# Hermite Subdivision with Shape Constraints on a Rectangular Mesh

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January 24, 2006

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**Abstract.**

We study a two parameter version of the Hermite subdivision scheme introduced in [7], which gives $C^1$ interpolants on rectangular meshes. We prove $C^1$-convergence for a range of the two parameters. By introducing a control grid we can choose the parameters in the scheme so that the interpolant inherits positivity and/or directional monotonicity from the initial data. Several examples are given showing that a desired shape can be achieved even if we use only very crude estimates for the initial slopes.

*Math Subject Classification:* 65D05, 65D17
*Keywords:* Interpolation, Subdivision, Hermite Data, Rectangular Mesh, Positivity, Monotonicity.

## 1 Introduction

Subdivision is a technique for constructing smooth curves or surfaces out of a sequence of successive refinements of polygons, or grids see [3]. Subdivision has found applications in areas such as geometric design [10, 21], and in computer games and animation [6]. Subdivision schemes can be of Lagrange type or Hermite type. In this last case derivatives are also used. This can be desirable since a Hermite scheme can be made more local, making it easier to obtain a desired shape. Moreover, as our examples show, we can achieve a required shape using only very crude estimates for the derivatives. For some classical methods for bimonotone interpolation on a rectangular grid see [1, 2, 4, 5].

The first Hermite scheme was introduced in [15]. This method has smoothness $C^1$ and we refer to it as the $HC^1$-scheme. A notion of control points for two subfamilies of the $HC^1$-scheme were introduced in [18]. In [14] some further studies of the $HC^1$-scheme where carried out. The calculation of values and derivatives was separated and this made it possible to simplify some of the proofs in [17]. It was also shown that a geometric formulation of the scheme has a totally positive transformation matrix, and algorithms for constructing curves satisfying local positivity, monotonicity, and convexity constraints were given and tested. For more references to Hermite subdivision see [8, 9, 12, 13, 16, 22].

In [11, 19] Hermite subdivision was studied on a rectangular mesh using tensor products of the $HC^1$-scheme and its control points. An algorithm for achieving a bimonotone interpolant was given.

A disadvantage using the tensor product construction is that mixed partial derivatives $\partial^2 f/\partial x \partial y$ is required as input data. In this paper we consider an alternative method the $HRC^1$-algorithm, where these mixed partial derivatives are not required. This scheme was introduced in [7]. It is a generalization of a $C^1$-quadratic finite element on a quadrilateral mesh ([20]). To describe the $HRC^1$-algorithm we start with values and gradients at the vertices of a rectangular grid $G$ in the plane. The algorithm is applied to each rectangle $R$ in turn by a local process. We divide $R$ into 4 rectangles by connecting midpoints of opposite edges and then compute values and derivatives at the vertices of the 4 sub-rectangles. Repeating this on each sub-rectangle we obtain in the limit a function defined on a dense subset of $R$. The scheme is interpolatory, i.e it retains the values at the vertices of the current rectangular grid. Moreover the value on an edge $E$ of $R$ only depends on the length of $E$ and on the values of $f$ and its derivatives at the endpoints of $E$. This makes it possible to obtain a global smooth surface by gluing together $HRC^1$-interpolants on neighboring sub-rectangles.

Our paper can be detailed as follows. In Section 2, we first recall the $HRC^1$-algorithm and some of its properties which were proved in [7]. We consider a simplified version of the scheme using only two parameters $\alpha$ and $\beta$. We show that this version simplifies further if we choose $\alpha = \beta/(4(1-\beta))$.

In Section 3 we show $C^1$-convergence of the $HRC^1$-algorithm for a range of the parameters $\alpha$ and $\beta$. This extends results in [7] where $C^1$-convergence was only shown for $\alpha = -1/8$. We also show Hölder continuity of the first order partial derivatives.

In Section 4 we define a control grid thereby giving a geometric formulation of the $HRC^1$-algorithm. This formulation is used in Section 5 to show how local shape constraints can be achieved in the limit function. We give several examples involving positivity and directional monotonicity constraints. We also show that a convexity preserving $HRC^1$-interpolant cannot be obtained in general.

## 2 Description of the algorithm $HRC^1$

We let $R := [a,b] \times [c,d]$ be a given rectangle. The algorithm $HRC^1$ which gives a $C^1$ Hermite interpolant on $R$ was proposed by Dubuc and Merrien [7]. The goal is to construct a bivariate function $f$ and its first partial derivatives $p := f_x$, $q := f_y$ on $R$ in such a way that $f, p, q$ are continuous.

The $HRC^1$-Algorithm can be formulated as follows. We start with Hermite data $f^0_{i,j}$, $p^0_{i,j}$, $q^0_{i,j}$ for $i,j = 0,1$ at the corners of the rectangle $[s^0_0, s^0_1] \times [t^0_0, t^0_1] := [a,b] \times [c,d]$. For $n = 0,1,2,\ldots$, let us denote by $\mathcal{P}_n$ the regular partition of $[a,b]$ into $2^n$ subintervals and by $\mathcal{Q}_n$ the similar regular partition of $[c,d]$. Also let $\mathcal{P} := \cup_{n\in\mathbb{N}}\mathcal{P}_n$ and $\mathcal{Q} := \cup_{n\in\mathbb{N}}\mathcal{Q}_n$. For $n = 0,1,2,\ldots$ the points in the partitions are denoted by $s^n_i := a + ih_n$ and $t^n_j := c + jk_n$, for $i,j = 0,\ldots,2^n$, where $h_n := 2^{-n}(b-a)$ and $k_n := 2^{-n}(d-c)$. What we compute at the grid points $(s^n_i, t^n_j)$ can be viewed either as point sequences $\{f^n_{ij}\}$, $\{p^n_{ij}\}$, $\{q^n_{ij}\}$ or as functions $f, p, q : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}$ defined by $f(s^n_i, t^n_j) := f^n_{ij}$, $p(s^n_i, t^n_j) := p^n_{ij}$, and $q(s^n_i, t^n_j) := q^n_{ij}$. We will find both the sequence point of view and the function point of view useful.

