Small time asymptotics in local limit theorems for Markov chains converging to diffusions. *

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20th February 2006

We consider triangular arrays of Markov chains that converge weakly to a diffusion process. Local limit theorems for transition densities are proved. The observation time $[0, T]$ may be fixed or $\lim_{n\to\infty} T = 0$, where $nh = T$ and h is a mesh between two neighboring observation points.

1991 MSC: primary 62G07, secondary 60G60

Keywords and phrases: Markov chains, diffusion processes, transition densities, Edgeworth expansions

*This research was supported by grant 436RUS113/845/0-1 from the Deutsche Forschungsgemeinschaft and by grants 04-01-00700 and 05-01-04004 from the Russian Foundation of Fundamental Researches. The author worked on the paper during a visit at the Laboratory of Probability Theory and Random Models of the University Paris VI in 2005/2006. He is grateful for the hospitality during his stay.

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1. Introduction and main results

Let $n \geq 2$ and $h > 0$ be such that $T = nh \leq 1$. Suppose that $q(t, x, \cdot)$, $(t, x) \in [0,1] \times \mathbb{R}^d$ is a given family of densities on $\mathbb{R}^d$ and m is a function from $[0,1] \times \mathbb{R}^d$ into $\mathbb{R}^d$. We shall impose the following conditions

A1 $\int_{\mathbb{R}^d} yq(t, x, y)\, dy = 0,\ 0 \leq t \leq 1,\ x \in \mathbb{R}^d$.

A2 There exists positive constants σ_* and σ^* such that the covariance matrix $\sigma(t, x) = \int_{\mathbb{R}^d} yy^T q(t, x, y)\, dy$ satisfies

$$\sigma_* \leq \theta^T \sigma(t, x)\, \theta \leq \sigma^*,$$

for all $\|\theta\| = 1$ and $t \in [0,1]$ $x \in \mathbb{R}^d$.

A3 There exists a positive integer S' and a real nonnegative function $\psi(y)$, $y \in \mathbb{R}^d$ satisfying $\sup_{y \in \mathbb{R}^d} \psi(y) < \infty$ and $\int_{\mathbb{R}^d} \|y\|^S \psi(y)\, dy < \infty$, with $S = 2dS' + 4$, such that

$$\left| D_y^\nu q(t, x, y) \right| \leq \psi(y),\ t \in [0,1],\ x, y \in \mathbb{R}^d\ \ |\nu| = 0, 1, 2, 3, 4$$

and

$$\left| D_x^\nu q(t, x, y) \right| \leq \psi(y),\ t \in [0,1],\ x, y \in \mathbb{R}^d\ \ |\nu| = 0, 1, 2.$$

Furthemore, for all $x \in \mathbb{R}^d$ it holds

$$\int_{\mathbb{R}^d} |q(t, x, y) - q(t', x, y)|\, dy \to 0,$$

as $|t - t'| \to 0$.

B1 The functions $m(t, x)$ and $\sigma(t, x)$ and their first and second derivatives w.r.t. t and x are continuous and bounded uniformly in t and x. All these functions are Lipschitz continuous with respect to x with a Lipschitz constant that does not depend on t. Furthemore, $D^\nu \sigma(t, x)$ exists for $|\nu| = 6$ and is Holder continuous w.r.t. x with positive exponent and a constant that does not depend on t.

Consider a family of Markov processes in $\mathbb{R}^d$ of the following form

(1) $$X_{k+1,h} = X_{k,h} + m(kh, X_{k,h})\, h + \sqrt{h}\xi_{k+1,h},\ X_{0,h} = x \in \mathbb{R}^d,\ k = 0, ..., n-1,$$

where $(\xi_{i,h})_{i=1,\ldots,n}$ is an innovation sequence satisfying the Markov assumption: the conditional distribution of $\xi_{k+1,h}$ given $X_{k,h} = x_k, ..., X_{0,h} = x_0$ depends only on $X_{k,h} = x_k$ and has conditional density $q(kh, x_k, \cdot)$. The conditional covariance matrix corresponding to this density is $\sigma(kh, x_k)$. The transition densities of $(X_{i,h})_{i=1,\ldots,n}$ are denoted by $p_h(0, kh, x, \cdot)$.

We shall consider the process (1) as an approximation to the following stochastic differential equation in $\mathbb{R}^d$:

$$dY_s = m(s, Y_s)\, ds + \Lambda(s, Y_s)\, dW_s,\ Y_0 = x \in \mathbb{R}^d,\ s \in [0, T],$$

where $(W_s)_{s\geq 0}$ is the standard Wiener process and Λ is a symmetic positive definite $d \times d$ matrix such that $\Lambda(s, y)\Lambda(s, y)^T = \sigma(s, y)$. The conditional density of Y_t, given $Y_0 = x$ is denoted by $p(0, t, x, \cdot)$.

