_{*,K}, \beta_{\#,K}\}\inf_{s_h\in\mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)}\sum_{L;L\cap K\neq\emptyset}\|p - s_h\|_L^2\,, \end{aligned}$$

where the constants $\alpha_{*,K}$, $\beta_{*,K}$, $\alpha_{\#,K}$, and $\beta_{\#,K}$, $K\in\mathcal{T}_h$, are defined in Section 4.3 and where the constant C_3 only depends on the space dimension d and on the shape regularity parameter $\kappa_{\mathcal{T}}$.

Proof:

Throughout this proof, let C denote a constant only depending on d and on $\kappa_{\mathcal{T}}$, not necessarily the same at each occurrence. We first show that

$$|||\,\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{*,K}^2 \le C\left(\alpha_{*,K}\sum_{\sigma;\sigma\cap K\neq\emptyset} h_\sigma^{-1}\|[\tilde{p}_h]\|_\sigma^2 + \beta_{*,K}\sum_{\sigma;\sigma\cap K\neq\emptyset} h_\sigma\|[\tilde{p}_h]\|_\sigma^2\right). \tag{7.3}$$

The first part of the estimate follows directly from Lemma 4.1 and the definition of $|||\cdot|||_{*,K}$. To estimate $\beta_{*,K}\|\tilde p_h-\mathcal{I}_{\rm MO}(\tilde p_h)\|_K^2$, we notice that the means of $\tilde p_h-\mathcal{I}_{\rm MO}(\tilde p_h)$ over all sides of a simplex $K\in\mathcal{T}_h$ are by the construction of the modified Oswald interpolation operator equal to 0. Hence

$$\|\tilde p_h-\mathcal{I}_{\rm MO}(\tilde p_h)\|_K^2\le C_{{\rm F},d}h_K^2\|\nabla(\tilde p_h-\mathcal{I}_{\rm MO}(\tilde p_h))\|_K^2$$

by the generalized Friedrichs inequality (2.2). The fact that h_K/h_σ for $K\cap\sigma\neq\emptyset$ only depends on $\kappa_{\mathcal{T}}$, which will be used in the sequel as well, and another use of Lemma 4.1 proves the second part of the estimate.

We will next use the inequality

$$h_\sigma^{-\frac12}\|[\tilde p_h]\|_\sigma\le C\sum_{L;\sigma\in\mathcal{E}_L}\|\nabla(\tilde p_h-\varphi)\|_L$$

established in [2, Theorem 10] for an arbitrary $\varphi\in H_0^1(\Omega)$. This inequality implies that

$$h_\sigma^\gamma\|[\tilde p_h]\|_\sigma^2\le C\frac{h_\sigma^{\gamma+1}}{\min_{L;\sigma\in\mathcal{E}_L}c_{\mathbf{S},L}}\sum_{L;\sigma\in\mathcal{E}_L}c_{\mathbf{S},L}\|\nabla(\tilde p_h-p)\|_L^2\,, \tag{7.4}$$

where we put $\gamma=-1,1$. Next, for an arbitrary $s_h\in\mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)$,

$$\begin{aligned} h_\sigma^{\frac12}\|[\tilde p_h]\|_\sigma &\le h_\sigma C\sum_{L;\sigma\in\mathcal{E}_L}\|\nabla(\tilde p_h-s_h)\|_L\le C\sum_{L;\sigma\in\mathcal{E}_L}h_L\|\nabla(\tilde p_h-s_h)\|_L\le C\sum_{L;\sigma\in\mathcal{E}_L}\|\tilde p_h-s_h\|_L\\ &\le C\sum_{L;\sigma\in\mathcal{E}_L}\|\tilde p_h-p\|_L+C\sum_{L;\sigma\in\mathcal{E}_L}\|p-s_h\|_L\,,\end{aligned}$$

using the inverse inequality given by [13, Theorem 3.2.6] and the triangle inequality. Hence

$$h_\sigma\|[\tilde p_h]\|_\sigma^2\le C\frac{1}{\min_{L;\sigma\in\mathcal{E}_L}c_{\mathbf{w},r,L}}\sum_{L;\sigma\in\mathcal{E}_L}c_{\mathbf{w},r,L}\|\tilde p_h-p\|_L^2+C\sum_{L;\sigma\in\mathcal{E}_L}\|p-s_h\|_L^2 \tag{7.5}$$

holds as well, which gives a sense when all $c_{\mathbf{w},r,L}$ for L such that $\sigma\in\mathcal{E}_L$ are nonzero. Combining estimates (7.3)–(7.5) while estimating $\min_{L;\sigma\in\mathcal{E}_L}\mu_L$ for a side σ such that $\sigma\cap K\neq\emptyset$ from below by $\min_{L;L\cap K\neq\emptyset}\mu_L$ concludes the proof for $|||\tilde p_h-\mathcal{I}_{\rm MO}(\tilde p_h)|||_{*,K}$. The proof for $|||\tilde p_h-\mathcal{I}_{\rm MO}(\tilde p_h)|||_{\#,K}$ is completely similar. □