To define $f$, $p$ and $q$ on $\mathcal{P} \times \mathcal{Q}$, we proceed by induction on $n$. For $n = 0,1,2,\ldots$ suppose we have computed $\{f^n_{i,j}\}$, $\{p^n_{i,j}\}$, and $\{q^n_{i,j}\}$ on the grid $\mathcal{P}_n \times \mathcal{Q}_n$. We set $h_n := 2^{-n}(b-a)$, $k_n := 2^{-n}(d-c)$ and compute $f^{n+1}_{i,j}$, $p^{n+1}_{i,j}$, and $q^{n+1}_{i,j}$ on the grid $\mathcal{P}_{n+1} \times \mathcal{Q}_{n+1}$ as follows:

$$\boxed{\begin{aligned}&\text{for } i = 2^n : -1 : 0, \text{ for } j = 2^n : -1 : 0\\ &\quad f^{n+1}_{2i,2j} := f^n_{i,j}, \quad p^{n+1}_{2i,2j} := p^n_{i,j}, \quad q^{n+1}_{2i,2j} := q^n_{i,j},\end{aligned}} \tag{2.1}$$

$$\boxed{\begin{aligned}&\text{for } i = 0 : 2^n - 1, \quad \text{for } j = 0 : 2^n\\ &\quad f^{n+1}_{2i+1,2j} := \frac{f^n_{i+1,j} + f^n_{i,j}}{2} + \alpha h_n \big(p^n_{i+1,j} - p^n_{i,j}\big),\\ &\quad p^{n+1}_{2i+1,2j} := (1-\beta)\frac{f^n_{i+1,j} - f^n_{i,j}}{h_n} + \beta\frac{p^n_{i+1,j} + p^n_{i,j}}{2},\\ &\quad q^{n+1}_{2i+1,2j} := \frac{q^n_{i+1,j} + q^n_{i,j}}{2},\end{aligned}} \tag{2.2}$$

$$\boxed{\begin{aligned}&\text{for } i = 0 : 2^n, \quad \text{for } j = 0 : 2^n - 1\\ &\quad f^{n+1}_{2i,2j+1} := \frac{f^n_{i,j+1} + f^n_{i,j}}{2} + \alpha k_n \big(q^n_{i,j+1} - q^n_{i,j}\big),\\ &\quad p^{n+1}_{2i,2j+1} := \frac{p^n_{i,j+1} + p^n_{i,j}}{2},\\ &\quad q^{n+1}_{2i,2j+1} := (1-\beta)\frac{f^n_{i,j+1} - f^n_{i,j}}{k_n} + \beta\frac{q^n_{i,j+1} + q^n_{i,j}}{2},\end{aligned}} \tag{2.3}$$

$$\boxed{\begin{aligned}&\text{for } i = 0 : 2^n - 1, \quad \text{for } j = 0 : 2^n - 1\\ &\quad f^{n+1}_{2i+1,2j+1} := \frac{f^n_{i,j} + f^n_{i+1,j} + f^n_{i,j+1} + f^n_{i+1,j+1}}{4}\\ &\qquad\qquad + \alpha h_n \frac{p^n_{i+1,j} - p^n_{i,j} + p^n_{i+1,j+1} - p^n_{i,j+1}}{2}\\ &\qquad\qquad + \alpha k_n \frac{q^n_{i,j+1} - q^n_{i,j} + q^n_{i+1,j+1} - q^n_{i+1,j}}{2}\\ &\quad p^{n+1}_{2i+1,2j+1} := (1-\beta)\frac{f^n_{i+1,j} - f^n_{i,j} + f^n_{i+1,j+1} - f^n_{i,j+1}}{2h_n}\\ &\qquad\qquad + \beta\frac{p^n_{i,j} + p^n_{i,j+1} + p^n_{i+1,j} + p^n_{i+1,j+1}}{4}\\ &\qquad\qquad + \beta k_n \frac{q^n_{i+1,j+1} - q^n_{i+1,j} + q^n_{i,j} - q^n_{i,j+1}}{4h_n},\\ &\quad q^{n+1}_{2i+1,2j+1} := (1-\beta)\frac{f^n_{i,j+1} - f^n_{i,j} + f^n_{i+1,j+1} - f^n_{i+1,j}}{2h_n}\\ &\qquad\qquad + \beta\frac{q^n_{i,j} + q^n_{i+1,j} + q^n_{i,j+1} + p^n_{i+1,j+1}}{4}\\ &\qquad\qquad + \beta k_n \frac{p^n_{i+1,j+1} - p^n_{i,j+1} + p^n_{i,j} - q^n_{i+1,j}}{4h_n}.\end{aligned}} \tag{2.4}$$

In (2.1) we simply redefine the functions at the points on $\mathcal{P}_n \times \mathcal{Q}_n$ as points on a subset of $\mathcal{P}_{n+1} \times \mathcal{Q}_{n+1}$. These points are marked by gray squares in Figure 2.1. In (2.3-2.4) we compute new values at the new points marked by black circles in Figure 2.1.

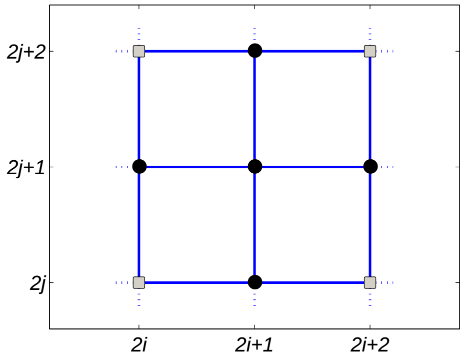


Figure 2.1: Recursive computation of $f, p$ and $q$.

For $\alpha = -1/8, \beta = -1$ it was shown in [7] that we obtain the Sibson-Thomson interpolant on $R$ proposed in [20]. In this case, the $HRC^1$-interpolant is a $C^1$ piecewise quadratic consisting of 16 individual pieces, see Figure 2.2. Moreover the cross boundary derivatives are linear functions along the outer boundary of $[a, b] \times [c, d]$.

It was shown in [7] that the $HRC^1$-algorithm is exact for bilinear functions for any value of $\alpha$ and $\beta$. It is exact for quadratic polynomials if and only if $\alpha = -1/8$ and exact for cubic polynomials if and only if $\alpha = -1/8$ and $\beta = -1/2$.

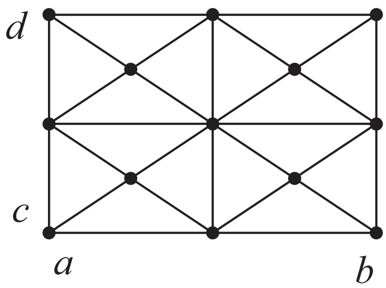


Figure 2.2: Sibson-Thomson subdivision of a rectangle.

We have simplified the construction of [7] and it depends on only two parameters $\alpha$ and $\beta$. This simplification gives new formulas for the computation of $f^{n+1}_{2i+1,2j+1}$, $p^{n+1}_{2i+1,2j+1}$ and $q^{n+1}_{2i+1,2j+1}$.

PROPOSITION 2.1.