Konakov and Mammen (2000) obtained a nonuniform rate of convergence for the difference $p_h(0, T, x, \cdot) - p(0, T, x, \cdot)$ as $n \to \infty$ in the case $T \asymp 1$. It is the goal of the present note to get an analogous result in the case $T = o(1)$. The following theorem is our main result.

Theorem 1 *Let $h > 0$, $n \geq 2$, $T = nh$. Assume (A1-A3) and (B1). Then, as $n \to \infty$ and $T \to 0$,*

$$\sup_{x,y} Q_{\sqrt{T}}(y - x)\, |p_h(0, T, x, y) - p(0, T, x, y)| = O\left(n^{-1/2}\right),$$

where the constant in $O(\cdot)$ does not depend on h and

$$Q_\delta(u) = \delta^d \left(1 + \left\|\frac{u}{\delta}\right\|^{2S'-2}\right).$$

2. Parametrix method for diffusions

For any $s \in (0, T)$, $x, y \in \mathbb{R}^d$ we consider an additional family of "frozen" diffusion processes

$$d\widetilde{Y}_t = m(t, y)\, dt + \Lambda(t, y)\, dW_t,\ \widetilde{Y}_s = x,\ s \leq t \leq T.$$

Let $\widetilde{p}^y(s, t, x, \cdot)$ be the conditional density of $\widetilde{Y}_t$, given $\widetilde{Y}_s = x$. In the sequel for any z we shall denote $\widetilde{p}(s, t, x, z) = \widetilde{p}^z(s, t, x, z)$, where the variable z acts here twice: as the arument of the density and as defining quantity of the process $\widetilde{Y}_t$.

The transition densities $\widetilde{p}$ can be computed explicitly

$$\begin{aligned}\widetilde{p}(s, t, x, y) &= (2\pi)^{-d/2} (\det \sigma(s, t, y))^{-1/2} \\ &\quad \times \exp\left(-\frac{1}{2}(y - x - m(s, t, y))^T \sigma^{-1}(s, t, y)(y - x - m(s, t, y))\right),\end{aligned}$$

where

$$\sigma(s, t, y) = \int_s^t \sigma(u, y)\, du, \qquad m(s, t, y) = \int_s^t m(u, y)\, du.$$

We need the following kernel

$$
\begin{aligned}
H(s,t,x,y) &= \frac{1}{2}\sum_{i,j=1}^{d}\left(\sigma_{ij}(s,x)-\sigma_{ij}(s,y)\right)\frac{\partial^2\widetilde{p}(s,t,x,y)}{\partial x_i\partial x_j} \\
&\quad+\sum_{i=1}^{d}\left(m_i(s,x)-m_i(s,y)\right)\frac{\partial\widetilde{p}(s,t,x,y)}{\partial x_i}.
\end{aligned}
$$

Introduce the convolution type binary operation $\otimes$:

$$f\otimes g(s,t,x,y)=\int_s^t du\int_{\mathbb{R}^d} f(s,u,x,z)\,g(u,t,z,y)\,dz.$$

k-fold convolution of H is denoted by $H^{(k)}$. The following results are taken from Konakov and Mammen (2000).

Lemma 2 *Let $0\le s<t\le T$. It holds*

$$p(s,t,x,y)=\sum_{r=0}^{\infty}\widetilde{p}\otimes H^{(r)}(s,t,x,y).$$

Lemma 3 *Let $0\le s<t\le T$. There exist constants C and C_1 such that*

$$|H(s,t,x,y)|\le C_1\rho^{-1}\phi_{C,\rho}(y-x)$$

and

$$\left|\widetilde{p}\otimes H^{(r)}(s,t,x,y)\right|\le C_1^{r+1}\frac{\rho^r}{\Gamma\left(1+\frac{r}{2}\right)}\phi_{C,\rho}(y-x),$$

where $\rho^2=t-s$, $\phi_{C,\rho}(u)=\rho^{-d}\phi_C(u/\rho)$ and

$$\phi_C(u)=\frac{\exp\left(-C\|u\|^2\right)}{\int\exp\left(-C\|v\|^2\right)dv}.$$

