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EXIT FROM A BASIN OF ATTRACTION FOR STOCHASTIC WEAKLY DAMPED NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider weakly damped nonlinear Schrödinger equations perturbed by a noise of small amplitude. The small noise is either complex and of additive type or real and of multiplicative type. It is white in time and colored in space. Zero is an asymptotically stable equilibrium point of the deterministic equations. We study the exit from a neighborhood of zero, invariant by the flow of the deterministic equation, in L^2 or in H^1. Due to noise, large fluctuations off zero occur. Thus, on a sufficiently large time scale, exit from these domains of attraction occur. A formal characterization of the small noise asymptotic of both the first exit times and the exit points is given.

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1. Introduction

The study of the first exit time from a neighborhood of an asymptotically stable equilibrium point, the exit place determination or the transition between two equilibrium points in randomly perturbed dynamical systems is important in several areas of physics among which statistical and quantum mechanics, chemical reactions, the natural sciences, macroeconomics as to model currency crises or escape in learning models...

For a fixed noise amplitude and for diffusions, the first exit time and the distribution of the exit points on the boundary of a domain can be characterized respectively by the Dirichlet and Poisson equations. However, when the dimension is larger than one, we may seldom solve explicitly these equations and large deviation techniques are precious tools when the noise is assumed to be small; see for example [11, 14]. The techniques used in the physics literature is often called optimal fluctuations or instanton formalism and are closely related to large deviations.

In that case, an energy generally characterizes the transition between two states and the exit from a neighborhood of an asymptotically stable equilibrium point of the deterministic equation. The energy is derived from the rate function of the sample path large deviation principle (LDP). When a LDP holds, the first order of the probability of rare events is that of the Boltzman theory and the square of the amplitude of the small noise acts as the temperature. The deterministic dynamics is sometimes interpreted as the evolution at temperature 0 and the small noise as the small temperature nonequilibrium case. The exit or transition problem

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is then related to a deterministic least-action principle. The paths that minimize the energy, also called minimum action paths, are the most likely exiting paths or transitions. When the infimum is unique, the system has a behavior which is almost deterministic even though there is noise. Indeed, other possible exiting paths, points or transitions are exponentially less probable. In the pioneering article [12], a nonlinear heat equation perturbed by a small noise of additive type is considered. Transitions in that case prove to be the instantons of quantum mechanics. The problem is studied again in [15] where a numerical scheme is presented to compute the optimal paths. In [20], mathematical and numerical predictions for a noisy exit problem are confirmed experimentally.

In this article, we consider the case of weakly damped nonlinear Schrödinger (NLS) equations in $\mathbb{R}^d$. These equations are a generic model for the propagation of the enveloppe of a wave packet in weakly nonlinear and dispersive media. They appear for example in nonlinear optics, hydrodynamics, biology, field theory, crystals Fermi-Pasta-Ulam chains of atoms. The equations are perturbed by a small noise. In optics, the noise corresponds to the spontaneous emission noise due to amplifiers placed along the fiber line in order to compensate for loss, corresponding to the weak damping, in the fiber. We shall consider here that there remains a small weak damping term. In the context of crystals or of Fermi-Pasta-Ulam chains of atoms, the noise accounts for thermal effects. The relevance of the study of the exit from a domain in nonlinear optics is discussed in [19]. The noise is of additive or multiplicative type. We define it as the time derivative in the sense of distributions of a Hilbert space-valued Wiener process $(W_t)_{t\geq 0}$. The evolution equation could be written in Itô form

$$(1) \qquad i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W,$$

where α and ϵ are positive and u_0 is an initial datum in L^2 or H^1. When the noise is of multiplicative type, the product is a Stratonovich product and the equation may be written

$$(2) \qquad i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda |u^{\epsilon,u_0}|^{2\sigma} u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}u^{\epsilon,u_0} \circ \mathrm{d}W.$$

Contrary to the Heat equation the linear part has no smoothing effects. In our case, it defines a linear group which is an isometry on the L^2 based Sobolev spaces. Thus, we cannot treat spatially rough noises and consider colored in space Wiener processes. This latter property is required to obtain *bona-fide* Wiener processes in infinite dimensions. The white noise often considered in Physics seems to give rise to ill-posed problems.

Results on local and global well-posedness and on the effect of a noise on the blow-up phenomenon are proved in [5, 6, 7, 8] in the case $\alpha = 0$. Mixing property and convergence to equilibrium is studied for weakly damped cubic one dimensional equations on a bounded domain in [10]. We consider these equations in the whole space $\mathbb{R}^d$ and assume that the power of the nonlinearity σ satisfies $\sigma < 2/d$. We may check that the above result still hold with the damping term and that for such powers of the nonlinearity the solutions do not exhibit blow-up.

In [16] and [17], we have proved sample paths LDPs for the two types of noises but without damping and deduced the asymptotic of the tails of the blow-up times. In [16], we also deduced the tails of the mass, defined later, of the pulse at the end of a fiber optical line. We have thus evaluated the error probabilities in optical soliton transmission when the receiver records the signal on an infinite time interval. In

[9] we have applied the LDPs to the problem of the diffusion in position of the soliton and studied the tails of the random arrival time of a pulse in optical soliton transmission for noises of additive and multiplicative types.

The flow defined by the above equations can be decomposed in a Hamiltonian, a gradient and a random component. The mass

$$\mathbf{N}(u) = \int_{\mathbb{R}^d} |u|^2 \, dx$$

characterizes the gradient component. The Hamiltonian denoted by $\mathbf{H}(u)$, defined for functions in H^1, has a kinetic and a potential term, it may be written

$$\mathbf{H}(u) = (1/2) \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - (\lambda/(2\sigma+2)) \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, dx.$$

Note that the vector fields associated to the mass and Hamiltonian are orthogonal. We could rewrite, for example equation (1), as

$$\mathrm{d}u^{\epsilon,u_0} = \left(\frac{\delta \mathbf{H}(u^{\epsilon,u_0})}{\delta \overline{u^{\epsilon,u_0}}} - (\alpha/2) \frac{\delta \mathbf{N}(u^{\epsilon,u_0})}{\delta u^{\epsilon,u_0}} \right) \mathrm{d}t - i\sqrt{\epsilon}\mathrm{d}W.$$

Also, the mass and Hamiltonian are invariant quantities of the equation without noise and damping. Other quantities like the linear or angular momentum are also invariant for nonlinear Schrödinger equations.

Without noise, solutions are uniformly attracted to zero in L^2 and in H^1. In this article we study the classical problem of exit from a bounded domain containing zero in its interior and invariant by the deterministic evolution. We prove that the behavior of the random evolution is completely different from the deterministic evolution. Though for finite times the probabilities of large excursions off neighborhoods of zero go to zero exponentially fast with ϵ, if we wait long enough - the time scale is exponential - such large fluctuations occur and exit from a domain takes place. We give two types of results depending on the topology we consider, L^2 or H^1. The L^2-setting is less involved than the H^1-setting. This is due to the structure of the NLS equation and the fact that the L^2 norm is conserved for deterministic non damped equations. We have chosen to also work in H^1 because it is the mathematical framework to study perturbations of solitons; a problem we hope to address in future research.

We give a formal characterization of the small noise asymptotic of the first exit time and exit points. The main tool is a uniform large deviation principle at the level of the paths of the solutions. The behavior of the process is proved to be exponentially equivalent to that of the process starting from a little ball around zero. Thus, if a multiplicative noise and the L^2 topology is considered such balls are invariant by the stochastic evolution as well and the exit problem is not interesting. In infinite dimensions we are faced with two major difficulties. Primarily, the domains under consideration are not relatively compact. In bounded domains of $\mathbb{R}^d$, it is sometimes possible to use compact embedding and the regularizing properties of the semi-group. In [13] where the case of the Heat semi-group and a space variable in a unidimensional torus is treated, these properties are at hand. Also, in [2], the neighborhood is defined for a strong topology of β-Hölder functions and is relatively compact for a weaker topology, the space variable is again in a bounded subset of $\mathbb{R}^d$. We are not able to use the above properties here since the Schrödinger linear group is an isometry on every Sobolev space based on L^2 and we work on the whole space $\mathbb{R}^d$. Another difficulty in infinite dimensions and with unbounded linear operators is that, unlike ODEs, continuity of the linear flow with respect to the initial data holds in a weak sense. The semi-group is strongly continuous and not in general uniformly continuous. We see that we may use other arguments than those used in the finite dimensional setting, some of which are taken from [3], and that the expected results still hold. We are also faced with particular difficulties arising from the nonlinear Schrödinger equation among which the fact that the nonlinearity is locally Lipschitz only in H^1 for $d = 1$. In this purpose, we use the hyper contractivity governed by the Strichartz inequalities which is related to the dispersive properties of the equation.

In this article, we do not address the control problems for the controlled deterministic PDE. We could expect that the upper and lower bound on the expected first exit time are equal and could be written in terms of the usual quasi-potential. The exit points could be related to solitary waves. These issues will be studied in future works.

The article is organized as follows. In the first section, we introduce the main notations and tools, the proof of the uniform large deviation principle is given in the annex. In the next section, we consider the exit off a domain in L^2 for equations with additive noise while in the last section we consider the exit off domains in H^1 for equations with an additive or multiplicative noise.

2. Preliminaries

Throughout the paper the following notations are used.

The set of positive integers and positive real numbers are denoted by $\mathbb{N}^*$ and $\mathbb{R}_+^*$. For $p \in \mathbb{N}^*$, L^p is the Lebesgue space of complex valued functions. For k in $\mathbb{N}^*$, $\mathrm{W}^{k,p}$ is the Sobolev space of L^p functions with partial derivatives up to level k, in the sense of distributions, in L^p. For $p = 2$ and s in $\mathbb{R}_+^*$, H^s is the Sobolev space of tempered distributions v of Fourier transform $\hat{v}$ such that $(1 + |\xi|^2)^{s/2}\hat{v}$ belongs to L^2. We denote the spaces by $\mathrm{L}^p_{\mathbb{R}}$, $\mathrm{W}^{k,p}_{\mathbb{R}}$ and $\mathrm{H}^s_{\mathbb{R}}$ when the functions are real-valued. The space L^2 is endowed with the inner product $(u, v)_{\mathrm{L}^2} = \mathfrak{Re} \int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1, \infty]$, then $\mathrm{L}^r(I; E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$.

The space of linear continuous operators from B into $\tilde{B}$, where B and $\tilde{B}$ are Banach spaces is $\mathcal{L}_c\left(B, \tilde{B}\right)$. When $B = H$ and $\tilde{B} = \tilde{H}$ are Hilbert spaces, such an operator is Hilbert-Schmidt when $\sum_{j\in\mathbb{N}} \|\Phi e_j^H\|^2_{\tilde{H}} < \infty$ for every $(e_j)_{j\in\mathbb{N}}$ complete orthonormal system of H. The set of such operators is denoted by $\mathcal{L}_2(H, \tilde{H})$, or $\mathcal{L}_2^{s,r}$ when $H = \mathrm{H}^s$ and $\tilde{H} = \mathrm{H}^r$. When $H = \mathrm{H}^s_{\mathbb{R}}$ and $\tilde{H} = \mathrm{H}^r_{\mathbb{R}}$, we denote it by $\mathcal{L}_{2,\mathbb{R}}^{s,r}$. When $s = 0$ or $r = 0$ the Hilbert space is L^2 or $\mathrm{L}^2_{\mathbb{R}}$.

We also denote by B_ρ^0 and S_ρ^0 respectively the open ball and the sphere centered at 0 of radius ρ in L^2. We denote these by B_ρ^1 and S_ρ^1 in H^1. We write $\mathcal{N}^0(A, \rho)$ for the $\rho-$neighborhood of a set A in L^2 and $\mathcal{N}^1(A, \rho)$ the neighborhood in H^1. In the following we impose that compact sets satisfy the Hausdorff property.

We use in Lemma 3.6 below the integrability of the Schrödinger linear group which is related to the dispersive property. Recall that $(r(p), p)$ is an admissible pair if p is such that $2 \le p < 2d/(d-2)$ when $d > 2$ ($2 \le p < \infty$ when $d = 2$ and $2 \le p \le \infty$ when $d = 1$) and $r(p)$ satisfies $2/r(p) = d\,(1/2 - 1/p)$.