(2.5)

$$\boxed{\begin{aligned}
&\textit{for } i = 0 : 2^n - 1, \quad \textit{for } j = 0 : 2^n - 1\\
&\quad f^{n+1}_{2i+1,2j+1} = \frac{f^{n+1}_{2i+2,2j+1} + f^{n+1}_{2i,2j+1}}{2} + \alpha h_n \big(p^{n+1}_{2i+2,2j+1} - p^{n+1}_{2i,2j+1}\big),\\
&\qquad\qquad\quad = \frac{f^{n+1}_{2i+1,2j+2} + f^{n+1}_{2i+1,2j}}{2} + \alpha k_n \big(q^{n+1}_{2i+1,2j+2} - q^{n+1}_{2i+1,2j}\big),\\
&\quad p^{n+1}_{2i+1,2j+1} = (1-\beta)\frac{f^{n+1}_{2i+2,2j+1} - f^{n+1}_{2i,2j+1}}{h_n} + \beta\frac{p^{n+1}_{2i+2,2j+1} + p^{n+1}_{2i,2j+1}}{2}\\
&\qquad\qquad + \frac{k_n}{h_n}\big((1-\beta)\alpha - \beta/4\big)\big(q^{n+1}_{2i+2,2j} - q^{n+1}_{2i+2,2j+2} + q^{n+1}_{2i,2j+2} - q^{n+1}_{2i,2j}\big),\\
&\quad q^{n+1}_{2i+1,2j+1} = (1-\beta)\frac{f^{n+1}_{2i+1,2j+2} - f^{n+1}_{2i+1,2j}}{k_n} + \beta\frac{q^{n+1}_{2i+1,2j+2} + q^{n+1}_{2i+1,2j}}{2}\\
&\qquad\qquad + \frac{h_n}{k_n}\big((1-\beta)\alpha - \beta/4\big)\big(p^{n+1}_{2i,2j+2} - p^{n+1}_{2i+2,2j+2} + p^{n+1}_{2i+2,2j} - p^{n+1}_{2i,2j}\big).
\end{aligned}}$$

PROOF. For $n \in \mathbb{N}$, $i, j \in \{0, \dots, 2^n - 1\}$, the first formula of (2.4) can be written:

$$\begin{aligned}
f^{n+1}_{2i+1,2j+1} =& \frac{1}{2}\Big[\frac{f^n_{i+1,j+1} + f^n_{i+1,j}}{2} + \alpha k_n (q^n_{i+1,j+1} - q^n_{i+1,j})\\
&+ \frac{f^n_{i,j+1} + f^n_{i,j})}{2} + \alpha k_n (q^n_{i,j+1} - q^n_{i,j})\Big]\\
&+ \alpha h_n \Big[\frac{p^n_{i+1,j+1} + p^n_{i+1,j}}{2} - \frac{p^n_{i,j+1} + p^n_{i,j}}{2}\Big].
\end{aligned}$$

Using (2.3) this gives the first formula in (2.5). The proof of the second formula is similar.

We write the second formula of (2.4)

$$\begin{aligned}
p^{n+1}_{2i+1,2j+1} =& \frac{1-\beta}{h_n}\Big[\frac{f^n_{i+1,j+1} + f^n_{i+1,j}}{2} + \alpha k_n (q^n_{i+1,j+1} - q^n_{i+1,j})\Big]\\
&- \frac{1-\beta}{h_n}\Big[\frac{f^n_{i,j+1} + f^n_{i,j}}{2} + \alpha k_n (q^n_{i,j+1} - q^n_{i,j})\Big]\\
&+ \frac{\beta}{2}\Big[\frac{p^n_{i+1,j+1} + p^n_{i+1,j}}{2} + \frac{p^n_{i,j+1} + p^n_{i,j}}{2}\Big]\\
&+ \frac{k_n}{h_n}\big[(1-\beta)\alpha - \beta/4\big]\big[q^n_{i+1,j} - q^n_{i+1,j+1} + q^n_{i,j+1} - q^n_{i,j}\big].
\end{aligned}$$

Thanks to (2.1) and (2.3), we obtain the result. The last formula is symmetrical from the previous one. □

## 3 $C^1$-convergence of the algorithm

We say that the scheme is $C^1$*-convergent* if, for any initial data, the functions $f$, $p$, and $q$ can be extended from $\mathcal{P} \times \mathcal{Q}$ to continuous functions on $[a, b] \times [c, d]$

with $p = f_x$ and $q = f_y$. We call $f$ defined either on $\mathcal{P} \times \mathcal{Q}$ or on $[a,b] \times [c,d]$ the $HRC^1$*-interpolant* to the data.

For the study of $C^1$-convergence it is enough to consider the construction on the unit square $[0,1]^2$. To see this, let $h = b-a$, $k = d-c$ and let again $(a,c)$ be the south-west vertex of the initial rectangle $[a,b] \times [c,d]$. On $[0,1]^2$, we define the initial data $g(u,v) := f(a+uh, c+vk)$, $g_x(u,v) := hf_x(a+uh, c+vk)$, $g_y(u,v) := kf_y(a+uh, c+vk)$, $(u,v) \in \{0,1\}^2$. The constructions of $f$ or $g$ by formulas (2.1), (2.2), (2.3) and (2.4) are equivalent and at each step, we obtain $g(u,v) = f(a+uh, c+vk)$, $g_x(u,v) = hf_x(a+uh, c+vk)$ and $g_y(u,v) = kf_y(a+uh, c+vk)$, $(u,v) \in \{0, 1/2^n, \ldots, \ell/2^n, \ldots, 1\}^2$. Thus the $C^1$-convergence of $f$ on $[a,b] \times [c,d]$ is equivalent to the $C^1$-convergence of $g$ on $[0,1]^2$.

So let us begin with data on the vertices of the unit square $[0,1]^2$. For $n \geq 0$ and $i,j = 0,1,\ldots,2^n-1$ we define vectors of differences $U_{ij}^n \in \mathbb{R}^{12}$ as follows:

$$U_{ij}^n := \begin{bmatrix} q_{i+1,j}^n - q_{i,j}^n \\ p_{i+1,j+1}^n - p_{i+1,j}^n \\ q_{i+1,j+1}^n - q_{i,j+1}^n \\ p_{i,j+1}^n - p_{i,j}^n \\ p_{i+1,j}^n - p_{i,j}^n \\ q_{i+1,j+1}^n - q_{i+1,j}^n \\ p_{i+1,j+1}^n - p_{i,j+1}^n \\ q_{i,j+1}^n - q_{i,j}^n \\ \dfrac{f_{i+1,j}^n - f_{i,j}^n}{h_n} - \dfrac{p_{i+1,j}^n + p_{i,j}^n}{2} \\ \dfrac{f_{i+1,j+1}^n - f_{i+1,j}^n}{h_n} - \dfrac{q_{i+1,j+1}^n + q_{i+1,j}^n}{2} \\ \dfrac{f_{i+1,j+1}^n - f_{i,j+1}^n}{h_n} - \dfrac{p_{i+1,j+1}^n + p_{i,j+1}^n}{2} \\ \dfrac{f_{i,j+1}^n - f_{i,j}^n}{h_n} - \dfrac{q_{i,j+1}^n + q_{i,j}^n}{2} \end{bmatrix}.$$

We then have

Lemma 3.1. *Suppose we can find a vector norm $\|\cdot\|$ on $\mathbb{R}^{12}$ and positive constants $c, \rho$ with $\rho < 1$ such that*

$$\|U_{ij}^n\| \leq c\rho^n, \text{ for } i,j = 0,\ldots,2^n-1.$$

*Then the $HRC^1$-algorithm is $C^1$-convergent.*

Proof. The proof of the $C^1$-convergence on $[0,1]^2$ is detailed in [7]. To summarize, with the hypothesis, it can be proved that $p$ and $q$ are uniformly continuous on the dyadic points so that they can be extended into continuous functions on $[0,1]^2$. Then we extend $f$ and prove that $f_x = p$ and $f_y = q$ using the four last components of $U_{i,j}^n$. □

To bound the vectors $U_{ij}^n$ we will use the following recurrence relations.