Lemma 7.7 ((Non)efficiency of the upwinding estimator). *Let η_σ, $\sigma\in\mathcal{E}_h$, be the upwinding estimators given by (4.7). Then*

$$\sum_{\sigma\in\mathcal{E}_h}\eta_\sigma^2\le C_4\max_{\sigma\in\mathcal{E}_h}\varrho_\sigma\max_{K\in\mathcal{T}_h}\tilde\varrho_K\min\left\{\frac12\sum_{K\in\mathcal{T}_h}\frac{\|f\|_K^2}{c_{\mathbf{w},r,K}},\|f\|_\Omega^2\frac{C_{\rm DF}}{c_{\mathbf{S},\Omega}}\right\},$$

where ϱ_σ, $\tilde\varrho_K$, and $c_{\mathbf{S},\Omega}$ are given by (4.8), $C_{\rm DF}$ is the constant from the discrete Friedrichs inequality (2.5), and where the constant C_4 only depends on the space dimension d and on the shape regularity parameter $\kappa_{\mathcal{T}}$.

PROOF:
Using the definition of $\hat p_\sigma$ for $\sigma\in\mathcal{E}_h^{\rm int}$ by (3.4), the fact that $0\le\nu_\sigma\le1/2$, and the estimate (9.2) from Lemma 9.2 below, we have

$$\begin{aligned}\|\hat p_\sigma-\tilde p_\sigma\|_\sigma &\le \|(1-\nu_\sigma)(p_K-\tilde p_\sigma)+\nu_\sigma(p_L-\tilde p_\sigma)\|_\sigma\\ &\le \max_{K;\sigma\in\mathcal{E}_K}\left\{\frac{C_{{\rm F},d}h_K}{(d-1)\kappa_K}\right\}^{\frac12}(\|\nabla\tilde p_h\|_K+\|\nabla\tilde p_h\|_L)\end{aligned}$$

for suitable denotation K, L of the two elements sharing σ. For $\sigma \in \mathcal{E}_h^{\rm ext}$, a similar estimate holds. Hence, for $\sigma \in \mathcal{E}_h^{\rm int}$,

$$\begin{aligned}
\eta_\sigma^2 \quad &\leq \quad 4(d+1)\left(\frac{C_{{\rm F},d}}{d-1}\right)^2 \max\left\{\frac{1}{\kappa_K}\right\}^2 \min\left\{\frac{\min h_K}{\max c_{\mathbf{S},K}}\frac{\max h_K}{\min h_K}\frac{\max c_{\mathbf{S},K}}{\min c_{\mathbf{S},K}},\right.\\
&\left.\frac{1}{\min h_K \max c_{\mathbf{w},r,K}}\frac{\max c_{\mathbf{w},r,K}}{\min c_{\mathbf{w},r,K}}\right\} \min\{C_{\mathbf{w},K}\}^2 \min h_K \frac{\max h_K}{\min h_K}(\|\nabla\tilde{p}_h\|_K^2 + \|\nabla\tilde{p}_h\|_L^2)\\
&\leq \quad 4(d+1)\left(\frac{C_{{\rm F},d}}{d-1}\right)^2 \max\left\{\frac{1}{\kappa_K}\right\}^2 \left(\frac{\max h_K}{\min h_K}\right)^2 \varrho_\sigma \min\left\{\min\{{\rm Pe}_K\}^2, \min\{\varrho_K\}^2 \frac{\max c_{\mathbf{w},r,K}}{\min c_{\mathbf{w},r,K}}\right\}\\
&\quad (c_{\mathbf{S},K}\|\nabla\tilde{p}_h\|_K^2 + c_{\mathbf{S},L}\|\nabla\tilde{p}_h\|_L^2)\,,
\end{aligned}$$

where, if the minimum or maximum is not specified, it is understood over $\{K; \sigma \in \mathcal{E}_K\}$, i.e. over the two elements sharing σ. This estimate holds true for $\sigma \in \mathcal{E}_h^{\rm ext}$ as well. The assertion of the lemma follows by observing that $\max h_K / \min h_K$ for neighboring elements only depends on $\kappa_\mathcal{T}$, that each simplex has $(d+1)$ sides, and reordering the sum over sides to a sum over elements, so that the term $\sum_{K\in\mathcal{T}_h} c_{\mathbf{S},K}\|\nabla\tilde{p}_h\|_K^2$ appeared, and by estimating this term using Lemma 6.5. □

Lemmas 7.5–7.7 together prove Theorem 4.4.