3. Parametrix method for Markov chains

For any $0\le jh\le T$, $x,y\in\mathbb{R}^d$ we consider an additional family of "frozen" Markov chains defined for $jh\le ih\le T$ as

$$\widetilde{X}_{i+1,h}=\widetilde{X}_{i,h}+m(ih,y)\,h+\sqrt{h}\widetilde{\xi}_{i+1,h},\ \ \widetilde{X}_{j,h}=x\in\mathbb{R}^d,\ \ j\le i\le n, \tag{2}$$

where $\widetilde{\xi}_{j+1,h},...,\widetilde{\xi}_{n,h}$ is an innovation sequence such that the conditional density of $\widetilde{\xi}_{i+1,h}$ given $\widetilde{X}_{i,h}=x_i,...,\widetilde{X}_{0,h}=x_0$ equals to $q(ih,y,\cdot)$. Let us introduce the infinitesimal operators corresponding to Markov chains (1) and (2) respectively,

$$
\begin{aligned}
&L_hf(jh,kh,x,y)\\
&=h^{-1}\left(\int p_h(jh,(j+1)h,x,z)\,f((j+1)h,kh,z,y)\,dz-f((j+1)h,kh,z,y)\right)
\end{aligned}
$$

and

$$\widetilde{L}_h f\left(jh,kh,x,y\right)$$
$$= h^{-1}\left(\int \widetilde{p}_h^y\left(jh,\left(j+1\right)h,x,z\right)f\left(\left(j+1\right)h,kh,z,y\right)dz - f\left(\left(j+1\right)h,kh,z,y\right)\right),$$

where $\widetilde{p}_h^y\left(jh,j'h,x,\cdot\right)$ denotes the conditional density of $\widetilde{X}_{j',h}$ given $\widetilde{X}_{j,h}=x$. As before for any z denote $\widetilde{p}_h\left(jh,j'h,x,z\right)=\widetilde{p}_h^z\left(jh,j'h,x,z\right)$, where the variable z acts here twise: as the arument of the density and as defining quantity of the process $\widetilde{X}_{i,h}$. For technical convenience the terms $f\left(\left(j+1\right)h,kh,z,y\right)$ on the right hand side of $L_h f$ and $\widetilde{L}_h f$ appear instead of $f\left(jh,kh,z,y\right)$.

In analogy with the definition of H we put, for $k>j$,

$$H_h\left(jh,kh,x,y\right)=\left(L_h-\widetilde{L}_h\right)\widetilde{p}_h\left(jh,kh,x,y\right).$$

We also shall use the convolution type binary operation $\otimes_h$:

$$g\otimes_h f\left(jh,kh,x,y\right)=\sum_{i=j}^{k-1} h\int_{\mathbb{R}^d} g\left(jh,ih,x,z\right)f\left(ih,kh,z,y\right)dz,$$

where $0\le j<k\le n$. Write $g\otimes_h H_h^{(0)}=g$ and $g\otimes_h H_h^{(r)}=\left(g\otimes_h H_h^{(r-1)}\right)\otimes_h H_h$ for $r=1,...,n$. For the higher order convolutions we use the convention $\sum_{i=j}^{l}=0$ for $l<j$. One can show the following analog of the "parametrix" expansion for p_h [see Konakov and Mammen (2000)].

Lemma 4 *Let $0\le jh<kh\le T$. It holds*

$$p_h\left(jh,kh,x,y\right)=\sum_{r=0}^{k-j}\widetilde{p}_h\otimes_h H_h^{(r)}\left(jh,kh,x,y\right),$$

where

$$p_h\left(jh,jh,x,y\right)=\widetilde{p}_h\left(kh,kh,x,y\right)=\delta\left(y-x\right)$$

and δ is the Dirac delta symbol.

4. Auxiliary statements

In this section we collect several bounds to be used later in the proofs. Throughout the proofs $C,C_1,C_2,...$ denote generic constants possibly different in different places.

We give some tools that are useful for comparison of the expansions of p and p_h (see Lemmas 2) and 4 respectively). Since p and p_h are written in terms of $\widetilde{p}^y$ and $\widetilde{p}_h^y$ it is essential first to bound the difference $\widetilde{p}_h^y-\widetilde{p}^y$. For this we use some bounds from

Konakov and Mammen (2000) which are proved for $T = 1$, but remain true also for $T \to 0$. The following lemmas are slight modifications of Lemmas 3.8, 3.9, 3.11, 3.12, 3.13 from Konakov and Mammen (2000) and therefore the proofs will be not detailes here.

To simplify the notation let $\rho = \sqrt{(k-j)\,h}$. Denote $\zeta_\rho(z) = \rho^{-p}\zeta(z/\rho)$, where

$$\zeta(z) = \frac{\left(1 + \|z\|^{S-4}\right)^{-1}}{\int \left(1 + \|z'\|^{S-4}\right)^{-1} dz'}.$$

Lemma 5 *Assume Conditions (A1-A3) and (B1). Then for* $0 \le j < k \le n$ *and all* $x, y \in \mathbb{R}^d$ *it holds*

$$\left|\widetilde{p}_h(jh, kh, x, y) - \widetilde{p}(jh, kh, x, y)\right| \le Ch^{1/2}\rho^{-1}\zeta_\rho(y - x).$$

