ACCURACY OF DIFFUSION APPROXIMATIONS FOR HIGH FREQUENCY MARKOV DATA. *

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We consider triangular arrays of Markov chains that converge weakly to a diffusion process. Edgeworth type expansions of third order for transition densities are proved. This is done for time horizons that converge to 0. For this purpose we represent the transition density as a functional of densities of sums of i.i.d. variables. This will be done by application of the parametrix method. Then we apply Edgeworth expansions to the densities. The resulting series gives our Edgeworth-type expansion for the transition density of Markov chains. The research is motivated by applications to high frequency data that are available on a very fine grid but are approximated by a diffusion model on a more rough grid.

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1 Introduction.

In this paper we study triangular arrays of Markov chains $X_{k,h}$ $(0 \leq k \leq n)$ that converge weakly to a diffusion process Y_s $(0 \leq s \leq T)$ for $n \to \infty$. Here $h = T/n$ denotes the discretization step. We allow that T depends on n. In particular, we consider the case that $T \to 0$ for $n \to \infty$. Our main result will give Edgeworth type expansions for the transition densities. The order of the expansions is $o(h^{-1-\delta}), \delta > 0$. This is done for time horizons that may converge to 0. The research is motivated by applications to high frequency data that are available on a very fine grid but are approximated by a diffusion model on a more rough grid. The work of this paper generalizes the results in Konakov and Mammen (2005) in two directions. The time horizon is allowed to converge to 0 and also cases are treated with nonhomogenous diffusion limits.

The theory of Edgeworth expansions is well developed for sums of independent random variables. For more general models approaches have been used where the expansion is reduced to models with sums of independent random variables. This is also the basic idea of our approach. We will make use of the parametrix method. In this approach the transition density is represented as a nested sum of functionals of densities of sums of independent variables. Plugging Edgeworth expansions into this representation will result in an expansion for the transition density.

Weak convergence of the distribution of scaled discrete time Markov processes to diffusions has been extensively studied in the literature [see Skorohod (1965) and Stroock and Varadhan (1979)]. Local limit theorems for Markov chains were given in Konakov and Molchanov (1984) and Konakov and Mammen (2000,2001,2002). In Konakov and Mammen (2000) it was shown that the transition density of a Markov chain converges with rate $O(n^{-1/2})$ to the transition density in the diffusion model. For the proof there an analytical approach was chosen that made essential use of the parametrix method. This method permits to obtain tractable representations of transition densities of diffusions that are based on Gaussian densities, see Lemma 1 below. Similar representations hold for discrete time Markov chains $X_{k,h}$, see Lemma 3 below. For a short exposition of the parametrix method, see Section 3 and Konakov and Mammen (2000). The parametrix method for Markov chains developped in Konakov and Mammen (2000) is exposed in Section 4. Applications to Markov random walks are given in Konakov and Mammen (2001). In Konakov and Mammen (2002) the approach is used to give Edgeworth-type expansions for Euler schemes for differential equations. Related treatments of Euler schemes can be found in [Bally and Talay (1996a,b), Protter and Talay (1997), Jacod and Protter (1998), Jacod (2004), Jacod, Kurtz, Méléard and Protter (2005) and Guyon (2006)]. Standard references for the parametrix method are the books by Friedman (1964) and Ladyženskaja, Solonnikov and Ural´ceva (1968) on parabolic PDEs [see also McKean and Singer (1967)].

The paper is organized as follows. In the next section we will present our model for the Markov chain and state our main result that gives an Edgeworth-type expansion for Markov chains. In Section 3 we will give a short introduction into the parametrix method for diffusions. In Section 4 we will recall the parametrix approach developed in Konakov and Mammen (2000) for Markov chains. Technical discussions, auxiliary results and proofs are given in Sections 5-7.

2 Results.

Let $n \geq 2$, $T = T(n) \leq 1$ and $h = T/n$. Suppose that $q(t, x, \cdot)$, $(t, x) \in [0, 1] \times \mathbb{R}^d$ is a given family of densities on $\mathbb{R}^d$, $\chi_\nu(t, x)$ is the $\nu - th$ cumulant corresponding to the density $q(t, x, \cdot)$ and m is a function from $[0, 1] \times \mathbb{R}^d$ into $\mathbb{R}^d$. We shall impose the following conditions

(A1) $\int_{\mathbb{R}^d} yq(t, x, y)\, dy = 0,\ 0 \leq t \leq 1,\ x \in \mathbb{R}^d.$

(A2) There exists positive constants σ_* and σ^* such that the covariance matrix $\sigma(t, x) = \int_{\mathbb{R}^d} yy^T q(t, x, y)\, dy$ satisfies

$$\sigma_* \leq \theta^T \sigma(t, x)\, \theta \leq \sigma^*,$$

for all $\|\theta\| = 1$ and $t \in [0, 1]$, $x \in \mathbb{R}^d$.

(A3) There exists a positive integer S' and a real nonnegative function $\psi(y)$, $y \in \mathbb{R}^d$ satisfying $\sup_{y \in \mathbb{R}^d} \psi(y) < \infty$ and $\int_{\mathbb{R}^d} \|y\|^S \psi(y)\, dy < \infty$ with $S = (S' + 2)d + 4$ such that

$$\left|D_y^\nu q(t, x, y)\right| \leq \psi(y),\ t \in [0, 1],\ x, y \in \mathbb{R}^d\ \ |\nu| = 0, 1, 2, 3, 4$$

$$|D_x^\nu q(t, x, y)| \leq \psi(y),\ t \in [0, 1],\ x, y \in \mathbb{R}^d\ \ |\nu| = 0, 1, 2.$$

It follows from **(A2),** Lemma 5 and (16),(17) below (applied with $h = 1$) that the following condition holds

(A3′) For all $x, y \in R$, $h > 0, 0 \leq t, t + jh \leq 1, j \geq j_0$, with j_0 does not depending on x, t

$$\left|D_x^\nu q^{(j)}(t, x, y)\right| \leq Cj^{-d/2}\psi\left(j^{-1/2}y\right), |\nu| = 0, 1, 2, 3$$

for a constant $C < \infty$. Here $q^{(j)}(t, x, y)$ denotes the $j-$ fold convolution of q for fixed x as a function of y.

$$q^{(j)}(t, x, y) = \int q^{(j-1)}(t, x, u) q(t + (j - 1)h, x, y - u) du,$$

$q^{(1)}(t, x, y) = q(t, x, y).$

It follows also from (A3) that for $1 \leq j \leq j_0$

$$\int \|y\|^S q^{(j)}(t,x,y)dy \leq C(j_0).$$

(B1) The functions $m(t,x)$ and $\sigma(t,x)$ and their first and second derivatives w.r.t. t and their derivatives up to the order six w.r.t. x are continuous and bounded uniformly in t and x. All these functions are Lipschitz continuous with respect to x with a Lipschitz constant that does not depend on t. The function $\chi_\nu(t,x)$, $|\nu| = 3, 4$, is Lipschitz continuous with respect to t with a Lipschitz constant that does not depend on x. A sufficient condition for this is the following inequality

$$\int_{\mathbb{R}^d} (1 + \|z\|^4) |q(t,x,z) - q(t',x,z)| \, dz \leq C |t - t'|, 0 \leq t, t' \leq 1,$$

with a constant that does not depend on $x \in \mathbb{R}^d$. Furthemore, $D_x^\nu \sigma(t,x)$ exists for $|\nu| = 6$ and is Holder continuous w.r.t. x with positive exponent and a constant that does not depend on t.

(B2) There exist $\varkappa < \frac{1}{5}$ such that $\liminf_{n\to\infty} T(n)n^\varkappa > 0$ (remind that we consider $T(n) \leq 1$).

Consider a family of Markov processes in $\mathbb{R}^d$ of the following form

$$(1) \qquad X_{k+1,h} = X_{k,h} + m(kh, X_{k,h})h + \sqrt{h}\xi_{k+1,h}, \ X_{0,h} = x \in \mathbb{R}^d, \ k = 0, ..., n-1,$$

where $(\xi_{i,h})_{i=1,\ldots,n}$ is an innovation sequence satisfying the Markov assumption: the conditional distribution of $\xi_{k+1,h}$ given $X_{k,h} = x_k, ..., X_{0,h} = x_0$ depends only on $X_{k,h} = x_k$ and has conditional density $q(kh, x_k, \cdot)$. The conditional covariance matrix corresponding to this density is $\sigma(kh, x_k)$. The transition densities of $(X_{i,h})_{i=1,\ldots,n}$ are denoted by $p_h(0, kh, x, \cdot)$.

We will consider the process (1) as an approximation to the following stochastic differential equation in $\mathbb{R}^d$:

$$dY_s = m(s, Y_s) ds + \Lambda(s, Y_s) dW_s, \ Y_0 = x \in \mathbb{R}^d, \ s \in [0, T],$$

where $(W_s)_{s\geq 0}$ is the standard Wiener process and Λ is a symmetic positive definite $d \times d$ matrix such that $\Lambda(s,y)\Lambda(s,y)^T = \sigma(s,y)$. The conditional density of Y_t, given $Y_0 = x$ is denoted by $p(0,t,x,\cdot)$. We shall consider the following differential operators L and $\widetilde{L}$:

$$Lf(s,t,x,y) = \frac{1}{2} \sum_{i,j=1}^d \sigma_{ij}(s,x) \frac{\partial^2 f(s,t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^d m_i(s,x) \frac{\partial f(s,t,x,y)}{\partial x_i},$$

$$(2) \qquad \tilde{L}f(s,t,x,y) = \frac{1}{2} \sum_{i,j=1}^d \sigma_{ij}(s,y) \frac{\partial^2 f(s,t,x,y)}{\partial x_i \partial x_j} + \sum_{i=1}^d m_i(s,y) \frac{\partial f(s,t,x,y)}{\partial x_i}.$$

To formulate our main result we need also the following operators

$$L'f(s,t,x,y)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial\sigma_{ij}(s,x)}{\partial s}\frac{\partial^2 f\,(s,t,x,y)}{\partial x_i\partial x_j}+\sum_{i=1}^{d}\frac{\partial m_i(s,x)}{\partial s}\frac{\partial f\,(s,t,x,y)}{\partial x_i}$$

$$\widetilde{L}'f(s,t,v,z)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial\sigma_{ij}(s,y)}{\partial s}\frac{\partial^2 f\,(s,t,x,y)}{\partial x_i\partial x_j}+\sum_{i=1}^{d}\frac{\partial m_i(s,y)}{\partial s}\frac{\partial f\,(s,t,x,y)}{\partial x_i}. \tag{3}$$

and the convolution type binary operation $\otimes$:

$$f\otimes g\,(s,t,x,y)=\int_s^t du\int_{R^d} f\,(s,u,x,z)\,g\,(u,t,z,y)\,dz.$$

Konakov and Mammen (2000) obtained a nonuniform rate of convergence for the difference $p_h\,(0,T,x,\cdot)-p\,(0,T,x,\cdot)$ as $n\to\infty$ in the case $T\asymp 1$. Konakov (2006) proved an analogous result for the case $T=o\,(1)$. Edgeworth type expansions for the case $T\asymp 1$ and homogenous diffusions were obtained in Konakov and Mammen (2005). The goal of the present paper is to obtain the Edgeworth type expansions for nonhomegenious case and for both cases $T\asymp 1$ or $T=o\,(1)$. The following theorem contains our main result. It gives Edgeworth type expansions for p_h. For the statement of the theorem we introduce the following differential operators

$$\begin{aligned}\mathcal{F}_1[f](s,t,x,y) &= \sum_{|\nu|=3}\frac{\chi_\nu(s,x)}{\nu!}D_x^\nu f(s,t,x,y),\\ \mathcal{F}_2[f](s,t,x,y) &= \sum_{|\nu|=4}\frac{\chi_\nu(s,y)}{\nu!}D_x^\nu f(s,t,x,y).\end{aligned}$$

Furthermore, we introduce two terms corresponding to the classical Edgeworth expansion

$$\widetilde{\pi}_1(s,t,x,y) = (t-s)\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(s,t,y)}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y), \tag{4}$$

$$\begin{aligned}\widetilde{\pi}_2(s,t,x,y) &= (t-s)\sum_{|\nu|=4}\frac{\overline{\chi}_\nu(s,t,y)}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y)\\ &\quad+\frac{1}{2}(t-s)^2\left\{\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(s,t,y)}{\nu!}D_x^\nu\right\}^2\widetilde{p}(s,t,x,y),\end{aligned} \tag{5}$$

where

$$\overline{\chi}_\nu(s,t,y)=\frac{1}{t-s}\int_s^t\chi_\nu(u,y)du,$$

and where $\chi_\nu(t,x)$ is the $\nu-th$ cumulant of the density of the innovations $q(t,x,\cdot)$. The gaussian transition densities $\widetilde{p}(s,t,x,y)$ are defined in (6). Note, that in the homogenous case $\chi_\nu(u,y)\equiv\chi_\nu(y)$ and $\overline{\chi}_\nu(s,t,y)\equiv\chi_\nu(y)$, where $\chi_\nu(y)$ is the $\nu-th$ cumulant of the density $q(y,\cdot)$.

Theorem 1. *Assume (A1)-(A3), (B1),(B2). Then there exists a constant $\delta>0$ such that the following expansion holds:*

$$\sup_{x,y\in R^d}\left[T^{d/2}\left(1+\left\|\frac{y-x}{\sqrt{T}}\right\|^{S'}\right)\times|p_h(0,T,x,y)-p(0,T,x,y)\right.$$
$$\left.-h^{1/2}\pi_1(0,T,x,y)-h\pi_2(0,T,x,y)|\right]=O(h^{1+\delta}),$$

where S' is defined in Assumption (A3) and where

$$\begin{aligned}\pi_1(0,T,x,y) &= p\otimes\mathcal{F}_1[p])(0,T,x,y),\\ \pi_2(0,T,x,y) &= (p\otimes\mathcal{F}_2[p])(0,T,x,y)+p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p]](0,T,x,y)\\ &\quad+\frac{1}{2}p\otimes(L_\star^2-L^2)p(0,T,x,y)-\frac{1}{2}p\otimes(L'-\widetilde{L}')p(0,T,x,y).\end{aligned}$$

Here $p(s,t,x,y)$ is a transition density of the diffusion $Y(t)$ and the operator L_* is defined analogously to $\widetilde{L}$ but with the coefficients "frozen" at the point x. The norm $\|\bullet\|$ is the usual Euclidean norm.

Discussion and remarks

1. It can be easily shown that

$$\begin{aligned}\left|h^{1/2}\pi_1(0,T,x,y)\right| &\leq C_1n^{-1/2}T^{-d/2}\exp\left[-C_2\left\|\frac{y-x}{\sqrt{T}}\right\|^2\right],\\ \left|h\pi_2(0,T,x,y)\right| &\leq C_1n^{-1}T^{-d/2}\exp\left[-C_2\left\|\frac{y-x}{\sqrt{T}}\right\|^2\right],\end{aligned}$$

and that by definition of h

$$\left|O(h^{1+\delta})T^{-d/2}\left[1+\left\|\frac{y-x}{\sqrt{T}}\right\|^{S'}\right]^{-1}\right|\leq C_1n^{-1-\delta}T^{1+\delta-d/2}\left[1+\left\|\frac{y-x}{\sqrt{T}}\right\|^{S'}\right]^{-1}$$

with some positive constants C_1 and C_2.

2. If the innovation density $q(t,x,\cdot)$ does not depend on x then $L_*=L, L'=\widetilde{L}'$ and $p(s,t,x,y)=\widetilde{p}(s,t,x,y)$ where $\widetilde{p}$ is defined in (6) with $\sigma(s,t,y)=\sigma(s,t)=\int_s^t\sigma(u)du$ and $m(s,t,y)=m(s,t)=\int_s^t m(u)du$. This gives

$$\begin{aligned}\pi_1(0,T,x,y) &= \int_0^T ds\int\widetilde{p}(0,s,x,v)\sum_{|\nu|=3}\frac{\chi_\nu(s)}{\nu!}D_v^\nu\widetilde{p}(s,T,v,y)dv\\ &= -\sum_{|\nu|=3}\int_0^T\frac{\chi_\nu(s)}{\nu!}dsD_y^\nu\int\widetilde{p}(0,s,x,v)\widetilde{p}(s,T,v,y)dv\\ &= -\sum_{|\nu|=3}\frac{T}{\nu!}\overline{\chi}_\nu(0,T)D_y^\nu\widetilde{p}(0,T,x,y)=\widetilde{\pi}_1(0,T,x,y),\end{aligned}$$

$$\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}](s,T,z,y)=\int_s^T du\int\widetilde{p}(s,u,z,w)\sum_{|\nu|=3}\frac{\chi_\nu(u)}{\nu!}D_w^\nu\widetilde{p}(u,T,w,y)dw$$

$$=-\sum_{|\nu|=3}D_y^\nu\int_s^T\frac{\chi_\nu(u)}{\nu!}\widetilde{p}(s,T,z,y)=(T-s)\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(s,T)}{\nu!}D_z^\nu\widetilde{p}(s,T,z,y),$$

$$\mathcal{F}_1[\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}](s,T,z,y)=(T-s)\sum_{|\nu|=3}\frac{\chi_\nu(s)}{\nu!}D_z^\nu\left[\sum_{|\nu'|=3}\frac{\overline{\chi}_{\nu'}(s,T)}{\nu'!}D_z^{\nu'}\widetilde{p}(s,T,z,y)\right]$$

$$=(T-s)\sum_{|\nu|=3,|\nu'|=3}\frac{\chi_\nu(s)}{\nu!}\frac{\overline{\chi}_{\nu'}(s,T)}{\nu'!}D_z^{\nu+\nu'}\widetilde{p}(s,T,z,y),$$

$$\widetilde{p}\otimes\mathcal{F}_2[\widetilde{p}](0,T,x,y)+\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}](0,T,x,y)=T\sum_{|\nu|=4}\frac{\overline{\chi}_\nu(0,T)}{\nu!}D_x^\nu\widetilde{p}(0,T,x,y)$$

$$+\int_0^T ds\int\widetilde{p}(0,s,x,z)(T-s)\sum_{|\nu|=3,|\nu'|=3}\frac{\chi_\nu(s)}{\nu!}\frac{\overline{\chi}_{\nu'}(s,T)}{\nu'!}D_y^{\nu+\nu'}\widetilde{p}(s,T,z,y)$$

$$=T\sum_{|\nu|=4}\frac{\overline{\chi}_\nu(0,T)}{\nu!}D_x^\nu\widetilde{p}(0,T,x,y)$$

$$+\sum_{|\nu|=3,|\nu'|=3}\frac{1}{\nu!}\frac{1}{\nu'!}\int_0^T\chi_\nu(s)\left(\int_s^T\chi_{\nu'}(u)du\right)dsD_x^{\nu+\nu'}\widetilde{p}(s,T,x,y).$$

For $\nu=\nu'$ we have

$$\int_0^T\chi_\nu(s)\left(\int_s^T\chi_{\nu'}(u)du\right)ds=\frac{1}{2}\int_0^T\int_0^T\chi_\nu(s)\chi_\nu(u)dsdu=\frac{T^2}{2}\overline{\chi}_\nu(0,T)\overline{\chi}_\nu(0,T).$$

For $\nu\neq\nu'$ we consider

$$\int_0^T\chi_\nu(s)\left(\int_s^T\chi_{\nu'}(u)du\right)ds+\int_0^T\chi_{\nu'}(s)\left(\int_s^T\chi_\nu(u)du\right)ds$$
$$=\int_0^T\int_s^T\left[\chi_\nu(s)\chi_{\nu'}(u)+\chi_{\nu'}(s)\chi_\nu(u)\right]dsdu$$
$$=\frac{1}{2}\int_0^T\int_0^T\left[\chi_\nu(s)\chi_{\nu'}(u)+\chi_{\nu'}(s)\chi_\nu(u)\right]dsdu$$
$$=\frac{T^2}{2}\overline{\chi}_\nu(0,T)\overline{\chi}_{\nu'}(0,T)+\frac{T^2}{2}\overline{\chi}_{\nu'}(0,T)\overline{\chi}_\nu(0,T).$$

From the last equations we obtain

$$\widetilde{p}\otimes\mathcal{F}_2[\widetilde{p}](0,T,x,y)+\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}\otimes\mathcal{F}_1[\widetilde{p}](0,T,x,y)$$
$$=T\sum_{|\nu|=4}\frac{\overline{\chi}_\nu(0,T)}{\nu!}D_x^\nu\widetilde{p}(0,T,x,y)+\frac{T^2}{2}\left\{\sum_{|\nu|=3}\frac{\overline{\chi}_\nu(0,T)}{\nu!}D_x^\nu\right\}\widetilde{p}(0,T,x,y)$$
$$=\widetilde{\pi}_2(0,T,x,y).$$

Thus for this case we get the first two terms of the classical Edgeworth expansion $h^{1/2}\widetilde{\pi}_1(0,T,x,y) + h\widetilde{\pi}_2(0,T,x,y)$ for the sums of independent non identically distributed random vectors.

3. If $\chi_\nu(t,x) = 0$ for $|\nu| = 3$ and for $t \in [0,T] \times R^d$ then it holds that $\mathcal{F}_1 \equiv 0$. The Theorem 1 holds with

$$\begin{aligned}
\pi_1(0,T,x,y) &= 0, \\
\pi_2(0,T,x,y) &= (p \otimes \mathcal{F}_2[p])(0,T,x,y) \\
&\quad + \frac{1}{2} p \otimes (L_\star^2 - L^2) p(0,T,x,y) - \frac{1}{2} p \otimes (L' - \widetilde{L}') p(0,T,x,y).
\end{aligned}$$

If in addition $\chi_\nu(t,x) = 0$ for $|\nu| = 4$ then the first four moments of the innovations coincide with the first four moments of a normal distribution with zero mean and covariance matrix $\sigma(t,x)$. In this case we have $\mathcal{F}_2 = 0$ and

$$\begin{aligned}
\pi_1(0,T,x,y) &= 0, \\
\pi_2(0,T,x,y) &= \frac{1}{2} p \otimes (L_\star^2 - L^2) p(0,T,x,y) - \frac{1}{2} p \otimes (L' - \widetilde{L}') p(0,T,x,y)
\end{aligned}$$

and the first two terms of the Edgeworth expansion do not depend on the innovation density. In particular the Edgeworth expansion for the homogeneous Euler scheme holds with the same π_1 and π_2 as in the last two equations. For the homogenous case

$$\begin{aligned}
\pi_1(0,T,x,y) &= 0, \\
\pi_2(0,T,x,y) &= \frac{1}{2} p \otimes (L_\star^2 - L^2) p(0,T,x,y).
\end{aligned}$$

This result for $T = [0,1]$ under a weaker condition on the diffusion matrix was obtained by Bally and Talay (1996).

3 Parametrix method for diffusions.

For any $s \in [0,T]$, $x, y \in \mathbb{R}^d$ we consider an additional family of "frozen" diffusion processes

$$d\widetilde{Y}_t = m(t,y)\,dt + \Lambda(t,y)\,dW_t,\ \widetilde{Y}_s = x,\ s \le t \le T.$$

Let $\widetilde{p}^y(s,t,x,\cdot)$ be the conditional density of $\widetilde{Y}_t$, given $\widetilde{Y}_s = x$. In the sequel for any z we shall denote $\widetilde{p}(s,t,x,z) = \widetilde{p}^z(s,t,x,z)$, where the variable z acts here twise: as the arument of the density and as defining quantity of the process $\widetilde{Y}_t$.

The transition densities $\widetilde{p}$ can be computed explicitly

$$\begin{aligned}
\widetilde{p}(s,t,x,y) &= (2\pi)^{-d/2} (\det \sigma(s,t,y))^{-1/2} \\
&\quad \times \exp\left(-\frac{1}{2} (y - x - m(s,t,y))^T \sigma^{-1}(s,t,y) (y - x - m(s,t,y)) \right), \qquad (6)
\end{aligned}$$

where

$$\sigma(s,t,y) = \int_s^t \sigma(u,y)\,du, \qquad m(s,t,y) = \int_s^t m(u,y)\,du.$$

Note that the differential operators L and $\tilde{L}$ corresponds to the infinitesimal operators of Y or of the frozen process $\widetilde{Y}_{s,x,y}$, respectively, i.e.

$$\begin{aligned} Lf(s,t,x,y) &= \lim_{h\to 0} h^{-1}\{E[f(s,t,Y(s+h),y) \mid Y(s)=x] - f(s,t,x,y)\}, \\ \tilde{L}f(s,t,x,y) &= \lim_{h\to 0} h^{-1}\{E[f(s,t,\tilde{Y}_{s,x,y}(s+h),y)] - f(s,t,x,y)\}. \end{aligned}$$

We put

$$H = (L - \tilde{L})\tilde{p}.$$

Then

$$\begin{aligned} H(s,t,x,y) &= \frac{1}{2}\sum_{i,j=1}^{d} \left(\sigma_{ij}(s,x) - \sigma_{ij}(s,y)\right) \frac{\partial^2 \widetilde{p}(s,t,x,y)}{\partial x_i \partial x_j} \\ &\quad + \sum_{i,j=1}^{d} \left(m_i(s,x) - m_i(s,y)\right) \frac{\partial \widetilde{p}(s,t,x,y)}{\partial x_i}. \end{aligned}$$

In the following lemmas k - fold convolution of H is denoted by $H^{(k)}$. The following results are taken from Konakov and Mammen (2000).