8 Numerical experiments

We test our a posteriori error estimates on two model problems in this section. The first problem contains a strongly inhomogeneous diffusion–dispersion tensor and the second one is convection-dominated. In both cases, the analytical solution is known.

8.1 Model problem with strongly inhomogeneous diffusion–dispersion tensor

This model problem is taken from [34, 18] and is motivated by the fact that in real applications, the diffusion–dispersion tensor $\mathbf{S}$ may be discontinuous and strongly inhomogeneous. We consider in particular $\Omega = (-1,1) \times (-1,1)$ and the equation (1.1a) with $\mathbf{w} = 0$, $r = 0$, and $f = 0$. We suppose that Ω is divided into four subdomains Ω_i corresponding to the axis quadrants (in the counterclockwise direction) and that $\mathbf{S}$ is constant and equal to $s_i\, Id$ in Ω_i. Under such conditions, analytical solution writing

$$p(r,\theta) = r^\alpha(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

in each Ω_i can be found. Here (r, θ) are the polar coordinates in Ω, a_i and b_i are constants depending on Ω_i, and α is a parameter. This solution is continuous across the interfaces but only the normal component of its flux $\mathbf{u} = -\mathbf{S}\nabla p$ is continuous; it finally exhibits a singularity in the origin. We assume Dirichlet boundary conditions given by this solution and consider two different sets of the coefficients:

$\alpha = 0.53544095$		$\alpha = 0.12690207$	
$a_1 = 0.44721360$	$b_1 = 1$	$a_1 = 0.1$	$b_1 = 1$
$a_2 = -0.74535599$	$b_2 = 2.33333333$	$a_2 = -9.60396040$	$b_2 = 2.96039604$
$a_3 = -0.94411759$	$b_3 = 0.55555556$	$a_3 = -0.48035487$	$b_3 = -0.88275659$
$a_4 = -2.40170264$	$b_4 = -0.48148148$	$a_4 = 7.70156488$	$b_4 = -6.45646175$.

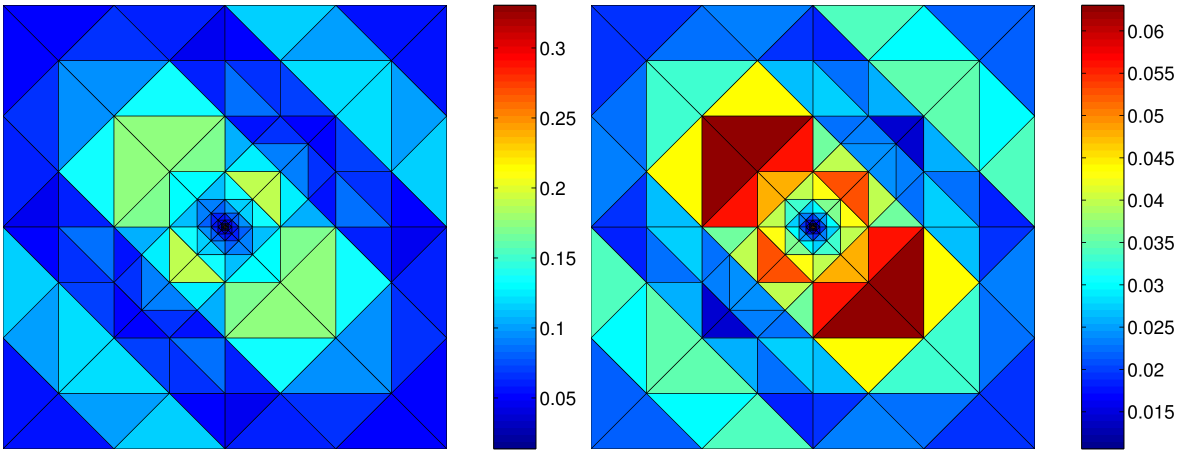

Figure 1: Estimated (left) and real (right) error distribution, $\alpha = 0.53544095$ (the maximum is attained at the origin)