For every $(r(p),p)$ admissible pair and T positive, we define the Banach spaces

$$Y^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{L}^2\right) \cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{L}^p\right),$$

and

$$X^{(T,p)} = \mathrm{C}\left([0,T];\mathrm{H}^1\right) \cap \mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}\right),$$

where the norms are the maximum of the norms in the two intersected Banach spaces. The Schrödinger linear group is denoted by $(U(t))_{t\geq 0}$; it is defined on L^2 or on H^1. Let us recall the Strichartz inequalities, see [1],

(i) There exists C positive such that for u_0 in L^2, T positive and $(r(p),p)$ admissible pair,

$$\|U(t)u_0\|_{Y^{(T,p)}} \leq C\,\|u_0\|_{\mathrm{L}^2},$$

(ii) For every T positive, $(r(p),p)$ and $(r(q),q)$ admissible pairs, s and ρ such that $1/s+1/r(q)=1$ and $1/\rho+1/q=1$, there exists C positive such that for f in $\mathrm{L}^s\left(0,T;\mathrm{L}^\rho\right)$,

$$\left\|\int_0^\cdot U(\cdot-s)f(s)ds\right\|_{Y^{(T,p)}} \leq C\|f\|_{\mathrm{L}^s(0,T;\mathrm{L}^\rho)}.$$

Similar inequalities hold when the group is acting on H^1, replacing L^2 by H^1, $Y^{(T,p)}$ by $X^{(T,p)}$ and $\mathrm{L}^s\left(0,T;\mathrm{L}^\rho\right)$ by $\mathrm{L}^s\left(0,T;\mathrm{W}^{1,\rho}\right)$.

It is known that, in the Hilbert space setting, only direct images of uncorrelated space wise Wiener processes by Hilbert-Schmidt operators are well defined. However, when the semi-group has regularizing properties, the semi-group may act as a Hilbert-Schmidt operator and a white in space noise may be considered. It is not possible here since the Schrödinger group is an isometry on the Sobolev spaces based on L^2. The Wiener process W is thus defined as ΦW_c, where W_c is a cylindrical Wiener process on L^2 and Φ is Hilbert-Schmidt. Then $\Phi\Phi^*$ is the correlation operator of $W(1)$, it has finite trace.

We consider the following Cauchy problems

$$(3)\qquad \left\{\begin{array}{ll} i\mathrm{d}u^{\epsilon,u_0} & = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W, \\ u^{\epsilon,u_0}(0) & = u_0 \end{array}\right.$$

with u_0 in L^2 and Φ in $\mathcal{L}_2^{0,0}$ or u_0 in H^1 and Φ in $\mathcal{L}_2^{0,1}$, and

$$(4)\qquad \left\{\begin{array}{ll} i\mathrm{d}u^{\epsilon,u_0} & = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0})\mathrm{d}t + \sqrt{\epsilon}u^{\epsilon,u_0}\circ\mathrm{d}W, \\ u^{\epsilon,u_0}(0) & = u_0 \end{array}\right.$$

with u_0 in H^1 and Φ in $\mathcal{L}_{2,\mathbb{R}}^{0,s}$ where $s>d/2+1$. When the noise is of multiplicative type, we may write the equation in terms of a Itô product,

$$i\mathrm{d}u^{\epsilon,u_0} = (\Delta u^{\epsilon,u_0} + \lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0} - i\alpha u^{\epsilon,u_0} - (i\epsilon/2)u^{\epsilon,u_0}F_\Phi)\mathrm{d}t + \sqrt{\epsilon}u^{\epsilon,u_0}\mathrm{d}W,$$

where $F_\Phi(x) = \sum_{j\in\mathbb{N}}\left(\Phi e_j(x)\right)^2$ for x in $\mathbb{R}^d$ and $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2. We consider mild solutions; for example the mild solution of (3) satisfy

$$\begin{array}{rl} u^{\epsilon,u_0}(t) = & U(t)u_0 - i\lambda\int_0^t U(t-s)(|u^{\epsilon,u_0}(s)|^{2\sigma}u^{\epsilon,u_0}(s) - i\alpha u^{\epsilon,u_0}(s))ds \\ & -i\sqrt{\epsilon}\int_0^t U(t-s)dW(s), \quad t>0. \end{array}$$

The Cauchy problems are globally well posed in L^2 and H^1 with the same arguments as in [6].

The main tools in this article are the sample paths LDPs for the solutions of the three Cauchy problems. They are uniform in the initial data. Unlike in [9, 16, 17], we use a Freidlin-Wentzell type formulation of the upper and lower bounds of the LDPs. Indeed, it seems that the restriction that initial data be in compact sets in [17] is a real limitation for stochastic NLS equations. The linear Schrödinger group is not compact due to the lack of smoothing effect and to the fact that we work on the whole space $\mathbb{R}^d$. This limitation disappears when we work with the Freidlin-Wentzell type formulation; we may now obtain bounds for initial data in balls of L^2 or H^1 for ϵ small enough. It is well known that in metric spaces and for non uniform LDPs the two formulations are equivalent. A proof is given in the Annex and we stress, in the multiplicative case, on the slight differences with the proof of the result in [17].

We denote by $\mathbf{S}(u_0, h)$ the skeleton of equation (3) or (4), *i.e.* the mild solution of the controlled equation

$$\begin{cases} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + \Phi h, \\ u(0) = u_0 \end{cases}$$

where u_0 belongs to L^2 or H^1 in the additive case and the mild solution of

$$\begin{cases} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda |u|^{2\sigma} u + u\Phi h, \\ u(0) = u_0 \end{cases}$$

where u_0 belongs to H^1 in the multiplicative case.
The rate functions of the LDPs are always defined as

$$I_T^{u_0}(w) = (1/2) \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2):\ \mathbf{S}(u_0,h)=w} \int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds.$$

We denote for T and a positive by $K_T^{u_0}(a) = \left(I_T^{u_0}\right)^{-1}([0,a])$ the sets

$$K_T^{u_0}(a) = \left\{ w \in \mathrm{C}\left([0,T];\mathrm{L}^2\right) :\ w = \mathbf{S}(u_0,h),\ (1/2)\int_0^T \|h(s)\|_{\mathrm{L}^2}^2 ds \le a \right\}.$$

We also denote by $d_{\mathrm{C}([0,T];\mathrm{L}^2)}$ the usual distance between sets of $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ and by $d_{\mathrm{C}([0,T];\mathrm{H}^1)}$ the distance between sets of $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$.

We write $\tilde{\mathbf{S}}(u_0, f)$ for the skeleton of equation (4) where we replace Φh by $\frac{\partial f}{\partial t}$ where f belongs to $\mathrm{H}_0^1\left(0,T;\mathrm{H}_{\mathbb{R}}^s\right)$, the subspace of $\mathrm{C}\left([0,T];\mathrm{H}_{\mathbb{R}}^s\right)$ of functions that vanishes at zero and whose time derivative is square integrable. Also C_a denotes the set

$$C_a = \left\{ f \in \mathrm{H}_0^1\left(0,T;\mathrm{H}_{\mathbb{R}}^s\right) :\ \frac{\partial f}{\partial t} \in \mathrm{im}\Phi,\ I_T^W(f) = (1/2) \left\| \Phi^{-1}_{|(\ker\Phi)^\perp} \frac{\partial f}{\partial t} \right\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le a \right\}$$

and $\mathcal{A}(d)$ the set $[2,\infty)$ when $d=1$ or $d=2$ and $[2, 2(3d-1)/(3(d-1)))$ when $d \ge 3$. The above I_T^W is the good rate function of the LDP for the Wiener process. The uniform LDP with the Freidlin-Wentzell formulation that we need in the remaining is then as follows. In the additive case we consider the L^2 and H^1 topologies while in the multiplicative case we consider the H^1 topology only. As it has been explained previously we do not consider the L^2 topology for multiplicative noises since then the L^2 norm remains invariant for the stochastic evolution.

Theorem 2.1. *In the additive case and in* L^2 *we have:*
for every a, ρ, T, δ *and* γ *positive,*

(i) there exists ϵ_0 *positive such that for every* ϵ *in* $(0,\epsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2}\le\rho$ *and* $\tilde a$ *in* $(0,a]$,

$$\mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0},K_T^{u_0}(\tilde a)\right)\ge\delta\right)<\exp\left(-(\tilde a-\gamma)/\epsilon\right),$$

(ii) there exists ϵ_0 *positive such that for every* ϵ *in* $(0,\epsilon_0)$, u_0 *such that* $\|u_0\|_{\mathrm{L}^2}\le\rho$ *and* w *in* $K_T^{u_0}(a)$,

$$\mathbb{P}\left(\|u^{\epsilon,u_0}-w\|_{\mathrm{C}([0,T];\mathrm{L}^2)}<\delta\right)>\exp\left(-(I_T^{u_0}(w)+\gamma)/\epsilon\right).$$

In H^1*, the result holds for additive and multiplicative noises replacing in the above* $\|u_0\|_{\mathrm{L}^2}$ *by* $\|u_0\|_{\mathrm{H}^1}$ *and* $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ *by* $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$.

The proof of this result is given in the annex.

Remark 2.2. *The extra condition "For every a positive and K compact in* L^2*, the set* $K_T^K(a)=\bigcup_{u_0\in K}K_T^{u_0}(a)$ *is a compact subset of* $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$*" often appears to be part of a uniform LDP. It is not used in the following.*

3. Exit from a domain of attraction in L^2

3.1. **Statement of the results.** In this section we only consider the case of an additive noise. Recall that for the real multiplicative noise the mass is decreasing and thus exit is impossible.

We may easily check that the mass $\mathbf{N}\left(\mathbf{S}(u_0,0)\right)$ of the solution of the deterministic equation satisfies

$$\mathbf{N}\left(\mathbf{S}(u_0,0)(t)\right)=\mathbf{N}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{5}$$

With noise though, the mass fluctuates around the deterministic decay. Recall how the Itô formula applies to the fluctuation of the mass, see [6] for a proof,

$$\begin{aligned}\mathbf{N}\left(u^{\epsilon,u_0}(t)\right)-\mathbf{N}\left(u_0\right)= & -2\sqrt{\epsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\overline{u}^{\epsilon,u_0}dWdx\\ & -2\alpha\left\|u^{\epsilon,u_0}\right\|^2_{\mathrm{L}^2(0,t;\mathrm{L}^2)}+\epsilon t\|\Phi\|^2_{\mathcal{L}_2^{0,0}}.\end{aligned} \tag{6}$$

We consider domains D which are bounded measurable subsets of L^2 containing 0 in its interior and invariant by the deterministic flow, *i.e.*

$$\forall u_0\in D,\ \forall t\ge 0,\ \mathbf{S}(u_0,0)(t)\in D.$$

It is thus possible to consider balls. There exists R positive such that $D\subset B_R$. We define by

$$\tau^{\epsilon,u_0}=\inf\left\{t\ge 0:\ u^{\epsilon,u_0}(t)\in D^c\right\}$$

the first exit time of the process u^{ϵ,u_0} off the domain D.

An easy information on the exit time is obtained as follows. The expectation of an integration via the Duhamel formula of the Itô decomposition, the process u^{ϵ,u_0} being stopped at the first exit time, gives $\mathbb{E}\left[\exp\left(-2\alpha\tau^{\epsilon,u_0}\right)\right]=1-2\alpha R/\left(\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}\right)$. Without damping we obtain $\mathbb{E}\left[\tau^{\epsilon,u_0}\right]=R/\left(\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}\right)$. To get more precise information for small noises we use LDP techniques.

Let us introduce

$$\overline{e}=\inf\left\{I_T^0(w):\ w(T)\in\overline{D}^c,\ T>0\right\}.$$

When ρ is positive and small enough, we set

$$e_\rho = \inf\left\{I_T^{u_0}(w) : \|u_0\|_{\mathrm{L}^2} \le \rho,\ w(T) \in (D_{-\rho})^c,\ T > 0\right\},$$

where $D_{-\rho} = D \setminus \mathcal{N}^0(\partial D, \rho)$ and ∂D is the the boundary of ∂D in L^2. We define then

$$\underline{e} = \lim_{\rho\to 0} e_\rho.$$

We shall denote in this section by $\|\Phi\|_c$ the norm of Φ as a bounded operator on L^2. Let us start with the following lemma.