Proposition 3.2. *We have*

$$U_{ij}^{n+1} = \Lambda^{(1)} U_{ij}^n \,,\, U_{i+1,j}^{n+1} = \Lambda^{(2)} U_{ij}^n,$$
$$U_{i+1,j+1}^{n+1} = \Lambda^{(3)} U_{ij}^n \,,\, U_{i,j+1}^{n+1} = \Lambda^{(4)} U_{ij}^n,$$

*where $\Lambda^{(1)}$,$\Lambda^{(2)}$,$\Lambda^{(3)}$,$\Lambda^{(4)}$ are 4 matrices in $\mathbb{R}^{12\times 12}$ depending only on the 2 parameters $\alpha, \beta$ of algorithm $HRC^1$. Explicit formulas for the matrices are as* *follows:*

$$\Lambda^{(i)} = \begin{bmatrix} \Lambda_{11}^{(i)} & \Lambda_{12}^{(i)} & \Lambda_{13}^{(i)} \\ 0 & \Lambda_{22}^{(i)} & \Lambda_{23}^{(i)} \\ 0 & \Lambda_{32}^{(i)} & \Lambda_{33}^{(i)} \end{bmatrix} = \begin{bmatrix} \Lambda_{11}^{(i)} & \dots \\ 0 & M^{(i)} \end{bmatrix}$$

*with* $\Lambda_{jk}^{(i)} \in \mathbb{R}^{4\times 4}$ *and* $M^{(i)} \in \mathbb{R}^{8\times 8}$. *More specifically:*

$$\Lambda_{11}^{(1)} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \Lambda_{11}^{(2)} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix},$$

$$\Lambda_{11}^{(3)} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}, \Lambda_{11}^{(4)} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$(\Lambda_{12}^{(1)}\Lambda_{13}^{(1)}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta/4 & 0 & -\beta/4 & \frac{\beta-1}{2} & 0 & \frac{1-\beta}{2} & 0 \\ -\beta/4 & 0 & \beta/4 & 0 & 0 & \frac{1-\beta}{2} & 0 & \frac{\beta-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$M^{(1)} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 & 0 & 0 \\ -\frac{\beta}{4} & \frac{1}{4} & \frac{\beta}{4} & \frac{1}{4} & 0 & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} \\ \frac{1}{4} & \frac{\beta}{4} & \frac{1}{4} & -\frac{\beta}{4} & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta \\ \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 & 0 \\ +\frac{\beta}{8}-\alpha & \frac{1}{8}+\alpha & \alpha-\frac{\beta}{8} & \frac{1}{8}+\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \\ \frac{1}{8}+\alpha & \alpha-\frac{\beta}{8} & \frac{1}{8}+\alpha & \frac{\beta}{8}-\alpha & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} \end{bmatrix}$$

$$M^{(2)} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \beta-1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 & 0 \\ \frac{1}{4} & -\frac{\beta}{4} & \frac{1}{4} & \frac{\beta}{4} & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} & 0 \\ -\frac{\beta}{4} & \frac{1}{4} & \frac{\beta}{4} & \frac{1}{4} & 0 & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} \\ -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 \\ -\frac{1}{8}-\alpha & -\frac{\beta}{8}+\alpha & -\frac{1}{8}-\alpha & -\alpha+\frac{\beta}{8} & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ \frac{\beta}{8}-\alpha & \frac{1}{8}+\alpha & \alpha-\frac{\beta}{8} & \frac{1}{8}+\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \end{bmatrix}$$

$$M^{(3)} = \begin{bmatrix} \frac{1}{4} & -\frac{\beta}{4} & \frac{1}{4} & \frac{\beta}{4} & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 & 0 \\ \frac{\beta}{4} & \frac{1}{4} & -\frac{\beta}{4} & \frac{1}{4} & 0 & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} \\ -\frac{1}{8}-\alpha & -\frac{\beta}{8}+\alpha & -\frac{1}{8}-\alpha & -\alpha+\frac{\beta}{8} & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 \\ -\alpha+\frac{\beta}{8} & -\frac{1}{8}-\alpha & -\frac{\beta}{8}+\alpha & -\frac{1}{8}-\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \end{bmatrix}$$

$$M^{(4)} = \begin{bmatrix} \frac{1}{4} & \frac{\beta}{4} & \frac{1}{4} & -\frac{\beta}{4} & \frac{1-\beta}{2} & 0 & \frac{1-\beta}{2} & 0 \\ \frac{\beta}{4} & \frac{1}{4} & -\frac{\beta}{4} & \frac{1}{4} & 0 & \frac{\beta-1}{2} & 0 & \frac{\beta-1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1-\beta & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \beta-1 \\ \frac{1}{8}+\alpha & \alpha-\frac{\beta}{8} & \frac{1}{8}+\alpha & \frac{\beta}{8}-\alpha & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} & 0 \\ -\alpha+\frac{\beta}{8} & -\frac{1}{8}-\alpha & -\frac{\beta}{8}+\alpha & -\frac{1}{8}-\alpha & 0 & \frac{1+\beta}{4} & 0 & \frac{1+\beta}{4} \\ 0 & 0 & \frac{1}{4}+2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4}-2\alpha & 0 & 0 & 0 & \frac{1+\beta}{2} \end{bmatrix}.$$

PROOF. These relations were shown in [7] using a a computer algebra system. □

We will not need the explicit form of the matrices $[\Lambda_{12}^{(i)}\Lambda_{13}^{(i)}]$ for $i \in \{1,2,3,4\}$. One can obtain all of them from $[\Lambda_{12}^{(1)}\Lambda_{13}^{(1)}]$ by permutations of rows and columns.

We mention that in [7] it was proved that the scheme is $C^1$-convergent if and only if the generalized spectral radius of $\Sigma = \{\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)}\}$ satisfies $\hat{\rho}(\Sigma) < 1$. The following analysis is maybe somewhat simpler. We start with a proposition.