Lemma 1. *Let $0 \le s < t \le T$. It holds*

$$p(s,t,x,y) = \sum_{r=0}^{\infty} \widetilde{p} \otimes H^{(r)}(s,t,x,y).$$

Lemma 2. *Let $0 \le s < t \le T$. There are constants C and C_1 such that*

$$\begin{aligned} |H(s,t,x,y)| &\le C_1 \rho^{-1} \phi_{C,\rho}(y-x) \\ \left|\widetilde{p} \otimes H^{(r)}(s,t,x,y)\right| &\le C_1^{r+1} \frac{\rho^r}{\Gamma(1+\frac{r}{2})} \phi_{C,\rho}(y-x), \end{aligned}$$

where $\rho^2 = t - s$, $\phi_{C,\rho}(u) = \rho^{-d}\phi_C(u/\rho)$ and

$$\phi_C(u) = \frac{\exp(-C\,\|u^2\|)}{\int \exp(-C\,\|v^2\|\,dv)}.$$

4 Parametrix method for Markov chains.

For any $0 \le jh \le T$, $x, y \in \mathbb{R}^d$ we consider an additional family of "frozen" Markov chains defined for $jh \le ih \le T$ as

(7) $$\widetilde{X}_{i+1,h} = \widetilde{X}_{i,h} + m(ih,y)\,h + \sqrt{h}\widetilde{\xi}_{i+1,h}, \ \widetilde{X}_{j,h} = x \in \mathbb{R}^d, \ j \le i \le n,$$

where $\widetilde{\xi}_{j+1,h}, ..., \widetilde{\xi}_{n,h}$ is an innovation sequence such that the conditional density of $\widetilde{\xi}_{i+1,h}$ given $\widetilde{X}_{i,h} = x_i, ..., \widetilde{X}_{0,h} = x_0$ equals to $q\left(ih, y, \cdot\right).$ Let us introduce the infinitesimal operators corresponding to Markov chains (1) and (7) respectively,

$$\begin{aligned} L_h f\left(jh, kh, x, y\right) &= h^{-1}\left(\int p_h\left(jh, (j+1)\,h, x, z\right) f\left((j+1)\,h, kh, z, y\right) dz \right. \\ &\quad \left. -f\left((j+1)\,h, kh, z, y\right)\right), \\ \widetilde{L}_h f\left(jh, kh, x, y\right) &= h^{-1}\left(\int \widetilde{p}_h^{y}\left(jh, (j+1)\,h, x, z\right) f\left((j+1)\,h, kh, z, y\right) dz \right. \\ &\quad \left. -f\left((j+1)\,h, kh, z, y\right)\right), \end{aligned}$$

where $\widetilde{p}_h^{y}\left(jh, j'h, x, \cdot\right)$ denotes the conditional density of $\widetilde{X}_{j',h}$ given $\widetilde{X}_{j,h} = x$. As before for any z denote $\widetilde{p}_h\left(jh, j'h, x, z\right) = \widetilde{p}_h^{z}\left(jh, j'h, x, z\right),$ where the variable z acts here twise: as the arument of the density and as defining quantity of the process $\widetilde{X}_{i,h}$. For technical convenience the terms $f\left((j+1)\,h, kh, z, y\right)$ on the right hand side of $L_h f$ and $\widetilde{L}_h f$ appear instead of $f\left(jh, kh, z, y\right).$

In analogy with the definition of H we put, for $k > j$,

$$H_h\left(jh, kh, x, y\right) = \left(L_h - \widetilde{L}_h\right) \widetilde{p}_h\left(jh, kh, x, y\right).$$

We also shall use the convolution type binary operation $\otimes_h$:

$$g \otimes_h f\left(jh, kh, x, y\right) = \sum_{i=j}^{k-1} h \int_{\mathbb{R}^d} g\left(jh, ih, x, z\right) f\left(ih, kh, z, y\right) dz,$$

where $0 \leq j < k \leq n$. Write $g \otimes_h H_h^{(0)} = g$ and $g \otimes_h H_h^{(r)} = \left(g \otimes_h H_h^{(r-1)}\right) \otimes_h H_h$ for $r = 1, ..., n$. For the higher order convolutions we use the convention $\sum_{i=j}^{l} = 0$ for $l < j$. One can show the following analog of the "parametrix " expansion for p_h [see Konakov and Mammen (2000)].

Lemma 3. *Let $0 \leq jh < kh \leq T$. It holds*

$$p_h(jh, kh, x, y) = \sum_{r=0}^{k-j} \widetilde{p}_h \otimes_h H_h^{(r)}(jh, kh, x, y),$$

where

$$p_h(jh, jh, x, y) = p_h(kh, kh, x, y) = \delta(y - x)$$

and δ is the Dirac delta symbol.

5 Bounds on $\widetilde{p}_h - \widetilde{p}$ based on Edgeworth expansions.

In this subsection we will develop some tools that are helpful for the comparison of the expansion of p (see Lemma 1) and the expansion of p_h (see Lemma 3). These expansions are simple expressions in $\tilde{p}$ or $\tilde{p}_h$, respectively. Recall that $\tilde{p}$ is a Gaussian density, see (6), and that $\tilde{p}_h$ is the density of a sum of independent variables. The densities $\tilde{p}$ and $\tilde{p}_h$ can be compared by application of the classical Edgeworth expansions. This is done in Lemma 5. This is the essential step for the comparison of the expansions of p and p_h. For the proof of Lemma 5 we need one additional lemma which is a technical tool for the further considerations. To formulate this lemma we need some additional notations. Suppose $X \in \mathbb{R}^d$ be a random vector having a density $q(\mathbf{x}), \mathbf{x} \in\mathbb{R}^d$, $EX = 0, Cov(X, X) = \Sigma$, where Σ be a positively definite $d \times d$ matrix . Denote $A = \|a_{ij}\| = \Sigma^{-1/2}$ and let $\chi_\nu(Z)$ be a cumulant of the order $\nu = (\nu_1, ..., \nu_d)$ of a random vector $Z \in \mathbb{R}^d$, $\phi(x)$ denotes a function in $\mathbb{R}^d$ such that $D_x^\nu \phi(x)$ exist and continuous for $|\nu| = 4$, and $A^{-1} = \|a^{ij}\| = \Sigma^{1/2}$.

Lemma 4. The following relation holds for $s = 3$ and for $s = 4$

$$\sum_{|\nu|=s} \frac{\chi_\nu(AX) D_z^\nu \phi(z)}{\nu!} = \sum_{|\nu|=s} \frac{\chi_\nu(X) D_x^\nu \phi(Ax)}{\nu!}$$

where $z = Ax$.

Proof of Lemma 4. For $|\nu| = 3, \nu = (\nu_1, ..., \nu_d)$, each cumulant $\chi_\nu(AX)$ is a linear combination of $\chi_\mu(X)$ with $|\mu| = 3$ and with coefficients depending only on a_{ij}. It follows from the following relation

$$\chi_\nu(AX) = \mu_\nu(AX) = \int (a_{11}x_1 + ... + a_{1d}x_d)^{\nu_1} \times ... \times (a_{11}x_1 + ... + a_{1d}x_d)^{\nu_d} q(\mathbf{x}) d\mathbf{x}.$$

Analogously, from the usual differentiation rule of a composite function and from the relation $\phi(z) = \phi(Ax)$, $x = A^{-1}z$, it follows that $D_z^\nu \phi(z) = D_z^\nu \phi(Ax)$ is a linear combination of $D_x^\nu \phi(Ax)$ with coefficients depending only on a^{ij}. As a result of such substitutions we obtain that

$$\sum_{|\nu|=3} \frac{\chi_\nu(AX) D_z^\nu \phi(z)}{\nu!} = \frac{1}{3!} \sum_{j=1}^{d} \left[\sum_{|\mu|=3} \frac{3!}{\mu_1!...\mu_d!} a_{1j}^{\mu_1} ... a_{dj}^{\mu_d} \chi_\mu(X) \right]$$

$$\times \left[\sum_{|\mu'|=3} \frac{3!}{\mu'_1!...\mu'_d!} (a^{j1})^{\mu'_1} ... (a^{jd})^{\mu'_d} D_x^{\mu'} \phi(AX) \right]$$

$$+ \frac{1}{2!1!} \sum_{\{i \neq, j\}} \left[\sum_{l=1}^{d} \sum_{|\mu|=2} \frac{2!}{\mu_1!...\mu_d!} a_{1j}^{\mu_1} ... a_{dj}^{\mu_d} a_{il} \chi_{\mu+e_l}(X) \right]$$

$$\times \left[\sum_{l'=1}^{d} \sum_{|\mu'|=2} \frac{2!}{\mu'_1!...\mu'_d!} (a^{j1})^{\mu'_1} ... (a^{jd})^{\mu'_d} a^{il'} D_x^{\mu'+e_{l'}} \phi(AX) \right]$$

$$+ \frac{1}{3!} \sum_{\{i \neq j \neq k\}} \left[\sum_{l,q=1}^{d} \sum_{|\mu|=1} \frac{1}{\mu_1!...\mu_d!} a_{1j}^{\mu_1} ... a_{dj}^{\mu_d} a_{il} a_{kq} \chi_{\mu+e_l+e_q}(X) \right]$$

$$\times \left[\sum_{l',q'=1}^{d} \sum_{|\mu'|=1} \frac{1}{\mu_1'!...\mu_d'!} (a^{j1})^{\mu_1'}...(a^{jd})^{\mu_d'} a^{il'} a^{kq'} D_x^{\mu'+e_{l'}+e_{q'}} \phi(AX) \right]$$

where $\sum_{\{i\neq,j\}}$ ($\sum_{\{i\neq j\neq k\}}$) denotes the sum over all different pairs (triples) of $i,j \in \{1,2,...,d\}$ (of $i,j,k \in \{1,2,...,d\}$) and $e_i \in \mathbb{R}^d$ denotes the vector whose $i-$th coordinate is equal to 1 and other coordinates are zero. Collecting the similar terms in the last equation we obtain that for $\nu = 3e_k$, $\nu' = 3e_l$ the coefficient before $\chi_\nu(X) D_x^{\nu'} \phi(AX)$ is equal to $\frac{1}{3!}(a_{1k}a^{l1} + ... + a_{dk}a^{ld})^3 = \frac{1}{3!}\delta_{kl}$, for $\nu = e_q + 2e_r, \nu' = e_l + 2e_n$, $q \neq r$, the coefficient before $\chi_\nu(X) D_x^{\nu'} \phi(AX)$ is equal to $\frac{1}{2!}(a_{1q}a^{l1} + ... + a_{dq}a^{ld})(a_{1r}a^{n1} + ... + a_{dr}a^{nd})^2 = \frac{1}{2!}\delta_{ql}\delta_{rn}$, in particular , for $l = n$ the last expression is equal to zero. For $\nu = e_q + e_r + e_n, \nu' = e_{q'} + e_{r'} + e_{n'}$ $q \neq r, q \neq n, r \neq n$, the coefficient before $\chi_\nu(X) D_x^{\nu'} \phi(AX)$ is equal to $(a_{1q}a^{q'1} + ... + a_{dq}a^{q'd}) \times (a_{1r}a^{r'1} + ... + a_{dr}a^{r'd}) \times (a_{1n}a^{n'1} + ... + a_{dn}a^{n'd}) = \delta_{qq'}\delta_{rr'}\delta_{nn'}$. This proves lemma for $|\nu| = 3$. The proof for $|\nu| = 4$ is quite similar. For this case we use the relation which enabes to express a cumulant $\chi_\nu(AX)$ as $\mu_\nu(AX)$ plus a second order polinomial of the moments $\mu_{\nu'}(AX)$, $|\nu'| = 2$. A necessary correction term for $\mu_\nu(X)$ to get a $\chi_\nu(X)$ comes from the derivation of $D_z^\nu \phi(z)$. This completes the proof of the lemma.

In Lemma 6 bounds will be given for derivatives of $\widetilde{p}_h$. The proof of this lemma also makes essential use of Edgeworth expansions. The following lemma is a higher order extension of the results in Section 3.3 in Konakov and Mammen (2000). Denote

(8) $$\mu_{j,k}(y) = h \sum_{i=j}^{k-1} m(ih, y), V_{j,k}(y) = h \sum_{i=j}^{k-1} \sigma(ih, y).$$

Lemma 5. *The following bound holds with a constant C for $\nu = (\nu_1, ...\nu_p)^T$ with $0 \le |\nu| \le 6$*

$$\Big| D_z^\nu \widetilde{p}_h(jh, kh, x, y) - D_z^\nu \widetilde{p}(jh, kh, x, y) - \sqrt{h} D_z^\nu \widetilde{\pi}_1(jh, kh, x, y) - h D_z^\nu \widetilde{\pi}_2(jh, kh, x, y) \Big| \le C h^{3/2} \rho^{-3} \zeta_\rho^{S-|\nu|}(y - x)$$

for all $j < k, x$ and y. Here D_z^ν denotes the partial differential operator of order ν with respect to $z = V_{j,k}^{-1/2}(y)(y - x - \mu_{j,k}(y))$. The quantity ρ denotes again the term $\rho = [h(k-j)]^{1/2}$ and the functions $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are defined in (4) and (5). We write $\zeta_\rho^k(\cdot) = \rho^{-p}\zeta^k(\cdot/\rho)$ where

$$\zeta^k(z) = \frac{[1 + \|z\|^k]^{-1}}{\int [1 + \|z'\|^k]^{-1} dz'}.$$

Proof of Lemma 5. We note first that $\tilde{p}_h(jh, kh, x, \bullet)$ is the density of the vector

$$x + \mu_{j,k}(y) + h^{1/2} \sum_{i=j}^{k-1} \widetilde{\xi}_{i+1,h},$$

where, as above in the definition of the "frozen" Markov chain $\widetilde{Y}_n$, $\widetilde{\xi}_{i+1,h}$ is a sequence of independent variables with densities $q(ih, y, \cdot)$, $\mu_{j,k}(y) = \sum_{i=j}^{k-1} hm(ih, y)$. Let $f_h(\cdot)$ be the density of the normalized sum

$$h^{1/2} \left[V_{j,k}(y)\right]^{-1/2} \sum_{i=j}^{k-1} \widetilde{\xi}_{i+1,h}.$$

Clearly, we have

$$\tilde{p}_h(jh, kh, x, \cdot) = \det\left[V_{j,k}(y)\right]^{-1/2} f_n\{[V_{j,k}(y)]^{-1/2}\left[\cdot - x - \mu_{j,k}(y)\right]\}.$$

We now argue that an Edgeworth expansion holds for f_h. This implies the following expansion for $\tilde{p}_h(jh, kh, x, \cdot)$

$$\tilde{p}_h(jh, kh, x, \cdot) \tag{9}$$

$$= \det\left[V_{j,k}(y)\right]^{-1/2} [\sum_{r=0}^{S-3} (k-j)^{-r/2} P_r(-\phi : \{\bar{\chi}_{\beta,r}\})\{[V_{j,k}(y)]^{-1/2}\left[\cdot - x - \mu_{j,k}(y)\right]\}$$

$$+[k-j]^{-(S-2)/2} O([1 + \left\|\{[V_{j,k}(y)]^{-1/2}[\cdot - x - \mu_{j,k}(y)]\}\right\|^S]^{-1})]$$

with standard notations, see Bhattacharya and Rao (1976), p. 53. In particular, P_r denotes a product of a standard normal density with a polynomial that has coefficients depending only on cumulants of order $\leq r+2$. Expansion (9) follows from Theorem 19.3 in Bhattacharya and Rao (1976). This can be seen as in the proof of Lemma 3.7 in Konakov and Mammen (2000a).

It follows from (9) and Condition (A3) that

$$\left|\tilde{p}_h(jh, kh, x, y) - \tilde{p}(jh, kh, x, y) - h^{1/2}\widehat{\pi}_1(jh, kh, x, y) - h\widehat{\pi}_2(jh, kh, x, y)\right|$$

$$\leq C h^{3/2} \rho^{-3} \zeta_\rho^{S-|\nu|}(y-x), \tag{10}$$

where

$$\tilde{p}(jh, kh, x, y) = \det\left[V_{j,k}(y)\right]^{-1/2} (2\pi)^{-p/2}$$
$$\exp\{-\frac{1}{2}(y - x - \mu_{j,k}(y))^T \left[V_{j,k}(y)\right]^{-1} (y - x - \mu_{j,k}(y))\},$$

$$\widehat{\pi}_1(jh, kh, x, y) = -\rho^{-1} \det\left[V_{j,k}(y)\right]^{-1/2} \sum_{|\nu|=3} \frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!} D_z^\nu \phi \left\{\left[V_{j,k}(y)\right]^{-1/2} (y - x - \mu_{j,k}(y))\right\},$$

$$\widehat{\pi}_2(jh, kh, x, y) = \rho^{-2} \det\left[V_{j,k}(y)\right]^{-1/2} \left[\sum_{|\nu|=4} \frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!} D_z^\nu \phi \left\{\left[V_{j,k}(y)\right]^{-1/2} (y - x - \mu_{j,k}(y))\right\}\right.$$
$$\left. + \frac{1}{2} \left\{\sum_{|\nu|=3} \frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!} D_z^\nu\right\}^2 \phi \left\{\left[V_{j,k}(y)\right]^{-1/2} (y - x - \mu_{j,k}(y))\right\}\right],$$

where $\overline{\chi}_{\nu,j,k}(y)=\frac{1}{k-j}\sum_{i=j}^{k-1}\chi_{\nu,j,k,i}(y)$, $\chi_{\nu,j,k,i}(y)=\nu-$th cumulant of $\rho\left[V_{j,k}(y)\right]^{-1/2}\widetilde{\xi}_{i+1,h}=\rho^{|\nu|}\times\{\nu-$th cumulant of $\left[V_{j,k}(y)\right]^{-1/2}\widetilde{\xi}_{i+1,h}\}$, and $D_z^\nu\phi(z)$ denotes the $\nu-$th derivative of ϕ with respect to $z=\left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))$. It follows from the (conditional) independence of $\widetilde{\xi}_{i+1,h}, i=j,...,k-1$, that $\overline{\chi}_{\nu,j,k}(y)=\frac{\rho^{|\nu|}}{k-j}h^{-|\nu|/2}\times\chi_\nu(AX)$, where $A=h^{1/2}\left[V_{j,k}(y)\right]^{-1/2}=\Sigma^{-1/2},\Sigma=Cov(X,X),X=\sum_{i=j}^{k-1}\widetilde{\xi}_{i+1,h}$. By Lemma 4 for $s=3,4$

$$\begin{aligned}
\sum_{|\nu|=s}\frac{\overline{\chi}_{\nu,j,k}(y)}{\nu!}D_z^\nu\phi(z) &= \rho^s\frac{1}{k-j}\sum_{|\nu|=s}\frac{\chi_\nu(AX)}{\nu!}D^\nu_{h^{1/2}z}\phi_h(h^{1/2}z)\\
&= (-1)^s\rho^s\sum_{|\nu|=s}\frac{\overline{\chi}_\nu(X)}{\nu!}D_x^\nu\phi_h(A(y-x-\mu_{j,k}(y)))\\
&= (-1)^s\rho^s\sum_{|\nu|=s}\frac{\overline{\chi}_\nu(X)}{\nu!}D_x^\nu\phi(\left[V_{j,k}(y)\right]^{-1/2}(y-x-\mu_{j,k}(y))), \quad (11)
\end{aligned}$$

where we put $\phi_h(z)=\phi(h^{-1/2}z),\overline{\chi}_\nu(X)=\frac{1}{k-j}\sum_{i=j}^{k-1}\chi_\nu(ih,y)$. It follows from (11) and the condition **B1** that up to the error term in the right hand side of (10) the functions $\widehat{\pi}_1$ and $\widehat{\pi}_2$ coincide with the functions $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ given at the beginning of Section 4. For $\nu=0$ the statement of the lemma immediately follows from (10). For $\nu>0$ one proceeds similarly. See the remark at the end of the proof of Lemma 3.7 in Konakov and Mammen (2000).

The following lemma was proved in Konakov and Molchanov (1984) (Lemma 4 on page 68).

Lemma 6. *Let $L(d)$ be the set of symmetric matrices, and let $D_{\lambda^+,\lambda^-}\subset L(d), 0<\lambda^-<\lambda^+<\infty$, be the open subset distinguished by the inequalities $\Lambda\in D_{\lambda^+,\lambda^-}\Leftrightarrow\lambda^-I\leq\Lambda\leq\lambda^+I$; $A=A(\Lambda)$ is a solution of the equation $A^2=\Lambda$ with $A=A^T$ and $a_{ij}(\Lambda)$ the corresponding matrix elements. Then for any $k,l\leq d,i,j\leq d$ and $\Lambda\subset D_{\lambda^+,\lambda^-}$ we have that*

$$\left|\frac{\partial a_{ij}(\Lambda)}{\partial\lambda_{kl}}\right|\leq\frac{1}{2\sqrt{\lambda^-}}.$$

From Lemma 5 we get the following corollary. The statement of the next lemma is an extension of Lemma 3.7 in Mammen and Konakov (2000) where the result has been shown for $0\leq|b|\leq2,a=0$.

Lemma 7. *The following bounds hold:*

$$\left|D_y^aD_x^b\widetilde{p}_h(jh,kh,x,y)\right|\leq C\rho^{-|a|-|b|}\varsigma_\rho^{S-|a|}(y-x)$$

for all $j<k$, for all x and y and for all a,b with $0\leq|a|+|b|\leq6$. Here, $\rho=[(k-j)h]^{1/2}$. The exponent S has been defined in Assumption **A3.**

Proof of Lemma 7. If $A=\|a_{ij}\|$ and $B=\|b_{kl}\|$ and elements $a_{ij}(B)$ are smooth functions of b_{kl} then an inequality $\left|\frac{\partial A}{\partial B}\right|\leq C$ will mean that $\left|\frac{\partial a_{ij}}{\partial b_{kl}}\right|\leq C$

for all $1 \leq i, j \leq d, 1 \leq k, l \leq d$. To obtain the assertion of the lemma we have to estimate the derivatives $D_y^a D_x^b z$,where $z = V_{j,k}^{-1/2}(y)(y - x - \mu_{j,k}(y))$. Note that $z = z(V_{j,k}^{-1/2}, \mu_{j,k}, x, y)$, where $V_{j,k}^{-1/2} = V_{j,k}^{-1/2}(y)$ and $\mu_{j,k} = \mu_{j,k}(y)$. It follows from the conditions (B1) and (8) that

$$\left| \frac{\partial \mu_{j,k}(y)}{\partial y} \right| \leq C\rho^2, \left| \frac{\partial V_{j,k}(y)}{\partial y} \right| \leq C\rho^2. \tag{12}$$

It follows from Lemma 6 that

$$\left| \frac{\partial V_{j,k}^{1/2}}{\partial V_{j,k}} \right| \leq C. \tag{13}$$

From inequalities (3.16) in Konakov and Mammen (2000) and from the representation of an inverse matrix in terms of cofactors divided by determinant we obtain that

$$\left| \frac{\partial V_{j,k}^{-1/2}}{\partial V_{j,k}^{1/2}} \right| \leq C\rho^{-2}. \tag{14}$$

From (12)-(14) and from the differentiation rule of a composite function we get

$$\left| \frac{\partial V_{j,k}^{-1/2}(y)}{\partial y} \right| \leq C. \tag{15}$$

Inequalities (3.16) in Konakov and Mammen (2000), (15) and the differentiation rule of a composite function imply

$$\left| \frac{\partial z}{\partial \mu_{j,k}} \right| \leq \frac{C}{\rho}, \left| \frac{\partial z}{\partial V_{j,k}^{-1/2}} \right| = (y - x - \mu_{j,k}(y)), \left| \frac{\partial z}{\partial y} \right| \leq \frac{C}{\rho}. \tag{16}$$

Inequalities (3.16) in Konakov and Mammen (2000) also imply

$$\left| \frac{\partial z}{\partial x} \right| \leq \frac{C}{\rho}. \tag{17}$$

The assertion of Lemma 7 for $a = e_i, b = e_j, 1 \leq i, j \leq d$, follows from Lemma 5 and from (16), (17). For other values of a and b one has to repeat this arguments.