Lemma 3.1. *$0 < \underline{e} \le \overline{e}$.*

Proof. It is clear that $\underline{e} \le \overline{e}$. Let us check that $\underline{e} > 0$. Let d denote the positive distance between 0 and ∂D. Take ρ small such that the distance between B_ρ^0 and $(D_{-\rho})^c$ is larger than $d/2$. Multiplying the evolution equation by $-i\overline{\mathbf{S}(u_0,h)}$, taking the real part, integrating over space and using the Duhamel formula we obtain

$$\begin{aligned}&\mathbf{N}\left(\mathbf{S}(u_0,h)(T)\right) - \exp(-2\alpha T)\,\mathbf{N}(u_0)\\ &= 2\int_0^T \exp(-2\alpha(T-s))\,\mathfrak{Im}\left(\int_{\mathbb{R}^d} \overline{\mathbf{S}(u_0,h)}\Phi h dx ds\right).\end{aligned}$$

If $\mathbf{S}(u_0,h)(T) \in (D_{-\rho})^c$ and correspond to the first escape off D then

$$\begin{aligned} d/2 &\le 2\|\Phi\|_c \int_0^T \exp(-2\alpha(T-s))\,\|\mathbf{S}(u_0,h)(s)\|_{\mathrm{L}^2}\,\|h(s)\|_{\mathrm{L}^2} ds\\ &\le 2R\|\Phi\|_c\left(\int_0^T \exp(-4\alpha(T-s))\,ds\right)^{1/2}\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2)},\end{aligned}$$

thus

$$\alpha d^2/\left(8R^2\|\Phi\|_c^2\right) \le \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}/2,$$

and the result follows. □

Remark 3.2. *We would expect $\underline{e}$ and $\overline{e}$ to be equal. We may check that it is enough to prove approximate controllability. The argument is however difficult since we are dealing with noises which are colored space wise, the Schrödinger group does not have global smoothing properties and because of the nonlinearity. If these two bounds were indeed equal, they would also correspond to*

$$\begin{aligned}\mathcal{E}(D) = &\ (1/2)\inf\left\{\|h\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2)} :\ \exists T > 0 :\ \mathbf{S}(0,h)(T) \in \partial D\right\}\\ = &\ \inf_{v\in\partial D} V(0,v)\end{aligned}$$

where the quasi-potential is defined as

$$V(u_0,u_f) = \inf\left\{I_T^{u_0}(w) :\ w \in \mathrm{C}\left([0,\infty);\mathrm{L}^2\right),\ w(0) = u_0,\ w(T) = u_f,\ T > 0\right\}.$$

We prove in this section the two following results. The first theorem characterizes the first exit time from the domain.

Theorem 3.3. *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0} \notin \left(\exp((\underline{e}-\delta)/\epsilon), \exp((\overline{e}+\delta)/\epsilon)\right)\right) \le -L, \tag{7}$$

and for every u_0 in D,

$$\underline{e} \le \underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right) \le \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right) \le \overline{e}. \tag{8}$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0} \ge \exp((\overline{e}+\delta)/\epsilon)\right) \le -L, \tag{9}$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\le \overline{e}. \tag{10}$$

The second theorem characterizes formally the exit points. We shall define for ρ positive small enough, N a closed subset of ∂D

$$e_{N,\rho}=\inf\left\{I_T^{u_0}(w):\|u_0\|_{\mathrm{L}^2}\le\rho,\ w(T)\in\left(D\setminus\mathcal{N}^0\left(N,\rho\right)\right)^c,\ T>0\right\}.$$

We then define

$$\underline{e}_N=\lim_{\rho\to 0}e_{N,\rho}.$$

Note that $e_\rho\le e_{N,\rho}$ and thus $\underline{e}\le\underline{e}_N$.

Theorem 3.4. *If $\underline{e}_N>\overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\le -L.$$

Thus the probability of an escape off D via points of N such that $e_\rho\le e_{N,\rho}$ goes to zero exponentially fast with ϵ.

Suppose that we are able to solve the previous control problem, then as the noise goes to zero, the probability of an exit via closed subsets of ∂D where the quasi-potential is not minimal goes to zero. As the expected exit time is finite, an exit occurs almost surely. It is exponentially more likely that it occurs via infima of the quasi-potential. When there are several infima, the exit measure is a probability measure on ∂D. When there is only one infimum we may state the following corollary.

Corollary 3.5. *Assume that v^* in ∂D is such that for every δ positive and $N=\{v\in\partial D:\ \|v-v^*\|_{\mathrm{L}^2}\ge\delta\}$ we have $\underline{e}_N>\overline{e}$ then*

$$\forall\delta>0,\ \forall u_0\in D,\ \exists L>0:\ \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\left\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)-v^*\right\|_{\mathrm{L}^2}\ge\delta\right)\le -L.$$

3.2. **Preliminary lemmas.** Let us define

$$\sigma_\rho^{\epsilon,u_0}=\inf\left\{t\ge 0: u^{\epsilon,u_0}(t)\in B_\rho^0\cup D^c\right\},$$

where $B_\rho^0\subset D$.

Lemma 3.6. *For every ρ and L positive with $B_\rho^0\subset D$, there exists T and ϵ_0 positive such that for every u_0 in D and ϵ in $(0,\epsilon_0)$,*

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0}>T\right)\le\exp\left(-L/\epsilon\right).$$

Proof. The result is straightforward if u_0 belongs to B_ρ^0. Suppose now that u_0 belongs to $D\setminus B_\rho^0$. From equation (5), the bounded subsets of L^2 are uniformly attracted to zero by the flow of the deterministic equation. Thus there exists a positive time T_1 such that for every u_1 in the $\rho/8-$neighborhood of $D\setminus B_\rho^0$ and $t\ge T_1$, $\mathbf{S}\left(u_1,0\right)(t)\in B_{\rho/8}^0$. We shall choose $\rho<8$ and follow three steps.

Step 1: Let us first recall why there exists $M'=M'(T_1,R,\sigma,\alpha)$ such that

$$\sup_{u_1\in\mathcal{N}^0\left(D\setminus B_\rho^0,\rho/8\right)}\|\mathbf{S}(u_1,0)\|_{Y^{(T_1,2\sigma+2)}}\le M'. \tag{11}$$

From the Strichartz inequalities, there exists C positive such that

$$\begin{aligned}\|\mathbf{S}(u_1,0)\|_{Y^{(t,2\sigma+2)}}\le\ & C\|u_1\|_{\mathrm{L}^2}+C\left\|\left|\mathbf{S}(u_1,0)\right|^{2\sigma+1}\right\|_{\mathrm{L}^{\gamma'}(0,t;\mathrm{L}^{s'})}\\ &+C\alpha\left\|\mathbf{S}\left(u_1,0\right)\right\|_{\mathrm{L}^1(0,t;\mathrm{L}^2)}\end{aligned}$$

where γ' and s' are such that $1/\gamma' + 1/r(\tilde{p}) = 1$ and $1/s' + 1/\tilde{p} = 1$ and $(r(\tilde{p}), \tilde{p})$ is an admissible pair. Note that the first term is smaller than $C(R+1)$. From the Hölder inequality, setting

$$\frac{2\sigma}{2\sigma+2} + \frac{1}{2\sigma+2} = \frac{1}{s'}, \quad \frac{2\sigma}{\omega} + \frac{1}{r(2\sigma+2)} = \frac{1}{\gamma'},$$

we can write

$$\left\| |\mathbf{S}(u_1,0)|^{2\sigma+1} \right\|_{\mathrm{L}^{\gamma'}(0,t;\mathrm{L}^{s'})} \leq C \left\| \mathbf{S}(u_1,0) \right\|_{\mathrm{L}^{r(2\sigma+2)}(0,t;\mathrm{L}^{2\sigma+2})} \left\| \mathbf{S}(u_1,0) \right\|^{2\sigma}_{\mathrm{L}^{\omega}(0,t;\mathrm{L}^{2\sigma+2})}.$$

It is easy to check that since $\sigma < 2/d$, we have $\omega < r(2\sigma+2)$. Thus it follows that

$$\|\mathbf{S}(u_1,0)\|_{Y^{(t,2\sigma+2)}} \leq C(R+1) + Ct^{\frac{\omega r(2\sigma+2)}{r(2\sigma+2)-\omega}} \|\mathbf{S}(u_1,0)\|^{2\sigma+1}_{Y^{(t,2\sigma+2)}} + C\alpha\sqrt{t}\,\|\mathbf{S}(u_1,0)\|_{Y^{(t,2\sigma+2)}}.$$

The function $x \mapsto C(R+1) + Ct^{\frac{\omega r(2\sigma+2)}{r(2\sigma+2)-\omega}} x^{2\sigma+1} + C\alpha\sqrt{t}x - x$ is positive on a neighborhood of zero. For $t_0 = t_0(R,\sigma,\alpha)$ small enough, the function has at least one zero. Also, the function goes to ∞ as x goes to ∞. Thus, denoting by $M(R,\sigma)$ the first zero of the above function, we obtain by a classical argument that $\|\mathbf{S}(u_1,0)\|_{Y^{(t_0,2\sigma+2)}} \leq M(R,\sigma)$ for every u_1 in $\mathcal{N}^0\left(D \setminus B^0_\rho, \rho/8\right)$.
Also, as for every t in $[0,T]$, $\mathbf{S}(u_1,0)(t)$ belongs to $\mathcal{N}^0\left(D \setminus B^0_\rho, \rho/8\right)$, repeating the previous argument, u_1 is replaced by $\mathbf{S}(u_1,0)(t_0)$ and so on, we obtain

$$\sup_{u_1 \in \mathcal{N}^0\left(D\setminus B^0_\rho, \rho/8\right)} \|\mathbf{S}(u_1,0)\|_{Y^{(T_1,p)}} \leq M',$$

where $M' = \lceil T_1/t_0 \rceil M$ proving (11).

Step 2: Let us now prove that for T large enough, to be defined later, and larger than T_1, we have
(12)
$$\mathcal{T}_\rho = \left\{ w \in \mathrm{C}\left([0,T];\mathrm{L}^2\right) : \ \forall t \in [0,T], \ w(t) \in \mathcal{N}^0\left(D \setminus B^0_\rho, \rho/8\right) \right\} \subset K^{u_0}_T(2L)^c.$$

Since $K^{u_0}_T(2L)$ is included in the image of $\mathbf{S}(u_0,\cdot)$ it suffices to consider w in $\mathcal{T}_\rho$ such that $w = \mathbf{S}(u_0,h)$ for some h in $\mathrm{L}^2(0,T;\mathrm{L}^2)$. Take h such that $\mathbf{S}(u_0,h)$ belongs to $\mathcal{T}_\rho$ we have

$$\|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} \geq \|\mathbf{S}(u_0,h)(T_1) - \mathbf{S}(u_0,0)(T_1)\|_{\mathrm{L}^2} \geq 3\rho/4,$$

but also, necessarily, for the admissible pair $(r(2\sigma+2), 2\sigma+2)$,

$$\|\mathbf{S}(u_0,h) - \mathbf{S}(u_0,0)\|_{Y^{(T_1,2\sigma+2)}} \geq 3\rho/4. \tag{13}$$

Denote by $\mathbf{S}^{M'+1}$ the skeleton corresponding to the following control problem

$$\begin{cases} i\left(\frac{\mathrm{d}u}{\mathrm{d}t} + \alpha u\right) = \Delta u + \lambda\theta\left(\frac{\|u\|_{Y^{(t,2\sigma+2)}}}{M'+1}\right)|u|^{2\sigma}u + \Phi h, \\ u(0) = u_1 \end{cases}$$

where θ is a C^∞ function with compact support, such that $\theta(x) = 0$ if $x \geq 2$ and $\theta(x) = 1$ if $0 \leq x \leq 1$. Then (13) implies that

$$\left\| \mathbf{S}^{M'+1}(u_0,h) - \mathbf{S}^{M'+1}(u_0,0) \right\|_{Y^{(T_1,2\sigma+2)}} \geq 3\rho/4.$$

We shall now split the interval $[0,T_1]$ in many parts. We shall denote here by $Y^{s,t,2\sigma+2}$ for $s < t$ the space $Y^{t,2\sigma+2}$ on the interval $[s,t]$. Applying the Strichartz inequalities on a small interval $[0,t]$ with the computations in the proof of Lemma 3.3 in [5], we obtain

$$\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(t,2\sigma+2)} \le C\alpha\sqrt{t}\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(t,2\sigma+2)}$$
$$+C_{M'+1}t^{1-d\sigma/2}\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(t,2\sigma+2)}+C\sqrt{t}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t;\mathrm{L}^2)}$$

where $C_{M'+1}$ is a constant which depends on $M'+1$. Take t_1 small enough such that $C_{M'+1}t_1^{1-d\sigma d/2}+C\alpha\sqrt{t_1}\le 1/2$. We obtain then

$$\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(t_1,2\sigma+2)} \le 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

In the case where $2t_1<T_1$, let us see how such inequality propagates on $[t_1,2t_1]$. We now have two different initial data $\mathbf{S}^{M'+1}(u_0,h)\,(t_1)$ and $\mathbf{S}^{M'+1}(u_0,0)\,(t_1)$. We obtain similarly

$$\begin{aligned}&\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(t_1,2t_1,2\sigma+2)}\\ &\le 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}+2\left\|\mathbf{S}^{M'+1}(u_0,h)\,(t_1)-\mathbf{S}^{M'+1}(u_0,0)\,(t_1)\right\|_{\mathrm{H}^1}\\ &\le 2C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}+2\left\|\mathbf{S}^{M'+1}(u_0,h)\,(t_1)-\mathbf{S}^{M'+1}(u_0,0)\,(t_1)\right\|_{Y(0,t_1,2\sigma+2)}.\end{aligned}$$

Then iterating on each interval of the form $[kt_1,(k+1)t_1]$ for k in $\{1,...,\lfloor T_1/t_1-1\rfloor\}$, the remaining term can be treated similarly, and using the triangle inequality we obtain that

$$\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(T_1,2\sigma+2)} \le 2^{\lceil T_1/t_1\rceil+1}C\sqrt{t_1}\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,t_1;\mathrm{L}^2)}.$$

We may then conclude that

$$\|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}/2\ge M''$$

where $M''=\rho^2/(8C\,(t_1,T_1)\,\|\Phi\|_c^2)$ and $C\,(t_1,T_1)$ is a constant which depends only on t_1 and T_1. Note that we have used for later purposes that $3\rho/2>\rho/2$.