For $j = 0, ..., k-2$, let

$$K_h(jh, kh, x, y) = \left(L - \widetilde{L}\right)\widetilde{p}_h(jh, kh, x, y)$$

and

$$\begin{aligned} M_h(jh, kh, x, y) &= 3h^{1/2} \sum_{|\nu|=3} \sum_{|\mu|=1} \int_{\mathbb{R}^d} d\theta \int_0^1 d\delta D_y^\mu q(jh, y, \theta)(x-y)^\mu \\ &\quad \times \frac{\theta^\nu}{\nu!} D_x^\nu \widetilde{p}_h\left((j+1)h, kh, x + \delta\theta h^{1/2}, y\right)(1-\delta)^2. \end{aligned}$$

If $j = k - 1$ define

$$K_h((k-1)h, kh, x, y) = 0 \quad \text{and} \quad M_h((k-1)h, kh, x, y) = 0.$$

Lemma 6 *Assume Conditions (A1-A3) and (B1). Then for* $0 \le j < k \le n$ *and all* $x, y \in \mathbb{R}^d$ *it holds*

$$\left|H_h(jh, kh, x, y) - K_h(jh, kh, x, y) - M_h(jh, kh, x, y)\right| \le Ch^{1/2}\rho^{-1}\zeta_\rho(y - x),$$

where ζ_p *is defined above.*

Denote $\xi_\rho(z) = \rho^{-p}\xi(z/\rho)$, where

$$\xi(z) = \frac{\left(1 + \|z\|^{2S'-2}\right)^{-1}}{\int \left(1 + \|z'\|^{2S'-2}\right)^{-1} dz'}.$$

Lemma 7 *Assume Conditions (A1-A3) and (B1). Then for* $r = 1, 2, ..., 0 \le j < k \le n$ *and all* $x, y \in \mathbb{R}^d$ *it holds*

$$\left|\widetilde{p}_h \otimes_h H_h^{(r)} (jh, kh, x, y)\right| \le \frac{C^{r+1} \rho^r}{\Gamma \left(1 + \frac{r}{2}\right)} \xi_\rho (x - y).$$

Lemma 8 *Assume Conditions (A1-A3) and (B1). Then for* $0 \le j < k \le n$ *and all* $x, y \in \mathbb{R}^d$ *it holds*

$$p_h (jh, kh, x, y) = \sum_{r=0}^{k-j} \left(\widetilde{p}_h \otimes_h (M_h + K_h)^{(r)}\right) (jh, kh, x, y) + R_1 (jh, kh, x, y),$$

where

$$|R_1 (jh, kh, x, y)| \le C h^{1/2} \rho \xi_\rho (y - x).$$

Lemma 9 *Assume Conditions (A1-A3) and (B1). Then for* $0 \le j < k \le n$ *and all* $x, y \in \mathbb{R}^d$ *it holds*

$$p_h (jh, kh, x, y) = \sum_{r=0}^{k-j} \left(\widetilde{p} \otimes_h (M_h + K_h)^{(r)}\right) (jh, kh, x, y) + R_2 (jh, kh, x, y),$$

where

$$|R_2 (jh, kh, x, y)| \le C h^{1/2} \rho^{-1} \xi_\rho (y - x).$$

5. Proof of the main result

From Lemmas 2 and 3 we get as $n \to \infty$ and $T = nh \to 0$

$$(3)\quad p (0, T, x, y) = \sum_{r=0}^{n} \left(\widetilde{p} \otimes H^{(r)}\right) (0, T, x, y) + T^{-d/2} \exp \left(-\frac{C \left\|y - x\right\|^2}{T}\right) o \left(T^n e^{-n}\right).$$

Furthemore, Lemma 9 implies that

$$(4)\qquad p_h (0, T, x, y) = \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h (M_h + K_h)^{(r)}\right) (0, T, x, y) + \xi_{\sqrt{T}} (y - x) O \left(n^{-1/2}\right).$$

Because of (3) and (4) for the statement of the theorem it remains to show that

$$\left|\sum_{r=0}^{n} \left(\widetilde{p} \otimes H^{(r)}\right) (0, T, x, y) - \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h (M_h + K_h)^{(r)}\right) (0, T, x, y)\right|$$
$$(5)\qquad = \xi_{\sqrt{T}} (y - x) O \left(n^{-1/2}\right).$$

For the proof of (5) note that

$$\left|\sum_{r=0}^{n} \left(\widetilde{p} \otimes H^{(r)}\right) (0, T, x, y) - \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h (M_h + K_h)^{(r)}\right) (0, T, x, y)\right|$$
$$(6)\qquad \le\ S_1 + S_2 + S_3,$$

where

$$S_1 = \left| \sum_{r=0}^{n} \left(\widetilde{p} \otimes H^{(r)}\right)(0,T,x,y) - \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h H^{(r)}\right)(0,T,x,y) \right|,$$
$$S_2 = \left| \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h H^{(r)}\right)(0,T,x,y) - \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h (M_h + H)^{(r)}\right)(0,T,x,y) \right|,$$
$$S_3 = \left| \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h (M_h + H)^{(r)}\right)(0,T,x,y) - \sum_{r=0}^{n} \left(\widetilde{p} \otimes_h (M_h + K_h)^{(r)}\right)(0,T,x,y) \right|.$$