6 Bounds on operator kernels used in the parametrix expansions.

In this section we will present bounds for operator kernels appearing in the expansions based on the parametrix method. In Lemma 8 we compare the infinitesimal operators

L_h and $\tilde{L}_h$ with the differential operators L and $\tilde{L}$. We give an approximation for the error if, in the definition of $H_h = (L_h - \tilde{L}_h)\tilde{p}_h$, the terms L_h and $\tilde{L}_h$ are replaced by L or $\tilde{L}$, respectively. We show that this term can be approximated by $K_h + M_h$, where $K_h = (L - \tilde{L})\tilde{p}_h$ and where M_h is defined in Remark 1 after Lemma 8 . Bounds on H_h, K_h, and M_h are given in Lemma 9. These bounds will be used in the proof of our theorem to show that in the expansion of p_h the terms $\tilde{p}_h \otimes_h H_h^{(r)}$ can be replaced by $\tilde{p}_h \otimes_h (K_h + M_h')^{(r)}$, M_h' is defined in Lemma 9.

Lemma 8. *The following bound holds with a constant C*

$$\begin{aligned}&|H_h(jh, kh, x, y) - K_h'(jh, kh, x, y) - M_h'(jh, kh, x, y) - R_h(jh, kh, x, y)| \\ &\quad \le Ch^{3/2}\rho^{-1}\zeta_\rho^S(y - x)\end{aligned}$$

with ζ_ρ^S as in Lemma 5 for all $j < k$, x and y . For $j < k - 1$ we define

$$\begin{aligned}
&K_h'(jh, kh, x, y) = (L - \widetilde{L})\lambda(x), M_h'(jh, kh, x, y) \\
&\quad = M_{h,1}(jh, kh, x, y) + M_{h,2}(jh, kh, x, y) + M_{h,3}'(jh, kh, x, y) \\
&M_{h,1}(jh, kh, x, y) = h^{1/2} \sum_{|\nu|=3} \frac{D_x^\nu \lambda(x)}{\nu!}(\chi_\nu(jh, x) - \chi_\nu(jh, y)), \\
&M_{h,2}(jh, kh, x, y) = h \sum_{|\nu|=4} \frac{D_x^\nu \lambda(x)}{\nu!}(\chi_\nu(jh, x) - \chi_\nu(jh, y)), \\
&M_{h,3}'(jh, kh, x, y) = \frac{h}{2}(L_*^2 - \widetilde{L}^2)\lambda(x), \\
&R_h(jh, kh, x, y) = h^{3/2} \sum_{|\nu|=4} \frac{D_x^\nu \lambda(x)}{\nu!} \sum_{r=1}^{d} \nu_r[m_r(jh, x)\mu_{\nu-e_r}(jh, x) - m_r(jh, y)\mu_{\nu-e_r}(jh, y)] \\
&\quad +5 \sum_{|\nu|=5} \frac{1}{\nu!} \sum_{k=1}^{d} (m_k(jh, x) - m_k(jh, y)) \left\{ \nu_k \int q(jh, x, \theta)\widetilde{h}^{\nu-e_k}(\theta) \right. \\
&\qquad \times \left[\int_0^1 (1-u)^4 D^\nu \lambda(x + u\widetilde{h}(\theta))du\right] d\theta \\
&\quad \left. + \int q(jh, x, \theta)\widetilde{h}^\nu(\theta) \left[\int_0^1 (1-u)^4 u D^{\nu+e_k} \lambda(x + u\widetilde{h}(\theta))du\right] d\theta \right\} \\
&\quad +h^2 \sum_{|\nu|=4} \frac{D_x^\nu \lambda(x)}{\nu!} \sum_{|\nu'|=2} \nu! N(\nu, \nu')[m^{\nu'}(jh, x)\mu_{\nu-\nu'}(jh, x) - m^{\nu'}(jh, y)\mu_{\nu-\nu'}(jh, y)].
\end{aligned}$$

Here L_* is defined analogously to $\widetilde{L}$ but with the coefficients "frozen" at the point x, e_r denotes a p - dimensional vector with $r - th$ element equal to 1 and with all other elements equal to 0. Furthermore, for $|\nu| = 4, |\nu'| = 2$ we define

$$N(\nu, \nu') = 2^{\chi[\nu'!=1] + \chi[(\nu-\nu')!=1] - 2}$$

where $\chi(\cdot)$ means an indicator function. We put $m(x)^\nu = m_1(x)^{\nu_1} \cdot ... \cdot m_p(x)^{\nu_p}$ and $m(x)^\nu = 0$, $\nu! = 0$ and $\mu_\nu(x) = 0$ if at least one of the coordinates of ν is negative.

We define also the following functions

$$\begin{aligned}\lambda(x) &= \widetilde{p}_h((j+1)h, kh, x, y),\\ \widetilde{h}(\theta) &= m(jh,y)h + \theta h^{1/2}.\end{aligned}$$

Here again ρ denotes the term $\rho = [h(k-j)]^{1/2}$. For $j = k-1$ and $l = 1, ..., 3$ we define

$$K_h'(jh,kh,x,y) = M_{h,1}(jh,kh,x,y) = M_{h,2}(jh,kh,x,y) = M_{h,3}'(jh,kh,x,y) = 0.$$

Proof of Lemma 8. As in the proof of Lemma 3.9 in Konakov and Mammen (2000) we have

$$H_h(jh,kh,x,y) = H_h^1(jh,kh,x,y) - H_h^2(jh,kh,x,y),$$

where

$$\begin{aligned}H_h^1(jh,kh,x,y) &= h^{-1}\int q(jh,x,\theta)[\lambda(x+h(\theta)) - \lambda(x)]d\theta \qquad (18)\\ H_h^2(jh,kh,x,y) &= h^{-1}\int q(jh,x,\theta)[\lambda(x+\widetilde{h}(\theta)) - \lambda(x)]d\theta, \qquad (19)\\ h(\theta) &= m(jh,x)h + \theta h^{1/2},\\ \widetilde{h}(\theta) &= m(jh,y)h + \theta h^{1/2}.\end{aligned}$$

For $[\lambda(x+h(\theta)) - \lambda(x)]$ and $[\lambda(x+\widetilde{h}(\theta)) - \lambda(x)]$ in (18), (19) we use now the Tailor expansion up to the order 5 with the remaining term in integral form. To pass from moments to cumulants we use the well known relations (see e.g. relation (6.11) on page 46 in Bhattacharya and Rao (1986)). After long but simple calculations we come to the conclusion of the lemma.

Remark 1. We show now that the function $K_h'(jh,kh,x,y) + M_{h,3}'(jh,kh,x,y)$ in Lemma 8 is equal to $K_h(jh,kh,x,y) + \frac{h}{2}(L_*^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) + M_{h,3}''(jh,kh,x,y)$ where

$$\begin{aligned}M_{h,3}''(jh,kh,x,y) = -h^2 \sum_{|\mu|=2} \frac{m^\mu(jh,y)}{\mu!}(L-\widetilde{L})D^\mu\lambda(x)\\ -3\sum_{|\mu|=3}\int_0^1 (1-\delta)^2 d\delta \int q(jh,y,\theta)\frac{\widetilde{h}(\theta)^\mu}{\mu!}(L-\widetilde{L})D^\mu\lambda(x+\delta\widetilde{h}(\theta))d\theta. \qquad (20)\end{aligned}$$

Thus in Lemma 8 we can replace $K_h'(jh,kh,x,y)+M_h'(jh,kh,x,y)$ by $K_h(jh,kh,x,y)+M_h(jh,kh,x,y)$ where $K_h(jh,kh,x,y) = (L-\widetilde{L})\widetilde{p}_h(jh,kh,x,y)$, $M_h(jh,kh,x,y) = \frac{h}{2}(L_*^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) + M_h''$, $M_h'' = M_{h,1}(jh,kh,x,y) + M_{h,2}(jh,kh,x,y) + M_{h,3}''(jh,kh,x,y)$ and

$$\max\{|M_h'(jh,kh,x,y)|, |M_h(jh,kh,x,y)|\} \le C\rho^{-1}\zeta_\rho(y-x),$$

$\rho^2 = kh - jh$. To show this we note that

$$\widetilde{p}_h(jh, kh, x, y) = \int q(jh, y, \theta)\lambda(x + \widetilde{h}(\theta))d\theta$$

where $\widetilde{h}(\theta) = m(jh, y)h + h^{1/2}\theta$. From the Tailor expansion we get

$$\widetilde{p}_h(jh, kh, x, y) = \lambda(x) + h\widetilde{L}\lambda(x) + h^2 \sum_{|\mu|=2} \frac{m^\mu(jh, y)}{\mu!} D^\mu \lambda(x)$$
$$+3 \sum_{|\mu|=3} \int_0^1 (1-\delta)^2 d\delta \int q(jh, y, \theta) \frac{\widetilde{h}(\theta)^\mu}{\mu!} D^\mu \lambda(x + \delta \widetilde{h}(\theta))d\theta$$

and, hence,

$$K'_h(jh, kh, x, y) = K_h(jh, kh, x, y) + (L - \widetilde{L})[\lambda(x) - \widetilde{p}_h(jh, kh, x, y)]$$

$$= K_h(jh, kh, x, y) + h(\widetilde{L}^2 - L\widetilde{L})\lambda(x) + M''_{h,3}(jh, kh, x, y). \tag{21}$$

Note that

$$\begin{aligned} h(\widetilde{L}^2 - L\widetilde{L})\lambda(x) + M'_{h,3}(jh, kh, x, y) &= h(\widetilde{L}^2 - L\widetilde{L})\lambda(x) + \frac{h}{2}(L_*^2 - \widetilde{L}^2)\lambda(x) \\ &= \frac{h}{2}(L_*^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) \end{aligned}$$

and from the definitions of the operators $L, \widetilde{L}$ and L_* and Lipschitz conditions on the coefficients $m(t, x)$ and $\sigma(t, x)$ we obtain that

$$\left| \frac{h}{2}(L_*^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x) \right| \le Ch\rho^{-3}\zeta_\rho(y - x). \tag{22}$$

Analogously, we have

$$\left| h^2 \sum_{|\mu|=2} \frac{m^\mu(jh, y)}{\mu!} (L - \widetilde{L}) D^\mu \lambda(x) \right| \le Ch^2\rho^{-3}\zeta_\rho(y - x), \tag{23}$$

$$\left| 3 \sum_{|\mu|=3} \int_0^1 (1-\delta)^2 d\delta \int q(jh, y, \theta) \frac{\widetilde{h}(\theta)^\mu}{\mu!} (L - \widetilde{L}) D^\mu \lambda(x + \delta\widetilde{h}(\theta))d\theta \right| \tag{24}$$
$$\le Ch^{3/2}\rho^{-4}\zeta_\rho(y - x).$$

Now (21)-(24) imply the assertion of this remark.

Lemma 9. *The following bounds holds:*

$$\left| \sum_{r=0}^n \widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0, T, x, y) - \sum_{r=0}^n \widetilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0, T, x, y) \right|$$
$$\le C(\varepsilon)hn^{-1/2+\varepsilon}\zeta^S_{\sqrt{T}}(y - x), \tag{25}$$

where $\lim_{\varepsilon \longrightarrow 0} C(\varepsilon) = +\infty$.

PROOF OF LEMMA 9. For $r = 1$ we will show that for any $\varepsilon > 0$ with $\rho^2 = kh$

$$
\begin{aligned}
(26) \qquad & |\widetilde{p}_h \otimes_h (K_h + M_h + R_h)(0, kh, x, y) - \widetilde{p}_h \otimes_h (K_h + M_h)(0, kh, x, y)| \\
& = |\widetilde{p}_h \otimes_h R_h(0, kh, x, y)| \leq C h^{3/2-\varepsilon} (kh)^{-1/2+\varepsilon} B(\frac{1}{2}, \varepsilon) \zeta_\rho^S (y - x).
\end{aligned}
$$

Clearly, for an estimate of $\widetilde{p}_h \otimes_h R_h(0, T, x, y)$ it suffices to estimate

$$
I_1 = h^{3/2} \sum_{j=0}^{k-2} h \int \widetilde{p}_h(0, kh, x, z)(f(jh, z) - f(jh, y)) D_z^\nu \widetilde{p}_h((j+1)h, kh, z, y) dz
$$

for ν with $|\nu| = 4$, and

$$
\begin{aligned}
I_2 &= h^2 \sum_{j=0}^{k-2} h \int \widetilde{p}_h(0, jh, x, z)(f(jh, z) - f(jh, y)) \int q(jh, z, \theta) \widetilde{h}^{\nu - e_k}(\theta) \\
& \qquad \times \int_0^1 (1-u)^4 D_z^\nu \lambda(z + u\widetilde{h}(\theta)) du d\theta dz
\end{aligned}
$$

for ν with $|\nu| = 5$, $1 \leq k \leq d$. Here $f(t, x)$ is a function whose first and second derivatives with respect to x are continuous and bounded, uniformly in t and x. After integration by parts we obtain for $1 \leq l, s \leq d$

$$
\begin{aligned}
I_1 &= -h^{3/2} \sum_{j=0}^{k-2} h \int D_z^{e_k} \widetilde{p}_h(0, jh, x, z)(f(jh, z) - f(jh, y)) D_z^{\nu - e_k} \widetilde{p}_h((j+1)h, kh, z, y) dz \\
& \quad + h^{3/2} \sum_{j=0}^{k-2} h \int D_z^{e_s} \widetilde{p}_h(0, jh, x, z) D_z^{e_k} f(jh, z) D_z^{\nu - e_k - e_s} \widetilde{p}_h((j+1)h, kh, z, y) dz \\
& \quad + h^{3/2} \sum_{j=0}^{k-2} h \int \widetilde{p}_h(0, jh, x, z) D_z^{e_k + e_s} f(jh, z) D_z^{\nu - e_k - e_s} \widetilde{p}_h((j+1)h, kh, z, y) dz.
\end{aligned}
$$

Hence,

$$
(27) \quad |I_1| \leq C h^{3/2} \sum_{j=0}^{k-2} h \frac{1}{\sqrt{jh}(kh - jh)} \zeta_\rho(y - x) \leq C h^{3/2-\varepsilon} (kh)^{-1/2+\varepsilon} B(\frac{1}{2}, \varepsilon) \zeta_{\sqrt{T}}^S (y - x).
$$

In the same way, we obtain after integration by parts with $1 \leq l, s \leq d$

$$
\begin{aligned}
I_2 &= -h^2 \sum_{j=0}^{k-2} h \int_0^1 (1-u)^4 du \int d\theta \left(m(jh, y) h^{1/2} + \theta \right)^{\nu - e_l} \\
& \qquad \int D_z^{e_l} \widetilde{p}_h(0, jh, x, z)(f(jh, z) - f(jh, y)) \\
& \qquad \times q(jh, z, \theta) D_z^{\nu - e_l} \widetilde{p}_h((j+1)h, kh, z + u\widetilde{h}(\theta), y) dz
\end{aligned}
$$

$$+h^2 \sum_{j=0}^{k-2} h \int_0^1 (1-u)^4 du \int d\theta (m(jh,y)h^{1/2}+\theta)^{\nu-e_l}$$
$$\times \int D_z^{e_s}[\widetilde{p}_h(0,jh,x,z) D^{e_l} f(jh,z) q(jh,z,\theta)]$$
$$\times D_z^{\nu-e_l-e_s} \widetilde{p}_h((j+1)h, kh, z+u\widetilde{h}(\theta), y) dz$$
$$-h^2 \sum_{j=0}^{k-2} h \int_0^1 (1-u)^4 du \int d\theta (m(jh,y)h^{1/2}+\theta)^{\nu-e_l}$$
$$\times \widetilde{p}_h(0,jh,x,z)(f(jh,z)-f(jh,y))$$
$$\times D_z^{e_l} q(jh,z,\theta) D_z^{\nu-e_l} \widetilde{p}_h((j+1)h, kh, z+u\widetilde{h}(\theta), y) dz.$$

It follows from this equation that

$$|I_2| \le C h^{3/2-\varepsilon}(kh)^{-1/2+\varepsilon} B(\frac{1}{2},\varepsilon)\zeta_\rho^S(y-x). \tag{28}$$

Claim (27) follows now from (27) and (28). For $r \ge 2$ we use the identity

$$\begin{aligned}(29)\ & \widetilde{p}_h \otimes_h (K_h+M_h+R_h)^{(r)}(0,T,x,y) - \widetilde{p}_h \otimes_h (K_h+M_h)^{(r)}(0,T,x,y) \\ &= \left[\widetilde{p}_h \otimes_h (K_h+M_h+R_h)^{(r-1)} - \widetilde{p}_h \otimes_h (K_h+M_h)^{(r-1)}\right] \otimes_h (K_h+M_h)(0,T,x,y) \\ &\quad + \widetilde{p}_h \otimes_h (K_h+M_h+R_h)^{(r-1)} \otimes_h R_h(0,T,x,y) \\ &= I + II.\end{aligned}$$

For $r=2$ we obtain from (27) and from the simple estimate $|(K_h+M_h)(jh,kh,z,y)| \le C\rho_2^{-1}\zeta_{\rho_2}^S(y-z)$ with $\rho_2^2 = kh - jh$, that

$$\begin{aligned} |I| &= |[\widetilde{p}_h \otimes_h (K_h+M_h+R_h) - \widetilde{p}_h \otimes_h (K_h+M_h)] \otimes_h (K_h+M_h)(0,kh,x,y)| \\ &\le C^2 h^{3/2-\varepsilon} B(\frac{1}{2},\varepsilon) \sum_{j=0}^{k-2} h (jh)^{-1/2+\varepsilon}(kh-jh)^{-1/2} \int \zeta_{\rho_1}^S(z-x)\zeta_{\rho_2}^S(y-z)dz \\ &\le C^2 h^{3/2-\varepsilon} B(\frac{1}{2},\varepsilon) B(\frac{1}{2},\varepsilon+\frac{1}{2})(kh)^{\varepsilon}\zeta_\rho^S(y-x), \rho^2 = kh.\end{aligned}$$

For $r \ge 3$ we obtain by induction

$$\begin{aligned} |I| &= \big|\left[\widetilde{p}_h \otimes_h (K_h+M_h+R_h)^{(r-1)} - \widetilde{p}_h \otimes_h (K_h+M_h)^{(r-1)}\right] \\ &\qquad \otimes_h (K_h+M_h)(0,kh,x,y)\big| \\ &\le C^r h^{3/2-\varepsilon} B(\frac{1}{2},\varepsilon) B(\frac{1}{2},\varepsilon+\frac{1}{2}) \ldots B(\frac{1}{2},\varepsilon+\frac{r-1}{2})(kh)^{\varepsilon+(r-2)/2}\zeta_\rho^S(y-x) \\ (30)\qquad &\le \Gamma(\varepsilon) h^{3/2-\varepsilon} \frac{[C\Gamma(1/2)]^r}{\Gamma(\varepsilon+\frac{r}{2})}(kh)^{\varepsilon+(r-2)/2}\zeta_\rho^S(y-x), \rho^2 = kh.\end{aligned}$$

To estimate II we use the following estimates

$$(31)\ \left|D_y^a D_x^b \widetilde{p}_h(jh,kh,x,y)\right| \le C\rho^{-|a|-|b|}\zeta_\rho^{S-|a|}(y-x), D_x^b \widetilde{p}_h(jh,kh,x,x+v) \le C\zeta_\rho^S(v),$$
$$(32)\ \left|D_x^b (K_h+M_h+R_h)(jh,kh,x+v,x)\right| \le C\rho^{-1}\zeta_\rho^S(v).$$

The inequalities (31) and (32) are obained by using the same arguments as is the proof of Lemma 7. Using these inequalities and mimicking the proof of Theorem 2.3 in Konakov and Mammen (2002) we obtain the following bounds for $r \geq 0$

$$\begin{aligned}
&\left|D_x^b D_y^a \widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0, kh, x, y)\right| \\
&\quad \leq C^r (kh)^{-|a|-|b|+r} B(\frac{1}{2}, \frac{1}{2}) B(1, \frac{1}{2}) ... B(\frac{r}{2}, \frac{1}{2}) \varsigma_\rho^{S-|a|}(y-x) \\
&\quad \leq \frac{[C\Gamma(1/2)]^r}{\Gamma(\frac{r+1}{2})} (kh)^{-|a|-|b|+r} \varsigma_\rho^{S-|a|}(y-x). \qquad (33)
\end{aligned}$$

Inequality (33) allows us to estimate $II = [\widetilde{p}_h \otimes_h (K_h + M_h'' + R_h)^{(r-1)}] \otimes_h R_h(0, kh, x, y)$. For this aim it suffices to estimate

$$\begin{aligned}
&h^{3/2} \sum h \int [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)}](0, jh, x, z) \\
&\quad \times D^\nu \widetilde{p}_h((j+1)h, kh, z, y)(f(jh, z) - f(jh, y)) dz \qquad (34)
\end{aligned}$$

for $r \geq 2$, $|\nu| = 4$, and

$$\begin{aligned}
&\sum_{j=0}^{n-2} h \int [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)}](0, jh, x, z)(f(jh, z) - f(jh, y)) \\
&\quad \times \int q(jh, z, \theta) \widetilde{h}^{\nu - e_l}(\theta) \int_0^1 (1-u)^4 D^\nu \widetilde{p}_h((j+1)h, kh, z + u\widetilde{h}(\theta), y) du d\theta dz \qquad (35)
\end{aligned}$$

for $r \geq 2$, $|\nu| = 5$, $1 \leq l \leq d$. Here $f(t, x)$ is a function whose first and second derivatives with respect to x are continuous and bounded, uniformly in t and x. The upper bound for (34) follows from (33) by integration by parts, exactly in the same way as it was done to obtain the upper bound for I_1 (see (27)). This gives an estimate

$$\begin{aligned}
&h^{3/2} \Big| \sum h \int [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)}](0, jh, x, z) \\
&\quad \times D^\nu \widetilde{p}_h((j+1)h, kh, z, y)(f(jh, z) - f(jh, y)) dz \Big| \\
&\quad \leq \Gamma(\varepsilon) h^{3/2-\varepsilon} \frac{[C\Gamma(1/2)]^r}{\Gamma(\frac{r+1}{2})} (kh)^{\varepsilon + (r-2)/2} \zeta_\rho^S(y-x). \qquad (36)
\end{aligned}$$

The upper bound for (35) also follows from (33) by integration by parts in the same way as it was done to obtain an upper bound for I_2 (see (28)). This gives for (35) the same estimate as in (36) and, hence,

$$|II| \leq C\Gamma(\varepsilon) h^{3/2-\varepsilon} \frac{[C\Gamma(1/2)]^r}{\Gamma(\frac{r+1}{2})} (kh)^{\varepsilon + (r-2)/2} \zeta_\rho^S(y-x). \qquad (37)$$

The assertion of the lemma follows now from (27), (29), (30) and (37).

Lemma 10. *Let* $A(s, t, x, y), B(s, t, x, y), C(s, t, x, y)$ some functions with the absolute value less than $C(t-s)^{-1/2} \zeta_{\sqrt{t-s}}(y-x)$ *for a constant* C. *Then*

$$\begin{aligned}
&\sum_{r=0}^{\infty} A \otimes_h (B + C)^{(r)}(ih, jh, x, y) - \sum_{r=0}^{\infty} A \otimes_h B^{(r)}(ih, jh, x, y) \\
&= \sum_{r=1}^{\infty} [A \otimes_h \Phi] \otimes_h [C \otimes_h \Phi]^{(r)} (ih, jh, x, y),
\end{aligned}$$

where $\Phi = \sum_{r=0}^{\infty} B^{(r)}$.

Proof of Lemma 10. Under the conditions of the lemma all series are absolutely convergent. The assertion of the lemma follows from linearity of the operation $\otimes_h$ and after permutation of the summands in the absolutely convergent series.

7 Proof of the main result.

We now come to the proof of Theorem 1. The main tools for the proof have been given in Sections 4 and 5. From Lemmas 1 and 2 we get that

$$p(0,T,x,y) = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0,T,x,y) + o(h^2T)\phi_{C,\sqrt{T}}(y-x).$$

With Lemma 3 this gives

$$p(0,T,x,y) - p_h(0,T,x,y) = T_1 + ... + T_7 + o(h^2T)\phi_{C,\sqrt{T}}(y-x), \tag{38}$$

where

$$T_1 = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h H^{(r)}(0,T,x,y),$$

$$T_2 = \sum_{r=0}^{n} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h'' + \sqrt{h}N_1)^{(r)}(0,T,x,y),$$

$$T_3 = \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h'' + \sqrt{h}N_1)^{(r)}(0,T,x,y)$$
$$- \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r)}(0,T,x,y),$$

$$T_4 = \sum_{r=0}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,T,x,y),$$

$$T_5 = \sum_{r=0}^{n} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0,T,x,y),$$

$$T_6 = \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y),$$

$$T_7 = \sum_{r=0}^{n} \tilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y) - \sum_{r=0}^{n} \tilde{p}_h \otimes_h H_h^{(r)}(0,T,x,y).$$

Here we put $N_1(s,t,x,y) = (L - \widetilde{L})\widetilde{\pi}_1(s,t,x,y)$.