Similarly replacing $[0,T_1]$ by $[T_1,2T_1]$ and u_0 respectively by $\mathbf{S}\,(u_0,h)\,(T_1)$ and $\mathbf{S}\,(u_0,0)\,(T_1)$ in (13), the inequality still holds true. Thus thanks to the inverse triangle inequality we obtain on $[T_1,2T_1]$

$$\begin{aligned}&\left\|\mathbf{S}^{M'+1}(u_0,h)-\mathbf{S}^{M'+1}(u_0,0)\right\|_{Y(T_1,2T_1,2\sigma+2)}\\ &=\left\|\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1),h\right)-\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,0)(T_1),0\right)\right\|_{Y(0,T_1,2\sigma+2)}\\ &\ge 3\rho/4\end{aligned}$$

Thus from the inverse triangle inequality along with the fact that for both $\mathbf{S}^{M'+1}(u_0,h)(T_1)$ and $\mathbf{S}^{M'+1}(u_0,0)(T_1)$ as initial data the deterministic solutions belong to the ball $B^0_{\rho/8}$, we obtain

$$\left\|\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1),h\right)-\mathbf{S}^{M'+1}\left(\mathbf{S}^{M'+1}(u_0,h)(T_1),0\right)\right\|_{Y(0,T_1,2\sigma+2)}\ge\rho/2.$$

We finally obtain the same lower bound

$$\|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)}/2\ge M''$$

as before.

Iterating the argument we obtain if $T>2T_1$,

$$\|h\|^2_{\mathrm{L}^2(0,2T_1;\mathrm{L}^2)}/2=\|h\|^2_{\mathrm{L}^2(0,T_1;\mathrm{L}^2)}/2+\|h\|^2_{\mathrm{L}^2(T_1,2T_1;\mathrm{L}^2)}/2\ge 2M''.$$

Thus for j positive and $T > jT_1$, we obtain, iterating the above argument, that

$$\|h\|^2_{\mathrm{L}^2(0,jT_1;\mathrm{L}^2)}/2 \geq jM''.$$

The result (12) is obtained for $T = jT_1$ where j is such that $jM'' > 2L$.

Step 3: We may now conclude from the (i) of Theorem 2.1 since,

$$\begin{aligned}\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T\right) &= \mathbb{P}\left(\forall t \in [0,T],\ u^{\epsilon,u_0}(t) \in D \setminus B_\rho^0\right)\\ &= \mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0}, \mathcal{T}_\rho^c\right) > \rho/8\right),\\ &\leq \mathbb{P}\left(d_{\mathrm{C}([0,T];\mathrm{L}^2)}\left(u^{\epsilon,u_0}, K_T^{u_0}(2L)\right) \geq \rho/8\right),\end{aligned}$$

taking $a = 2L$, $\rho = R$ where $D \subset B_R$, $\delta = \rho/8$ and $\gamma = L$.

Note that if $\rho \geq 8$, we should replace $R+1$ by $R+\rho/8$ and $M'+1$ by $M'+\rho/8$. Anyway, we will use the lemma for small ρ. □

Lemma 3.7. *For every ρ positive such that $B_\rho^0 \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \leq -L$$

Proof. Take ρ positive satisfying the assumptions of the lemma and take u_0 in D. When u_0 belongs to B_ρ^0 the result is straightforward. Suppose now that u_0 belongs to $D \setminus B_\rho^0$. Let T be defined as

$$T = \inf\left\{t \geq 0 :\ \mathbf{S}(u_0,0)(t) \in B_{\rho/2}^0\right\},$$

then since $\mathbf{S}(u_0,0)([0,T])$ is a compact subset of D, the distance d between $\mathbf{S}(u_0,0)([0,T])$ and D^c is well defined and positive. The conclusion follows then from the fact that

$$\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \leq \mathbb{P}\left(\|u^{\epsilon,u_0} - \mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T];\mathrm{L}^2)} \geq (\rho \wedge d)/2\right),$$

the LDP and the fact that, from the compactness of the sets $K_T^{u_0}(a)$ for a positive, we have

$$\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2):\ \|\mathbf{S}(u_0,h)-\mathbf{S}(u_0,0)\|_{\mathrm{C}([0,T];\mathrm{L}^2)} \geq (\rho\wedge d)/2} \|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} > 0.$$

We have used the fact that the upper bound of the LDP in the Freidlin-Wentzell formulation implies the classical upper bound. Note that this is a well known result for non uniform LDPs. Indeed we do not need a uniform LDP in this proof. □

The following lemma replaces Lemma 5.7.23 in [11]. Indeed, the case of a stochastic PDE is more intricate than that of a SDE since the linear group is only strongly and not uniformly continuous. However, it is possible to prove that the group on L^2 when acting on bounded sets of H^1 is uniformly continuous. We shall proceed in a different manner and thus we do not loose in regularity.

Lemma 3.8. *For every ρ and L positive such that $B_{2\rho}^0 \subset D$, there exists $T(L,\rho) < \infty$ such that*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in S_\rho^0} \mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]} \left(\mathbf{N}\left(u^{\epsilon,u_0}(t)\right) - \mathbf{N}(u_0)\right) \geq 3\rho^2\right) \leq -L$$

Proof. Take L and ρ positive. Note that for every ϵ in $(0,\epsilon_0)$ where $\epsilon_0 = \rho^2/\|\Phi\|^2_{\mathcal{L}_2^{0,0}}$, for $T(L,\rho) \leq 1$ we have $\epsilon T(L,\rho)\|\Phi\|^2_{\mathcal{L}_2^{0,0}} < \rho^2$. Thus from equation (6), we know that it is enough to prove that there exists $T(L,\rho)\le 1$ such that for ϵ_1 small enough, $\epsilon_1<\epsilon_0$, and all $\epsilon<\epsilon_0$,

$$\epsilon\log\sup_{u_0\in S_\rho^0}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\left(-2\sqrt{\epsilon}\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\overline{u}^{\epsilon,u_0,\tau}dWdx\right)\ge 2\rho^2\right)\le -L,$$

where $u^{\epsilon,u_0,\tau}$ is the process u^{ϵ,u_0} stopped at $\tau^{\epsilon,u_0}_{S^0_{2\rho}}$, the first time when u^{ϵ,u_0} hits $S^0_{2\rho}$. Setting $Z(t)=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\overline{u}^{\epsilon,u_0,\tau}dWdx$, it is enough to show that

$$\epsilon\log\sup_{u_0\in S_\rho^0}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)|\ge\rho^2/\sqrt{\epsilon}\right)\le -L,$$

and thus to show exponential tail estimates for the process $Z(t)$. Our proof now follows closely that of [21][Theorem 2.1]. We introduce the function $f_l(x)=\sqrt{1+lx^2}$, where l is a positive parameter. We now apply the Itô formula to $f_l(Z(t))$ and the process decomposes into $1+E_l(t)+R_l(t)$ where

$$E_l(t)=\int_0^t\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}dZ(t)-(1/2)\int_0^t\left(\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}\right)^2 d<Z>_t,$$

and

$$R_l(t)=(1/2)\int_0^t\left(\frac{2lZ(t)}{\sqrt{1+lZ(t)^2}}\right)^2 d<Z>_t+\int_0^t\frac{l}{(1+lZ(t)^2)^{3/2)}}d<Z>_t\ .$$

Moreover, given $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2,

$$<Z(t)>=\int_0^t\sum_{j\in\mathbb{N}}\left(u^{\epsilon,u_0,\tau},-i\Phi e_j\right)^2_{\mathrm{L}^2}(s)ds,$$

we prove with the Hölder inequality that $|R_l(t)|\le 12l\rho^2\|\Phi\|^2_{\mathcal{L}_2^{0,0}}t$, for every u_0 in D. We may thus write

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)|\ge\rho^2/\sqrt{\epsilon}\right)\\&=\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\exp\left(f_l(Z(t))\right)\ge\exp\left(f_l\left(\rho^2/\sqrt{\epsilon}\right)\right)\right)\\&\le\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}\exp\left(E_l(t)\right)\ge\exp\left(f_l\left(\rho^2/\sqrt{\epsilon}\right)-1-12l\rho^2\|\Phi\|^2_{\mathcal{L}_2^{0,0}}T(L,\rho)\right)\right).\end{aligned}$$

The Novikov condition is also satisfied and $E_l(t)$ is such that $\left(\exp\left(E_l(t)\right)\right)_{t\in\mathbb{R}^+}$ is a uniformly integrable martingale. The exponential tail estimates follow from the Doob inequality optimizing on the parameter l. We may then write

$$\sup_{u_0\in S_\rho^0}\mathbb{P}\left(\sup_{t\in[0,T(L,\rho)]}|Z(t)|\ge\rho^2/\sqrt{\epsilon}\right)\le 3\exp\left(-\frac{\rho^2}{48\epsilon\|\Phi\|^2_{\mathcal{L}_2^{0,0}}T(L,\rho)}\right).$$

We now conclude setting $T(L,\rho)=\rho^2/\left(50\|\Phi\|^2_{\mathcal{L}_2^{0,0}}L\right)$ and choosing $\epsilon_1<\epsilon_0$ small enough. $\square$

3.3. **Proof of Theorem 3.3 and Theorem 3.4.** We first prove Theorem 3.3.
Proof of Theorem 3.3. Let us first prove (10) and deduce (9). Fix δ positive and choose h and T_1 such that $\mathbf{S}(0,h)(T_1) \in \overline{D}^c$ and

$$I_{T_1}^0\left(\mathbf{S}(0,h)\right) = (1/2)\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)} \le \overline{e} + \delta/5.$$

Let d_0 denote the positive distance between $\mathbf{S}(0,h)\left(T_1\right)$ and $\overline{D}$. With similar arguments as in [6] or with a truncation argument we may prove that the skeleton is continuous with respect to the initial datum for the L^2 topology. Thus there exists ρ positive, a function of h which has been fixed, such that if u_0 belongs to B_ρ^0 then

$$\|\mathbf{S}\left(u_0,h\right) - \mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} < d_0/2.$$

We may assume that ρ is such that $B_\rho^0 \subset D$. From the triangle inequality and the (ii) of Theorem 2.1, there exists ϵ_1 positive such that for all ϵ in $(0,\epsilon_1)$ and u_0 in B_ρ^0,

$$\begin{array}{ll}\mathbb{P}\left(\tau^{\epsilon,u_0} < T_1\right) & \ge \mathbb{P}\left(\|u^{\epsilon,u_0} - \mathbf{S}(0,h)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} < d_0\right) \\ & \ge \mathbb{P}\left(\|u^{\epsilon,u_0} - \mathbf{S}\left(u_0,h\right)\|_{\mathrm{C}([0,T_1];\mathrm{L}^2)} < d_0/2\right) \\ & \ge \exp\left(-(I_{T_1}^{u_0}\left(\mathbf{S}(u_0,h)\right) + \frac{\delta}{5})/\epsilon\right).\end{array}$$

From Lemma 3.6, there exists T_2 and ϵ_2 positive such that for all ϵ in $(0,\epsilon_2)$,

$$\inf_{u_0\in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} \le T_2\right) \ge 1/2.$$

Thus, for $T = T_1 + T_2$, from the strong Markov property we obtain that for all $\epsilon < \epsilon_3 < \epsilon_1 \wedge \epsilon_2$.

$$\begin{array}{ll} q = \inf_{u_0\in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \le T\right) & \ge \inf_{u_0\in D} \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} \le T_2\right) \inf_{u_0\in B_\rho^0} \mathbb{P}\left(\tau^{\epsilon,u_0} \le T_1\right) \\ & \ge (1/2)\exp\left(-\left(I_{T_1}^{u_0}\left(\mathbf{S}(u_0,h)\right) + \delta/5\right)/\epsilon\right) \\ & \ge \exp\left(-\left(I_{T_1}^{u_0}\left(\mathbf{S}(u_0,h)\right) + 2\delta/5\right)/\epsilon\right).\end{array}$$

Thus, for any $k \ge 1$, we have

$$\begin{array}{ll}\mathbb{P}\left(\tau^{\epsilon,u_0} > (k+1)T\right) & = \left[1 - \mathbb{P}\left(\tau^{\epsilon,u_0} \le (k+1)T | \tau^{\epsilon,u_0} > kT\right)\right]\mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right) \\ & \le (1-q)\mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right) \\ & \le (1-q)^k.\end{array}$$

We may now compute, since $I_{T_1}^{u_0}\left(\mathbf{S}\left(u_0,h\right)\right) = I_{T_1}^0\left(\mathbf{S}\left(0,h\right)\right) = (1/2)\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2)}$

$$\begin{array}{ll}\sup_{u_0\in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right) & = \sup_{u_0\in D} \int_0^\infty \mathbb{P}\left(\tau^{\epsilon,u_0} > t\right) dt \\ & \le T\left[1 + \sum_{k=1}^\infty \sup_{x\in D} \mathbb{P}\left(\tau^{\epsilon,u_0} > kT\right)\right] \\ & \le T/q \\ & \le T\exp\left((\overline{e} + 3\delta/5)/\epsilon\right).\end{array}$$

It implies that there exists ϵ_4 small enough such that for ϵ in $(0,\epsilon_4)$,

$$\sup_{u_0\in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right) \le \exp\left((\overline{e} + 4\delta/5)/\epsilon\right). \tag{14}$$

Thus the Chebychev inequality gives that

$$\sup_{u_0\in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \ge \exp\left((\overline{e}+\delta)/\epsilon\right)\right) \le \exp\left(-(\overline{e}+\delta)/\epsilon\right) \sup_{u_0\in D} \mathbb{E}\left(\tau^{\epsilon,u_0}\right),$$

in other words

$$\sup_{u_0\in D} \mathbb{P}\left(\tau^{\epsilon,u_0} \ge \exp\left((\overline{e}+\delta)/\epsilon\right)\right) \le \exp\left(-\delta/(5\epsilon)\right). \tag{15}$$

Relations (14) and (15) imply (10) and (9).