For S_1, S_2, S_3 we will show the following estimates

$$S_k = Q_{\sqrt{T}}(y-x)\, O\left(h^{1/2}\right), \; k = 1,2,3. \tag{7}$$

We shall prove (7) for $k = 1$. The term S_1 corresponds to the passage from the continuous time to the lattice time. The fact that $T \to 0$ implies that integrands involved in the convolutions $\otimes$ and $\otimes_h$ become asymptotically degenerate and therefore more accurate estimates are required than those in Konakov and Mammen (2000). We will develop here the details for these bounds.

We start from the recurrence relations for $r = 1, 2, 3, ...$

$$\left(\widetilde{p} \otimes H^{(r)}\right)(0,jh,x,y) - \left(\widetilde{p} \otimes_h H^{(r)}\right)(0,jh,x,y)$$

$$= \left[\left(\widetilde{p} \otimes H^{(r-1)}\right) \otimes H - \left(\widetilde{p} \otimes H^{(r-1)}\right) \otimes_h H\right](0,jh,x,y)$$

$$+ \left[\left(\widetilde{p} \otimes H^{(r-1)}\right) - \left(\widetilde{p} \otimes_h H^{(r-1)}\right)\right] \otimes_h H(0,jh,x,y). \tag{8}$$

By summing up the identities in (8) from $r = 1$ to ∞ and by using the linearity of the operations $\otimes$ and $\otimes_h$ we get

$$\left(p - p^d\right)(0,jh,x,y) = \left(p \otimes H - p \otimes_h H\right)(0,jh,x,y)$$

$$+ (p - p^d) \otimes_h H(0,jh,x,y) \tag{9}$$

where we put

$$p^d(ih, i'h, x, y) = \sum_{r=0}^{\infty} (\widetilde{p} \otimes_h H^{(r)})(ih, i'h, x, y). \tag{10}$$

By iterative application of (9) we obtain

$$\left(p - p^d\right)(0,jh,x,y) = \left(p \otimes H - p \otimes_h H\right)(0,jh,x,y)$$

$$+ \left(p \otimes H - p \otimes_h H\right) \otimes_h \Phi\left(0, jh, x, y\right), \tag{11}$$

where $\Phi(ih, i'h, z, z') = H(ih, i'h, z, z') + H \otimes_h H(ih, i'h, z, z') + ... = \sum_{r=1}^{\infty} H^{(r)}(ih, i'h, z, z')$.

By the Taylor expansion we have

$$\left(p \otimes H - p \otimes_h H\right)(0, jh, x, z)$$

$$= \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} du \int_{R^d} \left[\lambda(u) - \lambda(ih)\right] dv$$

$$= \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u - ih) du \int_{R^d} \lambda'(ih) dv$$

$$+ \frac{1}{2} \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u - ih)^2 \int_0^1 (1 - \delta) \int_{R^d} \lambda''(s) \mid_{s=s_i} dv d\delta du, \tag{12}$$

where $\lambda(u) = p(0, u, x, v) H(u, jh, v, z)$, $s_i = s_i(u, i, \delta, h) = ih + \delta(u - ih)$.

Note that

$$\int_{R^d} \lambda'(ih) dv = \int_{R^d} \frac{\partial}{\partial s} p(0, s, x, v) \mid_{s=ih} H(ih, jh, v, z) dv$$

$$+ \int_{R^d} p(0, ih, x, v) \frac{\partial}{\partial s} H(s, jh, v, z) \mid_{s=ih} dv = \int_{R^d} L^t p(0, ih, x, v)$$

$$\times (L - \widetilde{L}) \widetilde{p}(ih, jh, v, z) dv - \int_{R^d} p(0, ih, x, v) [(L - \widetilde{L}) \widetilde{L} \widetilde{p}(ih, jh, v, z)$$

$$- H_1(ih, jh, v, z)] dv = \int_{R^d} p(0, ih, x, v) H_1(ih, jh, v, z) dv$$

$$+ \int_{R^d} p(0, ih, x, v) (L^2 - 2L\widetilde{L} + \widetilde{L}^2) \widetilde{p}(ih, jh, v, z) dv, \tag{13}$$

where $H_1(s, t, v, z)$ is defined below in (21). We get from (13)

$$\sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u - ih) du \int_{R^d} \lambda'(ih) dv = \frac{h}{2} (p \otimes_h H_1)(0, jh, x, z)$$

$$+ \frac{h}{2} (p \otimes_h A_0)(0, jh, x, z), \tag{14}$$

where $A_0(s, jh, v, z) = (L^2 - 2L\widetilde{L} + \widetilde{L}^2) \widetilde{p}(s, jh, v, z)$. The direct calculation shows that