We now discuss the asymptotic behavior of the terms $T_1, ..., T_7$.

Asymptotic treatment of the term T_1. Using Remark 1 in Konakov (2006) we get that

$$\begin{aligned} T_1 &= \frac{h}{2}[p \otimes_h (L^2 - 2L\widetilde{L} + \widetilde{L}^2)\widetilde{p} \otimes_h \Phi)](0,T,x,y) \\ &\quad + \frac{h}{2}[p \otimes_h (L' - \widetilde{L}')\widetilde{p} \otimes_h \Phi](0,T,x,y) + R_T(0,T,x,y), \end{aligned} \tag{39}$$

where for any $0 < \varepsilon < 1/2$

$$|R_T(0,T,x,y)| \le C(\varepsilon)(hn^{-1/2+\varepsilon} + h\sqrt{T})\phi_{C,\sqrt{T}}(y-x), \tag{40}$$

$\Phi(s,t,x,y) = \sum_{r=0}^{\infty} H^{(r)}(s,t,x,y)$. Here, the summand $H^{(0)}(s,t,x,y)$ was introduced to simplify notations. We define $g \otimes_h H^{(0)}(s,t,x,y) = g(s,t,x,y)$ for a function g. Note, that in the homogenous case it holds that $\sigma_{ij}(s,x) = \sigma_{ij}(x), m_i(s,x) = m_i(x)$ and that the second summand in (39) is equal to 0.

Asymptotic treatment of the term T_2. We will show that with a positive constant $\delta > 0$

$$\begin{aligned} &\Bigg| T_2 - 3\sum_{r=0}^{\infty} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) + \sum_{r=0}^{\infty} \tilde{p} \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r)}(0,T,x,y) \\ &+ \sum_{r=0}^{\infty} \tilde{p} \otimes_h (H + M_{h,2})^{(r)}(0,T,x,y) + \sum_{r=0}^{\infty} \tilde{p} \otimes_h (H + M''_{h,3})^{(r)}(0,T,x,y) \Bigg| \\ &\le Chn^{-\delta}\zeta_{\sqrt{T}}(y-x). \end{aligned} \tag{41}$$

It suffices to discuss the case $r \ge 2$ because for $r = 1, 2$ the left hand side of (41) is equal to zero. For $r \ge 2$, (41) immediately follows from the following bounds

$$\begin{aligned} &\Big| \tilde{p} \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r)}(0,T,x,y) \\ &\quad - \tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r)}(0,T,x,y) \\ &\quad - [\tilde{p} \otimes_h (H + M''_{h,3})^{(r)} - \tilde{p} \otimes_h H^{(r)}](0,T,x,y) \Big| \\ &\le C(\varepsilon)h^{3/2-2\varepsilon}\frac{C^r}{\Gamma(\frac{r-1}{2})}T^{3\varepsilon + \frac{r-3}{2}}\zeta_{\sqrt{kh}}(v-x), \end{aligned} \tag{42}$$

$$\begin{aligned} &\Big| \tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r)}(0,T,x,y) \\ &\quad - \tilde{p} \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r)}(0,T,x,y) \\ &\quad - [\tilde{p} \otimes_h (H + M_{h,2})^{(r)} - \tilde{p} \otimes_h H^{(r)}](0,T,x,y) \Big| \\ &\le C(\varepsilon)h^{3/2-2\varepsilon}\frac{C^r}{\Gamma(\frac{r-1}{2})}T^{3\varepsilon + \frac{r-4}{2}}\zeta_{\sqrt{T}}(v-x), \end{aligned} \tag{43}$$

for all sufficiently small $\varepsilon > 0$ with a function C that fulfils $\lim_{\varepsilon \longrightarrow 0} C(\varepsilon) = +\infty$. We will prove first the bound (42). Denote the expression under the sign of the absolute value in (42) by Γ_r. Note that $\Gamma_0 = \Gamma_1 = 0$. For $r \ge 2$ we make use of the following recurrence formula

$$(44)\quad \Gamma_r = \Gamma_{r-1} \otimes_h H + \Big[\tilde{p} \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r-1)}$$
$$-\tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)}\Big] \otimes_h (M''_h + \sqrt{h}N_1)$$
$$+ \Big[\tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M''_{h,3})^{(r-1)}\Big] \otimes_h M''_{h,3}$$
$$= I + II + III.$$

We start with the estimation of II. First we will estimate

$$(45)\qquad \Big|\tilde{p} \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)}\Big|.$$

For $r = 2$ we have $\tilde{p} \otimes_h M''_{h,3}(0, kh, x, y)$. It follows from (20) that it is enough to estimate

$$(46)\qquad J_1 = h^2 \sum_{i=0}^{k-2} h \int \widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y)) D^\nu_v \widetilde{p}_h((i+1)h, kh, v, y)dv$$

for $|\nu| = 4$ and

$$(47)\qquad J_2 = h^{3/2} \sum_{i=0}^{k-2} h \int \widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y)) \int q(ih, v, \theta)\theta^\nu \int_0^1 (1-\delta)^2$$
$$\times D^{\nu+e_l+e_q}_v \widetilde{p}_h((i+1)h, kh, v + \delta\widetilde{h}(\theta), y)d\delta d\theta dv$$

for $|\nu| = 3$. Here $f(t, x)$ is a function for which $D^\nu_x f(t, x)$ is bounded uniformly in (t, x) for $|\nu| = 0, 1, 2, 3$. An estimate forJ_1 follows from (27). This gives

$$(48)\qquad |J_1| \le Ch^{2-\varepsilon}(kh)^{\varepsilon-1/2} B(\frac{1}{2}, \varepsilon)\zeta^S_{\sqrt{kh}}(y - x).$$

The estimate for J_2 can be obtained analogously to the estimate of I_2 (see (28)). Integrating by parts we get

$$J_2 = h^{3/2} \sum_{j=0}^{k-2} h \int_0^1 (1-\delta)^2 d\delta \int d\theta \cdot \theta^\nu \int D^{e_l+e_q}_v [\widetilde{p}(0, ih, x, v)$$
$$\times(f(ih, v) - f(ih, y))q(ih, v, \theta)] D^\nu_v \widetilde{p}_h((i+1)h, kh, v + \delta\widetilde{h}(\theta), y)dv.$$

The derivative

$$D^{e_l+e_q}_v [\widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y))q(ih, v, \theta)]$$

is a sum of 9 summands. Integrating by parts once more for summands which contain $D^\mu_\nu \widetilde{p}(0, ih, x, v)$ with $|\mu| < 2$, we obtain

$$(49)\qquad |J_2| \le Ch^{3/2-2\varepsilon}\zeta^{S-3}_{\sqrt{kh}}(y - x) \int \psi(\theta) \|\theta\|^3 (h^{(S-3)/2} \|\theta\|^{S-3} + 1)d\theta$$
$$\sum_{j=0}^{k-2} h \frac{1}{(ih)^{1-\varepsilon}} \times \frac{1}{(kh - ih)^{1-\varepsilon}}$$
$$\le Ch^{3/2-2\varepsilon} B(\varepsilon, \varepsilon)(kh)^{2\varepsilon-1}\zeta^{S-3}_{\sqrt{kh}}(y - x)$$

for any $\varepsilon \in (0, 1/4)$. It follows from (48) and (49) that for $r = 2$ (45) does not exceed $Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)(kh)^{\varepsilon-1}\zeta^{S-3}_{\sqrt{kh}}(y-x)$. For $r \geq 3$ we use the recurrence relation

$$
\begin{aligned}
(50)\quad & \tilde p \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r-1)} - \tilde p \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-1)} \\
&= \Big[\tilde p \otimes_h (H + M''_h + \sqrt{h}N_1)^{(r-2)} - \tilde p \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-2)}\Big] \\
&\qquad \otimes_h (H + M''_h + \sqrt{h}N_1) + [\tilde p \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)^{(r-2)}] \otimes_h M''_{h,3} \\
&= I' + II'.
\end{aligned}
$$

From (50) we obtain for $r = 3$

$$
\begin{aligned}
|I'| &\leq Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)\zeta^{S-3}_{\sqrt{kh}}(y-x)\sum_{i=0}^{k-2} h(ih)^{\varepsilon-1}(kh-ih)^{-1/2} \\
&\leq Ch^{3/2-2\varepsilon}B(\varepsilon,\varepsilon)B(\frac{1}{2},\varepsilon)(kh)^{\varepsilon-1/2}\zeta^{S-3}_{\sqrt{kh}}(y-x).
\end{aligned}
$$

For the estimate of II' we use the following estimates

$$
(51)\qquad \left|D^a_v D^b_x (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)(jh, kh, x, v)\right| \leq C\rho^{-1-|a|-|b|}\zeta_\rho(v-x),
$$

$$
(52)\qquad \left|D^b_x (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1)(jh, kh, x, x+v)\right| \leq C\rho^{-1}\zeta_\rho(v-x).
$$

To prove (51) it is enough to get the corresponding estimates for summands in $M_{h,1}, M_{h,2}$ and $\sqrt{h}N_1$

$$
\begin{aligned}
(53)\quad & h^{1/2}D^a_v D^b_x \left[D^\nu_x \widetilde p_h((j+1)h, kh, x, v)(f(jh,x) - f(jh,v))\right] \\
& \leq C\rho^{-1-|a|-|b|}\zeta_\rho(v-x) \text{ for } |\nu| = 3, \\
(54)\quad & hD^a_v D^b_x \left[D^\nu_x \widetilde p_h((j+1)h, kh, x, v)(f(jh,x) - f(jh,v))\right] \\
& \leq C\rho^{-1-|a|-|b|}\zeta_\rho(v-x) \text{ for } |\nu| = 4, \\
(55)\quad & h^{1/2}D^a_v D^b_x \left[D^{\nu+e_p+e_q}_x \widetilde p_h((j+1)h, kh, x, v)\rho^2(f(jh,x) - f(jh,v))\right] \\
& \leq C\rho^{-1-|a|-|b|}\zeta_\rho(v-x) \text{ for } |\nu| = 3,
\end{aligned}
$$

for a function $f(t,x)$ with $|a| + |b|$ derivatives w.r.t. x that are uniformly bounded w.r.t. t. These estimates are direct consequences of Lemma 7. To prove (52) it is enough to show the corresponding estimates for summands in $M_{h,1}, M_{h,2}$ and $\sqrt{h}N_1$

$$
\begin{aligned}
(56)\quad & h^{1/2}D^b_x \left[D^\nu_x \widetilde p_h((j+1)h, kh, x, y)\,|_{y=x+v}\,(f(jh,x) - f(jh,x+v))\right] \\
& \leq C\rho^{-1}\zeta_\rho(v-x) \text{ for } |\nu| = 3, \\
(57)\quad & hD^b_x \left[D^\nu_x \widetilde p_h((j+1)h, kh, x, y)\,|_{y=x+v}\,(f(jh,x) - f(jh,x+v))\right] \\
& \leq C\rho^{-1}\zeta_\rho(v-x) \text{ for } |\nu| = 4, \\
(58)\quad & h^{1/2}D^b_x \left[D^{\nu+e_p+e_q}_x \widetilde p_h((j+1)h, kh, x, v)\,|_{y=x+v}\,\rho^2(f(jh,x) - f(jh,x+v))\right] \\
& \leq C\rho^{-1}\zeta_\rho(v-x) \text{ for } |\nu| = 3.
\end{aligned}
$$

These estimates also follow from the estimates obtained in the proof of Lemma 7. If $z(V^{-1/2}_{j,k}(y), \mu_{j,k}(y), x, y) = V^{-1/2}_{j,k}(y)(y - x - \mu_{j,k}(y))$ then

$$
(59)\qquad \left|\frac{\partial z}{\partial y}\right| \leq \frac{C}{\rho}, \left|\frac{\partial z}{\partial x}\right| \leq \frac{C}{\rho}
$$

and for $z(V_{j,k}^{-1/2}(x+v), \mu_{j,k}(x+v), x, x+v) = V_{j,k}^{-1/2}(x+v)(v-\mu_{j,k}(x+v))$ we have

$$\left|\frac{\partial z}{\partial x}\right| \leq \left|\frac{\partial z}{\partial V_{j,k}^{-1/2}}\right| \left|\frac{\partial V_{j,k}^{-1/2}}{\partial x}\right| + \left|\frac{\partial z}{\partial \mu_{j,k}}\right| \left|\frac{\partial \mu_{j,k}}{\partial x}\right| \leq C(\|v\|+1). \tag{60}$$

Now with the inequalities (51),(52) we can proceed like in the proof of Theorem 2.3 in Konakov and Mammen (2002). This gives the following estimate for $r \geq 3$

$$\begin{aligned} &\left|D_v^a D_x^b [\widetilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r-2)}](jh, kh, x, v)\right| \\ &\leq C^r B(1, \frac{1}{2}) \times ... \times B(\frac{r-1}{2}, \frac{1}{2}) \rho^{r-2-|a|-|b|} \zeta_\rho(v-x). \end{aligned} \tag{61}$$

Now we denote $\widetilde{p}_{1,r} = \widetilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r)}$ and $\widetilde{p}_0 = \widetilde{p}$. For an estimate of $\widetilde{p}_{1,r-2} \otimes_h M''_{h,3}$ it suffices to make the same calculations using integration by parts as it was done above for J_1 and J_2. This gives

$$\begin{aligned} |II'| &\leq \left|\widetilde{p}_{1,r-2} \otimes_h M''_{h,3}(0, kh, x, y)\right| \\ &\leq C^r h^{3/2-2\varepsilon} B(1, \frac{1}{2}) B(1 + \frac{1}{2}, \frac{1}{2}) \\ &\qquad \times ... \times B(1 + \frac{r-3}{2}, \frac{1}{2}) B(\frac{r-2}{2}, \varepsilon) (kh)^{\varepsilon + \frac{r-4}{2}} \zeta_{\sqrt{kh}}(v-x) \end{aligned} \tag{62}$$

and by induction for $r \geq 3$

$$\begin{aligned} |I'| \leq\ & C^r h^{3/2-2\varepsilon} B(\varepsilon, \frac{1}{2}) B(\varepsilon + \frac{1}{2}, \frac{1}{2}) \\ & \times ... \times B(\varepsilon + \frac{r-3}{2}, \frac{1}{2}) B(\varepsilon, \varepsilon) (kh)^{\varepsilon + \frac{r-4}{2}} \zeta_{\sqrt{kh}}(v-x). \end{aligned} \tag{63}$$

Comparing (62) and (63) we obtain that for $r \geq 3$

$$\begin{aligned} &\left|\tilde{p} \otimes_h (H + M''_h + \sqrt{h} N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r-1)}\right| \\ &\leq C^r h^{3/2-2\varepsilon} B(\varepsilon, \frac{1}{2}) B(\varepsilon + \frac{1}{2}, \frac{1}{2}) \\ &\qquad \times ... \times B(\varepsilon + \frac{r-3}{2}, \frac{1}{2}) B(\varepsilon, \varepsilon) (kh)^{\varepsilon + \frac{r-4}{2}} \zeta_{\sqrt{kh}}(v-x). \end{aligned} \tag{64}$$

From (64) we get the following estimate for II

$$|II| \leq C^r h^{3/2-2\varepsilon} B(\varepsilon, \varepsilon) B(\varepsilon, \frac{1}{2}) B(\varepsilon + \frac{1}{2}, \frac{1}{2}) \times ... \times B(\varepsilon + \frac{r-2}{2}, \frac{1}{2}) T^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{T}}(v-x). \tag{65}$$

To estimate III note that the same inequalities (51), (52),(61) hold for $H + M''_{h,3}$, namely,

$$\left|D_v^a D_x^b (H + M''_{h,3})(jh, kh, x, v)\right| \leq C \rho^{-1-|a|-|b|} \zeta_\rho(v-x) \tag{66}$$

$$\left|D_x^b (H + M''_{h,3})(jh, kh, x, x+v)\right| \leq C \rho^{-1} \zeta_\rho(v-x) \tag{67}$$

$$\begin{aligned} &\left|D_v^a D_x^b [\widetilde{p} \otimes_h (H + M''_{h,3})^{(r)}](jh, kh, x, v)\right| \\ &\qquad \leq C^r B(1, \frac{1}{2}) \times ... \times B(\frac{r+1}{2}, \frac{1}{2}) \rho^{r-|a|-|b|} \zeta_\rho(v-x). \end{aligned} \tag{68}$$

To prove (66)- (68) it is enough to get the corresponding estimates for summands in $M''_{h,3}$ (see (20)). These estimates can be proved by the same arguments as used in the proof of (51), (52) and (61). To estimate III we have now to estimate $\widetilde{p}_{1,r} \otimes_h M''_{h,3}$ and $\widetilde{p}_{2,r} \otimes_h M''_{h,3}$ where

$$\widetilde{p}_{2,r} = \widetilde{p} \otimes_h (H + M''_{h,3})^{(r)}.$$

Using integration by parts and inequality (68), we obtain for $\widetilde{p}_{2,r-1} \otimes_h M''_{h,3}$ the same estimate as for $\widetilde{p}_{1,r} \otimes_h M''_{h,3}$. It holds for $i = 1, 2$

$$\begin{aligned}&\left|\widetilde{p}_{2,r} \otimes_h M''_{h,3}(0, kh, x, y)\right| \\ &\quad \leq C^r h^{3/2-2\varepsilon} B(1, \frac{1}{2}) \times ... \times B(\frac{r+1}{2}, \frac{1}{2}) B(\frac{r}{2}, \varepsilon)(kh)^{\varepsilon + \frac{r-2}{2}} \zeta_{\sqrt{kh}}(y - x).\end{aligned}$$

Hence it holds for $r \geq 2$

$$\begin{aligned}(69) \qquad |III| &\leq \left|\widetilde{p}_{1,r-1} \otimes_h M''_{h,3}(0, kh, x, y)\right| + \left|\widetilde{p}_{2,r-1} \otimes_h M''_{h,3}(0, kh, x, y)\right| \\ &\leq C^r h^{3/2-2\varepsilon} B(1, \frac{1}{2}) \times ... \times B(\frac{r}{2}, \frac{1}{2}) B(\frac{r-1}{2}, \varepsilon)(kh)^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{kh}}(y - x).\end{aligned}$$

From (44), (65) and (69) we get for $r \geq 2$

$$|\Gamma_r(0, kh, x, y)| \leq C^r h^{3/2-2\varepsilon} B(\varepsilon, \varepsilon) B(\varepsilon, \frac{1}{2}) \times ... \times B(\varepsilon + \frac{r-2}{2}, \frac{1}{2})(kh)^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{kh}}(v - x).$$

In particular

$$(70) \qquad |\Gamma_r(0, T, x, y)| \leq h^{3/2-2\varepsilon} \frac{\Gamma^3(\varepsilon)}{\Gamma(2\varepsilon)} \frac{C^r}{\Gamma(\varepsilon + \frac{r-1}{2})} T^{\varepsilon + \frac{r-3}{2}} \zeta_{\sqrt{kh}}(v - x)$$

for any $\varepsilon \in (0, 1/4)$ and $r \geq 2$. Now we shall estimate the left hand side of (43). Denote the expression under the sign of the absolute value in (43) by $\mathcal{F}_r$. Note that $\mathcal{F}_0 = \mathcal{F}_1 = 0$. For $r \geq 2$ we make use of the following recurrence formula

$$\begin{aligned}\mathcal{F}_r &= \mathcal{F}_{r-1} \otimes_h H + \Big[\widetilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r-1)} \\ &\qquad - \widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)}\Big] \otimes_h (M_{h,1} + M_{h,2} + \sqrt{h} N_1) \\ &\quad + \Big[\widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)} - \widetilde{p} \otimes_h (H + M_{h,1})^{(r-1)}\Big] \otimes_h M_{h,2} \\ &= I + II + III.\end{aligned}$$

We start again from the estimation of

$$A_{r-1} = \widetilde{p} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h} N_1)^{(r-1)} - \widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h} N_1)^{(r-1)}.$$

For $r = 2$ we have $A_1 = (\widetilde{p} \otimes_h M_{h,2})(0, kh, x, y)$. It is enough to estimate

$$J_3 = h \sum_{i=0}^{k-2} h \int \widetilde{p}(0, ih, x, v)(f(ih, v) - f(ih, y)) D_v^{\nu} \widetilde{p}_h((i+1)h, kh, v, y) dv$$

for $|\nu| = 4$. Analogously to (27) we obtain that

$$|J_3| \le Ch^{1-\varepsilon}(kh)^{-1/2+\varepsilon}B(\frac{1}{2},\varepsilon)\zeta^S_{\sqrt{kh}}(y-x)$$

and, hence,

$$|A_1| \le Ch^{1-\varepsilon}(kh)^{-1/2+\varepsilon}B(\frac{1}{2},\varepsilon)\zeta^S_{\sqrt{kh}}(y-x). \tag{71}$$

For $r \ge 3$ we use the recurrence relation

$$\begin{aligned} A_{r-1} &= A_{r-2} \otimes_h (H + M_{h,1} + M_{h,2} + \sqrt{h}N_1) \\ &\quad + \left[\widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r-2)}\right] \otimes_h M_{h,2} \\ &= I' + II'. \end{aligned} \tag{72}$$

From (71) and (72) we obtain for $r = 3$

$$\begin{aligned} |I'| &\le Ch^{1-\varepsilon}B(\frac{1}{2},\varepsilon)\zeta^S_{\sqrt{kh}}(y-x)\sum_{i=0}^{k-2} h(ih)^{\varepsilon-1/2}(kh-ih)^{-1/2} \\ &\le Ch^{1-\varepsilon}B(\frac{1}{2},\varepsilon)B(\frac{1}{2},\varepsilon+\frac{1}{2})(kh)^{\varepsilon}\zeta^S_{\sqrt{kh}}(y-x). \end{aligned} \tag{73}$$

To estimate II' we use the inequality for $r \ge 3$

$$\begin{aligned} &\left|D^a_v D^b_x[\widetilde{p} \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r-2)}](jh,kh,x,v)\right| \\ &\qquad \le C^r B(1,\frac{1}{2}) \times \ldots \times B(\frac{r-1}{2},\frac{1}{2})\rho^{r-2-|a|-|b|}\zeta_\rho(v-x). \end{aligned} \tag{74}$$

This inequality is a direct consequence of (61). We have

$$|II'| \le Ch^{1-\varepsilon}B(1,\varepsilon)(kh)^{\varepsilon}\zeta^S_{\sqrt{kh}}(y-x). \tag{75}$$

Comparing (73) and (75) we obtain that $|A_2| \le C^2h^{1-\varepsilon}B(\frac{1}{2},\varepsilon)B(\frac{1}{2},\varepsilon+\frac{1}{2})(kh)^{\varepsilon}\zeta^S_{\sqrt{kh}}(y-x)$. By induction we easily get that for $r \ge 2$

$$|A_{r-1}(0,kh,x,y)| \le C^r h^{1-\varepsilon}B(\frac{1}{2},\varepsilon) \times \ldots \times B(\frac{1}{2},\varepsilon+\frac{r-2}{2})(kh)^{\varepsilon+\frac{r-3}{2}}\zeta^S_{\sqrt{kh}}(y-x). \tag{76}$$

For a bound of $A_{r-1} \otimes_h (M_{h,1} + M_{h,2} + \sqrt{h}N_1)$ it is enough to estimate

$$\begin{aligned} J_4 = h^{1/2}\sum_{i=0}^{k-2} h \int A_{r-1}(0,ih,x,v)(f(ih,v) - f(ih,y)) \\ D^\nu_v \widetilde{p}_h((i+1)h,kh,v,y)dv \text{ for } |\nu| = 3, \end{aligned} \tag{77}$$

$$\begin{aligned} J_5 = h\sum_{i=0}^{k-2} h \int A_{r-1}(0,ih,x,v)(f(ih,v) - f(ih,y)) \\ D^\nu_v \widetilde{p}_h((i+1)h,kh,v,y)dv \text{ for } |\nu| = 4, \end{aligned} \tag{78}$$

$$\begin{aligned} J_6 = h^{1/2}\sum_{i=0}^{k-2} h \int A_{r-1}(0,ih,x,v)(f(ih,v) - f(ih,y))(kh-ih) \\ D^{\nu+e_p+e_q}_v \widetilde{p}(ih,kh,v,y)dv \text{ for } |\nu| = 3. \end{aligned} \tag{79}$$

It follows from (76) that

$$|J_4| \le C^r h^{3/2-2\varepsilon} B(\frac{1}{2},\varepsilon) \times ... \times B(\frac{1}{2},\varepsilon+\frac{r-2}{2})$$
$$\times B(\varepsilon,\varepsilon+\frac{r-1}{2})(kh)^{2\varepsilon+\frac{r-3}{2}}\zeta^S_{\sqrt{kh}}(y-x).$$