Let us now prove the lower bound on τ^{ϵ,u_0}. Take δ positive. Remind that we have proved that $\underline{e} > 0$. Take ρ positive small enough such that $\underline{e} - \delta/4 \leq e_\rho$ and $B^0_{2\rho} \subset D$. We define the following sequences of stopping times, $\theta_0 = 0$ and for k in $\mathbb{N}$,

$$\begin{array}{ll} \tau_k & = \inf\left\{t \geq \theta_k : \ u^{\epsilon,u_0}(t) \in B^0_\rho \cup D^c\right\}, \\ \theta_{k+1} & = \inf\left\{t > \tau_k : \ u^{\epsilon,u_0}(t) \in S^0_{2\rho}\right\}, \end{array}$$

where $\theta_{k+1} = \infty$ if $u^{\epsilon,u_0}(\tau_k) \in \partial D$. Fix $T_1 = T\left(\underline{e} - 3\delta/4, \rho\right)$ given in Lemma 3.8. We know that there exists ϵ_1 positive such that for all ϵ in $(0, \epsilon_1)$, for all $k \geq 1$ and u_0 in D,

$$\mathbb{P}\left(\theta_k - \tau_{k-1} \leq T_1\right) \leq \exp\left(-(\underline{e} - 3\delta/4)/\epsilon\right).$$

For u_0 in D and an m in $\mathbb{N}^*$, we have

$$\begin{array}{rl} \mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \leq & \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_0\right) + \sum_{k=1}^m \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \\ & +\mathbb{P}\left(\exists k \in \{1,...,m\} : \ \theta_k - \tau_{k-1} \leq T_1\right) \\ = & \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_0\right) + \sum_{k=1}^m \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \\ & + \sum_{k=1}^m \mathbb{P}\left(\theta_k - \tau_{k-1} \leq T_1\right). \end{array} \tag{16}$$

In other words the escape before mT_1 can occur either as an escape without passing in the small ball B^0_ρ (if u_0 belongs to $D \setminus B^0_\rho$) or as an escape with k in $\{1, ...m\}$ significant fluctuations off B^0_ρ, *i.e.* crossing $S^0_{2\rho}$, or at least one of the m first transitions between S^0_ρ and $S^0_{2\rho}$ happens in less than T_1. The latter is known to be arbitrarily small. Let us prove that the remaining probabilities are small enough for small ϵ.

For every $k \geq 1$ and T_2 positive, we may write

$$\mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \leq \mathbb{P}\left(\tau^{\epsilon,u_0} \leq T_2; \tau^{\epsilon,u_0} = \tau_k\right) + \mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_2\right).$$

Fix T_2 as in Lemma 3.6 with $L = \underline{e} - 3\delta/4$. Thus there exists ϵ_2 small enough such that for ϵ in $(0, \epsilon_2)$,

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T_2\right) \leq \exp\left(-(\underline{e} - 3\delta/4)/\epsilon\right).$$

Also, from the (i) of Theorem 2.1, we obtain that there exists ϵ_3 positive such that for every u_1 in B^0_ρ and ϵ in $(0, \epsilon_3)$,

$$\begin{array}{rl} \mathbb{P}\left(\tau^{\epsilon,u_1} \leq T_2\right) & \leq \mathbb{P}\left(d_{C([0,T_2];L^2)}\left(u^{\epsilon,u_1}, K^{u_1}_{T_2}\left(e_\rho - \delta/4\right)\right) \geq \rho\right) \\ & \leq \exp\left(-(e_\rho - \delta/2)/\epsilon\right) \\ & \leq \exp\left(-(\underline{e} - 3\delta/4)/\epsilon\right). \end{array}$$

Thus the above bound holds for $\mathbb{P}\left(\tau^{\epsilon,u_0} \leq T_2; \tau^{\epsilon,u_0} = \tau_k\right)$ replacing u_1 by $u^{\epsilon,u_0}\left(\tau_{k-1}\right)$ since as $k \geq 1$, $u^{\epsilon,u_0}\left(\tau_{k-1}\right)$ belongs to B^0_ρ and $\tau_k - \tau_{k-1} \leq T_2$ and using the Markov property. The inequality (16) gives that for all ϵ in $(0, \epsilon_0)$ where $\epsilon_0 = \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$,

$$\mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + 3m\exp\left(-(\underline{e} - 3\delta/4)/\epsilon\right).$$

Fix $m = \lceil (1/T_1)\exp\left((\underline{e} - \delta)/\epsilon\right) \rceil$, then for all ϵ in $(0, \epsilon_0)$,

$$\begin{array}{rl} \mathbb{P}\left(\tau^{\epsilon,u_0} \leq \exp\left((\underline{e} - \delta)/\epsilon\right)\right) & \leq \mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \\ & \leq \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) + (3/T_1)\exp\left(-\delta/(4\epsilon)\right). \end{array}$$

We may now conclude with Lemma 3.7 and obtain the expected lower bound on $\mathbb{E}\left(\tau^{\epsilon,u_0}\right)$ from the Chebychev inequality. □

Let us now prove Theorem 3.4.

Proof of Theorem 3.4. Let N be closed subset of ∂D. When $\underline{e}_N = \infty$ we shall replace in the proof that follows $\underline{e}_N$ by an increasing sequence of positive numbers.

Take δ such that $0<\delta<(\underline{e}_N-\overline{e})/3$, ρ positive such that $\underline{e}_N-\delta/3\leq e_{N,\rho}$ and $B^0_{2\rho}\subset D$. Define the same sequences of stopping times $(\tau_k)_{k\in\mathbb{N}}$ and $(\theta_k)_{k\in\mathbb{N}}$ as in the proof of Theorem 3.3.
Take $L=\underline{e}_N-\delta$ and T_1 and $T_2=T(L,\rho)$ as in Lemma 3.6 and 3.8. Thanks to Lemma 3.6 and the uniform LDP, with a computation similar to the one following inequality (16), we obtain that for ϵ_0 small enough and $\epsilon\leq\epsilon_0$,

$$\begin{array}{l}\sup_{u_0\in S^0_{2\rho}}\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma^{\epsilon,u_0}_\rho\right)\in N\right)\\ \leq\sup_{u_0\in S^0_{2\rho}}\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma^{\epsilon,u_0}_\rho\right)\in N,\sigma^{\epsilon,u_0}_\rho\leq T_1\right)+\sup_{u_0\in S^0_{2\rho}}\mathbb{P}\left(\sigma^{\epsilon,u_0}_\rho>T_1\right)\\ \leq\sup_{u_0\in B^0_{2\rho}}\mathbb{P}\left(d_{\mathrm{C}([0,T_1];\mathrm{L}^2)}\left(u^{\epsilon,u_0},K^{u_0}_{T_1}\left(e_{N,\rho}-\delta/3\right)\right)\geq\rho\right)\\ \quad+\sup_{u_0\in D}\mathbb{P}\left(\sigma^{\epsilon,u_0}_\rho>T_1\right)\\ \leq 2\exp\left(-(\underline{e}_N-\delta)/\epsilon\right).\end{array}$$

Possibly choosing ϵ_0 smaller, we may assume that for every positive integer l and every $\epsilon\leq\epsilon_0$,

$$\begin{array}{rl}\sup_{u_0\in D}\mathbb{P}\left(\tau_l\leq lT_2\right)\leq & l\sup_{u_0\in S^0_\rho}\mathbb{P}\left(\sup_{t\in[0,T_2]}\left(\mathbf{N}\left(u^{\epsilon,u_0}(t)\right)-\mathbf{N}\left(u_0\right)\right)\geq\rho\right)\\ \leq & l\exp\left(-(\underline{e}_N-\delta)/\epsilon\right).\end{array}$$

Thus if u_0 belongs to B^0_ρ

$$\begin{array}{rl}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq & \mathbb{P}\left(\tau^{\epsilon,u_0}>\tau_l\right)+\sum_{k=1}^l\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N,\tau^{\epsilon,u_0}=\tau_k\right)\\ \leq & \mathbb{P}\left(\tau^{\epsilon,u_0}>lT_2\right)+\mathbb{P}\left(\tau_l\leq lT_2\right)\\ & +l\sup_{u_0\in S^0_{2\rho}}\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma^{\epsilon,u_0}_\rho\right)\in N\right)\\ \leq & \mathbb{P}\left(\tau^{\epsilon,u_0}>lT_2\right)+3l\exp\left(-(\underline{e}_N-\delta)/\epsilon\right).\end{array}$$

Take now $l=\lceil(1/T_2)\exp\left((\overline{e}+\delta)/\epsilon\right)\rceil$ and use the upper bound (15), possibly choosing ϵ_0 smaller, we obtain that for $\epsilon\leq\epsilon_0$

$$\begin{array}{rl}\sup_{u_0\in B^0_\rho}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq & \exp\left(-\delta/(5\epsilon)\right)+(4/T_2)\exp\left(-(\underline{e}_N-\overline{e}+2\delta)/\epsilon\right)\\ \leq & \exp\left(-\delta/(5\epsilon)\right)+(4/T_2)\exp\left(-\delta/\epsilon\right).\end{array}$$

Finally, when u_0 is any function in D, we conclude thanks to

$$\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq\mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma^{\epsilon,u_0}_\rho\right)\in\partial D\right)+\sup_{u_0\in B^0_\rho}\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)$$

and to Lemma 3.7. □

Remark 3.9. *It is proposed in* [22] *to introduce control elements to reduce or enhance exponentially the expected exit time or to act on the exiting points, for a limited cost. We could then optimize on these external fields. However, the problem is computationally involved since the optimal control problem requires double optimisation.*

4. Exit from a domain of attraction in H^1

4.1. **Preliminaries.** We now consider a measurable bounded subset D of H^1 invariant by the flow of the deterministic equation and which contains zero in its interior. We choose R such that $D\subset B^1_R$. We consider both (3) and (4) where the noise is either of additive or of multiplicative type. In this section we are interested in both the fluctuation of the L^2 norm and that of the L^2 norm of the gradient. The Hamiltonian and a modified Hamiltonian are thus of particular interest. We first distinguish the case where the nonlinearity is defocusing ($\lambda=-1$) where the

Hamiltonian takes non negative values from the case where the nonlinearity is focusing ($\lambda = 1$) where the Hamiltonian may take negative values.