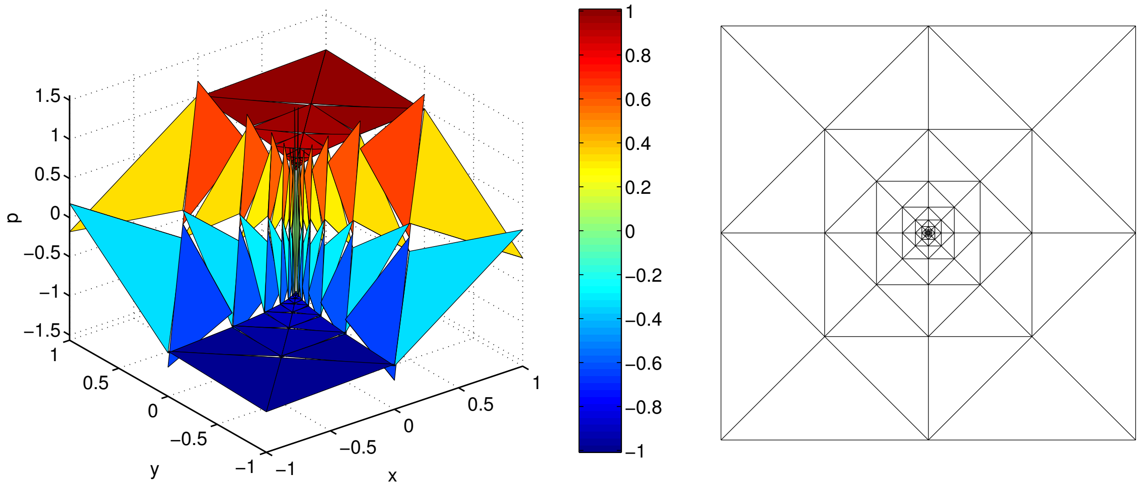

Figure 2: Approximate solution and the corresponding adaptively refined mesh, $\alpha = 0.12690207$

In the first case, $s_1 = s_3 = 5$, $s_2 = s_4 = 1$, whereas in the second one, $s_1 = s_3 = 100$, $s_2 = s_4 = 1$. The original grid consisted of 24 right-angled triangles and we have refined it either uniformly (up to 5 refinements) or adaptively on the basis of our estimator. In the latter case, we refine each element where the estimated $|||\cdot|||_\Omega$-error is greater than the half of the maximum of the estimators regularly into four sub-elements and than use the "longest edge" refinement to recover an admissible mesh. The estimate (5.3) for pure diffusion problems was used. In fact, in the given case, the residual estimators η_K (5.4) are zero for each $K \in \mathcal{T}_h$ (recall that this would be the case for general piecewise constant f, cf. Section 5.4), and hence the a posteriori error estimate is entirely given by the nonconformity estimators ζ_K (5.5).

We give in Figure 1 an example of our a posteriori estimate on the error and its distribution and the actual error and its distribution on an adaptively refined mesh for the first test case. We can see that the predicted distribution is excellent and that in particular even in this case where the solution is smoother, the singularity is well recognized. Next, Figure 2 gives an example of the approximate solution on an adaptively refined mesh and this mesh in the second test case. Here, the singularity is much more important and consequently the grid is highly refined around the

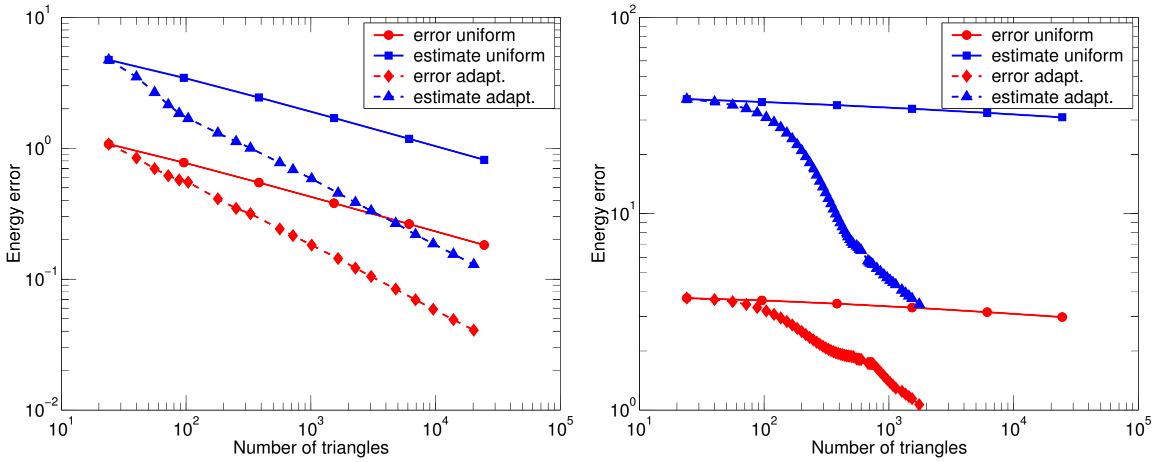

Figure 3: Estimated and real error against the number of elements in uniformly/adaptively refined meshes for $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right)