Clearly, the same estimate holds for J_5 and J_6. Thus we obtain

$$(80)\quad |II| \le C^r h^{3/2-2\varepsilon} B(\frac{1}{2},\varepsilon)\times...\times B(\frac{1}{2},\varepsilon+\frac{r-2}{2})B(\varepsilon,\varepsilon+\frac{r-1}{2})(kh)^{2\varepsilon+\frac{r-3}{2}}\zeta^S_{\sqrt{kh}}(y-x).$$

Now we estimate III. We denote

$$B_{r-1} = \widetilde{p}\otimes_h (H+M_{h,1}+\sqrt{h}N_1)^{(r-1)} - \widetilde{p}\otimes_h (H+M_{h,1})^{(r-1)}.$$

Using the recurrence equation $B_0 = 0$ and

$$B_{r-1} = B_{r-2}\otimes_h (H+M_{h,1}+\sqrt{h}N_1) + \widetilde{p}\otimes_h (H+M_{h,1})^{(r-2)}\otimes_h \sqrt{h}N_1$$

we obtain that

$$(81)\qquad III = \sum_{l=0}^{r-2} \widetilde{p}_{3,l}\otimes_h \sqrt{h}N_1 \otimes_h (H+M_{h,1}+\sqrt{h}N_1)^{(r-l-2)}\otimes_h M_{h,2}(0,T,x,y),$$

where $\widetilde{p}_{3,l} = \widetilde{p}\otimes_h (H+M_{h,1})^{(l)}$. To estimate III it's enough to estimate a typical term in the last sum. Thus we need a bound for

$$(82)\qquad h^{3/2}\sum_{k=0}^{n-2} h\int\left\{\sum_{j=0}^{k-1} h\left[\int\sum_{i=0}^{j-1} h\int \widetilde{p}_{3,l}(0,ih,x,w)(jh-ih)\right.\right.$$
$$\left.\times D_w^{\mu+e_n+e_m}\widetilde{p}(ih,jh,w,z)(g(ih,w)-g(ih,z))dw\right]$$
$$\left.\times(H+M_{h,1}+\sqrt{h}N_1)^{(r-l-2)}(ih,kh,z,v)dz\right\}(f(kh,v)-f(kh,y))$$
$$\times D_v^{\nu}\widetilde{p}_h((k+1)h,T,v,y)dv.$$

For an estimate of (82) we apply two times integration by parts in the internal integral $\int ...dw$ and then we make two times an integration by parts in $\int ...dv$. We use also the following estimates for $0\le l\le r-3$

$$(83)\quad \left|D_w^a D_x^b \widetilde{p}_{3,l}(0,ih,x,w)\right|$$
$$\le C^l B(1,\frac{1}{2})\times ...\times B(\frac{l+1}{2},\frac{1}{2})(ih)^{\frac{l-|a|-|b|}{2}}\zeta_{\sqrt{ih}}(w-x),$$

$$(84)\quad \left|D_v^a D_z^b (H+M_{h,1}+\sqrt{h}N_1)^{(r-l-2)}(ih,kh,z,v)\right|$$
$$\le C^{r-l-2}B(\frac{1}{2},\frac{1}{2})\times ...\times B(\frac{1}{2},\frac{r-l-3}{2})(kh-ih)^{\frac{r-l-4-|a|-|b|}{2}}\zeta_{\sqrt{kh-ih}}(v-z).$$

We put here $B(\frac{1}{2},0)=1$. This gives the following estimate for $0\le l\le r-3, r\ge 2$

$$(85)\qquad \left|\widetilde{p}_{3,l}\otimes_h \sqrt{h}N_1\otimes_h (H+M_{h,1}+\sqrt{h}N_1)^{(r-l-2)}\otimes_h M_{h,2}(0,T,x,y)\right|$$
$$\le C^r h^{3/2-3\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(3\varepsilon+\frac{r-1}{2})}T^{3\varepsilon+\frac{r-3}{2}}\zeta_{\sqrt{T}}(y-x).$$

For $l = r-2$ we have to estimate

$$\widetilde{p}_{3,r-2} \otimes_h \sqrt{h} N_{1h} \otimes_h M_{h,2}(0,T,x,y).$$

This is a finite sum of terms corresponding to the different summands in $N_{1,h}$ and $M_{h,2}$. A typical term can be bounded by

$$(86) \quad h^{3/2} \sum_{k=0}^{n-2} h \int \left\{ \sum_{j=0}^{k-1} h \int \widetilde{p}_{3,r-2}(0,jh,x,w)(kh-jh) D_w^{\mu+e_n+e_m} \widetilde{p}(jh,kh,w,v) \right. \\ \left. \times (g(jh,w) - g(jh,v)) dw \right\} (f(kh,v) - f(kh,y)) D_v^{\nu} \widetilde{p}_h((k+1)h,T,v,y) dv.$$

We apply again integration by parts and after direct calculations we obtain the following estimate for $r \geq 2$

$$(87) \quad \left| \widetilde{p}_{3,r-2} \otimes_h \sqrt{h} N_{1h} \otimes_h M_{h,2}(0,T,x,y) \right| \\ \leq C^r h^{3/2-3\varepsilon} \frac{\Gamma(\varepsilon + \frac{r-2}{2})}{\Gamma(3\varepsilon + \frac{r-2}{2})} \Gamma^2(\varepsilon) \frac{1}{\Gamma(\frac{r}{2})} T^{3\varepsilon + \frac{r-4}{2}} \zeta_{\sqrt{T}}(y-x).$$

The inequalities (42), (43) follow from (70), (85) and (87).

Asymptotic treatment of the term T_3. We will show that

$$(88) \quad \left| T_3 - \left[\sum_{r=0}^{\infty} \tilde{p} \otimes_h (H+A)^{(r)}(0,T,x,y) - \sum_{r=0}^{\infty} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) \right] \right| \\ \leq C h n^{-\delta} \zeta_{\sqrt{T}}(y-x),$$

where $A = M_h'' - M_h = -\frac{h}{2}(L_*^2 - 2L\widetilde{L} + \widetilde{L}^2)\lambda(x)$. Denote

$$\begin{aligned} C_r &= \tilde{p} \otimes_h (H + M_h'' + \sqrt{h} N_1)^{(r)}(0,T,x,y) \\ &\quad - \tilde{p} \otimes_h (H + M_h + \sqrt{h} N_1)^{(r)}(0,T,x,y) \\ &\quad - [\tilde{p} \otimes_h (H+A)^{(r)} - \tilde{p} \otimes_h H^{(r)}](0,T,x,y). \end{aligned}$$

Analogously to (44) we have the recurrence relation

$$(89) \quad \begin{aligned} C_r &= C_{r-1} \otimes_h H + \Big[\tilde{p} \otimes_h (H + M_h'' + \sqrt{h} N_1)^{(r-1)} \\ &\quad - \tilde{p} \otimes_h (H + M_h + \sqrt{h} N_1)^{(r-1)} \Big] \otimes_h (M_h'' + \sqrt{h} N_1) \\ &\quad + \Big[\tilde{p} \otimes_h (H + M_h + \sqrt{h} N_1)^{(r-1)} - \tilde{p} \otimes_h (H+A)^{(r-1)} \Big] \otimes_h A \\ &= I + II + III. \end{aligned}$$

Denote

$$D_{r-1} = \tilde{p} \otimes_h (H + M_h + \sqrt{h} N_1)^{(r-1)} - \tilde{p} \otimes_h (H+A)^{(r-1)}.$$

Clearly

$$D_{r-1} = D_{r-2} \otimes_h (H + M_h + \sqrt{h} N_1) + \tilde{p}_h \otimes_h (H+A)^{(r-2)} \otimes_h (M_h - A + \sqrt{h} N_1).$$

Iterating we obtain

$$
\begin{aligned}
(90) \quad III &= D_{r-1} \otimes_h A \\
&= \sum_{l=0}^{r-2} \widetilde{p}_{4,l} \otimes_h (M_h - A + \sqrt{h} N_1) \otimes_h (H + M_h + \sqrt{h} N_1)^{(r-l-2)} \otimes_h A(0,T,x,y),
\end{aligned}
$$

where $\widetilde{p}_{4,l} = \widetilde{p} \otimes_h (H+A)^{(l)}$. The sum (90) can be estimated exactly in the same way as the sum (81). This gives

$$
(91) \qquad |III| \le C(\varepsilon) h^{3/2-2\varepsilon} \frac{C^r}{\Gamma(\frac{r-1}{2})} T^{3\varepsilon + \frac{r-4}{2}} \zeta_{\sqrt{T}}(v-x), r = 2,3...
$$

To estimate II denote

$$
E_{r-1} = \tilde{p} \otimes_h (H + M_h'' + \sqrt{h} N_1)^{(r-1)} - \tilde{p} \otimes_h (H + M_h + \sqrt{h} N_1)^{(r-1)}.
$$

For $r=2$ we have $E_1 = \tilde{p} \otimes_h A$ and analogously to (48)

$$
(92) \qquad |E_1| \le C h^{1-\varepsilon} (kh)^{\varepsilon - 1/2} B(\frac{1}{2}, \varepsilon) \zeta^S_{\sqrt{kh}}(y-x).
$$

Analogously to (50) for $r \ge 3$ we use the recurrence relation

$$
\begin{aligned}
(93) \ E_{r-1} &= E_{r-2} \otimes_h (H + M_h'' + \sqrt{h} N_1) + [\tilde{p} \otimes_h (H + M_h + \sqrt{h} N_1)^{(r-2)}] \otimes_h A \\
&= I' + II'.
\end{aligned}
$$

Both terms in (93) are analogous to the corresponding terms in (72) and may be estimated analogously. This gives the following estimates for $r \ge 3$

$$
\begin{aligned}
|E_{r-1}| &\le C^r h^{1-\varepsilon} B(\frac{1}{2}, \varepsilon) \times ... \times B(\frac{1}{2}, \varepsilon + \frac{r-2}{2}) (kh)^{\varepsilon + \frac{r-3}{2}} \zeta^S_{\sqrt{kh}}(y-x), \\
(94) \qquad |II| &= \Big| E_{r-1} \otimes_h (M_h'' + \sqrt{h} N_1)(0,T,x,y) \Big| \\
&\le C(\varepsilon) h^{3/2-2\varepsilon} \frac{C^r}{\Gamma(\frac{r-1}{2})} T^{3\varepsilon + \frac{r-4}{2}} \zeta_{\sqrt{T}}(v-x).
\end{aligned}
$$

The desired estimate (88) follows from (89), (91) and (94).

Asymptotic treatment of the term T_4. We will show that

$$
\begin{aligned}
(95) \qquad T_4 &= \sum_{r=1}^{\infty} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) \\
&\quad - \sum_{r=1}^{\infty} \tilde{p} \otimes_h [H + hN_2]^{(r)}(0,T,x,y) + R_h^*(x,y),
\end{aligned}
$$

with $N_2(s,t,x,y) = (L - \widetilde{L})\widetilde{\pi}_2(s,t,x,y)$, $|R_h^*(x,y)| \le Chn^{-\delta} \zeta^S_{\sqrt{T}}(y-x)$ for $\delta > 0$ small enough and with a constant C depending on δ. For the proof of (95) it suffices to show that for δ small enough

$$\left|\sum_{r=1}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(r)}(0,T,x,y) - \sum_{r=1}^{n} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,T,x,y)\right| \leq \left[\sum_{k=1}^{n} \frac{C^k}{\Gamma(\frac{k}{2})}\right] hn^{-\delta}\zeta^S_{\sqrt{T}}(y-x), \tag{96}$$

$$\left|\sum_{r=1}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1)^{(r)}(0,T,x,y) - \sum_{r=1}^{n} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(r)}(0,T,x,y) - \left[\sum_{r=1}^{n} \tilde{p} \otimes_h H^{(r)}(0,T,x,y) - \sum_{r=1}^{n} \tilde{p} \otimes_h [H + hN_2]^{(r)}(0,T,x,y)\right]\right| \leq \left[\sum_{k=1}^{n} \frac{C^k}{\Gamma(\frac{k}{2})}\right] Chn^{-\delta}\zeta^S_{\sqrt{T}}(y-x). \tag{97}$$

Denote $D_{3,0} \equiv 0$ and

$$\begin{aligned} D_{3,m}(0,jh,x,y) &= \sum_{r=1}^{m} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,jh,x,y) \\ &\quad - \sum_{r=1}^{m} \tilde{p} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(r)}(0,jh,x,y). \end{aligned}$$

Then (96) can be rewritten as

$$|D_{3,n}(0,T,x,y)| \leq Chn^{-\delta}\zeta^S_{\sqrt{T}}(y-x).$$

We now make iterative use of

$$D_{3,m} = D_{3,m-1} \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2) + g_{m-1}, \tag{98}$$

for $m = 1, 2, ...,$ where

$$\begin{aligned} g_m(0,jh,x,y) &= -\left[\sum_{r=0}^{m} \tilde{p} \otimes_h (K_h + M_h)^{(r)}\right] \otimes_h (H - K_h + \sqrt{h}N_1 + hN_2)(0,jh,x,y) \\ &= S_{h,m} \otimes_h (L - \widetilde{L})d_h(0,jh,x,y) \end{aligned}$$

with

$$\begin{aligned} g_0(0,jh,x,y) &= -\tilde{p} \otimes_h (H - K_h + \sqrt{h}N_1 + hN_2)(0,jh,x,y), \\ d_h &= \widetilde{p}_h - \widetilde{p} - \sqrt{h}\widetilde{\pi}_1 - h\widetilde{\pi}_2, \\ S_{h,m}(0,ih,x,y) &= \sum_{r=0}^{m} \tilde{p} \otimes_h (K_h + M_h)^{(r)}(0,ih,x,y). \end{aligned}$$

Iterative application of (98) gives

$$D_{3,n}(0,T,x,y) = \sum_{r=0}^{n-1} g_r \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(0,T,x,y).$$

To prove (96) we will show that

$$\left| g_r \otimes_h (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(0,T,x,y) \right| \le \frac{C^{n-r}}{\Gamma(\frac{n-r}{2})} hn^{-\delta} \zeta^S_{\sqrt{T}}(y-x). \tag{99}$$

For this purpose we decompose the left handside of (100) into four terms

$$\begin{aligned}
a_{r,1} &= \sum_{0\le i\le n/2} h \int g_r(0,ih,x,u)(H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)du, \\
a_{r,2} &= \sum_{n/2<i\le n} h^2 \sum_{0\le k\le i/2} \int\int S_{h,r}(0,kh,x,v)(L - \widetilde{L})d_h(kh,ih,v,u) \\
&\quad \times (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)dvdu, \\
a_{r,3} &= \sum_{n/2<i\le n} h^2 \sum_{i/2<k\le i-n^{\delta'}} \int\int (L^T - \widetilde{L}^T)S_{h,r}(0,kh,x,v)d_h(kh,ih,v,u) \\
&\quad \times (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)dvdu, \\
a_{r,4} &= \sum_{n/2<i\le n} h^2 \sum_{i-n^{\delta'}<k\le i-1} \int\int (L^T - \widetilde{L}^T)S_{h,r}(0,kh,x,v)d_h(kh,ih,v,u) \\
&\quad \times (H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)dvdu.
\end{aligned}$$

Here L^T and $\widetilde{L}^T$ denote the adjoint operators of L and $\widetilde{L}$, and δ' satisfies inequalities $2\varkappa < \delta' < \frac{3}{5}(1-\varkappa)$, where $\varkappa$ is defined in (B2). For the proof of (100) it suffices to show for $l = 1,2,3,4$

$$\begin{aligned}
|a_{r,l}| &\le hn^{-\delta} C^{n-r} B(1,\frac{1}{2}) \times \ldots \times B(\frac{n-r-1}{2},\frac{1}{2}) \zeta^S_{\sqrt{T}}(y-x) \\
&\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r}{2})}\right] hn^{-\delta} \zeta^S_{\sqrt{T}}(y-x)
\end{aligned} \tag{100}$$

for some $\delta > 0$.

Proof of (100) for $l = 2$. Note that $k \le i/2, i > n/2$ imply $ih - kh \ge \frac{T}{4}$. The claim follows from the inequalities

$$\begin{aligned}
&\max\{|K_h(ih,jh,x,y)|, |M_h(ih,jh,x,y)|, \left|\sqrt{h}N_1(ih,jh,x,y)\right| \\
&\qquad |hN_2(ih,jh,x,y)|, |H(ih,jh,x,y)|\} \\
&\le C\rho^{-1}\zeta_\rho(y-x) \text{ with } \rho^2 = jh - ih \text{ for } 0 \le i < j \le n,
\end{aligned} \tag{101}$$

$$(102)\qquad |S_{h,m}(0,kh,x,v)| \le C\zeta^{S-2}_{\sqrt{kh}}(v-x),$$

$$(103)\qquad \left|(L-\widetilde{L})d_h(kh,ih,v,u)\right| \le Ch^{3/2}(ih-kh)^{-2}\zeta^{S-8}_{\sqrt{ih-kh}}(u-v)$$
$$= O(hn^{-1/2+3/2\varkappa})\zeta^{S-8}_{\sqrt{ih-kh}}(u-v),$$

$$(104)\qquad \left|(H+M_h+\sqrt{h}N_1+hN_2)^{(n-r-1)}(ih,T,u,y)\right|$$
$$\le C^{n-r}\rho^{n-r-3}B(\frac{1}{2},\frac{1}{2})\times ... \times B(\frac{n-r-2}{2},\frac{1}{2})\zeta^{S-2}_{\sqrt{T-ih}}(y-u)$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right](T-ih)^{-1/2}\zeta^{S-2}_{\sqrt{T-ih}}(y-u)$$

for $n-r-3=-1,0,1,...,n-3$ with $\rho^2=T-ih$. We put $B(\frac{1}{2},0)=1)$. Inequality (101) follows from the definitions of the functions $K_h,...,H$. Inequalities (102) and (104) can be proved by the same method as used in the proof of Theorem 2.3 in Konakov and Mammen (2002) (pp. 282 - 284). Inequality (103) follows from the inequality $ih-kh \ge \frac{T}{4}$, Lemma 5 and the arguments used in the proof of Lemma 7.

Proof of (100) for $l=3$. Note that $n/2<i, k>i/2$ imply $kh>\frac{T}{4}$. We use the following inequalities

$$(105)\qquad |d_h(kh,ih,v,u)| \le Ch^{3/2}(ih-kh)^{-3/2}\zeta^{S-6}_{\sqrt{ih-kh}}(u-v),$$
$$\left|(L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\right| \le CT^{-1}\zeta^{S-2}_{\sqrt{kh}}(v-x),$$

$$(106)\qquad \left|h\sum_{i/2<k\le i-n^{\delta'}}\int (L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)d_h(kh,ih,v,u)dv\right|$$
$$\le Ch^{3/2}T^{-1}\sum_{i/2<k\le i-n^{\delta'}} h\frac{1}{(ih-kh)^{3/2}}\zeta_{\sqrt{ih}}(u-x)$$
$$\le Ch^{3/2}T^{-1}\int_{ih/2}^{ih-n^{\delta'}h}\frac{du}{(ih-u)^{3/2}}\zeta_{\sqrt{ih}}(u-x) \le Ch^{3/2}T^{-3/2}n^{(1-\delta')/2}\zeta_{\sqrt{ih}}(u-x)$$
$$\le Chn^{-\delta''}\zeta_{\sqrt{ih}}(u-x),$$

where $\delta''=\delta'/2-\varkappa>0$. Claim (100) for $l=3$ now follows from (106) and (104).

Proof of (100) for $l=4$. For $i-n^{\delta'}<k\le i-1, n/2<i$ we have $ih>T/2$, $kh>T/3$, $(i-k)<n^{\delta'}$ for sufficiently large n. The integral

$$\int (L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\widetilde{p}_h(kh,ih,v,u)dv$$

is a finite sum of integrals. We show how to estimate a typical term of this sum. The other terms can be estimated analogously. We consider for fixed j,l

$$\begin{aligned}
(107)\qquad & \int \frac{\partial^2 S_{h,r}(0,kh,x,v)}{\partial v_j \partial v_l}(\sigma_{jl}(kh,v)-\sigma_{jl}(kh,u))h^{-d/2} \\
& \times q^{(i-k)}[kh,u,h^{-1/2}(u-v-h\sum_{l=k}^{i-1} m(lh,u))]dv \\
&= \int \frac{\partial^2 S_{h,r}(0,kh,x,v)}{\partial v_j \partial v_l}\,|_{v=u^*-\sqrt{h}w} \\
& \times[\sigma_{jl}(kh,u^*-\sqrt{h}w)-\sigma_{jl}(kh,u)]q^{(i-k)}(kh,u,w)dw,
\end{aligned}$$

where $u^* = u - h\sum_{l=k}^{i-1} m(lh,u)$. Now using a Tailor expansion we obtain that the right hand side of (107) is equal to

$$\begin{aligned}
& \int \left[\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l} - \sqrt{h}\sum_{|\nu|=1}\frac{w^\nu}{\nu!}\int_0^1 D_v^\nu \frac{\partial^2 S_{h,r}(0,kh,x,u^*-\delta\sqrt{h}w)}{\partial v_j \partial v_l}d\delta\right] \\
& \times\left[-\sqrt{h}\sum_{|\nu|=1}\frac{[w+\sqrt{h}\sum_{l=k}^{i-1} m(lh,u)]^\nu}{\nu!}D_u^\nu \sigma_{jl}(kh,u)\right. \\
& \quad +2h\sum_{|\nu|=2}\frac{[w+\sqrt{h}\sum_{l=k}^{i-1} m(lh,u)]^\nu}{\nu!} \\
& \quad \left.\times\int_0^1 D_u^\nu \sigma_{jl}(kh,u-\delta\sqrt{h}w-\delta h\sum_{l=k}^{i-1} m(lh,u))d\delta\right] q^{(i-k)}(kh,u,w)dw.
\end{aligned}$$

Note that

$$\begin{aligned}
& -\sqrt{h}\int \frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l}(w_p+\sqrt{h}\sum_{l=k}^{i-1} m_p(lh,u))q^{(i-k)}(kh,u,w)dw \\
& = -h\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l}\sum_{l=k}^{i-1} m_p(lh,u), \\
(108)\qquad & h\int \frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l}(w_p+\sqrt{h}\sum_{l=k}^{i-1} m_p(lh,u))(w_q+\sqrt{h}\sum_{l=k}^{i-1} m_q(lh,u)) \\
& \quad \times\left\{\int_0^1 D_u^\nu \sigma_{jl}(kh,u)d\delta + \int_0^1\left[D_u^\nu \sigma_{jl}(kh,u-\delta\sqrt{h}w\right.\right. \\
& \quad \left.\left. -\delta h\sum_{l=k}^{i-1} m(lh,u)) - D_u^\nu \sigma_{jl}(kh,u)\right]d\delta\right\} q^{(i-k)}(kh,u,w)dw \\
& = h\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l}D_u^\nu \sigma_{jl}(kh,u)\int w_p w_q q^{(i-k)}(kh,u,w)dw \\
& \quad +h^2\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l}\sum_{l=k}^{i-1} m_p(lh,u)\sum_{l=k}^{i-1} m_q(lh,u) + R,
\end{aligned}$$

where by (A3′) we have for $j_0 < (i-k) < n^{\delta'}$, $w' = (i-k)^{-1/2}w$

$$
\begin{aligned}
|R| &\leq Ch^{3/2}\left|\frac{\partial^2 S_{h,r}(0,kh,x,u^*)}{\partial v_j \partial v_l}\right| \int \left(n^{\delta'/2}\|w'\| + O(T^{1/2}n^{-1/2+\delta'})\right)^3 \psi(w')dw' \\
&\leq Ch\zeta_{\sqrt{ih}}(u-x)(h^S n^{S\delta'}+1)T^{-3/2}n^{-1/2+3\delta'/2}\int \|w'\|^3 \psi(w')dw' \qquad (109)\\
&\leq Chn^{-1/2+\varkappa/2+3\delta'/2}\zeta_{\sqrt{ih}}(u-x) \leq Chn^{-1/2(1-3\delta'-\varkappa)}\zeta_{\sqrt{ih}}(u-x),
\end{aligned}
$$