We may prove, see for example [18], that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}\left(\mathbf{S}(u_0,0)(t)\right) + 2\alpha\mathbf{\Psi}\left(\mathbf{S}(u_0,0)\right) = 0,$$

where $\mathbf{S}(u_0,0)$ is the solution of the deterministic weakly damped nonlinear Schrödinger equation with initial datum u_0 in H^1 and

$$\mathbf{\Psi}\left(\mathbf{S}(u_0,0)\right) = \|\nabla\mathbf{S}(u_0,0)\|_{\mathrm{L}^2}^2/2 - \lambda\int_{\mathbb{R}^d}|\mathbf{S}(u_0,0)(x)|^{2\sigma+2}\,dx/2.$$

Thus, when the nonlinearity is defocusing we have

$$0 \le \mathbf{H}\left(\mathbf{S}(u_0,0)(t)\right) \le \mathbf{H}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{17}$$

As it is done in [10], we consider in the focusing case a modified Hamiltonian denoted by $\tilde{\mathbf{H}}(u)$ defined for u in H^1 by

$$\tilde{\mathbf{H}}(u) = \mathbf{H}(u) + \beta(\sigma,d)C\,\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)}$$

where the constant C is that of the third inequality in the following sequence of inequalities where we use the Gagliardo-Nirenberg inequality

$$\|u\|_{\mathrm{L}^{2\sigma+2}}^{2\sigma+2}/(2\sigma+2) \le C\|u\|_{\mathrm{L}^2}^{2\sigma+2-\sigma d}\|\nabla u\|_{\mathrm{L}^2}^{\sigma d} \le \|\nabla u\|_{\mathrm{L}^2}^2/4 + C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)},$$

and $\beta(\sigma,d) = \frac{2\sigma(2-\sigma d)}{(\sigma+2)(2-\sigma d)+2\sigma(4\sigma+3)} \vee 2$. When evaluated at the deterministic solution, the modified Hamiltonian satisfies

$$0 \le \tilde{\mathbf{H}}\left(\mathbf{S}(u_0,0)(t)\right) \le \tilde{\mathbf{H}}\left(u_0\right)\exp\left(-2\alpha\frac{3(\sigma+1)}{4\sigma+3}t\right). \tag{18}$$

Also, when the nonlinearity is defocusing we now have, for every β positive,

$$0 \le \tilde{\mathbf{H}}\left(\mathbf{S}(u_0,0)(t)\right) \le \tilde{\mathbf{H}}\left(u_0\right)\exp\left(-2\alpha t\right). \tag{19}$$

From the Sobolev inequalities, for ρ positive, the sets

$$\tilde{\mathbf{H}}_\rho = \left\{u \in \mathrm{H}^1 : \ \tilde{\mathbf{H}}(u) = \rho\right\} = \tilde{\mathbf{H}}^{-1}\left(\{\rho\}\right), \quad \rho > 0$$

are closed subsets of H^1 and

$$\tilde{\mathbf{H}}_{<\rho} = \left\{u \in \mathrm{H}^1 : \ \tilde{\mathbf{H}}(u) < \rho\right\} = \tilde{\mathbf{H}}^{-1}\left([0,\rho)\right) \quad \rho > 0$$

are open subsets of H^1.

Also, $\tilde{\mathbf{H}}$ is such that

$$\|\nabla u\|_{\mathrm{L}^2}^2/2 + \beta C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \le \tilde{\mathbf{H}}(u) \le 3\|\nabla u\|_{\mathrm{L}^2}^2/4 + (\beta+1)C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \tag{20}$$

when the nonlinearity is defocusing and

$$\|\nabla u\|_{\mathrm{L}^2}^2/4 + C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \le \tilde{\mathbf{H}}(u) \le \|\nabla u\|_{\mathrm{L}^2}^2/2 + \beta(\sigma,d)C\|u\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)} \tag{21}$$

when it is focusing. Thus the sets $\tilde{\mathbf{H}}_{<\rho}$ for ρ positive are bounded in H^1 and a bounded set in H^1 is bounded for $\tilde{\mathbf{H}}$. Note that the domain D of attraction may be a domain of the form $\tilde{\mathbf{H}}_{<\rho}$.

We no longer distinguish the focusing and defocusing cases and take the same value of β, *i.e.* $\beta(\sigma,d)$. Also to simplify the notations we now sometimes drop the dependence of the solution in ϵ and u_0.

The fluctuation of $\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right)$ is of particular interest. We have the following result when the noise is of additive type.

Proposition 4.1. *When u denotes the solution of equation (3), $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2, the following decomposition holds*

$$\begin{aligned}\tilde{\mathbf{H}}(u(t)) = \ & \tilde{\mathbf{H}}(u_0)\\ & -2\alpha\int_0^t \mathbf{\Psi}(u(s))\,ds - 2\beta C\left(1+2\sigma/(2-\sigma d)\right)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)}ds\\ & +\sqrt{\epsilon}\left(\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \nabla\overline{u}(s)\nabla dW(s)dx - \lambda\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t |u(s)|^{2\sigma}\,\overline{u}(s)dW(s)dx\right.\\ & \qquad\left.+2\beta C\left(1+2\sigma/(2-\sigma d)\right)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\,\overline{u}(s)dW(s)dx\right)\\ & -(\lambda\epsilon/2)\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}\left[|u(s)|^{2\sigma}\,|\Phi e_j|^2 + 2\sigma\,|u(s)|^{2\sigma-2}\left(\mathfrak{Re}(\overline{u}(s)\Phi e_j)\right)^2\right]dxds\\ & +(\epsilon/2)\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2 t + \epsilon\beta C\left(1+2\sigma/(2-\sigma d)\right)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2\int_0^t \|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}ds\\ & +\epsilon\beta C\left(4\sigma/(2-\sigma d)\right)\left(1+2\sigma/(2-\sigma d)\right)\sum_{j\in\mathbb{N}}\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2(2\sigma/(2-\sigma d)-1)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_j dx\right)^2 ds\end{aligned}$$

Proof. The result follows from the Itô formula. The main difficulty is in justifying the computations. We may proceed as in [6]. □

Also, when the noise is of multiplicative type we obtain the following proposition.

Proposition 4.2. *When u denotes the solution of equation (4), $(e_j)_{j\in\mathbb{N}}$ a complete orthonormal system of L^2, the following decomposition holds*

$$\begin{aligned}\tilde{\mathbf{H}}(u(t)) = \ & \tilde{\mathbf{H}}(u_0)\\ & -2\alpha\int_0^t \mathbf{\Psi}(u(s))\,ds - 2\beta C\left(1+2\sigma/(2-\sigma d)\right)\alpha\int_0^t \|u(s)\|_{\mathrm{L}^2}^{2+4\sigma/(2-\sigma d)}ds\\ & +\sqrt{\epsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t u(s)\nabla\overline{u}(s)\nabla dW(s)dx\\ & +(\epsilon/2)\sum_{j\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2 dxds.\end{aligned}$$

The first exit time τ^{ϵ,u_0} from the domain D in H^1 is defined as in Section 2. We also define

$$\overline{e} = \inf\left\{I_T^0(w):\ w(T)\in\overline{D}^c,\ T>0\right\},$$

and for ρ positive small enough

$$e_\rho = \inf\left\{I_T^{u_0}(w):\ \tilde{\mathbf{H}}(u_0)\le\rho,\ w(T)\in\left(D_{-\rho}\right)^c,\ T>0\right\},$$

where $D_{-\rho} = D\setminus\mathcal{N}^1(\partial D,\rho)$. Then we set

$$\underline{e} = \lim_{\rho\to 0} e_\rho.$$

Also, for ρ positive small enough, N a closed subset of the boundary of D, we define

$$e_{N,\rho} = \inf\left\{I_T^{u_0}(w):\ \tilde{\mathbf{H}}(u_0)\le\rho,\ w(T)\in\left(D\setminus\mathcal{N}^1(N,\rho)\right)^c,\ T>0\right\}$$

and

$$\underline{e}_N = \lim_{\rho\to 0} e_{N,\rho}.$$

We finally also introduce

$$\sigma_\rho^{\epsilon,u_0} = \inf\left\{t\ge 0: u^{\epsilon,u_0}(t)\in\tilde{\mathbf{H}}_{<\rho}\cup D^c\right\},$$

where $\tilde{\mathbf{H}}_{<\rho}\subset D$.

Again we have the following inequalities.

Lemma 4.3. $0<\underline{e}\le\overline{e}$.

Proof. We only have to prove the first inequality. Integrating the equation describing the evolution of $\tilde{\mathbf{H}}\left(\mathbf{S}\left(u_0,h\right)(t)\right)$ via the Duhamel formula where the skeleton is that of the equation with an additive noise we obtain

$$\begin{aligned}
&\tilde{\mathbf{H}}\left(\mathbf{S}(u_0,h)(T)\right)-\exp\left(-2\alpha\tfrac{3(\sigma+1)}{4\sigma+3}T\right)\tilde{\mathbf{H}}\left(u_0\right)\\
&\leq\int_0^T\exp\left(-2\alpha\tfrac{3(\sigma+1)}{4\sigma+3}(T-s)\right)\left[\mathfrak{Im}\int_{\mathbb{R}^d}\left(\nabla\mathbf{S}(u_0,h)\nabla\overline{\Phi h}\right)(s,x)dx\right.\\
&\quad-\lambda\mathfrak{Im}\int_{\mathbb{R}^d}\left(|\mathbf{S}(u_0,h)|^{2\sigma}\mathbf{S}(u_0,h)\overline{\Phi h}\right)(s,x)dx\\
&\quad\left.-2C\beta\left(1+2\sigma/(2-\sigma d)\right)\mathfrak{Im}\int_{\mathbb{R}^d}\left(\mathbf{S}\left(u_0,h\right)\overline{\Phi h}\right)(s,x)dx\right]ds,
\end{aligned}$$

with a focusing or defocusing nonlinearity. Let d denote the positive distance between 0 and ∂D. Take ρ such that the distance between B_ρ^1 and $\left(D_{-\rho}\right)^c$ is larger than $d/2$. We then have, from the fact that the Sobolev injection from H^1 into $\mathrm{L}^{2\sigma+2}$,

$$\begin{aligned}
d/2\leq&\int_0^T\exp\left(-2\alpha\tfrac{3(\sigma+1)}{4\sigma+3}(T-s)\right)\left[R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2}\right.\\
&+CR^{2\sigma+1}\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{H}^1)}\|h\|_{\mathrm{L}^2}\\
&\left.+2C\beta\left(1+2\sigma/(2-\sigma d)\right)R\|\Phi\|_{\mathcal{L}_c(\mathrm{L}^2,\mathrm{L}^2)}\|h\|_{\mathrm{L}^2}\right]ds,
\end{aligned}$$

We conclude as in Lemma 3.1 and use that from the choice of β the complementary of a ball is included in the complementary of a set $\tilde{\mathbf{H}}_{<a}$. In the case of the skeleton of the equation with a multiplicative noise, it is enough to replace the term in bracket in the right hand side of the above formula by $\mathfrak{Im}\int_{\mathbb{R}^d}\left(\nabla\mathbf{S}\left(u_0,h\right)\overline{\mathbf{S}\left(u_0,h\right)\nabla\Phi h}\right)(s,x)dx$. Recall that we can proceed as in the additive case since we have imposed that Φ belongs to $\mathcal{L}_{2,\mathbb{R}}^{0,s}$ where $s>d/2+1$, in particular Φ belongs to $\mathcal{L}_c\left(\mathrm{L}^2,\mathrm{W}^{1,\infty}\right)$. □

4.2. **Statement of the results.** The theorems of Section 2 still hold for a domain of attraction in H^1 and a noise of additive and multiplicative type.

Theorem 4.4. *For every u_0 in D and δ positive, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(\tau^{\epsilon,u_0}\notin\left(\exp\left((\underline{e}-\delta)/\epsilon\right),\exp\left((\overline{e}+\delta)/\epsilon\right)\right)\right)\leq-L,\tag{22}$$

and for every u_0 in D,

$$\underline{e}\leq\underline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{e}.\tag{23}$$

Moreover, for every δ positive, there exists L positive such that

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\sup_{u_0\in D}\mathbb{P}\left(\tau^{\epsilon,u_0}\geq\exp\left((\overline{e}+\delta)/\epsilon\right)\right)\leq-L,\tag{24}$$

and

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\sup_{u_0\in D}\mathbb{E}\left(\tau^{\epsilon,u_0}\right)\leq\overline{e}.\tag{25}$$

Remark 4.5. *Again the control argument to prove that $\underline{e}=\overline{e}$ seems difficult. It should be even more difficult for multiplicative noises.*

Theorem 4.6. *If $\underline{e}_N>\overline{e}$, then for every u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)\in N\right)\leq-L.$$

Again we may deduce the corollary

Corollary 4.7. *Assume that v^* in ∂D is such that for every δ positive and $N=\{v\in\partial D:\ \|v-v^*\|_{\mathrm{L}^2}\geq\delta\}$ we have $\underline{e}_N>\overline{e}$ then*

$$\forall\delta>0,\ \forall u_0\in D,\ \exists L>0:\ \overline{\lim}_{\epsilon\to0}\epsilon\log\mathbb{P}\left(\|u^{\epsilon,u_0}\left(\tau^{\epsilon,u_0}\right)-v^*\|_{\mathrm{L}^2}\geq\delta\right)\leq-L.$$

4.3. **Proof of the results.** The proof of these results still rely on three lemmas and the uniform LDP. Let us now state the lemmas for both a noise of additive and of multiplicative type.