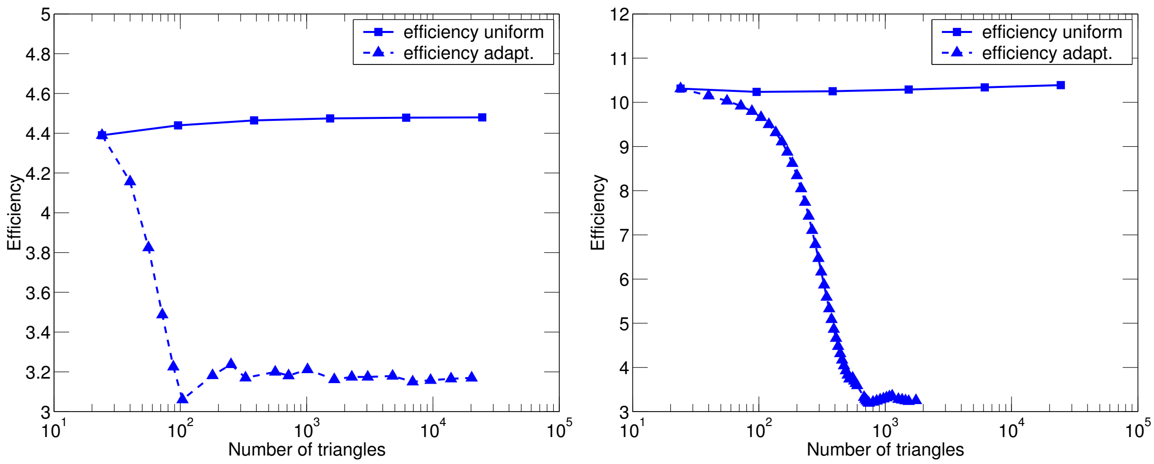

Figure 4: Overall efficiency of the a posteriori error estimates against the number of elements in uniformly/adaptively refined meshes for $\alpha = 0.53544095$ (left) and $\alpha = 0.12690207$ (right)

origin (for an adaptively refined grid of 1800 triangles, the diameter of the smallest triangles near the origin is 10^{-16} and 73% of the triangles are contained in the circle of radius 0.1). Figure 3 then reports the estimated and actual errors of the numerical solutions on uniformly/adaptively refined grids in the two test cases. The energy norm (2.7) was approximated with a 7-point quadrature formula in each triangle. It can be seen from these plots that one can substantially reduce the number of unknowns necessary to attain the prescribed precision using the derived a posteriori error estimates and adaptively refined grids. Finally, we can see in Figure 4 the efficiency plots for the two cases, giving the ratio of the estimated $|||\cdot|||_\Omega$-error to the real $|||\cdot|||_\Omega$-error. This quantity simply expresses how many times we have overestimated the actual error—recall that there are no undetermined multiplicative constants in our estimates. These plots show that our estimator is almost asymptotically exact, and this even for the cases with strong inhomogeneities. Recall that we are able to prove this property theoretically for the alternative form of the nonconformity estimate, cf. Section 5.3. In the present case, instead of evaluating the infimum (5.2), we simply use an interpolate (the modified Oswald one, cf. Section 4.2), but it shows to be sufficient.