PROPOSITION 3.3. *If there exists a vector norm $\|\cdot\|$ on $\mathbb{R}^{12}$ and a number $\rho < 1$ such that the associated matrix operator norm satisfies $\|\Lambda^{(i)}\| \le \rho$ for $i = 1,2,3,4$ then the scheme is $C^1$-convergent. Moreover the functions $p$ and $q$ are Hölder continuous with exponent $-log_2(\rho)$.*

PROOF. It is enough to prove the Proposition on the square $[0,1]^2$. That the scheme is $C^1$-convergent follows immediately from Lemma 3.1.

The proof that $p$ and $q$ are Hölder continuous is similar to a proof in dimension one in [17]. In the following proof we will use the function notation for the sequences $\{U_{ij}^n\}_{i,j}, \{f_{ij}^n\}_{i,j}, \{p_{ij}^n\}_{i,j}$, and $\{q_{ij}^n\}_{i,j}$ . Thus if $x := i2^{-n}$ and $y := j2^{-n}$ then we write $U_{ij}^n$ and $p_{ij}^n$ as $U^n(x,y)$ and $p^n(x,y)$. We recall that $\mathcal{P}_\ell = \{k2^{-\ell}, k = 0,\dots,2^\ell\}, \ell \in \mathbb{N}$ is the set of dyadic points at step $\ell$ on $[0,1]$ and we write $h_\ell = 1/2^\ell$. With the hypothesis, for $(x,y) \in {\mathcal{P}_\ell}^2, x \ne 1, y \ne 1$, we have $\|U^\ell(x,y)\| \le \rho^\ell \|U^0(0,0)\|$ for $\ell \ge 0$. Using the equivalence of the norms in $\mathbb{R}^{12}$, this implies that $\|U^\ell(x,y)\|_\infty \le c_2\rho^\ell$ for some positive constant $c_2$ independent of $U^\ell$. In particular, this holds for components 4 and 5 of $U^\ell(x,y)$ and we deduce that

$$\begin{aligned} &|p(x \pm h_\ell, y) - p(x,y)| \le c_2\rho^\ell, \text{ with } (x,y),(x \pm h_\ell, y) \in {\mathcal{P}_\ell}^2 \\ &|p(x, y \pm h_\ell) - p(x,y)| \le c_2\rho^\ell, \text{ with } (x,y),(x, y \pm h_\ell) \in {\mathcal{P}_\ell}^2. \end{aligned} \tag{3.1}$$

Suppose that $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are 2 points in $[0,1]^2$. Let $n$ be the unique nonnegative integer such that $2^{-n-1} < \|P_1 - P_2\|_\infty \le 2^{-n}$. Then $|x_2 - x_1| \le 2^{-n}$ and $|y_2 - y_1| \le 2^{-n}$ and there exist $x, y \in \mathcal{P}_n$ such that $|x_j - x| \le 2^{-n}$ and $|y_j - y| \le 2^{-n}$ for $j = 1,2$. Thus $P := (x,y) \in {\mathcal{P}_n}^2$ is such that $\|P_j - P\|_\infty \le 2^{-n}$ for $j = 1,2$.

To prove that $|p(P_1) - p(P)| \le c_3\rho^n$ for some constant $c_3$ we write $P_1 = P + \sum_{i=1}^\infty (u_i,v_i)2^{-i-n}$ with $u_i$ and $v_i$ in $\{0,1,-1\}$. We define the sequence $\{\hat{P}_j\} := \{(\hat{x}_j,\hat{y}_j)\}$ by $\hat{P}_0 = P$ and $\hat{P}_j = \hat{P}_{j-1} + (u_j,v_j)2^{-j-n}$, for $j \ge 1$. Then $\hat{P}_j \in {\mathcal{P}_{n+j}}^2$ and

$$|p(\hat{P}_j) - p(\hat{P}_{j-1})| \le |p(\hat{x}_j,\hat{y}_j) - p(\hat{x}_j,\hat{y}_{j-1})| + |p(\hat{x}_j,\hat{y}_{j-1}) - p(\hat{x}_{j-1},y_{j-1})|.$$

Since $(\hat{x}_j, \hat{y}_j), (\hat{x}_j, \hat{y}_{j-1})$ and $(\hat{x}_{j-1}, \hat{y}_{j-1})$ are in ${\mathcal{P}_{n+j}}^2$ we can bound them using (3.1) with $\ell = n+j$ and we obtain $|p(\hat{P}_j) - p(\hat{P}_{j-1})| \leq 2c_2\rho^{n+j}$ so that $|p(P_1) - p(P)| \leq \sum_{j=1}^{\infty} 2c_2\rho^{n+j} = \frac{2c_2}{1-\rho}\rho^{n+1}$.

With the same upper bound for $|p(P_2)-p(P)|$, we deduce that $|p(P_2)-p(P_1)| \leq c_4\rho^{n+1}$ with $c_4 = \frac{4c_2}{1-\rho}$.

To conclude, notice that since $\|P_1 - P_2\|_\infty > 2^{-n-1}$ then

$$|p(P_2) - p(P_1)| \leq c_4\rho^{n+1} = c_4 2^{(-n-1)(-\log_2\rho)} < c_4\|P_1 - P_2\|_\infty^{-\log_2(\rho)}.$$

A similar inequality holds for the function $q$. □

To find a good norm on $\mathbb{R}^{12}$, we use the following well known result:

LEMMA 3.4. *Corresponding to a positive integer $d$, a nonsingular matrix $P \in \mathbb{R}^{d\times d}$ and a vector norm $\|\cdot\|$ on $\mathbb{R}^d$ we define a vector norm on $\mathbb{R}^d$ by $\|V\|_1 := \|P^{-1}V\|$. Then the associated matrix operator norm $\|\cdot\|$ is given by $\|A\|_1 = \|P^{-1}AP\|$ for any matrix $A \in \mathbb{R}^{d\times d}$.*

PROOF. Clearly $\|\cdot\|_1$ defines a norm on $\mathbb{R}^d$. Now if $A \in \mathbb{R}^{d\times d}$ then

$$\|A\|_1 := \max_{V\neq 0}\frac{\|AV\|_1}{\|V\|_1} = \max_{V\neq 0}\frac{\|P^{-1}AV\|}{\|P^{-1}V\|} = \max_{U\neq 0}\frac{\|P^{-1}APU\|}{\|U\|} = \|P^{-1}AP\|.$$

□

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two vector norms on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. For a matrix $A \in \mathbb{R}^{d_1\times d_2}$ we write $\|A\|_{12}$ for the associated mixed matrix operator norm $\|A\|_{12} := \max_{V\in\mathbb{R}^{d_2}, V\neq 0}\frac{\|AV\|_1}{\|V\|_2}$.