Lemma 4.8. *For every ρ and L positive with $\tilde{\mathbf{H}}_{<\rho} \subset D$, there exists T and ϵ_0 positive such that for every u_0 in D and ϵ in $(0, \epsilon_0)$,*

$$\mathbb{P}\left(\sigma_\rho^{\epsilon,u_0} > T\right) \leq \exp\left(-L/\epsilon\right).$$

Proof. We proceed as in the proof of Lemma 3.6.
Let d denote the positive distance between 0 and $D \setminus \tilde{\mathbf{H}}_{<\rho}$. Take α positive such that $\alpha\rho < d$. The domain D is uniformly attracted to 0, thus there exists a time T_1 such that for every initial datum u_1 in $\mathcal{N}^1\left(D \setminus \tilde{\mathbf{H}}_{<\rho}, \alpha\rho/8\right)$, for $t \geq T_1$, $\mathbf{S}\left(u_1, 0\right)(t)$ belongs to $B^1_{\alpha\rho/8}$.

We could also prove, see [6], that there exists a constant M' which depends on T_1, R, σ and α such that

$$\sup_{u_1 \in \mathcal{N}^1\left(D \setminus \tilde{\mathbf{H}}_{<\rho}, \alpha\rho/8\right)} \left\|\mathbf{S}\left(u_1, 0\right)\right\|_{X^{(T_1, 2\sigma+2)}} \leq M'. \tag{26}$$

The Step 2, corresponding to that of Lemma 3.6, in the proof in the additive case uses the truncation argument, upper bounds similar to that in [6] derived from the Strichartz inequalities on smaller intervals; we shall also replace in the proof of Lemma 3.6 $\rho/8$ by $\alpha\rho/8$.

In Step 2 for the multiplicative case, we also introduce the truncation in front of the term $u\Phi h$ in the controlled PDE.

The end of the proof is identical to that of Lemma 3.6, the LDP is the LDP in $\mathrm{C}\left([0,T]; \mathrm{H}^1\right)$, for additive or multiplicative noises. □

Lemma 4.9. *For every ρ positive such that $\tilde{\mathbf{H}}_\rho \subset D$ and u_0 in D, there exists L positive such that*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(u^{\epsilon,u_0}\left(\sigma_\rho^{\epsilon,u_0}\right) \in \partial D\right) \leq -L$$

Proof. It is the same proof as for Lemma 3.7. We only have to replace $B^0_{\rho/2}$ by any ball in H^1 centered at 0 and included in $\tilde{\mathbf{H}}_{<\rho}$ and use the LDP in $\mathrm{C}\left([0,T]; \mathrm{H}^1\right)$.□

Lemma 4.10. *For every ρ and L positive such that $\tilde{\mathbf{H}}_{2\rho} \subset D$, there exists $T(L,\rho) < \infty$ such that*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \sup_{u_0 \in \tilde{\mathbf{H}}_\rho} \mathbb{P}\left(\sup_{t \in [0, T(L,\rho)]} \left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \tilde{\mathbf{H}}\left(u_0\right)\right) \geq \rho\right) \leq -L$$

Proof. Integrating the Itô differential relation using the Duhamel formula allows to get rid of the drift term that is not originated from the bracket. Indeed, the event

$$\left\{\sup_{t \in [0, T(L,\rho)]} \left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \tilde{\mathbf{H}}\left(u_0\right)\right) \geq \rho\right\}$$

is included in

$$\left\{\sup_{t \in [0, T(L,\rho)]} \left(\tilde{\mathbf{H}}\left(u^{\epsilon,u_0}(t)\right) - \exp\left(-2\alpha\left(\frac{3(\sigma+1)}{4\sigma+3}\right) T(L,\rho)\right) \tilde{\mathbf{H}}\left(u_0\right)\right) \geq \rho\right\}.$$

Then, setting $c(\sigma)=\frac{3(\sigma+1)}{4\sigma+3}$ and $m(\sigma,d)=1+2\sigma/(2-\sigma d)$, dropping the exponents ϵ and u_0 to have more concise formulas, we obtain in the additive case

$$\begin{aligned}
&\tilde{\mathbf{H}}\,(u(t))-\exp\left(-2\alpha c(\sigma)t\right)\tilde{\mathbf{H}}\,(u_0)\\
&\le\sqrt{\epsilon}\Big(\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\nabla\overline{u}(s)\nabla dW(s)dx\\
&-\lambda\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)|u(s)|^{2\sigma}\,\overline{u}(s)dW(s)dx\\
&+2\beta Cm(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\,\overline{u}(s)dW(s)dx\Big)\\
&-(\lambda\epsilon/2)\sum_{j\in\mathbb{N}}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\int_{\mathbb{R}^d}\Big[|u(s)|^{2\sigma}\,|\Phi e_j|^2\\
&\qquad\qquad+2\sigma\,|u(s)|^{2\sigma-2}\left(\mathfrak{Re}(\overline{u}(s)\Phi e_j)\right)^2\Big]\,dxds\\
&+\left(\epsilon/(4\alpha c(\sigma))\right)\left(1-\exp\left(-2\alpha c(\sigma)t\right)\right)\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2\\
&+\epsilon\beta Cm(\sigma,d)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}ds\\
&+\epsilon\beta Cm(\sigma,d)\left(4\sigma/(2-\sigma d)\right)\sum_{j\in\mathbb{N}}\int_0^t\exp\left(-2\alpha c(\sigma)(t-s)\right)\|u(s)\|_{\mathrm{L}^2}^{2(2\sigma/(2-\sigma d)-1)}\left(\mathfrak{Re}\int_{\mathbb{R}^d}\overline{u}(s)\Phi e_jdx\right)^2ds.
\end{aligned}$$

We again use a localization argument and replace the process u by the process u^τ stopped at the first exit time off $\tilde{\mathbf{H}}_{<2\rho}$. We use (20) and (21) and obtain

$$\|u^\tau\|_{\mathrm{H}^1}^2\le 8\rho+\left(2\rho/(C\sigma)\right)^{\frac{1}{1+2\sigma/(2-\sigma d)}}.$$

We denote the right hand side of the above by $b(\rho,\sigma,d)$.
From the Hölder inequality along with the Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$ we obtain the following upper bound for the drift

$$\begin{aligned}
&\left(\epsilon/(4\alpha c(\sigma))\right)\left[(1+2\sigma)c(1,2\sigma+2)^{2\sigma+2}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2b(\rho,\sigma,d)^{2\sigma}+\|\nabla\Phi\|_{\mathcal{L}_2^{0,0}}^2\right]\\
&+m(\sigma,d)\left(\epsilon\beta C/(2\alpha c(\sigma))\right)\left(1+4\sigma/(2-\sigma d)\right)\|\Phi\|_{\mathcal{L}_2^{0,0}}^2b(\rho,\sigma,d)^{4\sigma/(2-\sigma d)}
\end{aligned}$$

where we denote by $c(1,2\sigma+2)$ the norm of the continuous injection of H^1 into $\mathrm{L}^{2\sigma+2}$.
Thus, choosing ϵ small enough, it is enough to show the result for the stochastic integral replacing ρ by $\rho/2$. Also it is enough to show the result for each of the three stochastic integrals replacing $\rho/2$ by $\rho/6$. With the same one parameter families and similar computations as in the proof of Lemma 3.8, we know that it is enough to obtain upper bounds of the brackets of the stochastic integrals

$$\begin{aligned}
Z_1(t)&=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)\nabla\overline{u^\tau}(s)\nabla dW(s)dx\\
Z_2(t)&=\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)|u^\tau(s)|^{2\sigma}\,\overline{u^\tau}(s)dW(s)dx\\
Z_3(t)&=2\beta Cm(\sigma,d)\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t\exp\left(2\alpha c(\sigma)s\right)\|u^\tau(s)\|_{\mathrm{L}^2}^{4\sigma/(2-\sigma d)}\,\overline{u^\tau}(s)dW(s)dx.
\end{aligned}$$

We then obtain

$$\begin{aligned}
&d<Z_1>_t\le\exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(\nabla u^\tau(t),-i\nabla\Phi e_j\right)_{\mathrm{L}^2}^2dt\\
&d<Z_2>_t\le\exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(|u^\tau(t)|^{2\sigma}u^\tau(t),-i\Phi e_j\right)_{\mathrm{L}^2}^2dt\\
&d<Z_3>_t\le 4\beta^2C^2m(\sigma,d)^2\exp\left(4\alpha c(\sigma)t\right)\|u^\tau(t)\|_{\mathrm{L}^2}^{8\sigma/(2-\sigma d)}\sum_{j\in\mathbb{N}}\left(u^\tau(t),-i\Phi e_j\right)_{\mathrm{L}^2}^2dt.
\end{aligned}$$

Using the Hölder inequality and, for Z_2, the continuous Sobolev injection of H^1 into $\mathrm{L}^{2\sigma+2}$ we obtain

$$\begin{aligned}
&d<Z_1>_t\le\exp\left(4\alpha c(\sigma)t\right)\|\Phi\|_{\mathcal{L}_2^{0,1}}^2b(\rho,\sigma,d)dt\\
&d<Z_2>_t\le\exp\left(4\alpha c(\sigma)t\right)c(1,2\sigma+2)^{2(2\sigma+2)}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2b(\rho,\sigma,d)^{2\sigma+1}dt\\
&d<Z_3>_t\le 4\beta^2C^2m(\sigma,d)^2\exp\left(4\alpha c(\sigma)t\right)b(\rho,\sigma,d)^{(1+4\sigma/(2-\sigma d))}\|\Phi\|_{\mathcal{L}_2^{0,1}}^2dt.
\end{aligned}$$

We can then bound each of the three remainders $\left(R_l^i(t)\right)_{i=1,2,3}$ similar to that of Lemma 3.8 using the inequality $R_l^i(t) \leq 3l \int_0^t d < Z_i >_t$.
We conclude that it is possible to choose $T(L,\rho)$ equal to

$$\frac{1}{4\alpha c(\sigma)} \log\left(\frac{\alpha c(\sigma)\rho^2}{90 b(\rho,\sigma,d)\|\Phi\|^2_{\mathcal{L}_2^{0,1}} \max\left(1, c(1,2\sigma+2)^{2(2\sigma+1)} b(\rho,\sigma,d)^{2\sigma}, 4\beta^2 C^2 m(\sigma,d)^2 b(\rho,\sigma,d)^{4\sigma/(2-\sigma d)}\right)}\right).$$

When the noise is of multiplicative type we obtain

$$\begin{array}{l} \tilde{\mathbf{H}}\left(u(t)\right) - \exp\left(-2\alpha c(\sigma)t\right)\tilde{\mathbf{H}}\left(u_0\right) \\ \leq \sqrt{\epsilon}\,\mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right) u(s)\nabla\overline{u}(s)\nabla dW(s)dx \\ \quad +(\epsilon/2)\sum_{j\in\mathbb{N}}\int_0^t \exp\left(-2\alpha c(\sigma)(t-s)\right)\int_{\mathbb{R}^d}|u(s)|^2|\nabla\Phi e_j|^2 dxds. \end{array}$$

Again we use a localization argument and consider the process u stopped at the exit off $\tilde{\mathbf{H}}_{2\rho}$. As Φ is Hilbert-Schmidt from L^2 into $\mathrm{H}^s_{\mathbb{R}}$, the second term of the right hand side is less than $\frac{\epsilon}{4\alpha c(\sigma)}\|\Phi\|^2_{\mathcal{L}_2^{0,s}} b(\rho,\sigma,d)$ and for ϵ small enough, it is enough to prove the result for the stochastic integral replacing ρ by $\rho/2$. We know that it is enough to obtain an upper bound of the bracket of

$$Z(t) = \mathfrak{Im}\int_{\mathbb{R}^d}\int_0^t \exp\left(2\alpha c(\sigma)s\right) u^\tau(s)\nabla\overline{u}^\tau(s)\nabla dW(s)dx.$$

We obtain

$$d < Z >_t \leq \exp\left(4\alpha c(\sigma)t\right)\sum_{j\in\mathbb{N}}\left(\nabla u^\tau(t), -iu^\tau(t)\nabla\Phi e_j\right)^2_{\mathrm{L}^2} dt.$$

Denoting by $c(s,\infty)$ the norm of the Sobolev injection of $\mathrm{H}^s_{\mathbb{R}}$ into $\mathrm{W}^{1,\infty}_{\mathbb{R}}$ we deduce that

$$d < Z >_t \leq \exp\left(4\alpha c(\sigma)t\right) c(s,\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}} b(\rho,\sigma,d)^2 dt.$$

Finally, we conclude that we may choose

$$T(L,\rho) = \frac{1}{4\alpha c(\sigma)}\log\left(\frac{\alpha c(\sigma)\rho^2}{10 b(\rho,\sigma,d)^2 c(s,\infty)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}} L}\right).$$

□

We may now prove Theorem 22 and 23.
Elements for the proof of Theorem 22. There is no difference in the proof of the upper bound on τ^{ϵ,u_0}. Let us thus focus on the lower bound. Take δ positive. Since $\underline{e} > 0$, we now choose ρ positive such that $\underline{e} - \delta/4 \leq e_\rho$, $\tilde{\mathbf{H}}_{2\rho} \subset D$ and $\tilde{\mathbf{H}}_{2\rho} \subset D^c_{-\rho}$. We define the sequences of stopping times $\theta_0 = 0$ and for k in $\mathbb{N}$,

$$\begin{aligned} \tau_k &= \inf\left\{t \geq \theta_k :\ u^{\epsilon,u_0}(t) \in \tilde{\mathbf{H}}_{<\rho} \cup D^c\right\}, \\ \theta_{k+1} &= \inf\left\{t > \tau_k :\ u^{\epsilon,u_0}(t) \in \tilde{\mathbf{H}}_{2\rho}\right\}, \end{aligned}$$

where $\theta_{k+1} = \infty$ if $u^{\epsilon,u_0}(\tau_k) \in \partial D$. Let us fix $T_1 = T\left(\underline{e} - 3\delta/4, \rho\right)$ given by Lemma 4.10. We now use that for u_0 in D and m a positive integer,

$$\begin{array}{rl} \mathbb{P}\left(\tau^{\epsilon,u_0} \leq mT_1\right) \leq & \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_0\right) + \sum_{k=1}^m \mathbb{P}\left(\tau^{\epsilon,u_0} = \tau_k\right) \\ & +\sum_{k=1}^m \mathbb{P}\left(\theta_k - \tau_{k-1} \leq T_1\right) \end{array} \qquad (27)$$

and conclude as in the proof of Theorem 3.3. □

We may check that the proof of Theorem 3.4 also applies to Theorem 4.6, the LDPs are those in H^1 and the sequences of stopping times are those defined above.