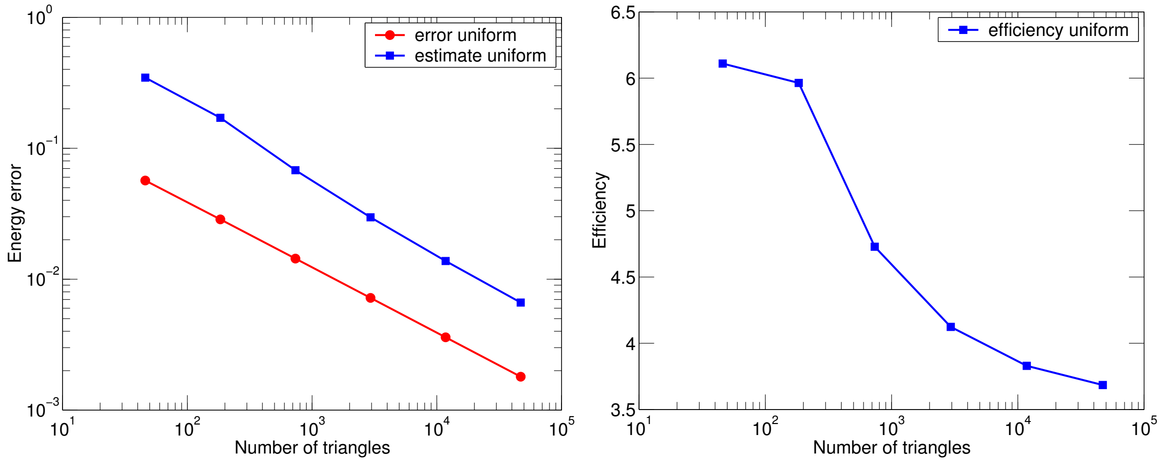

Figure 5: Estimated and real error (left) and overall efficiency (right) against the number of elements, $\varepsilon = 1$, $a = 0.5$

8.2 Convection-dominated model problem

This problem is a modification of a problem considered in [20]. We put $\Omega = (0, 1) \times (0, 1)$, $\mathbf{w} = (0, 1)$, and $r = 1$ in (1.1a) and consider three cases with $\mathbf{S} = \varepsilon\, Id$ and ε equal to, respectively, 1, 10^{-2}, and 10^{-4}. The right-hand side term f, Neumann boundary conditions on the upper side, and Dirichlet boundary conditions elsewhere are chosen such that the solution was

$$p(x, y) = 0.5 \left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right) .$$

This solution is in fact one-dimensional and possesses an internal layer of width a which we set, respectively, equal to 0.5, 0.05, and 0.02. We start the computations from an unstructured grid of Ω consisting of 46 triangles and refine it either uniformly (up to 5 refinements) or adaptively. We use the scheme described in Section 5.5.

For $\varepsilon = 1$ and $a = 0.5$ (diffusion-dominated regime), our estimator reproduced very precisely the distribution of the error. Estimated and actual errors in the energy norm (2.7) as well as the efficiency are reported in Figure 5. For $\varepsilon = 10^{-2}$ and $a = 0.05$ (convection-dominated regime on coarse meshes and diffusion-dominated regime with progressive refinement), still the distribution of the error is predicted very well, cf. Figure 6. Note in particular the correct localization of the error away from the center of the shock, as well as the sensitivity of our estimator to the shape of the elements. Next, an example of an adaptively refined mesh for $\varepsilon = 10^{-4}$ and $a = 0.02$ is given in Figure 7. For the two last test cases, the estimated and real errors are plotted against the number of elements in uniformly/adaptively refined meshes in Figure 8. Again, one can see that we can substantially reduce the number of unknowns necessary to attain the prescribed precision using the derived a posteriori error estimates and adaptively refined grids. For the given examples, our estimator tends to slightly overestimate the error in the shock region in the strongly convection-dominated regime, in great part thanks to the upwinding estimator. Once the local Péclet number (4.3) gets smaller than 2, the upwinding estimator disappears (thanks to using the combination of the centered and upwind-weighted schemes, cf. Section 5.5), and the biggest part of the estimated error passes to the nonconformity estimator, which itself is again asymptotically almost optimal (the residual estimator is only significant on rough grids). The efficiency for $\varepsilon = 10^{-2}$ and $a = 0.05$ and the finest grids is approximately 20 (and continues to