$$A_0(s, jh, v, z) = \frac{1}{4} \sum_{p,q,r,l=1}^{d} (\sigma_{pq}(s, v) - \sigma_{pq}(s, z))(\sigma_{rl}(s, v) - \sigma_{rl}(s, z))$$

$$\times \frac{\partial^4 \widetilde{p}(s,jh,v,z)}{\partial v_p \partial v_q \partial v_r \partial v_l} + \sum_{p,q,r=1}^{d} (\sigma_{pq}(s,v) - \sigma_{pq}(s,z))(m_r(s,v) - m_r(s,z))$$

$$\times \frac{\partial^3 \widetilde{p}(s,jh,v,z)}{\partial v_p \partial v_q \partial v_r} + \frac{1}{2} \sum_{p,q,r,l=1}^{d} \sigma_{pq}(s,v) \frac{\partial \sigma_{rl}(s,v)}{\partial v_p} \frac{\partial^3 \widetilde{p}(s,jh,v,z)}{\partial v_q \partial v_r \partial v_l} + (\leq 2), \tag{15}$$

where we denote by (≤ 2) the sum of terms containing the derivatives of $\widetilde{p}(s,jh,v,z)$ of the order less or equal than 2. Note that for a constant $C < \infty$ and any $0 < \varepsilon < \frac{1}{2}$

$$\left| \frac{h}{2} (p \otimes_h H_1)(0,jh,x,z) \right| \leq Ch \phi_{C,\sqrt{jh}}(z-x)\,,$$

$$\left| \frac{h}{2} (p \otimes_h A_0)(0,jh,x,z) \right| \leq C(\varepsilon) h^{1/2} j^{-(1/2-\varepsilon)} \phi_{C,\sqrt{jh}}(z-x)\,. \tag{16}$$

First inequality (16) follows from (B1) and the well know estimates for the diffusion density p and for the kernel H_1 . The second inequality (16) follows from (B1), (15) and the following estimate

$$\frac{h}{2} \sum_{i=0}^{j-1} h \left| \int_{R^d} p(0,ih,x,v) \frac{\partial^3 \widetilde{p}(ih,jh,v,z)}{\partial v_q \partial v_r \partial v_l} dv \right| = \frac{h}{2} \sum_{i=0}^{j-1} h \left| \int_{R^d} \frac{\partial p(0,ih,x,v)}{\partial v_q} \right.$$

$$\left. \frac{\partial^2 \widetilde{p}(ih,jh,v,z)}{\partial v_r \partial v_l} dv \right| \leq Ch^{1-\varepsilon} \sum_{i=0}^{j-1} h \frac{1}{\sqrt{ih}} \frac{1}{(jh-ih)^{1-\varepsilon}} \phi_{C,\sqrt{jh}}(z-x) \leq Ch^{1/2}$$

$$\times j^{-(1/2-\varepsilon)} B(\frac{1}{2},\varepsilon) \phi_{C,\sqrt{jh}}(z-x)\,. \tag{17}$$

Now we shall estimate the second summand in the right hand side of (12). Clearly

$$\lambda''(s) = \frac{\partial^2}{\partial s^2} p(0,s,x,v) H(s,jh,v,z) + 2 \frac{\partial}{\partial s} p(0,s,x,v)$$

$$\times \frac{\partial}{\partial s} H(s,jh,v,z) + p(0,s,x,v) \frac{\partial^2}{\partial s^2} H(s,jh,v,z). \tag{18}$$

Using forward and backward Kolmogorov equations we get from (18) after long but simple calculations

$$\frac{1}{2} \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u-ih)^2 \int_0^1 (1-\delta) \int_{R^d} \lambda''(s) \,|_{s=s_i} \, dv d\delta du$$

$$= \frac{1}{2} \sum_{i=0}^{j-1} \int_{ih}^{(i+1)h} (u-ih)^2 \int_0^1 (1-\delta) \sum_{k=1}^{4} \int_{R^d} p(0,s,x,v) A_k(s,jh,v,z) \,|_{s=s_i} \, dv d\delta du, \tag{19}$$

where

$$A_1(s,jh,v,z) = (L^3 - 3L^2\widetilde{L} + 3L\widetilde{L}^2 - \widetilde{L}^3)\widetilde{p}(s,jh,v,z),$$
$$A_2 = (L_1 H + 2LH_1)(s,jh,v,z),$$
$$A_3(s,jh,v,z) = [(L-\widetilde{L})\widetilde{L}_1 + 2(L_1 - \widetilde{L}_1)\widetilde{L}]\widetilde{p}(s,jh,v,z),$$