We obtain analogously

$$
\begin{aligned}
& h\int w_p(w_q + \sqrt{h}\sum_{l=k}^{i-1} m_q(lh,u))D_u^{e_q}\sigma_{jl}(kh,u)\int_0^1 \frac{\partial^3 S_{h,r}(0,kh,x,u^*-\delta\sqrt{h}w)}{\partial v_p \partial v_j \partial v_l}d\delta \\
&\quad \times q^{(i-k)}(kh,u,w)dw \\
&= h\frac{\partial \sigma_{jl}(kh,u)}{\partial u_q}\frac{\partial^3 S_{h,r}(0,kh,x,u^*)}{\partial v_p \partial v_j \partial v_l}\int w_p w_q q^{(i-k)}(kh,u,w)dw + R,
\end{aligned}
$$

where

$$|R| \leq Chn^{-1/2(1-3\delta'-3\varkappa)}\zeta_{\sqrt{ih}}(u-x), 1-3\delta'-3\varkappa > 0$$

and, for $1-3\delta'-2\varkappa > 0$

$$
\begin{aligned}
&\Bigg| h^{3/2}\int w_p \int_0^1 \frac{\partial^3 S_{h,r}(0,kh,x,u^*-\delta\sqrt{h}w)}{\partial v_p \partial v_j \partial v_l}d\delta \qquad (110)\\
&\quad \int_0^1 \frac{\partial^2 \sigma_{jl}(kh, u-\delta\sqrt{h}w - \delta h\sum_{l=k}^{i-1} m_q(lh,u))}{\partial u_r \partial u_s}d\delta \\
&\quad \times(w_r + \sqrt{h}\sum_{l=k}^{i-1} m_r(lh,u))(w_s + \sqrt{h}\sum_{l=k}^{i-1} m_s(lh,u))q^{(i-k)}(kh,u,w)dw\Bigg| \\
&\leq Chn^{-1/2(1-3\delta'-2\varkappa)}\zeta_{\sqrt{ih}}(u-x).
\end{aligned}
$$

For $1 \leq i-k \leq j_0$ the same estimates remain true because the following bound holds

$$\int \|w\|^S q^{(j)}(t,x,w)dw \leq C(j_0). \qquad (111)$$

The same estimates hold for $\widetilde{p}(kh,ih,v,u)$ with $\phi^{(i-k)}(kh,u,w)$ instead of $q^{(i-k)}(kh,u,w)$, where $\phi(kh,u,w)$ is a gaussian density with the mean 0 and with the covariance matrix equal to $\sigma(kh,u)$. The first two moments of $q^{(i-k)}$ and $\phi^{(i-k)}$ coinside so after substraction we obtain uniformly for $i - n^{\delta'} < k \leq i-1$

$$
\begin{aligned}
&\Bigg|\sum_{n/2<i\leq n} h^2 \sum_{i-n^{\delta'}<k\leq i-1}\int\int (L^T - \widetilde{L}^T)S_{h,r}(0,kh,x,v) \qquad (112)\\
&\quad \times(\widetilde{p}_h(kh,ih,v,u) - \widetilde{p}(kh,ih,v,u))dv \\
&\quad \times(H + M_h + \sqrt{h}N_1 + hN_2)^{(n-r-1)}(ih,T,u,y)du\Bigg| \\
&\leq \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hT^{3/2}n^{-3/2(1-\varkappa-5\delta'/3)}\zeta_{\sqrt{ih}}(u-x).
\end{aligned}
$$

To estimate the other terms in $d_h(kh, ih, v, u)$ we need bounds for the following expressions

$$h \sum_{i-n^{\delta'}<k\le i-1} \int (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) \sqrt{h}(ih - kh) \times D_v^\nu \widetilde{p}(kh, ih, v, u) dv \text{ for } |\nu| = 3,$$

$$h \sum_{i-n^{\delta'}<k\le i-1} \int (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) h(ih - kh) \times D_v^\nu \widetilde{p}(kh, ih, v, u) dv \text{ for } |\nu| = 4,$$

$$h \sum_{i-n^{\delta'}<k\le i-1} \int (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) h(ih - kh)^2 \times D_v^\nu \widetilde{p}(kh, ih, v, u) dv \text{ for } |\nu| = 6.$$

We have

$$\begin{aligned}
(113) \qquad & \left| h \sum_{i-n^{\delta'}<k\le i-1} \int (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) \sqrt{h}(ih - kh) \right. \\
& \quad D_v^\nu \widetilde{p}(kh, ih, v, u) dv \Big| \\
& = \left| h \sum_{i-n^{\delta'}<k\le i-1} \int D^{e_p+e_q} (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) \sqrt{h}(ih - kh) \right. \\
& \quad D_v^{\nu-e_p-e_q} \widetilde{p}(kh, ih, v, u) dv \Big| \\
& \le C T^{-2} n^{\delta'} h^{3/2} \sum_{i-n^{\delta'}<k\le i-1} \frac{h}{\sqrt{ih - kh}} \zeta_{\sqrt{ih}}(u - x) \\
& \le C h n^{-(1-\varkappa-3\delta'/2)} \zeta_{\sqrt{ih}}(u - x).
\end{aligned}$$

Clearly, the same estimate (113) holds for $|\nu| = 4$ and $|\nu| = 6$. Now (100) for $l = 4$ follows from this remark and (112) and (113).

Proof of (100) for $l = 1$. Note that for this case $T - ih \ge T/2$.

$$\begin{aligned}
(114) \qquad a_{r,1} &= \sum_{0\le i\le n/2} h^2 \sum_{0\le k\le i-1} \int\int (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) \\
& \quad \times d_h(kh, ih, v, u) \Psi_{h,r}(ih, T, u, y) dv du \\
&= \sum_{0\le k\le n/2-1} h \int (L^T - \widetilde{L}^T) S_{h,r}(0, kh, x, v) \\
& \quad \times \left\{ \sum_{k+1\le i\le k+n^{\delta'}} h \int d_h(kh, ih, v, u) \Psi_{h,r}(ih, T, u, y) du \right. \\
& \quad \left. + \sum_{k+n^{\delta'}<i\le n/2} h \int d_h(kh, ih, v, u) \Psi_{h,r}(ih, T, u, y) du \right\} dv,
\end{aligned}$$

where we denote

$$\Psi_{h,r}(ih,T,u,y)=(H+M_h+\sqrt{h}N_1+hN_2)^{(n-r-1)}(ih,T,u,y).$$

We consider

$$\sum_{k+1\le i\le k+n^{\delta'}} h\int h^{-d/2}q^{(i-k)}(kh,u,h^{-1/2}[u-v-h\sum_{l=k}^{i-1}m(lh,u)])\Psi_{h,r}(ih,T,u,y)du$$

$$\begin{aligned}
&=\sum_{k+1\le i\le k+n^{\delta'}} h\int\Bigg\{q^{(i-k)}(kh,v,w)+\sqrt{h}\sum_{|\nu|=1}(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}D_v^{\nu}q^{(i-k)}(kh,v,w)\\
&\quad+h\sum_{|\nu|=2}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}D_v^{\nu}q^{(i-k)}(kh,v,w)\\
&\quad+3h^{3/2}\sum_{|\nu|=3}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}\\
&\quad\times\int_0^1(1-\delta)^2D_v^{\nu}q^{(i-k)}(kh,v+\delta h^{1/2}w+\delta h\sum_{l=k}^{i-1}m(lh,u),w)d\delta\Bigg\}\\
&\quad\times\Bigg\{\Psi_{h,r}(ih,T,v,y)+\sqrt{h}\sum_{|\nu|=1}(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}D_v^{\nu}\Psi_{h,r}(ih,T,v,y)\\
&\quad+h\sum_{|\nu|=2}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}D_v^{\nu}\Psi_{h,r}(ih,T,v,y)\\
&\quad+3h^{3/2}\sum_{|\nu|=3}\frac{(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}}{\nu!}\\
&\quad\times\int_0^1(1-\delta)^2D_v^{\nu}\Psi_{h,r}(ih,T,v+\delta h^{1/2}w+\delta h\sum_{l=k}^{i-1}m(lh,u),y)d\delta\Bigg\}dw
\end{aligned}$$

This integral is a sum of $4\times4=16$ integrals. We estimate only two of them. Other integrals can be estimated by similar methods. First, we estimate

$$\sum_{k+1\le i\le k+n^{\delta'}} h\int q^{(i-k)}(kh,v,w)\Psi_{h,r}(ih,T,v,y)dw=\sum_{k+1\le i\le k+n^{\delta'}} h\Psi_{h,r}(ih,T,v,y)dw.$$

Note that we get the same term when we replace $q^{(i-k)}(kh,v,w)$ by $\phi^{(i-k)}(kh,v,w)$.

After the replacement this term disappears. Second, we estimate

$$\sum_{k+1\le i\le k+n^{\delta'}} h\int q^{(i-k)}(kh,v,w)\sqrt{h}\sum_{|\nu|=1}(w+\sqrt{h}\sum_{l=k}^{i-1}m(lh,u))^{\nu}D_v^{\nu}\Psi_{h,r}(ih,T,v,y)dw$$

$$= h^{3/2}\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\int q^{(i-k)}(kh,v,w)[w_j+\sqrt{h}\sum_{l=k}^{i-1}m_j(lh,v)$$
$$+O(hn^{\delta'}\|w\|+h^{3/2}n^{2\delta'})]dw$$
$$= h^2\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\sum_{l=k}^{i-1}m_j(lh,v)$$
$$+O\left(h^2n^{2\delta'}\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} h\left|D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\right|\right)$$
$$+O\left(h^{3/2}n^{\delta'}\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} h\left|D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\right|\int q^{(i-k)}(kh,v,w)\|w\|\,dw\right)$$
$$= h^2\sum_{j=1}^{d}\sum_{k+1\le i\le k+n^{\delta'}} D_v^{e_j}\Psi_{h,r}(ih,T,v,y)\sum_{l=k}^{i-1}m_j(lh,v)+R,$$

where

$$|R|\le \frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}T^{1/2}hn^{-3/2+2\delta'}\zeta_{\sqrt{T-kh}}(y-v).$$

The first term in the right hand side of this equation will be the same if we replace $q^{(i-k)}(kh,v,w)$ by $\phi^{(i-k)}(kh,v,w)$. After the replacement this term disappears. For a proof of this equation we consider the function $u(w)$ that is defined as an implicit function and we used the following change of variables

$$h^{1/2}w = u - v - h\sum_{l=k}^{i-1}m(lh,u)$$

to obtain

$$\sqrt{h}\sum_{l=k}^{i-1}m(lh,u(w)) = \sqrt{h}\sum_{l=k}^{i-1}m(lh,v) + O\left(h(i-k)\|w\| + h^{3/2}(i-k)^2\right)$$

because of $(i-k)\le n^{\delta'}$. By similar methods we get

$$\left|\sum_{k+1\le i\le k+n^{\delta'}} h\int[\sqrt{h}\widetilde{\pi}_1(kh,ih,v,u)+h\widetilde{\pi}_2(kh,ih,v,u)]\Psi_{h,r}(ih,T,u,y)du\right| \tag{115}$$
$$\le \frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}hn^{-3/2+2\delta'+\varkappa/2}\zeta_{\sqrt{T-kh}}(y-v).$$

It remains to estimate

$$\sum_{k+n^{\delta'}<i\le n/2} h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du.$$

From (104) and (105) we obtain

$$(116)\qquad \left|\sum_{k+n^{\delta'}<i\le n/2} h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\right|$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] T^{-1/2}h^{3/2}\int_{kh+n^{\delta'}h}^{T/2}\frac{du}{(u-kh)^{3/2}}\zeta_{\sqrt{T-kh}}(y-v)$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] T^{-1/2}hn^{-\delta'/2}\zeta_{\sqrt{T-kh}}(y-v)$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-1/2(\delta'-\varkappa)}\zeta_{\sqrt{T-kh}}(y-v).$$

Now we substitute the estimate (116) into (114). This gives the following estimate for any $0<\varepsilon<\varkappa$

$$(117)\qquad \left|\sum_{k=1}^{n/2-1} h\int (L^T-\widetilde{L}^T)S_{h,r}(0,kh,x,v)\right.$$
$$\left.\sum_{k+n^{\delta'}<i\le n/2} h\int d_h(kh,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\right|$$
$$\le \left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-1/2(\delta'-\varkappa)}h^{-\varepsilon}\sum_{k=1}^{n/2} h(kh)^{\varepsilon-1}\zeta_{\sqrt{T}}(y-x)$$
$$\le C(\varepsilon)\left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-1/2(\delta'-\varkappa-\varepsilon)}\zeta_{\sqrt{T}}(y-x).$$

For $k=0$ we get with $S_{h,r}(0,0,x,v)=\delta(x-v)$ where $\delta(\bullet)$ is the Dirac function that

$$\left|\sum_{1\le i\le i/2} h^2\int\int S_{h,r}(0,0,x,v)(L-\widetilde{L})d_h(0,ih,v,u)\Psi_{h,r}(ih,T,u,y)du\right|$$
$$\le C(\varepsilon)\left[\frac{C^{n-r}}{\Gamma(\frac{n-r-1}{2})}\right] hn^{-(1/2-\varepsilon)}\zeta_{\sqrt{T}}(y-x).$$

This completes the proof (100) for $l=1$. The estimate (97) may be proved by the same arguments as were used to prove (88).

Asymptotic treatment of the term T_5. We will show that,

$$(118)\qquad T_5=-\sqrt{h}\sum_{r=0}^{\infty}\widetilde{\pi}_1\otimes_h(H+M_{h,1}+\sqrt{h}N_1)^{(r)}(0,T,x,y)$$
$$-h\sum_{r=0}^{\infty}\widetilde{\pi}_2\otimes_h H^{(r)}(0,T,x,y)+R_h(x,y),$$

where $|R_h(x,y)| \le Chn^{-\gamma}\zeta^{S-2}_{\sqrt{T}}(y-x)$ for some $\gamma > 0$. Note that with $S_h(s,t,x,y) = \sum_{r=1}^{n}(K_h + M_h)^{(r)}(s,t,x,y)$ the term T_5 can be rewritten as

$$T_5 = (\widetilde{p} - \widetilde{p}_h)(0,T,x,y) + (\widetilde{p} - \widetilde{p}_h) \otimes_h S_h(0,T,x,y).$$

We start by showing that for $\varkappa < \delta < \frac{1-\varkappa}{4}$ uniformly for $x, y \in R$

$$\left| h \sum_{1 \le j \le n^\delta} \int (\widetilde{p}_h - \widetilde{p})(0, jh, x, u) S_h(jh, T, u, y) du \right| \le O(hn^{-1/2(1-\varkappa-4\delta)})\zeta^{S-2}_{\sqrt{T}}(y-x) \tag{119}$$

for δ small enough. For the proof of (119) we will show that uniformly for $1 \le j \le n^\delta$ and for $x, y \in R^d$

$$\int \widetilde{p}_h(0, jh, x, u) S_h(jh, T, u, y) du = S_h(jh, T, x, y) + O[h^{1/2}T^{-1/2}n^{-1/2+\delta} + h^{1/2}T^{-1} + n^{\delta/2}h^{1/2}]\zeta^{S-2}_{\sqrt{T}}(y-x), \tag{120}$$

$$\int \widetilde{p}(0, jh, x, u) S_h(jh, T, u, y) du = S_h(jh, T, x, y) + O[h^{1/2}T^{-1/2}n^{-1/2+\delta} + h^{1/2}T^{-1} + n^{\delta/2}h^{1/2}]\zeta^{S-2}_{\sqrt{T}}(y-x). \tag{121}$$

Claim (119) immediately follows from (120)-(121). For the proof we will make use of the fact that for all $1 \le j \le n^\delta$ and for all $x, y \in R^d$ and $|\nu| = 1$

$$|D^\nu_x S_h(jh, T, x, y)| \le C(T - jh)^{-1}\zeta^{S-2}_{\sqrt{T-jh}}(y-x). \tag{122}$$

Claim (122) can be shown with the same arguments as the proof of (5.7) in Konakov and Mammen (2002). Note that the function Φ in that paper has a similar structure as S_h. For $1 \le j \le n^\delta$ the bound (122) immediately implies for a constant C'

$$|D^\nu_x S_h(jh, T, x, y)| \le C' T^{-1}\zeta^{S-2}_{\sqrt{T}}(y-x). \tag{123}$$

We have $\widetilde{p}_h(0, jh, x, u) = h^{-d/2}q^{(j)}[0, u, h^{-1/2}(u - x - h\sum_{i=0}^{j-1} m(ih, u))]$. Denote the determinant of the Jacobian matrix of $u - h\sum_{i=0}^{j-1} m(ih, u)$ by Δ_h. Because of condition

(A3) and (123) it holds that for $1 \leq j \leq n^\delta$

$$
\begin{aligned}
&\int \widetilde{p}_h(0, jh, x, u) S_h(jh, T, u, y) du \\
&= \int h^{-d/2} q^{(j)}[0, u, h^{-1/2}(u - x - h \sum_{i=0}^{j-1} m(ih, u))] S_h(jh, T, u, y) du \\
&= \int q^{(j)}(0, x + h^{1/2} w + h \sum_{i=0}^{j-1} m(ih, u(w)), w) \left|\Delta_h^{-1}\right| \\
&\qquad S_h(jh, T, x + h^{1/2} w + h \sum_{i=0}^{j-1} m(ih, u(w)), y) dw \\
&= \int [q^{(j)}(0, x, w) + O(j^{-d/2} h^{1/2})(\|w\| + 1)\psi(j^{-1/2} w)][1 + O(jh)][S_h(jh, T, x, y) \\
&\qquad + O(h^{1/2} T^{-1}) \zeta_{\sqrt{T}}^{S-2}(y - x)(1 + h^{(S-2)/2} \|w\|^{S-2})(\|w\| + 1)] dw \\
&= S_h(jh, T, x, y) + O[h^{1/2} T^{-1/2} n^{-1/2+\delta} + h^{1/2} T^{-1} + h^{1/2} n^{\delta/2}] \zeta_{\sqrt{T}}^{S-2}(y - x)
\end{aligned}
$$

with $u = u(w)$ in $\sum_{i=0}^{j-1} m(ih, u)$ defined by the Inverse Function Theorem with the substitution $w = h^{-1/2}(u - x - h \sum_{i=0}^{j-1} m(ih, u))$. This proves (120). The proof of (121) is the same with obvious modifications. From (119) we get that for $\delta < \frac{1-\varkappa}{4}$ (where $\varkappa$ is defined in (B2))

$$
T_5 = (\widetilde{p} - \widetilde{p}_h)(0, T, x, y) + h \sum_{n^\delta < j < n} \int (\widetilde{p} - \widetilde{p}_h)(0, jh, x, u) S_h(jT, u, y) du + R_h(x, y)
$$

with $|R_h(x, y)| \leq O(h n^{-1/2(1-\varkappa-4\delta)}) \zeta_{\sqrt{T}}^{S-2}(y - x)$. We now make use of the expansion of $\widetilde{p}_h - \widetilde{p}$ given in Lemma 5. We have with $\rho = (jh)^{1/2} \geq h^{1/2} n^{\delta/2}$

$$
\begin{aligned}
(124) \qquad &\left| h \sum_{j=n^\delta}^{n} h^{3/2} \rho^{-3} \int \zeta_\rho^S(u - x) S_h(jh, T, u, y) du \right| \\
&\qquad \leq C h^2 T^{-\delta'} n^{-\delta''} \sum_{j=n^\delta}^{n} \rho^{-2+2\delta'} \int \left|\zeta_\rho^S(u - x) S_h(jh, T, u, y)\right| du,
\end{aligned}
$$

where $\delta' < \frac{1}{2}\delta(1-\delta)^{-1}, 2\ \delta'' = \delta + 2\delta\delta' - 2\delta'$. Now we get that

$$
(125) \quad h \sum_{j=n^\delta}^{n} \rho^{-2+2\delta'} \int \left|\zeta_\rho^S(u - x) S_h(jh, T, u, y)\right| du \leq C B(\delta', 1/2) T^{\delta'-1/2} \zeta_{\sqrt{T}}^{S-2}(y - x)
$$

for a constant C. This shows that for $\delta' > 0$ small enough

$$
T_5 = -[\sqrt{h}\widetilde{\pi}_1 + h\widetilde{\pi}_2](0, T, x, y)
$$

$$
-h \sum_{n^\delta < j < n} \int [\sqrt{h}\widetilde{\pi}_1 + h\widetilde{\pi}_2](0, jh, x, u) S_h(jh, T, u, y) du + R'_h(x, y),
$$

with $|R'_h(x,y)| \le O(hn^{-(\delta''-\varkappa/2)})\zeta^{S-2}_{\sqrt{T}}(y-x)$ with a constant in $O(\cdot)$ depending on δ'. It follows from (119), (124) and (125) that

$$T_5 = -\sum_{r=0}^{\infty}[\sqrt{h}\widetilde{\pi}_1 + h\widetilde{\pi}_2] \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) + R''_h(x,y), \tag{126}$$

where $|R''_h(x,y)| \le O(hn^{-(\delta''-\varkappa/2)})\varsigma^{S-2}_{\sqrt{T}}(y-x)$. Now we apply Lemma 10 with $A = \sqrt{h}\widetilde{\pi}_1$, $B = H + M_{h,1} + \sqrt{h}N_1, C = (K_h - H - \sqrt{h}N_1) + (M_h - M_{h,1})$ to

$$\begin{aligned} &-\sum_{r=0}^{\infty}\sqrt{h}\widetilde{\pi}_1 \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) \\ &\quad + \sum_{r=0}^{\infty}\sqrt{h}\widetilde{\pi}_1 \otimes_h (H + M_{h,1} + \sqrt{h}N_1)^{(r)}(0,T,x,y) \end{aligned} \tag{127}$$

and with $A = h\widetilde{\pi}_2, B = H, C = (K_h - H) + M_h$ to

$$-\sum_{r=0}^{\infty} h\widetilde{\pi}_2 \otimes_h (K_h + M_h)^{(r)}(0,T,x,y) + \sum_{r=0}^{\infty} h\widetilde{\pi}_2 \otimes_h H^{(r)}(0,T,x,y). \tag{128}$$

The estimate (118) follows from (125), (127), (128), Lemma 10, Lemma 5 , (16) and (17).

Asymptotic treatment of the term T_6. By Lemma 9

$$|T_6| \le C(\varepsilon)hn^{-1/2+\varepsilon}\zeta^{S}_{\sqrt{T}}(y-x).$$

Asymptotic treatment of the term T_7. We use the recurrence relation for $r = 2, 3, ...$

$$\begin{aligned} &\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r)}(0,T,x,y) - \widetilde{p}_h \otimes_h {H_h}^{(r)}(0,T,x,y) \\ &= \left[\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} - \widetilde{p}_h \otimes_h {H_h}^{(r-1)}\right] \otimes_h H_h(0,T,x,y) \\ &+ [\widetilde{p}_h \otimes_h (K_h + M_h + R_h)^{(r-1)} \otimes_h (K_h + M_h + R_h - H_h)](0,T,x,y) \end{aligned}$$

and we apply Lemma 8 for $r = 1$. We get in the same way as in the proof of Lemma 9

$$|T_7| \le Ch^{3/2}T^{-1/2}\zeta^{S}_{\sqrt{T}}(y-x) = Chn^{-1/2}\zeta^{S}_{\sqrt{T}}(y-x).$$

Plugging in the asymptotic expansions of $T_1, ..., T_7$. We now plug the asymptotic expansions of $T_1, ..., T_7$ into (38). Using Lemma 10, Theorem 2.1 in Konakov and Mammen (2002) we get

$$\begin{aligned} &p_h(0,T,x,y) - p(0,T,x,y) \\ &= \sqrt{h}\left[\widetilde{\pi}_1 + p^d \otimes_h \Re_1\right] \otimes_h \Phi(0,T,x,y) \\ &\quad + h\Big\{\left[\widetilde{\pi}_2 + \widetilde{\pi}_1 \otimes_h \Phi \otimes_h \Re_1 + p^d \otimes_h \Re_2 + p^d \otimes_h \Re_3\right] \otimes_h \Phi(0,T,x,y) \\ &\quad + p^d \otimes_h (\Re_1 \otimes_h \Phi)^{(2)}(0,T,x,y) \\ &\quad + \frac{1}{2}p \otimes_h (L_*^2 - L^2)p^d(0,T,x,y) - \frac{1}{2}p \otimes_h (L' - \widetilde{L}')p^d(0,T,x,y)\Big\} \\ &\quad + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)), \end{aligned} \tag{129}$$

where

$$\Re_1(s,t,x,y) = N_1(s,t,x,y) + M_1(s,t,x,y) - \widetilde{M}_1(s,t,x,y),$$
$$\Re_2(s,t,x,y) = N_2(s,t,x,y) + \Pi_1(s,t,x,y) - \widetilde{\Pi}_1(s,t,x,y),$$
$$\Re_3(s,t,x,y) = \sum_{|\nu|=4} \frac{(\chi_\nu(s,x) - \chi_\nu(s,y))}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y),$$

$$M_1(s,t,x,y) = \sum_{|\nu|=3} \frac{\chi_\nu(s,x)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y), \widetilde{M}_1(s,t,x,y) = \sum_{|\nu|=3} \frac{\chi_\nu(s,y)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y),$$

$$\Pi_1(s,t,x,y) = \sum_{|\nu|=3} \frac{\chi_\nu(s,x)}{\nu!} D_x^\nu \widetilde{\pi}_1(s,t,x,y), \widetilde{\Pi}_1(s,t,x,y) = \sum_{|\nu|=3} \frac{\chi_\nu(s,y)}{\nu!} D_x^\nu \widetilde{\pi}_1(s,t,x,y).$$

For the homogenous case and $T=[0,1]$ (129) coinsides with formula (53) on page 623 in Konakov and Mammen (2005).