LEMMA 3.5. *Suppose for positive integers $d_1$ and $d_2$ that $\Sigma$ is a set of square matrices $\{A\}$ of order $d := d_1 + d_2$ that are written by blocks as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.2}$$

*with diagonal blocks $A_{ii} \in \mathbb{R}^{d_i\times d_i}$ for $i = 1, 2$. For $i = 1, 2$, let $\|\cdot\|_i$ be two vector norms on $\mathbb{R}^{d_i}$ and for $i, j = 1, 2$, let $\gamma_{ij}$ be positive constants such that for any $A \in \Sigma$ the estimates $\|A_{ij}\|_{ij} \leq \gamma_{ij}$ hold. If*

$$\gamma_{11} < 1, \quad \gamma_{22} < 1, \ \text{and } \gamma_{21}\gamma_{12} < (1-\gamma_{11})(1-\gamma_{22})$$

*then we can find a matrix norm on $\mathbb{R}^{d\times d}$ such that any $A \in \Sigma$ has norm less than 1.*

PROOF. On $\mathbb{R}^d$, we define a norm $\|\cdot\|_\theta$ depending on a parameter $\theta > 0$. If $V = (X, Y)^T$ with $X \in \mathbb{R}^{d_1}$ and $Y \in \mathbb{R}^{d_2}$ then $\|V\|_\theta := \|X\|_1 + \theta\|Y\|_2$.

Then for any matrix $A \in \mathbb{R}^{d\times d}$, we have:

$$\begin{aligned}
\|AV\|_\theta &= \|A_{11}X + A_{12}Y\|_1 + \theta\|A_{21}X + A_{22}Y\|_2 \\
&\leq \|A_{11}\|_{11}\|X\|_1 + \|A_{12}\|_{12}\|Y\|_2 + \theta\|A_{21}\|_{21}\|X\|_1 + \theta\|A_{22}\|_{22}\|Y\|_2 \\
&= (\|A_{11}\|_{11} + \theta\|A_{21}\|_{21})\|X\|_1 + (\|A_{12}\|_{12}/\theta + \|A_{22}\|_{22})(\theta\|Y\|_2) \\
&\leq \max(\|A_{11}\|_{11} + \theta\|A_{21}\|_{21}, \|A_{12}\|_{12}/\theta + \|A_{22}\|_{22})\|V\|_\theta.
\end{aligned}$$

We deduce that

$$\|A\|_\theta \le \max(\|A_{11}\|_{11} + \theta\|A_{21}\|_{21}, \|A_{12}\|_{12}/\theta + \|A_{22}\|_{22}), A \in \mathbb{R}^{d\times d}. \tag{3.3}$$

$\|A\|_\theta < 1$, as soon as $\|A_{11}\|_{11} + \theta\|A_{21}\|_{21} < 1$ and $\|A_{12}\|_{12}/\theta + \|A_{22}\|_{22} < 1$. Since, for any $A \in \Sigma$, $\|A_{ij}\| \le \gamma_{ij}, i, j = 1, 2$, it suffices that $\gamma_{11} + \theta\gamma_{21} < 1$ and $\gamma_{12}/\theta + \gamma_{22} < 1$. If $\gamma_{11} < 1$ and $\gamma_{22} < 1$, these conditions are satisfied whenever there exists a real number $\theta > 0$ such that $\frac{\gamma_{12}}{1-\gamma_{22}} < \theta < \frac{1-\gamma_{11}}{\gamma_{21}}$. Since $\gamma_{21}\gamma_{12} < (1-\gamma_{11})(1-\gamma_{22})$ we can find such a $\theta$. □

LEMMA 3.6. *Suppose in Lemma 3.5 that $\Sigma$ is a finite family of matrices of the form* (3.2) *with $A_{21} = 0$. If there exists a real number $c > 0$ such that for any $A \in \Sigma$, $\|A_{11}\|_{11} \le c$ and $\|A_{22}\|_{22} < c$ then there exists a matrix norm such that for all $A \in \Sigma$, we have $\|A\| \le c$ .*

PROOF. Using (3.3) in the previous Lemma, $\|A\|_\theta \le \max(\|A_{11}\|_{11}, \|A_{12}\|_{12}/\theta + \|A_{22}\|_{22})$. Now $\|A_{11} \le c$ and $\|A_{22}\|_{22} < c$. Since the set $\Sigma$ is finite, we can find a real number $\theta > 0$ such that for any $A \in \Sigma$, $\|A_{12}\|_{12}/\theta$ is small enough to get $\|A\|_\theta \le c$. □

Now we have the tools to study the $C^1$-convergence of the algorithm.

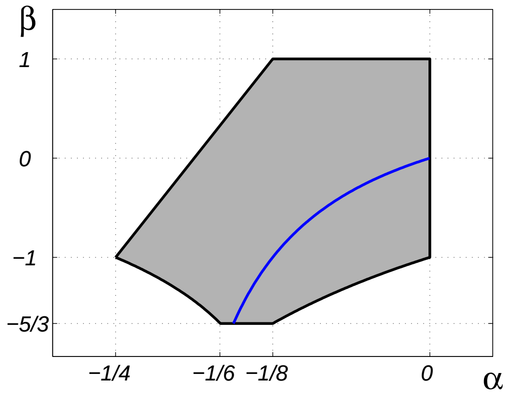


Figure 3.1: The region $\mathcal{R}$ in Theorem 3.7 together with the curve $\alpha = \beta/(4(1-\beta))$.

THEOREM 3.7. *The algorithm $HRC^1$ is $C^1$ convergent if $(\alpha, \beta)$ belongs to the region*

$$\mathcal{R} := \{(\alpha, \beta) : -\frac{1}{4} < \alpha < 0 \text{ and } l(\alpha) < \beta < u(\alpha)\}, \tag{3.4}$$

*where*

$$l(\alpha) := \begin{cases} 8\alpha - 2 + \sqrt{(8\alpha+1)(8\alpha-7)} & \text{if } -\frac{1}{4} \le \alpha < -\frac{1}{6}, \\ -\frac{5}{3} & \text{if } -\frac{1}{6} \le \alpha < -\frac{1}{8}, \\ \frac{2\alpha-1}{2\alpha+1} & \text{if } -\frac{1}{8} \le \alpha \le 0, \end{cases} \tag{3.5}$$

$$u(\alpha) := \begin{cases} 16\alpha + 3, & \text{if } -\frac{1}{4} \le \alpha < -\frac{1}{8}, \\ 1 & \text{if } -\frac{1}{8} \le \alpha \le 0, \end{cases} \tag{3.6}$$

Proof. Let $\|A\|_\infty = \max_{i=1,\dots,d}(\sum_{j=1}^d |a_{ij}|)$ be the matrix norm on $\mathbb{R}^{d\times d}$ associated with the vector norm $\|V\|_\infty = \max_{k=1,\dots,d}(|v_k|)$ on $\mathbb{R}^d$. By Lemma 3.6, since $\|\Lambda_{11}^{(\ell)}\|_\infty = 1/2$, with we get a matrix norm such that $\|\Lambda^{(\ell)}\| < 1$ as soon as there exists a matrix norm such that $\|M^{(\ell)}\| < 1$, $\ell = 1,\dots,4$.