$$A_4(s,jh,v,z) = H_2(s,jh,v,z). \tag{20}$$

and

$$H_l(s,t,v,z) = (L_l - \widetilde{L}_l)\widetilde{p}(s,t,v,z)$$
$$= \frac{1}{2}\sum_{i,j=1}^{d}\left(\frac{\partial^l \sigma_{ij}(s,v)}{\partial s^l} - \frac{\partial^l \sigma_{ij}(s,z)}{\partial s^l}\right)\frac{\partial^2 \widetilde{p}(s,t,v,z)}{\partial v_i \partial v_j}$$

$$+\sum_{i=1}^{d}\left(\frac{\partial^l m_i(s,v)}{\partial s^l} - \frac{\partial^l m_i(s,z)}{\partial s^l}\right)\frac{\partial \widetilde{p}(s,t,v,z)}{\partial v_i}, \quad l = 1,2. \tag{21}$$

Using integration by parts and the definition (20) of A_2, A_3 and A_4 it is easy to get that for any $0 < \varepsilon < 1/2$ and for $k = 2,3,4$

$$\frac{1}{2}\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d} p(0,s,x,v)A_k(s,jh,v,z)\mid_{s=s_i} dv d\delta du\right|$$

$$\leq C(\varepsilon)h^{3/2-\varepsilon}\phi_{C,\sqrt{jh}}(z-x)\,. \tag{22}$$

For $k = 1$ we shall prove the following estimate for any $0 < \varepsilon < \frac{1}{2}$

$$\frac{1}{2}\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d} p(0,s,x,v)A_1(s,jh,v,z)\mid_{s=s_i} dv d\delta du\right|$$

$$\leq C(\varepsilon)hj^{-(1/2-\varepsilon)}\phi_{C,\sqrt{jh}}(z-x)\,. \tag{23}$$

Note that the function $A_1(s,jh,v,z)$ can be written as the following sum

$$A_1(s,jh,v,z) = \frac{1}{8}\sum_{i,j,p,q,l,r=1}^{d}(\sigma_{ij}(s,v) - \sigma_{ij}(s,z))(\sigma_{pq}(s,v) - \sigma_{pq}(s,z))(\sigma_{lr}(s,v)$$

$$-\sigma_{lr}(s,z))\frac{\partial^6 \widetilde{p}(s,jh,v,z)}{\partial v_i \partial v_j \partial v_p \partial v_q \partial v_l \partial v_r} + \frac{3}{4}\sum_{i,j,p,q,l=1}^{d}(\sigma_{ij}(s,v) - \sigma_{ij}(s,z))(\sigma_{pq}(s,v)$$

$$-\sigma_{pq}(s,z))(m_l(s,v)-m_l(s,z))\frac{\partial^5\widetilde{p}(s,jh,v,z)}{\partial v_i\partial v_j\partial v_p\partial v_q\partial v_l}+\frac{3}{4}\sum_{i,j,p,q,l,r=1}^{d}\sigma_{ij}(s,v)\frac{\partial\sigma_{pq}(s,v)}{\partial v_i}$$

$$(24)\quad (\sigma_{lr}(s,v)-\sigma_{lr}(s,z))\times\frac{\partial^5\widetilde{p}(s,jh,v,z)}{\partial v_j\partial v_p\partial v_q\partial v_l\partial v_r}+(\leq 4),$$

where we denote by (≤ 4) the sum of terms containing the derivatives of $\widetilde{p}(s,jh,v,z)$ of the order less or equal than 4. By (B1) and (24) it is clear that the estimate for the left hand side of (22) for $k=1$ will be the same (up to a constant) as for the following sum for fixed p,q,r,l

$$\frac{1}{2}\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}p(0,s,x,v)\frac{\partial^4\widetilde{p}(s,jh,v,z)}{\partial v_p\partial v_q\partial v_l\partial v_r}\mid_{s=s_i}dvd\delta du\right|$$

After integration by parts w.r.t. v_p and with the substitution $hw=(u-ih)$ in each integral we obtain

$$\frac{1}{2}\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}p(0,s,x,v)\frac{\partial^4\widetilde{p}(s,jh,v,z)}{\partial v_p\partial v_q\partial v_l\partial v_r}\mid_{s=s_i}dvd\delta du\right|$$

$$=\frac{1}{2}\left|\sum_{i=0}^{j-1}\int_{ih}^{(i+1)h}(u-ih)^2\int_0^1(1-\delta)\int_{R^d}\frac{\partial p(0,s,x,v)}{\partial v_p}\frac{\partial^3\widetilde{p}(s,jh,v,z)}{\partial v_q\partial v_l\partial v_r}\mid_{s=s_i}dvd\delta du\right|$$

$$\leq Ch^2\phi_{C,\sqrt{jh}}(z-x)\int_0^1 w^2\int_0^1(1-\delta)\sum_{i=0}^{j-1}h\frac{1}{\sqrt{ih+\delta hw}}\frac{1}{[(j-i)h-\delta hw]^{3/2}}d\delta dw.$$