Asymptotic replacement of p^d by p. It follows from (11), (26) and (27) in Konakov (2006) that

$$\left|(p^d - p)(ih, jh, x, z)\right| \le C(\varepsilon) h^{1-\varepsilon} (jh - ih)^{\varepsilon - 1/2} \phi_{\sqrt{(j-i)h}}(z-x) \tag{130}$$

for any $0 < \varepsilon < 1/2$. Using (130) and making an integration by parts we can replace p^d by p in (129). For example the operator $L_*^2 - L^2$ is the operator of the third order. Making an integration by parts we have for $|\nu| = 3$

$$\begin{aligned} &\left|\sum_{i=1}^{n-1} h \int D_z^\nu p(0, ih, x, z)(p^d - p)(ih, T, z, y) dz\right| \\ &\le C(\varepsilon) h^{1-\varepsilon} \sum_{i=1}^{n-1} h \frac{1}{(ih)^{3/2}} \frac{1}{(T-ih)^{1/2-\varepsilon}} \phi_{\sqrt{T}}(y-x) \\ &\le C(\varepsilon) h^{1/2-2\varepsilon} T^{2\varepsilon - 1/2} B(\varepsilon, \varepsilon + \frac{1}{2}) \phi_{\sqrt{T}}(y-x). \end{aligned} \tag{131}$$

By (B2) we have $0 < \varkappa < 1 - 4\varepsilon$. This implies

$$\begin{aligned} \left|\frac{h}{2} p \otimes_h (L_*^2 - L^2)(p^d - p)(0,T,x,y)\right| &\le C(\varepsilon) h T^{1/2} n^{-(1/2 - 2\varepsilon - \varkappa/2)} \phi_{\sqrt{T}}(y-x) \\ &\le C(\varepsilon) h^{1+\delta} \phi_{\sqrt{T}}(y-x) \end{aligned}$$

for some $0 < \delta < 1/2$. The other terms in (129) containing p^d can be estimated analogously. Thus we get the following representation

$$\begin{aligned} &p_h(0,T,x,y) - p(0,T,x,y) \\ &= \sqrt{h}\,[\widetilde{\pi}_1 + p \otimes_h \Re_1] \otimes_h \Phi(0,T,x,y) \\ &\quad + h\,\{[\widetilde{\pi}_2 + \widetilde{\pi}_1 \otimes_h \Phi \otimes_h \Re_1 + p \otimes_h \Re_2 + p \otimes_h \Re_3] \otimes_h \Phi(0,T,x,y) \\ &\quad + p \otimes_h (\Re_1 \otimes_h \Phi)^{(2)}(0,T,x,y) \\ &\quad + \frac{1}{2} p \otimes_h (L_*^2 - L^2) p(0,T,x,y) - \frac{1}{2} p \otimes_h (L' - \widetilde{L}') p(0,T,x,y) \Big\} \\ &\quad + O(h^{1+\delta} \zeta_{\sqrt{T}}(y-x)). \end{aligned} \tag{132}$$

For the further analysis we need a generalization of a binary operation $\otimes$ introduced in Konakov and Mammen (2005). Recall a corresponding definition. Suppose that $s \in [0, t-h]$ and $t \in \{h, 2h, ..., T\}$. Then the binary type operation $\otimes_h'$ is defined as follows

$$f \otimes_h' g(s,t,x,y) = \sum_{s \le jh \le t-h} h \int f(s, jh, x, z) g(jh, t, z, y) dz.$$

Note that for $s \in \{0, h, 2h, ..., T\}$ these two operations coincide, that is $\otimes_h' \equiv \otimes_h$.

Asymptotic replacement of $(p \otimes_h \Re_i) \otimes_h \Phi(0,T,x,y)$ *by* $p \otimes (\Re_i \otimes_h' \Phi)(0,T,x,y) = (p \otimes \Re_i) \otimes_h \Phi(0,T,x,y), i = 1,2,3,$ $[p \otimes_h (\Re_1 \otimes_h \Phi)] \otimes_h (\Re_1 \otimes_h \Phi)(0,T,x,y)$ *by* $p \otimes [(\Re_i \otimes_h' \Phi) \otimes_h' (\Re_i \otimes_h' \Phi)](0,T,x,y)$, $p \otimes_h (L_*^2 - L^2) p(0,T,x,y)$ *by* $p \otimes (L_*^2 - L^2) p(0,T,x,y)$ *and* $p \otimes_h (L' - \widetilde{L}') p(0,T,x,y)$ *by* $p \otimes (L' - \widetilde{L}') p(0,T,x,y)$.

These replacements follow from the definitions of $\Re_i, i = 1,2,3,$ and can be proved by the same method as in Konakov (2006), pp. 9-12, where we estimate the replacement error of $p \otimes_h H$ by $p \otimes H$. Linearity of the operation $\otimes_h$ implies that it is enough to consider the functions $p \otimes_h \Im$ where $\Im(u,t,z,v)$ is a function of one of the followin forms:

$$\frac{\chi_\nu(u,z) - \chi_\nu(u,v)}{\nu!} D_x^\nu \widetilde{p}(u,t,z,v), |\nu| = 3,4, \frac{\chi_\nu(u,z) - \chi_\nu(u,v)}{\nu!} D_x^\nu \widetilde{\pi}_1(u,t,z,v), |\nu| = 3$$

$$(L - \widetilde{L})\widetilde{\pi}_1(u,t,z,v), (L - \widetilde{L})\widetilde{\pi}_2(u,t,z,v).$$

We consider the case $\Im(u,t,z,v) = (L - \widetilde{L})\widetilde{\pi}_1(u,t,z,v)$. The other cases can be treated similarly. It is enough to consider a typical term of $(L - \widetilde{L})\widetilde{\pi}_1(u,t,z,v)$, namely, we shall estimate

$$\int_0^{jh} du \int p(0,u,x,z) \left(\int_u^{jh} \chi_\nu(w,v) dw \right) D_z^\nu (L - \widetilde{L}) \widetilde{p}(u, jh, z, v) dz$$

$$- \sum_{i=0}^{j-1} h \int p(0, ih, x, z) \left(\int_{ih}^{jh} \chi_\nu(w,v) dw \right) D_z^\nu (L - \widetilde{L}) \widetilde{p}(ih, jh, z, v) dz$$

$$\sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} du \int [\lambda(u) - \lambda(ih)] dz = \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u - ih) du \int \lambda'(ih) dz$$

$$+\frac{1}{2} \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u-ih)^2 du \int_0^1 (1-\delta) \int \lambda''(s) |_{s=s_i} dz d\delta du, \tag{133}$$

where $\lambda(u) = p(0,u,x,z) \left(\int_u^{jh} \chi_\nu(w,v) dw \right) D_z^\nu H(u, jh, z, v), s_i = ih + \delta(u - ih)$. As in Konakov (2006), p.9, we obtain that

$$\sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u - ih) du \int \lambda'(ih) dz$$

$$= \frac{h}{2}\sum_{i=0}^{j-1} h \int_{ih}^{jh} \chi_\nu(s,v)ds \int p(0,ih,x,z) D_z^\nu A_0(ih,jh,z,v)dz$$

$$+ \frac{h}{2}\sum_{i=0}^{j-1} h \int_{ih}^{jh} \chi_\nu(s,v)ds \int p(0,ih,x,z) D_z^\nu H_1(ih,jh,z,v)dz$$

$$-\frac{h}{2}\sum_{i=0}^{j-1} h\chi_\nu(ih,v) \int p(0,ih,x,z) D_z^\nu H(ih,jh,z,v)dz, \tag{134}$$

where

$$A_0(s,jh,z,v) = (L^2 - 2L\widetilde{L} + \widetilde{L}^2)\widetilde{p}(s,jh,z,v),$$

$$H_l(s,t,z,v) = \frac{1}{2}\sum_{i,j=1}^{d}\left(\frac{\partial^l \sigma_{ij}(s,z)}{\partial s^l} - \frac{\partial^l \sigma_{ij}(s,v)}{\partial s^l}\right)\frac{\partial^2 \widetilde{p}(s,t,z,v)}{\partial z_i \partial z_j}$$

$$\sum_{i=1}^{d}\left(\frac{\partial^l m_i(s,z)}{\partial s^l} - \frac{\partial^l m_i(s,v)}{\partial s^l}\right)\frac{\partial \widetilde{p}(s,t,z,v)}{\partial z_i}, l = 0,1,2, H_0 \equiv H.$$

The differential operator A_0 was calculated in Konakov (2006), p.9. A_0 is a forth order differential operator. From the structure of this operator and from (134) it is clear that it'is enough to estimate

$$I \triangleq \frac{h}{2}\sum_{j=0}^{n-1} h \int \sum_{i=0}^{j-1} h \int_{ih}^{jh} \chi_\nu(s,v)ds \int p(0,ih,x,z) D_z^{\nu+\mu}\widetilde{p}(ih,jh,z,v)dz\Phi(jh,T,v,y)dv \tag{135}$$

for $|\nu| = 3, |\mu| = 3$. To estimate (135) we consider three possible cases: a) $jh > T/2, ih \le jh/2 \Longrightarrow jh - ih > T/4$ b) $jh > T/2, ih > jh/2 \Longrightarrow ih > T/4$ c) $jh < T/2 \Longrightarrow T - jh > T/2$. In the case a) we make an integration by parts transferring two derivatives to $p(0,ih,x,z)$. This gives

$$\left|\frac{h}{2}\sum_{i=0}^{j-1} h \int_{ih}^{jh} \chi_\nu(s,v)ds \int D_z^{e_k+e_l} p(0,ih,x,z) D_z^{\nu+\mu-e_k-e_l}\widetilde{p}(ih,jh,z,v)dz\right|$$

$$\le Ch^{1-2\varepsilon}\int_0^{jh} \frac{1}{u^{1-\varepsilon}}\frac{du}{(jh-u)^{1-\varepsilon}}\phi_{\sqrt{jh}}(v-x)$$

$$\le C(\varepsilon)h^{1-2\varepsilon}(jh)^{2\varepsilon-1}\phi_{\sqrt{jh}}(v-x)$$

and

$$|I| \le C(\varepsilon)h^{1-2\varepsilon}\int_0^T \frac{du}{u^{1-2\varepsilon}(T-u)^{1/2}}\phi_{\sqrt{T}}(y-x)$$

$$\le C(\varepsilon)h^{3/4}T^{1/2}n^{-(1/4-2\varepsilon-\varkappa/2)}\phi_{\sqrt{T}}(y-x) \le C(\varepsilon)T^{1/2-\delta}h^{3/4+\delta}\phi_{\sqrt{T}}(y-x), \tag{136}$$

where $\delta = (1/4 - 2\varepsilon - \varkappa/2) > 0$ if $\varkappa < 1/2 - 4\varepsilon$, $0 < \varepsilon < 0,05$ (see the condition (B2)). In the case b) we make an integration by parts transferring four derivatives to

$p(0,ih,x,z)$. This gives the same estimate (136). At last, in the case c) we make an integration by parts transferring three derivatives to $\Phi(jh,T,v,y)$ and one derivative to $p(0,ih,x,z)$. This gives the same estimate (136). To pass from $D_z^{\mu}\widetilde{p}(ih,jh,z,v)$ to $D_v^{\mu}\widetilde{p}(ih,jh,z,v)$ we used the following estimate

$$|D_z^{\mu}\widetilde{p}(ih,jh,z,v)+D_v^{\mu}\widetilde{p}(ih,jh,z,v)|\le C\phi_{\sqrt{jh-ih}}(v-z).$$

Clearly, the same estimate (136) hold true for the other summands in the right hand side of (134)

$$\frac{h}{2}\left|\sum_{j=0}^{n-1}h\int\sum_{i=0}^{j-1}h\int_{ih}^{jh}\chi_{\nu}(s,v)ds\int p(0,ih,x,z)D_z^{\nu}H_1(ih,jh,z,v)dz\Phi(jh,T,v,y)dv\right|$$
$$\le C(\varepsilon)T^{1/2-\delta}h^{3/4+\delta}\phi_{\sqrt{T}}(y-x),$$
$$\frac{h}{2}\sum_{i=0}^{j-1}h\int\chi_{\nu}(ih,v)\sum_{i=0}^{j-1}h\int p(0,ih,x,z)D_z^{\nu}H(ih,jh,z,v)dz\Phi(jh,T,v,y)dv$$
$$\le C(\varepsilon)T^{1/2-\delta}h^{3/4+\delta}\phi_{\sqrt{T}}(y-x).$$

Now we shall estimate the second summand in the right hand side of (134). Analogously to (19) in Konakov (2006) we obtain

$$\frac{1}{2}\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2du\int_0^1(1-\delta)\int\lambda''(s)\mid_{s=s_i}dzd\delta du \tag{137}$$
$$=\frac{1}{2}\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\sum_{k=1}^{4}\int_s^{jh}\chi_{\nu}(\tau,v)d\tau$$
$$\times\int p(0,s,x,z)D_z^{\nu}A_k(s,jh,z,v)\mid_{s=s_i}dzd\delta du$$
$$-\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\chi_{\nu}(s,v)$$
$$\times\int p(0,s,x,z)D_z^{\nu}A_0(s,jh,z,v)\mid_{s=s_i}dzd\delta du$$
$$-\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\chi_{\nu}(s,v)$$
$$\times\int p(0,s,x,z)D_z^{\nu}H_1(s,jh,z,v)\mid_{s=s_i}dzd\delta du$$
$$-\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\frac{\partial\chi_{\nu}(s,v)}{\partial s}$$
$$\times\int p(0,s,x,z)D_z^{\nu}H(s,jh,z,v)\mid_{s=s_i}dzd\delta du,$$

where the operators $A_i, i = 1, 2, 3, 4$, are defined as follows:

$$A_1(s, jh, z, v) = (L^3 - 3L^2\widetilde{L} + 3L\widetilde{L}^2 - \widetilde{L}^3)\widetilde{p}(s, jh, z, v),$$

$$A_2(s, jh, z, v) = (L_1 H + 2LH_1)(s, jh, z, v),$$

$$A_3(s, jh, z, v) = [(L - \widetilde{L})\widetilde{L}_1 + 2(L_1 - \widetilde{L}_1)\widetilde{L}]\widetilde{p}(s, jh, z, v),$$

$$A_4(s, jh, z, v) = H_2(s, jh, z, v). \tag{138}$$

The operator A_1 is given in (24) in Konakov (2006). As in Konakov (2006) it is enough to estimate for fixed p, q, r, l

$$\frac{1}{2}\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u - ih)^2\int_0^1(1-\delta)\sum_{k=1}^{4}\int_s^{jh}\chi_\nu(\tau, v)d\tau$$
$$\int p(0, s, x, z)D_z^\nu\left(\frac{\partial^4\widetilde{p}(s, jh, z, v)}{\partial z_p\partial z_q\partial z_l\partial z_r}\right)|_{s=s_i}\, dzd\delta du.$$

As in (25) in Konakov (2006) we obtain that (139) does not exceed

$$C(\varepsilon)h^{3/2-\varepsilon}(jh)^{2\varepsilon-1}\phi_{\sqrt{jh}}(v - x). \tag{139}$$

It follows from the explicit form of these operators (138) that the same estimate (139) holds for A_2, A_3 and A_4. The other three terms in the right hand side of (137) do not contain the factor $\int_s^{jh}\chi_\nu(\tau, v)d\tau$ and they should be estimated separately. Clearly, it's enough to estimate the term containing A_0. The remaining two summands are less singular. From the explicit form of A_0 (formula (15) in Konakov (2006)) we obtain that it is enough to estimate for fixed q, l, r

$$\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u - ih)^2\int_0^1(1-\delta)\chi_\nu(s, v) \tag{140}$$
$$\int p(0, s, x, z)D_z^\nu\left(\frac{\partial^3\widetilde{p}(s, jh, z, v)}{\partial z_q\partial z_l\partial z_r}\right)(s, jh, z, v)\,|_{s=s_i}\, dzd\delta du.$$

Analogously to (25) in Konakov (2006) we get that (140) does not exceed

$$C(\varepsilon)h^{1-2\varepsilon}(jh)^{2\varepsilon-1}\phi_{\sqrt{jh}}(v - x). \tag{141}$$

Now from (133), (136), (137), (139) and (141) we obtain that

$$\left|[p \otimes_h (L - \widetilde{L})\widetilde{\pi}_1] \otimes_h \Phi(0, T, x, y) - p \otimes [(L - \widetilde{L})\widetilde{\pi}_1 \otimes_h' \Phi](0, T, x, y)\right|$$

$$\le Ch^{3/4+\delta}\phi_{\sqrt{T}}(y - x) \tag{142}$$

for some $\delta > 0$. The other replacements can be shown analogously. Thus we come to the following representation

$$p_h(0,T,x,y) - p(0,T,x,y)$$

$$= \sqrt{h}\,[\widetilde{\pi}_1 \otimes_h' \Phi(0,T,x,y) + p \otimes (\Re_1 \otimes_h' \Phi)(0,T,x,y)]$$

$$+h\,[\widetilde{\pi}_2 \otimes_h' \Phi(0,T,x,y) + p \otimes (\Re_2 \otimes_h' \Phi)(0,T,x,y) + p \otimes_h (\Re_3 \otimes_h' \Phi)(0,T,x,y)]$$

$$+h\,[\widetilde{\pi}_1 \otimes_h' \Phi + p \otimes (\Re_1 \otimes_h' \Phi)] \otimes_h' (\Re_1 \otimes_h' \Phi)(0,T,x,y)$$

$$+\frac{h}{2} p \otimes (L_*^2 - L^2) p(0,T,x,y) - \frac{h}{2} p \otimes (L' - \widetilde{L}') p(0,T,x,y) + O(h^{1+\delta} \zeta_{\sqrt{T}}(y-x)). \tag{143}$$

Now we furhter simplify our expansion of $p_h - p$. We now show the following expansion

$$p_h(0,T,x,y) - p(0,T,x,y)$$

$$= \sqrt{h}(p \otimes \mathcal{F}_1[p_\Delta])(0,T,x,y) + h\,(p \otimes \mathcal{F}_2[p_\Delta])\,(0,T,x,y)$$

$$+h\,(p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p_\Delta]])\,(0,T,x,y)$$

$$+\frac{h}{2} p \otimes (L_*^2 - L^2) p(0,T,x,y) - \frac{h}{2} p \otimes (L' - \widetilde{L}') p(0,T,x,y) + O(h^{1+\delta} \zeta_{\sqrt{T}}(y-x)), \tag{144}$$

where for $s \in [0, t-h], t \in \{h, 2h, ..., T\}$

$$p_\Delta(s,t,z,y) = (\widetilde{p} \otimes_h' \Phi)(s,t,z,y)$$

$$= \widetilde{p}(s,t,z,y) + \sum_{s \le jh \le t-h} h \int \widetilde{p}(s,jh,z,v) \Phi_1(jh,t,v,y) dv.$$

Here $\Phi_1 = H + H \otimes_h' H + H \otimes_h' H \otimes_h' H + ...$We start from the calculation of $p \otimes \widetilde{L}\widetilde{\pi}_1(s,t,x,y)$.

$$p \otimes \widetilde{L}\widetilde{\pi}_1(s,t,x,y) = \int_s^t d\tau \int p(s,\tau,x,v)(t-\tau) \sum_{|\nu|=3} \frac{\overline{\chi}_\nu(\tau,t,y)}{\nu!} D_v^\nu (\widetilde{L}_v \widetilde{p}(\tau,t,v,y)) dv$$

$$= -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \left[\int_s^t p(s,\tau,x,v) \left(\int_\tau^t \chi_\nu(u,y) du \right) \frac{\partial}{\partial \tau} D_v^\nu \widetilde{p}(\tau,t,v,y) d\tau \right]$$

$$= -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \int_s^{\frac{s+t}{2}} ... - \sum_{|\nu|=3} \frac{1}{\nu!} \int dv \int_{\frac{s+t}{2}}^t ... = I + II. \tag{145}$$

Integrating by parts w.r.t. time variable we obtain

$$I = -\sum_{|\nu|=3} \frac{1}{\nu!} \int dv \left[p(s,\tau,x,v) \left(\int_\tau^t \chi_\nu(u,y) du \right) D_v^\nu \widetilde{p}(\tau,t,v,y) \,|_{\tau=s}^{\tau=(s+t)/2} \right.$$

$$-\int_s^{\frac{s+t}{2}} D_v^\nu \widetilde{p}(\tau,t,v,y)\left(\frac{\partial p(s,\tau,x,v)}{\partial \tau}\int_\tau^t \chi_\nu(u,y)du - p(s,\tau,x,v)\chi_\nu(\tau,y)\right)d\tau\Bigg]$$

$$= -\sum_{|\nu|=3}\frac{1}{\nu!}\int dv\left[p(s,\frac{s+t}{2},x,v)\left(\int_{\frac{s+t}{2}}^t \chi_\nu(u,y)du\right)D_v^\nu\widetilde{p}(\frac{s+t}{2},t,v,y)\right.$$

$$\left.+\sum_{|\nu|=3}\frac{1}{\nu!}\left(\int_s^t \chi_\nu(u,y)du\right)D_v^\nu\widetilde{p}(s,t,x,y)\right]+\sum_{|\nu|=3}\frac{1}{\nu!}\int_s^{\frac{s+t}{2}} d\tau\left(\int_\tau^t \chi_\nu(u,y)du\right)$$

(146)

$$\times\int L^T p(s,\tau,x,v)D_v^\nu\widetilde{p}(\tau,t,v,y)dv - \sum_{|\nu|=3}\frac{1}{\nu!}\int_s^{\frac{s+t}{2}}\chi_\nu(\tau,y)d\tau\int p(s,\tau,x,v)D_v^\nu\widetilde{p}(\tau,t,v,y)dv.$$

Analogously we get

$$II = \sum_{|\nu|=3}\frac{1}{\nu!}\int p(s,\frac{s+t}{2},x,v)\left(\int_{\frac{s+t}{2}}^t \chi_\nu(u,y)du\right)D_v^\nu\widetilde{p}(\frac{s+t}{2},t,v,y)dv$$

$$+\sum_{|\nu|=3}\frac{1}{\nu!}\int_{\frac{s+t}{2}}^t d\tau\left(\int_\tau^t \chi_\nu(u,y)du\right)\int L^T p(s,\tau,x,v)D_v^\nu\widetilde{p}(\tau,t,v,y)dv$$

$$-\sum_{|\nu|=3}\frac{1}{\nu!}\int_{\frac{s+t}{2}}^t \chi_\nu(\tau,y)d\tau\int p(s,\tau,x,v)D_v^\nu\widetilde{p}(\tau,t,v,y)dv. \tag{147}$$

From (145)- (147) we have

$$p\otimes\widetilde{L}\widetilde{\pi}_1(s,t,x,y) = \widetilde{\pi}_1(s,t,x,y) + p\otimes L\widetilde{\pi}_1(s,t,x,y) - p\otimes\widetilde{M}_1(s,t,x,y). \tag{148}$$

and from (148) we obtain

$$\widetilde{\pi}_1(s,t,x,y) + p\otimes\Re_1(s,t,x,y)$$

$$= \widetilde{\pi}_1(s,t,x,y) + p\otimes L\widetilde{\pi}_1(s,t,x,y) - p\otimes\widetilde{L}\widetilde{\pi}_1(s,t,x,y) + p\otimes M_1(s,t,x,y)$$

$$-p\otimes\widetilde{M}_1(s,t,x,y) = p\otimes M_1(s,t,x,y). \tag{149}$$

It follows from (149) and the definitions of the operations $\otimes$ and $\otimes'_h$that $(\chi[s,jh]$

below denotes the indicator of the interval $[s, jh]$)

$$
\begin{aligned}
(150)\qquad & \sqrt{h}\,[\widetilde{\pi}_1 \otimes_h' \Phi(s,t,x,y) + (p \otimes \Re_1) \otimes_h' \Phi(s,t,x,y)] \\
&= \sqrt{h}(\widetilde{\pi}_1 + p \otimes \Re_1) \otimes_h' \Phi(s,t,x,y) = \sqrt{h}(p \otimes M_1) \otimes_h' \Phi(s,t,x,y) \\
&= \sqrt{h} \sum_{0 \le jh \le t-h} h \int (p \otimes M_1)(s, jh, x, z)\Phi(jh, t, z, y) dz \\
&= \sqrt{h} \sum_{0 \le jh \le t-h} h \int \Big[\int_s^{jh} du \int p(s,u,x,v) \\
&\quad \times M_1(u, jh, v, z) dv\Big]\, \Phi(jh, t, z, y) dz \\
&= \sqrt{h} \sum_{0 \le jh \le t-h} h \int \Big[\int_s^{t} du \chi[s, jh] \int p(s,u,x,v) \\
&\quad \times M_1(u, jh, v, z) dv\Big]\, \Phi(jh, t, z, y) dz \\
&= \sqrt{h} \int_s^t du \int p(s,u,x,v) \sum_{|\nu|=3} \frac{\chi_\nu(u,v)}{\nu!} \\
&\quad \times D_v^\nu \left[\sum_{0 \le jh \le t-h} h\chi[s,jh] \int \widetilde{p}(u, jh, v, z)\Phi(jh, t, z, y) dz \right] dv \\
&= \sqrt{h} \int_s^t du \int p(s,u,x,v) \\
&\quad \times \sum_{|\nu|=3} \frac{\chi_\nu(u,v)}{\nu!} D_v^\nu p_\Delta(u,t,v,y) dv = \sqrt{h}(p \otimes \mathcal{F}_1)[p_\Delta](s,t,x,y).
\end{aligned}
$$