Let $P_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}$. We compute $N^{(\ell)} := P^{-1}M^{(\ell)}P = \begin{bmatrix} N_{11}^{(\ell)} & N_{12}^{(\ell)} \\ N_{21}^{(\ell)} & N_{22}^{(\ell)} \end{bmatrix}$ for $\ell = 1,\dots,4$. By Lemma 3.4 we know that it suffices to find a matrix norm such that $\|N^{(\ell)}\| < 1$, for $\ell = 1,2,3,4$. The computation gives:

$$N_{11}^{(1)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 1 & \beta \\ 0 & 2 & -\beta & -1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & -\beta & 1 \end{bmatrix} \qquad N_{12}^{(1)} = \frac{1-\beta}{2}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{21}^{(1)} = \begin{bmatrix} 1/4+2\alpha & 0 & 1/8+\alpha & \alpha-\beta/8 \\ 0 & 1/4+2\alpha & -\alpha+\beta/8 & -1/8-\alpha \\ 0 & 0 & 1/8+\alpha & -\alpha+\beta/8 \\ 0 & 0 & -\alpha+\beta/8 & 1/8+\alpha \end{bmatrix} \qquad N_{22}^{(1)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{11}^{(2)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 1 & -\beta \\ 0 & 2 & -\beta & 1 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix} \qquad N_{12}^{(2)} = \frac{1-\beta}{2}\begin{bmatrix} -2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{21}^{(2)} = \begin{bmatrix} -1/4-2\alpha & 0 & -1/8-\alpha & \alpha-\beta/8 \\ 0 & 1/4+2\alpha & -\alpha+\beta/8 & 1/8+\alpha \\ 0 & 0 & -1/8-\alpha & -\alpha+\beta/8 \\ 0 & 0 & \alpha-\beta/8 & 1/8+\alpha \end{bmatrix} \qquad N_{22}^{(2)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{11}^{(3)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & -1 & -\beta \\ 0 & 2 & \beta & 1 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & -\beta & 1 \end{bmatrix} \qquad N_{12}^{(3)} = \frac{1-\beta}{2}\begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$N_{21}^{(3)} = \begin{bmatrix} -1/4-2\alpha & 0 & 1/8+\alpha & \alpha-\beta/8 \\ 0 & -1/4-2\alpha & -\alpha+\beta/8 & -1/8-\alpha \\ 0 & 0 & -1/8-\alpha & \alpha-\beta/8 \\ 0 & 0 & \alpha-\beta/8 & -1/8-\alpha \end{bmatrix} \qquad N_{22}^{(3)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_{11}^{(4)} = \frac{1}{4}\begin{bmatrix} 2 & 0 & -1 & \beta \\ 0 & 2 & \beta & -1 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{bmatrix} \qquad N_{12}^{(4)} = \frac{1-\beta}{2}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$N_{21}^{(4)} = \begin{bmatrix} 1/4+2\alpha & 0 & -1/8-\alpha & \alpha-\beta/8 \\ 0 & -1/4-2\alpha & -\alpha+\beta/8 & 1/8+\alpha \\ 0 & 0 & 1/8+\alpha & \alpha-\beta/8 \\ 0 & 0 & -\alpha+\beta/8 & -1/8-\alpha \end{bmatrix} \qquad N_{22}^{(4)} = \frac{1+\beta}{4}\begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In $\mathbb{R}^4$, we use the norm $\|\cdot\|_\theta, \theta > 0$ defined at the beginning of the proof of Lemma 3.5, using $\|\cdot\|_\infty$ in $\mathbb{R}^4$, i.e $\|U\|_\theta = \|X\|_\infty + \theta\|Y\|_\infty$ where $U =$

$\begin{bmatrix} X \\ Y \end{bmatrix}$, $X, Y \in \mathbb{R}^2$. Using (3.3), we deduce that for $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \in \mathbb{R}^{4\times4}$ with $A_{ij} \in \mathbb{R}^{2\times2}$, we have $\|A\|_\theta \le \max(\|A_{11}\|_\infty, \|A_{12}\|_\infty/\theta + \|A_{22}\|_\infty)$.

For $\ell = 1, 2, 3, 4$, we can then bound $\|N_{ij}^{(\ell)}\|_\theta$. Let $\mu := 1 + \frac{1}{\theta} > 1$ and assume $\mu < 2$. Then

$$\begin{aligned}
&\|N_{11}^{(\ell)}\|_\theta \le \tfrac{1}{4}\max(2, \mu(1-\beta)) =: \gamma_{11},\\
&\|N_{12}^{(\ell)}\|_\theta \le |1-\beta| =: \gamma_{12},\\
&\|N_{21}^{(\ell)}\|_\theta \le \max(|1/4 + 2\alpha|, \mu(|-\alpha+\beta/8| + |1/8+\alpha|)) =: \gamma_{21},\\
&\|N_{22}^{(\ell)}\|_\theta \le \tfrac{|1+\beta|}{2} =: \gamma_{22}.
\end{aligned}$$

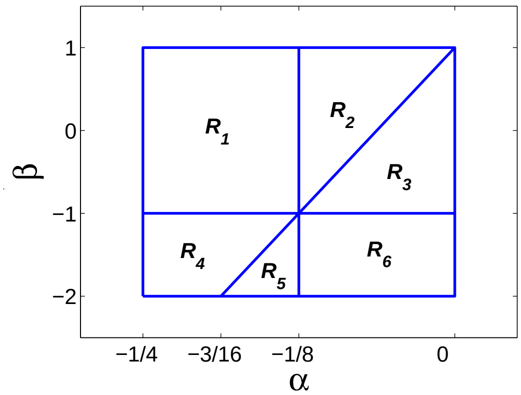


Figure 3.2: The subsets $R_1, \ldots, R_6$ used in the proof of Theorem 3.7.