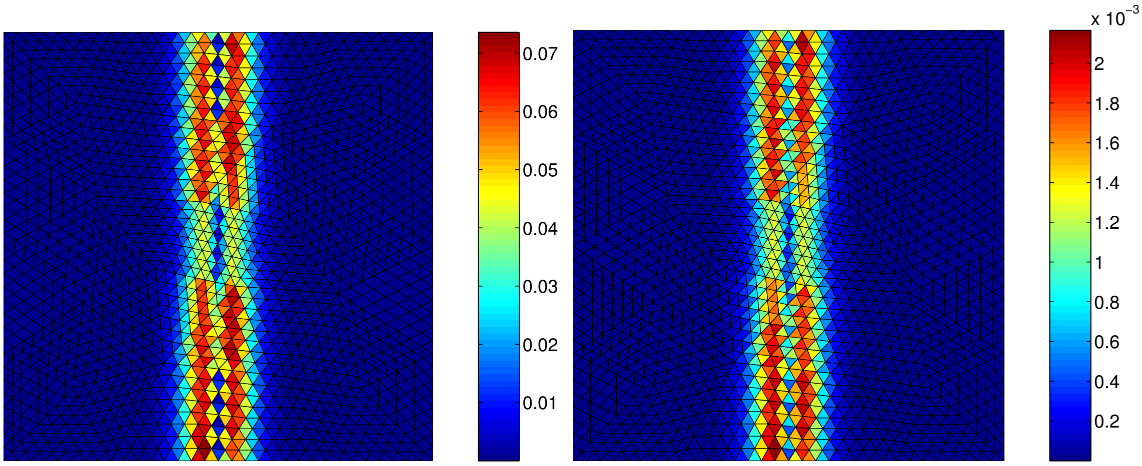

Figure 6: Estimated (left) and real (right) error distribution, $\varepsilon = 10^{-2}$, $a = 0.05$

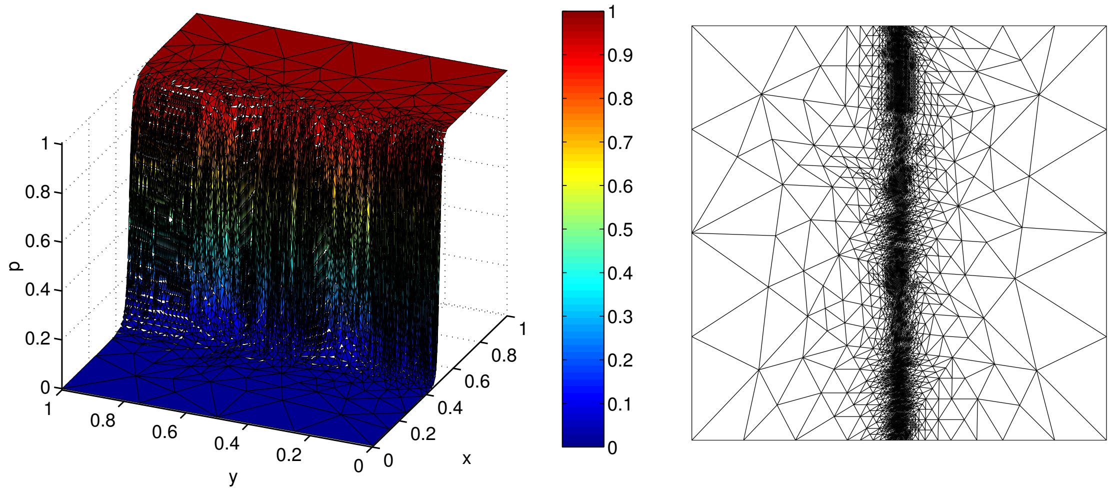

Figure 7: Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$

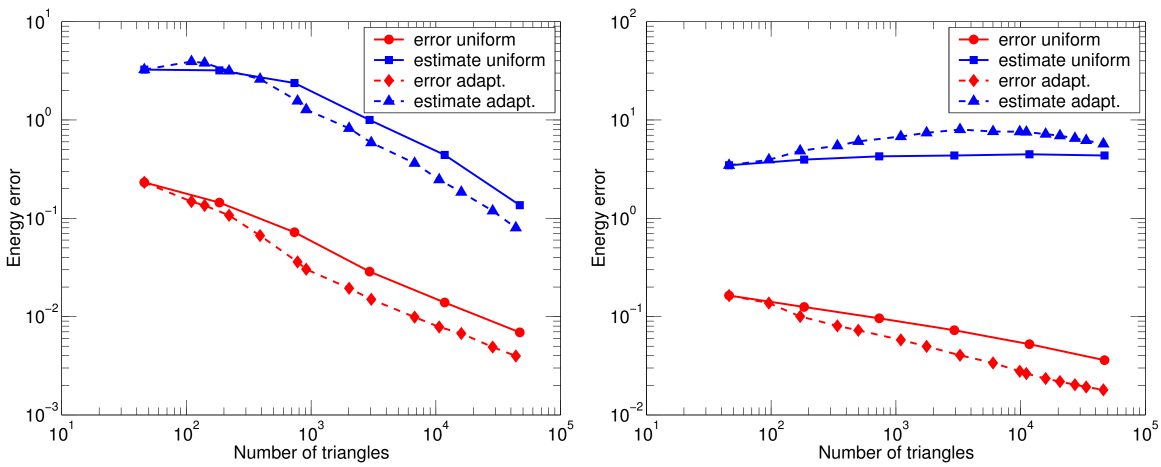

Figure 8: Estimated and real error against the number of elements in uniformly/adaptively refined meshes for $\varepsilon = 10^{-2}$, $a = 0.05$ (left) and $\varepsilon = 10^{-4}$, $a = 0.02$ (right)

decrease), whereas the efficiency for $\varepsilon = 10^{-4}$ and $a = 0.02$ only starts to decrease for the finest adaptively refined grids, where it is roughly equal to 320, as the elements in the shock region start to leave the convection-dominated regime.