Using similar arguments as in the proof of (150) one can show that

$$
h\,[\widetilde{\pi}_2 \otimes_h' \Phi(s,t,x,y) + (p \otimes \Re_2) \otimes_h' \Phi(s,t,x,y)
$$

$$
(151)\quad + p \otimes_h (\Re_3 \otimes_h' \Phi)(s,t,x,y)] = h(p \otimes \mathcal{F}_2)[p_\Delta](s,t,x,y) + hp \otimes \Pi_1 \otimes_h' \Phi(s,t,x,y).
$$

For the first two terms in the right hand side of (143) we obtain from (150) and (151)

$$
\sqrt{h}\,[\widetilde{\pi}_1 \otimes_h' \Phi(0,T,x,y) + (p \otimes \Re_1) \otimes_h' \Phi(0,T,x,y)]
$$

$$
+ h\,[\widetilde{\pi}_2 \otimes_h' \Phi(0,T,x,y) + p \otimes (\Re_2 \otimes_h' \Phi)(0,T,x,y) + p \otimes_h (\Re_3 \otimes_h' \Phi)(0,T,x,y)]
$$

$$
(152)\quad = \sqrt{h}(p \otimes \mathcal{F}_1)[p_\Delta](0,T,x,y) + h(p \otimes \mathcal{F}_2)[p_\Delta](s,t,x,y) + hp \otimes \Pi_1 \otimes_h' \Phi(s,t,x,y).
$$

Using (150) we also have

$$
h\,[\widetilde{\pi}_1 \otimes_h' \Phi + p \otimes (\Re_1 \otimes_h' \Phi)] \otimes_h' (\Re_1 \otimes_h' \Phi)(0,T,x,y)
$$

$$
= h(p \otimes \mathcal{F}_1[p_\Delta]) \otimes_h' (\Re_1 \otimes_h' \Phi)(0,T,x,y)
$$

$$= hp \otimes \mathcal{F}_1 \left[p_\Delta \otimes_h' (\Re_1 \otimes_h' \Phi)\right](0, T, x, y).$$

Note that

$$hp \otimes \Pi_1 \otimes_h' \Phi(s,t,x,y) = h \int_s^t du \int p(s,u,x,v) \sum_{|\nu|=3} \frac{\chi_\nu(u,v)}{\nu!} D_v^\nu [\widetilde{\pi}_1 \otimes_h' \Phi](u,t,v,y)$$

$$= hp \otimes \mathcal{F}_1[\widetilde{\pi}_1 \otimes_h' \Phi](s,t,x,y).$$

For the proof of (144) it remains to show that

$$hp \otimes \mathcal{F}_1[\widetilde{\pi}_1 \otimes_h' \Phi + p_\Delta \otimes_h' (\Re_1 \otimes_h' \Phi)](0,T,x,y)$$

$$= h \left(p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p_\Delta]]\right)(0,T,x,y) + O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)). \tag{153}$$

We will show that

$$hp \otimes \mathcal{F}_1[(p - p_\Delta) \otimes_h' (\Re_1 \otimes_h' \Phi)](0,T,x,y) = O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)), \tag{154}$$

and

$$hp \otimes \mathcal{F}_1[p \otimes_h' (\Re_1 \otimes_h' \Phi)](0,T,x,y)$$

$$-hp \otimes \mathcal{F}_1[p \otimes (\Re_1 \otimes_h' \Phi)](0,T,x,y) = O(h^{1+\delta}\zeta_{\sqrt{T}}(y-x)). \tag{155}$$

Then (153) will follow from (154), (155) and (150). The estimate (155) can be shown analogously to (142). An additional singularity arising from the derivatives in the operator $\mathcal{F}_1[\cdot]$ is neglected by the factor h in (155). To estimate (154) note that from the definition of $\Re_1$ and Φ

$$|(\Re_1 \otimes_h' \Phi)(jh,T,x,y)| \le C(\varepsilon) h^{-\varepsilon}(T-jh)^{\varepsilon-1}\phi_{\sqrt{T-jh}}. \tag{156}$$

Then we use the following estimate which can be proved by the same method as in Konakov (2006), pp.8-12, where an estimate for $(p-p^d)(0,jh,x,y)$ was obtained.

$$|(p-p_\Delta)(u,jh,v,z)| \le Ch^{1/2}\phi_{\sqrt{jh-u}}(z-v). \tag{157}$$

From (156) and (157)

$$|(p-p_\Delta) \otimes_h' (\Re_1 \otimes_h' \Phi)(u,T,v,y)| \le C(\varepsilon)h^{1/2-\varepsilon}(T-u)^{\varepsilon}\phi_{\sqrt{T-u}}(y-v) \tag{158}$$

Now to estimate (154) it's enough to estimate a typical term of a corresponding sum, namely, for $|\nu| = 3$ we have to estimate

$$h \int_0^T du \int p(0,u,x,v) \frac{\chi_\nu(u,v)}{\nu!} D_v^\nu [\sum_{\{j: u \le jh \le T-h\}} h \int (p-p_\Delta)(u,jh,v,z)$$

$$\times(\Re_1 \otimes'_h \Phi)(jh,T,z,y)dz]dv = h\int_0^{T/2} ... + h\int_{T/2}^{T} ... = I + II.$$

To estimate II we make an integration by parts transferring three derivatives to $p(0,u,x,v)\frac{\chi_\nu(u,v)}{\nu!}$. Using (158) we obtain the following estimate

$$|II| \le C(\varepsilon)h^{3/2-\varepsilon}\int_{T/2}^{T}\frac{(T-u)^\varepsilon}{u^{3/2}}du\phi_{\sqrt{T}}(y-x)$$

$$\le C(\varepsilon)h^{3/2-\varepsilon}T^\varepsilon\phi_{\sqrt{T}}(y-x). \tag{159}$$

To estimate I we consider two cases, namely, a) $jh-u \ge T/4$ and b) $jh-u \le T/4$ $\Longrightarrow T-jh \ge T/4$. Analogously to (11) in Konakov (2006) the difference $h(p-p_\Delta)$ can be represented as

$$h(p-p_\Delta)(u,jh,v,z) = h(p\otimes H - p\otimes'_h H)(u,jh,v,z)$$

$$+h(p\otimes H - p\otimes'_h H)\otimes'_h \Phi_1(u,jh,v,z)$$

$$= h\int_u^{j^*h}d\tau\int p(u,\tau,v,z')H(\tau,jh,z',z)dz'$$

$$+h\sum_{i=j^*}^{j-1}\int_{ih}^{(i+1)h}d\tau\int(\lambda(\tau,z')-\lambda(ih,z'))dz' + h(p\otimes H - p\otimes'_h H)\otimes'_h \Phi_1(u,jh,v,z)$$

$$= I' + II' + III'. \tag{160}$$

where $\lambda(\tau,z') = p(u,\tau,v,z')H(\tau,jh,z',z)$, $\Phi_1(ih,jh,z'z) = H(ih,jh,z'z)+ H\otimes'_h H(ih,jh,z'z) + ...$, $j^* = j^*(u) = [\frac{u}{h}]+1$ (with a convention $[x] = x-1$ for integer x). For I', case a), we have $jh-\tau > T/5$ for n large enough. With a substitution $v+v' = z'$ we obtain

$$\left|D_v^\nu h\int_u^{j^*h}d\tau\int p(u,\tau,v,z')H(\tau,jh,z',z)dz'\right|$$

$$= \left|D_v^\nu h\int_u^{j^*h}d\tau\int p(u,\tau,v,v+v')H(\tau,jh,v+v',z)dv'\right|$$

$$\le Ch\int_u^{j^*h}\frac{d\tau}{(jh-\tau)^2}\phi_{\sqrt{jh-u}}(z-v) \le Ch^2T^{-2}\phi_{\sqrt{jh-u}}(z-v)$$

$$\le Cn^{-2}\phi_{\sqrt{jh-u}}(z-v) = CT^2h^2\phi_{\sqrt{jh-u}}(z-v) \tag{161}$$

To get (161) we used an estimate from Friedman (1964) (Theorem 7, page 260)

$$|D_v^\nu p(u,\tau,v,v+v')| \le C(\tau-u)^{-d/2}\exp\left(\frac{C|v'|}{\tau-u)}\right).$$

For $\int I'(u,jh,v,z)(\Re_1 \otimes_h' \Phi)(jh,T,z,y)dz$, case b), it is enough to estimate for $|\nu| = 3$

$$h\int_u^{j^*h} d\tau \int [p(u,\tau,v,v+v')(\sigma_{lk}(\tau,v+v')-\sigma_{lk}(\tau,z))]D_{v'}^{\nu}\frac{\partial^2\widetilde{p}(\tau,jh,v+v',z)}{\partial v_l'\partial v_k'}dv'$$

$$\times(\Re_1 \otimes_h' \Phi)(jh,T,z,y)dz. \tag{162}$$

Transferring five derivatives from $\widetilde{p}$ to $(\Re_1 \otimes_h' \Phi)(jh,T,z,y)$ and using the following estimate for $|\mu| = 5$

$$|D_{v'}^{\mu}\widetilde{p}(\tau,jh,v+v',z)+D_z^{\mu}\widetilde{p}(\tau,jh,v+v',z)|$$

$$\leq C(jh-\tau)^{-d/2}\phi_{\sqrt{jh-\tau}}(z-v-v')$$

we obtain that (162) does not exceed

$$C(j^*h-\tau)hT^{-7/2}\phi_{\sqrt{T-u}}(y-v) \leq Ch^{1+\delta}T^{1-\delta}n^{-(1-\delta-7\varkappa/2)}\phi_{\sqrt{T-u}}(y-v)$$

$$= o(h^{1+\delta}T^{1-\delta})\phi_{\sqrt{T-u}}(y-v). \tag{163}$$

We used that for any $0 < \delta < 1$ $\varkappa < \frac{2-2\delta}{7}$ (see condition (B2)). To estimate $\int II'(u,jh,v,z)(\Re_1 \otimes_h' \Phi)(jh,T,z,y)dz$ we use a decomposition (12) in Konakov (2006). The estimate for terms in II' containing the first derivatives $\lambda'(ih,z')$ we use identity (14) from Konakov (2006) and similar arguments to already used in estimation of $\int I'(u,jh,v,z)(\Re_1 \otimes_h' \Phi)(jh,T,z,y)dz$. The estimate for terms in II' containing second derivatives $\lambda''(ih,z')$ follows from (22) and (23) in Konakov (2006). At last for III' the same estimates hold because of smoothing properties of the convolution $...\otimes_h' \Phi_1(u,jh,v,z)$. This implies (154) and, hence, the expansion (144).

Asymptotic replacement of p_Δ *by* p. We shall compare $hp\otimes\mathcal{F}_2[p_\Delta](0,T,x,y)$ and $hp\otimes\mathcal{F}_2[p](0,T,x,y)$. Note that for $2\varkappa < \delta < \frac{2}{5}, |\nu| = 4$

$$\left|h\int_0^{h^\delta} du \int p(0,u,x,z)\chi_\nu(u,z)D_z^\nu p(u,T,z,y)dz\right| \leq Ch^{1+\delta}(T-h^\delta)^{-2}\phi_{\sqrt{T}}(y-x)$$

$$\leq Ch^{1+\delta}\frac{n^{2\varkappa}}{(Tn^\varkappa - n^\varkappa h^\delta)^2} \leq Ch^{1+(\delta-2\varkappa)}T^{2\varkappa}\phi_{\sqrt{T}}(y-x), \tag{164}$$

and, analogously,

$$\left|h\int_{T-h^\delta}^{T} du \int D_z^\nu[p(0,u,x,z)\chi_\nu(u,z)]p(u,T,z,y)dz\right| \leq Ch^{1+(\delta-2\varkappa)}T^{2\varkappa}\phi_{\sqrt{T}}(y-x). \tag{165}$$

The same estimates hold for $p_\Delta(u,T,z,y)$. Hence, it is enough to consider $u \in [h^\delta, T-h^\delta]$. We consider

$$h\int_{h^\delta}^{T-h^\delta} du \int p(0,u,x,z)\chi_\nu(u,z)D_z^\nu(p-p_\Delta)(u,T,z,y)dz$$

(166) $$= h\int_{h^\delta}^{T/2} ... + h\int_{T/2}^{T-h^\delta} ... = I + II.$$

By (157)

$$|II| = \left| h\int_{T/2}^{T-h^\delta} du \int D_z^\nu[p(0,u,x,z)\chi_\nu(u,z)](p-p_\Delta)(u,T,z,y)dz \right|$$

$$\leq Ch^{3/2}n^\varkappa \phi_{\sqrt{T}}(y-x) = Ch^{3/2-\varkappa}T^\varkappa \phi_{\sqrt{T}}(y-x)$$

(167) $$= Ch^{1+\gamma}\phi_{\sqrt{T}}(y-x), \gamma > 0$$

For $u \in [h^\delta, T/2]$

$$|I| = \left| h\int_{h^\delta}^{T/2} du \int D_z^\nu[p(0,u,x,z)\chi_\nu(u,z)](p-p_\Delta)(u,T,z,y)dz \right|$$

(168) $$\leq Ch^{3/2-\delta}\phi_{\sqrt{T}}(y-x),$$

where condition $\delta < \frac{2}{5}$ implies $3/2-\delta > 1$. It follows from (164)-(168) that

(169) $$hp \otimes \mathcal{F}_2[p_\Delta](0,T,x,y) - hp \otimes \mathcal{F}_2[p](0,T,x,y) = O(h^{1+\gamma}\phi_{\sqrt{T}}(y-x)).$$

To prove that

$$hp \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p-p_\Delta]] = O(h^{1+\delta}\phi_{\sqrt{T}}(y-x))$$

we consider a typical summand for fixed $\nu, |\nu| = 3$,

(170)
$$h\int_0^T du \int p(0,u,x,z)\chi_\nu(u,z)D_z^\nu \left[\int_u^T d\tau \int p(u,\tau,z,v)\chi_\nu(\tau,v)D_v^\nu(p-p_\Delta)(\tau,T,v,y)dv\right] dz.$$

As before it's enough to consider $u \in [h^\delta, T-h^\delta]$. The integral in (170) is a sum of four integrals

$$I_1 = h\int_{h^\delta}^{T/2} du \int ...D_z^\nu \int_u^{(T+u)/2} d\tau \int ...$$

$$I_2 = h\int_{h^\delta}^{T/2} du \int ...D_z^\nu \int_{(T+u)/2}^{T} d\tau \int ...$$

$$I_3 = h\int_{T/2}^{T-h^\delta} du \int ...D_z^\nu \int_u^{(T+u)/2} d\tau \int ...$$

$$(171) \qquad I_4 = h\int_{T/2}^{T-h^\delta} du \int ...D_z^\nu \int_{(T+u)/2}^{T} d\tau \int ....$$

Note that $\tau - u \geq T/4$ in the integrand in I_2 . Integrating by parts w.r.t. v we obtain by (157)

$$(172) \qquad |I_2| \leq Ch^{3/2-\varkappa}T^\varkappa \phi_{\sqrt{T}}(y-x)).$$

Furthermore, $u \geq T/2, \tau - u \geq h^\delta/2, T-u \geq h^\delta$ in the integrand in I_4. Integrating by parts w.r.t. z we obtain

$$(173) \qquad |I_4| \leq Ch^{3/2-\delta}T^{-1/2}\phi_{\sqrt{T}}(y-x)) \leq CT^{\varkappa/2}h^{3/2-\varkappa/2-\delta}\phi_{\sqrt{T}}(y-x)),$$

where, by our choice of δ, $3/2-\varkappa/2-\delta > 1$. To estimate I_3 we use the representation

$$(p-p_\Delta)(\tau,T,v,y) = (p\otimes H - p\otimes_h' H)(\tau,T,v,y)$$

$$+(p\otimes H - p\otimes_h' H)\otimes_h' \Phi_1(\tau,T,v,y)$$

$$= \int_\tau^{j^*h} ds \int p(\tau,s,v,w)H(s,T,w,y)dw$$

$$+\frac{h}{2}[p\otimes_h' (H_1+A_0)](\tau,T,v,y)$$

$$+\frac{1}{2}\sum_{i=j^*}^{n-1}\int_{ih}^{(i+1)h}(t-ih)^2\int_0^1(1-\gamma)\sum_{k=1}^{4}\int p(\tau,s,v,w)A_k(s,T,w,y)\mid_{s=ih+\gamma(t-ih)} dwd\gamma dt,$$

$$(174) \qquad +(p\otimes H - p\otimes_h' H)\otimes_h' \Phi_1(\tau,T,v,y)$$

where $j^* = j^*(\tau) = [\tau/h]+1$ (with a convention $[x] = x-1$ for integer x) , H_1 and $A_k, k = 0,1,2,3,4$, are defined in Konakov (2006) and

$$\Phi_1(ih,i'h,z,z') = H(ih,i'h,z,z') + H\otimes_h' H(ih,i'h,z,z') + ...$$

To estimate $D_v^\nu(p-p_\Delta)(\tau,T,v,y)$ we note that

$$h\left|D_v^\nu\int_\tau^{j^*h} ds\int p(\tau,s,v,w)H(s,T,w,y)dw\right|$$

$$= h\left|D_v^\nu\int_\tau^{j^*h} ds\int p(\tau,s,v,v+w')H(s,T,v+w',y)dw'\right|$$

$$(175) \qquad \leq Ch\int_\tau^{j^*h}\frac{ds}{(T-s)^2}\phi_{\sqrt{T-\tau}}(y-v) \leq Ch^{2-2\delta}\phi_{\sqrt{T-\tau}}(y-v).$$

Furthermore,

$$\left|\frac{h^2}{2}D_v^{\nu}[p\otimes_h' H_1](\tau,T,v,y)\right| = \left|\frac{h^2}{2}D_v^{\nu}\sum_{\tau\le jh\le T-h} h\int p(\tau,jh,v,w)H_1(jh,T,w,y)dw\right|$$

$$\le \frac{h^2}{2}\left|\sum_{\tau\le jh\le T-h^{\delta}/2} h\int D_v^{\nu}[p(\tau,jh,v,v+w')H_1(jh,T,v+w',y)]dw'\right|$$

$$+\frac{h^2}{2}\left|C\sum_{i,k=1}^{d}\sum_{T-h^{\delta}/2<jh\le T-h} h\int D_{w'}^{\nu+e_i+e_k}[p(\tau,jh,v,v+w')]\widetilde{p}(jh,T,v+w',y)dw'\right|$$

$$\le Ch^{2-2\delta}\phi_{\sqrt{T-\tau}}(y-v)+Ch^{2-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \tag{176}$$

Because of a structure of the operator A_0 (see Konakov (2006)) it is enough to estimate for fixed i,l,k

$$\frac{h^2}{2}D_v^{\nu}\sum_{\tau\le jh\le T-h} h\int D_{w'}^{e_k}p(\tau,jh,v,v+w')\frac{\partial^2\widetilde{p}(jh,T,v+w',y)}{\partial w_i'\partial w_l'}dw' \tag{177}$$

With the same decomposition as in (176) we obtain that (177) does not exceed

$$Ch^{2-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \tag{178}$$

From (176) and (178) we obtain that

$$\frac{h^2}{2}\left|D_v^{\nu}[p\otimes_h'(H_1+A_0)](\tau,T,v,y)\right|\le Ch^{1+\gamma}\phi_{\sqrt{T-\tau}}(y-v) \tag{179}$$

for some $\gamma>0$. It remains to estimate the last summand in (174). It follows from the structure of the operators $A_k, k=1,2,3,4,$ (see Konakov (2006)) that it is enough to estimate

$$\frac{1}{2}\sum_{i=j^*}^{n-1}\int_{ih}^{(i+1)h}(t-ih)^2\int_0^1(1-\gamma) \tag{180}$$

$$\sum_{k=1}^{4}\int D_v^{\nu}\left[p(\tau,s,v,v+w')\frac{\partial^4\widetilde{p}(s,T,v+w',y)}{\partial w_i'\partial w_l'\partial w_p'\partial w_q'}\right]|_{s=ih+\gamma(t-ih)}\,dw'd\gamma dt$$

for fixed i,j,p,q. As in Konakov (2006), p. 12, we obtain that (180) does not exceed

$$Ch^2\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{i=j^*}^{n-1}h\frac{1}{[(ih-\tau)+\gamma hz]^{3/2}}\frac{1}{[(n-\gamma z)h-ih]^2}d\gamma dz$$

$$=Ch^2\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{\{i:j^*h\le ih\le\tau+h^{\delta}/4\}}\cdots$$

$$+Ch^2\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{\{i:\tau+h^\delta/4<ih\le T-h\}} ... = I''+II''. \tag{181}$$

$$|I''|\le Ch^2h^{-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1(1-\gamma)\sum_{\{i:j^*h\le ih\le\tau+h^\delta/4\}} h\frac{1}{[(ih-\tau)+\gamma hz]}d\gamma dz$$

$$\le Ch^{2-\varepsilon}h^{-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^{2-\varepsilon}\int_0^1\frac{(1-\gamma)}{\gamma^\varepsilon}\sum_{\{i:j^*h\le ih\le\tau+h^\delta/4\}} h\frac{1}{[(ih-\tau)+\gamma hz]^{1-\varepsilon}}d\gamma dz$$

$$\le C(\varepsilon)h^{2-\varepsilon-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \tag{182}$$

Using inequality $(h-\gamma z)h-ih=(n-i)h-\gamma zh\ge h(1-\gamma z)\ge h(1-\gamma)$ we obtain that

$$|II''|\le Chh^{-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v)\int_0^1 z^2\int_0^1 d\gamma\sum_{\{i:\tau+h^\delta/4<ih\le T-h\}} h$$

$$\le Ch^{1-5\delta/2}\phi_{\sqrt{T-\tau}}(y-v). \tag{183}$$

Now from (174), (175), (179), (180), (182) and (182) we obtain that

$$|D_v^\nu(p\otimes H-p\otimes_h' H)(\tau,T,v,y)|\le Ch^\gamma\phi_{\sqrt{T-\tau}}(y-v) \tag{184}$$

for some positive γ. The last summand in the right hand side of (174) admits the same estimate (184) because of the smoothing properties of the operation $\otimes_h'$. Hence,

$$|D_v^\nu(p-p_\Delta)(\tau,T,v,y)|\le Ch^\gamma\phi_{\sqrt{T-\tau}}(y-v). \tag{185}$$

Making a chaige of variables $v=z+v'$ in (170) we get that the integral w.r.t. v is equal to

$$D_z^\nu\left[\int_u^T d\tau\int p(u,\tau,z,z+v')\chi_\nu(\tau,v)D_v^\nu(p-p_\Delta)(\tau,T,z+v',y)dv'\right]. \tag{186}$$

Taking into account (185) and making an integration by parts in (186) we obtain that (186) does not exceed

$$Ch^\gamma\int_u^T\frac{d\tau}{(\tau-u)^{3/2}}\phi_{\sqrt{T-u}}(y-z)\le\frac{Ch^\gamma}{\sqrt{T-u}}\phi_{\sqrt{T-u}}(y-z) \tag{187}$$

From (170) and (187) we obtain that

$$|I_3|\le Ch^{1+\gamma}\phi_{\sqrt{T}}(y-x).$$

The estimate for I_1 may be proved absolutely analogously to the estimate for I_3. Thus, we proved that

$$hp\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p-p_\Delta]]=O(h^{1+\delta}\phi_{\sqrt{T}}(y-x)) \tag{188}$$

The estimate

$$h^{1/2}p\otimes\mathcal{F}_1[p-p_\Delta]=O(h^{1+\delta}\phi_{\sqrt{T}}(y-x))$$

may be proved by using the same decomposition of $p-p_\Delta$. This completes the proof of Theorem 1.

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