Remark 4.11. *In* [13]*, reaction-diffusion equations perturbed by an additive white noise are considered. When the space dimension is larger than one, the case where the vector field can decomposed in a gradient and a second field which is orthogonal is treated. The quasi-potential is then equal to the potential at the end point. It again involves a control argument. In our case, since we consider colored noises and nonlinear equations, the orthogonality is lost for the geometry of the reproducing kernel Hilbert space of the law of $W(1)$. We thus obtain extra commutator terms. Under suitable assumptions on the space correlations of the noise, going to zero, it is possible that we obtain a non trivial minimisation problem. Recall that solitary waves are solutions of variational problem where we minimize the Hamiltonian for fixed levels of the mass.*

5. Annex - proof of Theorem 2.1

The following lemma is at the core of the proof of the uniform LDPs. It is often called Azencott lemma or Freidlin-Wentzell inequality. The differences with the result of [17] are that here the initial data are the same for the random process and the skeleton and that the "for every ρ positive" stands before "there exists ϵ_0 and γ positive". We shall only stress on the differences in the proof.

Lemma 5.1. *For every a, L, T, δ and ρ positive, f in C_a, p in $\mathcal{A}(d)$, there exists ϵ_0 and γ positive such that for every ϵ in $(0, \epsilon_0)$, $\|u_0\|_{\mathrm{H}^1} \leq \rho$,*

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \geq \delta;\ \|\sqrt{\epsilon}W - f\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma\right) \leq -L.$$

Elements of proof. There are still three steps in the proof of this result. The first step is a change of measure to center the process around f. It uses the Girsanov theorem and is the same as in [17].
The second step is a reduction to estimates for the stochastic convolution. It strongly involves the Strichartz inequalities but it is slightly different than in [17]. The truncation argument has to hold for all $\|u_0\|_{\mathrm{H}^1} \leq \rho$. Thus we use the fact that there exists $M = M(T, \rho, \sigma)$ positive such that

$$\sup_{u_1 \in B^1_\rho} \left\|\tilde{\mathbf{S}}(u_1, f)\right\|_{X^{(T,p)}} \leq M.$$

The proof of this fact follows from the computations in [6], we have recalled the arguments in L^2 in the proof of Lemma 3.6. The result in H^1 is again be used in the proof of Lemma 4.8. As the initial data are the same for the random process and the skeleton, the remaining of the argument does not require restrictions on ρ.
The third step corresponds to estimates for the stochastic convolution. It is the same as in [17].
The extra damping term in the drift is treated easily thanks to the Strichartz inequalities. □

Elements for the proof of Theorem 2.1. Let us start with the case of an additive noise. Recall that, in that case, the mild solution of the stochastic equation could be written as a function of the perturbation in the convolution form. Let

$v^{u_0}(Z)$ denote the solution of

$$\left\{\begin{array}{l} i\frac{\partial v}{\partial t}-\left(\Delta v+|v-iZ|^{2\sigma}(v-iZ)-i\alpha(v-iZ)\right)=0,\\ v(0)=u_0,\end{array}\right.$$

or equivalently a fixed point of the functional $\mathcal{F}_Z$ such that

$$\begin{array}{rl}\mathcal{F}_Z(v)(t)= & U(t)u_0-i\lambda\int_0^t U(t-s)\left(|(v-iZ)(s)|^{2\sigma}(v-iZ)(s)\right)ds\\ & -\alpha\int_0^t U(t-s)(v-iZ)(s)ds,\end{array}$$

where Z belongs to $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$). If u^{ϵ,u_0} is defined as $u^{\epsilon,u_0}=v^{u_0}\left(Z^\epsilon\right)-iZ^\epsilon$ where Z^ϵ is the stochastic convolution $Z^\epsilon(t)=\sqrt{\epsilon}\int_0^t U(t-s)dW(s)$ then u^{ϵ,u_0} is a solution of the stochastic equation. Consequently, if $\mathcal{G}\left(\cdot,u_0\right)$ denotes the mapping from $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$) to $\mathrm{C}\left([0,T];\mathrm{L}^2\right)$ (respectively $\mathrm{C}\left([0,T];\mathrm{H}^1\right)$) defined by $\mathcal{G}\left(Z,u_0\right)=v^{u_0}(Z)-iZ$, we obtain $u^{\epsilon,u_0}=\mathcal{G}\left(Z^\epsilon,u_0\right)$. We may also check with arguments similar to that of [6, 16], involving the Strichartz inequalities that the mapping $\mathcal{G}$ is equicontinuous in its first arguments for second arguments in bounded sets of L^2 (respectively H^1). The result now follows from Proposition 5 in [23].

Let us now consider the case of a multiplicative noise. Initial data belong to H^1 and we consider paths in H^1. The proof is very close to that in [17].

The main tool is again the Azencott lemma or almost continuity of the Itô map. We need the slightly different result from that in [17].
Let us see how the above lemma implies (i) and (ii).

We start with the upper bound (i). Take a, ρ, T and δ positive. Take $L>a$. For $\tilde{a}$ in $(0,a]$, we denote by

$$A_{\tilde{a}}^{u_0}=\left\{v\in\mathrm{C}\left([0,T];\mathrm{H}^1\right):\ d_{\mathrm{C}([0,T];\mathrm{H}^1)}\left(v,K_T^{u_0}(\tilde{a})\right)\geq\delta\right\}.$$

Note that we have $A_a^{u_0}\subset A_{\tilde{a}}^{u_0}$ and $C_{\tilde{a}}\subset C_a$. Take $\tilde{a}\in(0,a]$ and f such that $I_T^W(f)<\tilde{a}$.

We shall now apply the Azencott lemma and choose $p=2$. We obtain $\epsilon_{\rho,f,\delta}$ and $\gamma_{\rho,f,\delta}$ positive such that for every $\epsilon\leq\epsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1}\leq\rho$,

$$\epsilon\log\mathbb{P}\left(\left\|u^{\epsilon,u_0}-\tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}}\geq\delta;\left\|\sqrt{\epsilon}W-f\right\|_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)}<\gamma_{\rho,f,\delta}\right)\leq -L.$$

Let us denote by $O_{\rho,f,\delta}$ the set $O_{\rho,f,\delta}=B_{\mathrm{C}\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)}(f,\gamma_{\rho,f,\delta})$. The family $\left(O_{\rho,f,\delta}\right)_{f\in C_a}$ is a covering by open sets of the compact set C_a, thus there exists a finite subcovering of the form $\bigcup_{i=1}^N O_{\rho,f_i,\delta}$. We can now write

$$\begin{array}{rl}\mathbb{P}\left(u^{\epsilon,u_0}\in A_{\tilde{a}}^{u_0}\right)\leq & \mathbb{P}\left(\left\{u^{\epsilon,u_0}\in A_{\tilde{a}}^{u_0}\right\}\cap\left\{\sqrt{\epsilon}W\in\bigcup_{i=1}^N O_{\rho,f_i,\delta}\right\}\right)\\ & +\mathbb{P}\left(\sqrt{\epsilon}W\notin\bigcup_{i=1}^N O_{\rho,f_i,\delta}\right)\\ \leq & \sum_{i=1}^N\mathbb{P}\left(\left\{u^{\epsilon,u_0}\in A_{\tilde{a}}^{u_0}\right\}\cap\left\{\sqrt{\epsilon}W\in O_{\rho,f_i,\delta}\right\}\right)\\ & +\mathbb{P}\left(\sqrt{\epsilon}W\notin C_a\right)\\ \leq & \sum_{i=1}^N\mathbb{P}\left(\left\{\left\|u^{\epsilon,u_0}-\tilde{\mathbf{S}}(u_0,f)\right\|_{X^{(T,p)}}\geq\delta\right\}\cap\left\{\sqrt{\epsilon}W\in O_{\rho,f_i,\delta}\right\}\right)\\ & +\exp\left(-a/\epsilon\right),\end{array}$$

for $\epsilon\leq\epsilon_0$ for some ϵ_0 positive. We used that

$$d_{\mathrm{C}([0,T];\mathrm{H}^1)}\left(\tilde{\mathbf{S}}(u_0,f),A_{\tilde{a}}^{u_0}\right)\geq\delta,$$

which is a consequence of the definition of the sets $A_{\tilde{a}}^{u_0}$.

As a consequence, for $\epsilon \le \epsilon_0 \wedge (\min_{i=1,..,N} \epsilon_{u_0,f_i})$ we obtain for u_0 in B_ρ^1,

$$\mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \le \ N \exp\left(-L/\epsilon\right) + \exp\left(-a/\epsilon\right),$$

and for ϵ_1 small enough, for every $\epsilon \in (0, \epsilon_1)$,

$$\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \le \epsilon \log 2 + (\epsilon \log N - L) \vee (-a).$$

If ϵ_1 is also chosen such that $\epsilon_1 < \frac{\gamma}{\log(2)} \wedge \frac{L-a}{\log(N)}$ we obtain

$$\epsilon \log \mathbb{P}\left(u^{\epsilon,u_0} \in A_{\tilde{a}}^{u_0}\right) \le -\tilde{a} - \gamma,$$

which holds for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$.

We consider now the lower bound (ii). Take a, ρ, T and δ positive. The continuity of $\tilde{\mathbf{S}}(u_0, \cdot)$, to be proved as in [17], along with the compactness of C_a give that for u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$ and w in $K_T^{u_0}(a)$, there exists f such that $w = \tilde{\mathbf{S}}(u_0, f)$ and $I_T^{u_0}(w) = I_T^W(f)$. Take $L > I^{u_0}(w)$. Choose $\epsilon_{\rho,f,\delta}$ positive and $O_{\rho,f,\delta}$, the ball centered at f of radius $\gamma_{\rho,f,\delta}$ defined as previously, such that for every $\epsilon \le \epsilon_{\rho,f,\delta}$ and u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$,

$$\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \ge \delta; \left\|\sqrt{\epsilon} W - f\right\|_{C\left([0,T];\mathrm{H}^s_{\mathbb{R}}\right)} < \gamma_{\rho,f,\delta}\right) \le -L.$$

We obtain

$$\begin{aligned} \exp\left(-I_T^W(f)/\epsilon\right) \ &\le \mathbb{P}\left(\sqrt{\epsilon} W \in O_{\rho,f,\delta}\right) \\ &\le \mathbb{P}\left(\left\{\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} \ge \delta\right\} \cap \left\{\sqrt{\epsilon} W \in O_{\rho,f,\delta}\right\}\right) \\ &\quad + \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} < \delta\right). \end{aligned}$$

Thus, for $\epsilon \le \epsilon_{\rho,f,\delta}$, for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$,

$$-I^{u_0}(w) \le \epsilon \log 2 + \left(\epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} < \delta\right)\right) \vee (-L)$$

and for ϵ_1 small enough and such that $\epsilon_1 \log(2) < \gamma$, for every ϵ positive such that $\epsilon < \epsilon_1$, for every u_0 such that $\|u_0\|_{\mathrm{H}^1} \le \rho$,

$$-I^{u_0}(w) - \gamma \le \epsilon \log \mathbb{P}\left(\left\|u^{\epsilon,u_0} - \tilde{\mathbf{S}}(u_0, f)\right\|_{X^{(T,p)}} < \delta\right).$$

It ends the proof of (i) and (ii). □

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