9 Auxiliary results

We give in this section several auxiliary results that were needed in the paper.

Lemma 9.1. *Let $K \in \mathcal{T}_h$, let $\varphi \in H^1(K)$, and let φ_K be the mean of φ over K given by $\varphi_K := \int_K \varphi \,\mathrm{d}\mathbf{x}/|K|$. Then*

$$\|\varphi - \varphi_K\|_K^2 \le \min\left\{C_{\mathrm{P}}\frac{h_K^2}{c_{\mathbf{S},K}}, \frac{2}{c_{\mathbf{w},r,K}}\right\} |||\,\varphi\,|||_K^2 \ .$$

PROOF:
The Poincaré inequality (2.1) and the definition of $|||\cdot|||_K$ by (2.7) imply

$$\|\varphi - \varphi_K\|_K^2 \le C_{\mathrm{P}} h_K^2 \|\nabla\varphi\|_K^2 \le C_{\mathrm{P}}\frac{h_K^2}{c_{\mathbf{S},K}} |||\,\varphi\,|||_K^2 \ . \tag{9.1}$$

Next, the estimate

$$\|\varphi - \varphi_K\|_K^2 \le 2\|\varphi\|_K^2 \le \frac{2}{c_{\mathbf{w},r,K}} |||\,\varphi|||_K^2$$

follows from the triangle and Cauchy–Schwarz inequalities and the definition of $|||\cdot|||_K$ by (2.7). □

Lemma 9.2. *Let $K \in \mathcal{T}_h$, let $\varphi \in H^1(K)$, and let φ_K be the mean of φ over K given by $\varphi_K := \int_K \varphi \,\mathrm{d}\mathbf{x}/|K|$ and φ_σ be the mean of φ over $\sigma \in \mathcal{E}_K$ given by $\varphi_\sigma := \int_\sigma \varphi \,\mathrm{d}\gamma(\mathbf{x})/|\sigma|$, respectively. Then*

$$\|\varphi_K - \varphi_\sigma\|_\sigma \le \left\{\frac{C_{\mathrm{F},d} h_K}{(d-1)\kappa_K c_{\mathbf{S},K}}\right\}^{\frac{1}{2}} |||\,\varphi|||_K$$

and

$$\|\varphi_K\|_\sigma \le \left\{\frac{1}{(d-1)\kappa_K h_K c_{\mathbf{w},r,K}}\right\}^{\frac{1}{2}} |||\,\varphi\,|||_K \ .$$

PROOF:
We have

$$\|\varphi_K - \varphi_\sigma\|_\sigma = |\varphi_K - \varphi_\sigma||\sigma|^{\frac{1}{2}} \le h_K \left\{C_{\mathrm{F},d}\frac{|\sigma|}{|K|}\right\}^{\frac{1}{2}} \|\nabla\varphi\|_K \le h_K \left\{\frac{C_{\mathrm{F},d}}{(d-1)h_K\kappa_K}\right\}^{\frac{1}{2}} \|\nabla\varphi\|_K \ , \tag{9.2}$$

using the generalized Friedrichs inequality (2.2), the fact that $|\sigma| \le h_K^{d-1}/(d-1)$, and the definition of κ_K from Assumption (A). Using the definition of $|||\cdot|||_K$ by (2.7) concludes the proof of the first estimate.

For the second estimate, we have

$$\|\varphi_K\|_\sigma = |\varphi_K||\sigma|^{\frac{1}{2}} \le \|\varphi\|_K \left\{\frac{|\sigma|}{|K|}\right\}^{\frac{1}{2}} \le \left\{\frac{1}{(d-1)h_K\kappa_K}\right\}^{\frac{1}{2}} \|\varphi\|_K$$

by virtue of $|\sigma| \le h_K^{d-1}/(d-1)$ and of the definition of κ_K from Assumption (A). Using the definition of $|||\cdot|||_K$ by (2.7) concludes the proof of the second estimate. □

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