$$\leq Ch^{3/2-\varepsilon}\phi_{C,\sqrt{jh}}(z-x)\int_0^1 w^2\int_0^1(1-\delta)^{1/2-\varepsilon}\sum_{i=0}^{j-1}h\frac{1}{\sqrt{ih+\delta hw}}\frac{1}{[(j-\delta w)h-ih]^{1-\varepsilon}}d\delta dw$$

$$\leq Ch^{3/2-\varepsilon}\phi_{C,\sqrt{jh}}(z-x)\int_0^1 w^2dw\int_0^1(1-\delta)^{1/2-\varepsilon}d\delta\int_0^{(j-1)h}\frac{dt}{\sqrt{t}[(j-1)h-t]^{1-\varepsilon}}$$

$$(25)\quad \leq Chj^{-(1/2-\varepsilon)}B(\frac{1}{2},\varepsilon)\phi_{C,\sqrt{jh}}(z-x),$$

where $B(p,q)$ is a Beta function and $\phi_{C,\rho}(z-x)$ is defined in Lemma 3 . As we mentioned above (23) follows now from (25). By (12), (14), (16), (22) and (23) we obtain for any $0<\varepsilon<\frac{1}{2}$ and $j=1,2,...n$

$$(26)\qquad |(p\otimes H-p\otimes_h H)(0,jh,x,z)|\leq C(\varepsilon)h^{1/2}j^{-(1/2-\varepsilon)}\phi_{C,\sqrt{jh}}(z-x)$$

We use now the following estimate for $\Phi(ih,i'h,z,z')$ proved in Konakov and Mammen (2002)

$$(27)\qquad |\Phi(ih,i'h,z,z')|\leq C\frac{1}{\sqrt{i'h-ih}}\phi_{C,\sqrt{i'h-ih}}(z'-z)$$

From (11), (26) and (27) we obtain

$$\left|\left(p-p^{d}\right)(0,nh,x,y)\right| \leq C(\varepsilon)h^{1/2}n^{\varepsilon-1/2}\phi_{C,\sqrt{T}}(y-x)\,. \tag{28}$$

The last inequality proves (7) for $k=1$.The terms S_2 and S_3 can be handled in the same way as the terms T_2 and T_3 in Konakov and Mammen (2000). This completes the proof of our main result.

Remark 1. In fact for S_1 we proved stronger result than the estimate (7). We get the following representation

$$\left(p-p^{d}\right)(0,T,x,y)=\frac{h}{2}(p\otimes_h H_1)(0,T,x,y)+\frac{h}{2}(p\otimes_h A_0)(0,T,x,y)$$

$$+\frac{h}{2}\left(p\otimes_h H_1\otimes_h \Phi\right)(0,T,x,y)+\frac{h}{2}(p\otimes_h A_0\otimes_h \Phi)(0,T,x,y)$$

$$+R(0,T,x,y), \tag{29}$$

where for any $0<\varepsilon<1/2$

$$|R(0,T,x,y)|\leq C(\varepsilon)(h^{3/2-\varepsilon}+hn^{-(1/2-\varepsilon)}+h\sqrt{T})\phi_{C,\sqrt{T}}(y-x)$$

$$=C(\varepsilon)\phi_{C,\sqrt{T}}(y-x)\,o(h).$$

This representation is useful to obtain a small time Edgeworth type expansions for our model (see Konakov and Mammen (2005))

Remark 2. If $T=const.$ (without loss of generality we assume $T=1$) than we can easily avoid the difficulties connected with singularity by splitting the time interval [0,1] and by using an integration by parts. For example

$$\int_0^1 du\int_{R^d} p(0,u,x,v)\frac{\partial^4\widetilde{p}(u,1,v,z)}{\partial v_p\partial v_q\partial v_l\partial v_r}dv=\int_0^{1/2}...+\int_{1/2}^1...$$

$$=\int_0^{1/2} du\int_{R^d} p(0,u,x,v)\frac{\partial^4\widetilde{p}(u,1,v,z)}{\partial v_p\partial v_q\partial v_l\partial v_r}dv$$

$$+\int_{1/2}^1 du\int_{R^d}\frac{\partial^4 p(0,u,x,v)}{\partial v_p\partial v_q\partial v_l\partial v_r}\widetilde{p}(u,1,v,z)dv \tag{30}$$

and the derivatives in the right hand side of (30) are not singular. The representation (29) remains true. All summands in the right hand side of (29) are estimated from above in absolute value by $Ch\phi_{C,1}(y-x)$ and for the remainder term $R(0,1,x,y)$ the following estimate holds

$$|R(0,1,x,y)|\leq Ch^2\phi_{C,1}(y-x)\,. \tag{31}$$

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