We need to bound the $\gamma's$. The analysis below shows that it is enough to consider $(\alpha, \beta)$ in the rectangle $[-\frac{1}{4}, 0] \times [-2, 1]$. To compute $\gamma_{21}$, which is the most difficult, we divide the rectangle $[-\frac{1}{4}, 0] \times [-2, 1]$ into open subsets $R_1, \ldots, R_6$ as shown in Figure 3.2. In these regions a lengthy, but straightforward calculation gives the following values for the numbers $\gamma_{ij}$ and the quantity $\pi_\gamma := (1-\gamma_{11})(1-\gamma_{22})$:

| | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $R_6$ |
|---|---|---|---|---|---|---|
| $\gamma_{11}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{\mu}{4}(1-\beta)$ | $\frac{\mu}{4}(1-\beta)$ | $\frac{\mu}{4}(1-\beta)$ |
| $\gamma_{22}$ | $\frac{1}{2}(1+\beta)$ | $\frac{1}{2}(1+\beta)$ | $\frac{1}{2}(1+\beta)$ | -$\frac{1}{2}(1+\beta)$ | -$\frac{1}{2}(1+\beta)$ | -$\frac{1}{2}(1+\beta)$ |
| $\gamma_{12}$ | $1-\beta$ | $1-\beta$ | $1-\beta$ | $1-\beta$ | $1-\beta$ | $1-\beta$ |
| $\gamma_{21}$ | $-\mu(2\alpha + \frac{1-\beta}{8})$ | $\frac{\mu}{8}(1+\beta)$ | $2\alpha + \frac{1}{4}$ | $-2\alpha - \frac{1}{4}$ | -$\frac{\mu}{8}(1+\beta)$ | $\mu(2\alpha + \frac{1-\beta}{8})$ |
| $\pi_\gamma$ | $\frac{1}{4}(1-\beta)$ | $\frac{1}{4}(1-\beta)$ | $\frac{1}{4}(1-\beta)$ | $\nu$ | $\nu$ | $\nu$ |

where $\nu := \frac{1}{8}(3+\beta)^2 - \epsilon$, and where $\epsilon > 0$ can be made arbitrary small by choosing $\theta$ sufficiently big.

We need to compute subsets $S_j$ of $R_j$ so that $\gamma_{12}\gamma_{21} < \pi_\gamma$ for $(\alpha, \beta) \in S_j$, $j = 1, \ldots, 6$.

- On $R_1$, we need $-\mu(1-\beta)(2\alpha + \frac{1-\beta}{8}) < \frac{1-\beta}{4}$. This is satisfied if $\beta < 16\alpha + 3$.

- On $R_2$, the condition is $\mu(1-\beta)\frac{1+\beta}{8} < \frac{1-\beta}{4}$ which holds if $\beta < 1$.
- On $R_3$, we should have $(1-\beta)(\frac{1}{4}+2\alpha) < \frac{1-\beta}{4}$. This is true for $\alpha < 0$.
- On $R_4$, the inequality is $(1-\beta)(-\frac{1}{4}-2\alpha) < \frac{(3+\beta)^2}{8} - \epsilon$. Since this should hold for all $\epsilon > 0$ we can drop the $\epsilon$ (this is also true for $R_5$, and $R_6$) and we obtain
$$\alpha < \frac{11+4\beta+\beta^2}{16(1-\beta)} \text{ or } \beta < 8\alpha - 2 + \sqrt{(8\alpha+1)(8\alpha-7)}.$$
- The condition on $R_5$ takes the form $(1-\beta)\frac{1+\beta}{8} < \frac{(3+\beta)^2}{8}$ which holds if $\beta > -\frac{5}{3}$.
- Finally on $R_6$, the inequality $\mu(1-\beta)(2\alpha + \frac{1-\beta}{8}) < \frac{(3+\beta)^2}{8}$ is true for $\beta < \frac{2\alpha-1}{2\alpha+1}$.

This defines the subregions $S_j$ of $R_j$ for $j = 1, \ldots, 6$. It remains to show that the result also holds on the curve segments forming the interior boundaries between the regions. These curves can be identified as $\beta = -1$, $\alpha = -1/8$, and $\beta = 16\alpha + 1$. We divide these curves into segments as follows:

$$C_{11} := \{(\alpha,\beta) : -\frac{1}{4} < \alpha < \frac{1}{8},\ \beta = -1\}, \quad C_{12} := \{(\alpha,\beta) : -\frac{1}{8} \le \alpha < 0,\ \beta = -1\},$$
$$C_{21} := \{(\alpha,\beta) : \alpha = -\frac{1}{8},\ -\frac{5}{3} < \beta < -1\}, \quad C_{22} := \{(\alpha,\beta) : \alpha = -\frac{1}{8},\ -1 \le \beta < \psi\},$$
$$C_{23} := \{(\alpha,\beta) : \alpha = -\frac{1}{8},\ \psi \le \beta < 1\}, \quad C_{31} := \{(\alpha, 16\alpha+1) : -\frac{1}{6} < \alpha < -\frac{1}{8}\},$$
$$C_{32} := \{(\alpha, 16\alpha+1) : -\frac{1}{8} \le \alpha < -\frac{1}{8\mu}\}, \quad C_{33} := \{(\alpha, 16\alpha+1) : -\frac{1}{8\mu} \le \alpha < 0\},$$

where $\psi := -1 + \frac{2}{1+\theta}$. The values of $\gamma_{ij}$ and $\delta := (1-\gamma_{11})(1-\gamma_{22}) - \gamma_{12}\gamma_{21}$ on the different segments are shown in the following table:

| Segment | $\gamma_{11}$ | $\gamma_{22}$ | $\gamma_{12}$ | $\gamma_{21}$ | $\delta$ |
|---|---|---|---|---|---|
| $C_{11}$ | $\mu/2$ | $0$ | $2$ | $-2\mu(\alpha+\frac{1}{8})$ | $1+4\mu\alpha$ |
| $C_{12}$ | $\mu/2$ | $0$ | $2$ | $2\mu(\alpha+\frac{1}{8})$ | $1-\mu-4\mu\alpha$ |
| $C_{21}$ | $\frac{\mu}{4}(1-\beta)$ | $-\frac{1+\beta}{2}$ | $1-\beta$ | $-\frac{\mu}{8}(\beta+1)$ | $\frac{1}{4}(6-\mu+\beta(\mu+2))$ |
| $C_{22}$ | $\frac{\mu}{4}(1-\beta)$ | $\frac{1+\beta}{2}$ | $1-\beta$ | $\frac{\mu}{8}(\beta+1)$ | $\frac{1}{4}(1-\beta)(2-\mu)$ |
| $C_{23}$ | $\frac{1}{2}$ | $\frac{1+\beta}{2}$ | $1-\beta$ | $\frac{\mu}{8}(\beta+1)$ | $\frac{1}{8}(1-\beta)(2-\mu-\mu\beta)$ |
| $C_{31}$ | $-4\mu\alpha$ | $-1-8\alpha$ | $-16\alpha$ | $-\frac{\mu}{4}(1+8\alpha)$ | $2+4\alpha(2+\mu)$ |
| $C_{32}$ | $-4\mu\alpha$ | $1+8\alpha$ | $-16\alpha$ | $\frac{\mu}{4}(1+8\alpha)$ | $-4\alpha(2-\mu)$ |
| $C_{33}$ | $\frac{1}{2}$ | $1+8\alpha$ | $-16\alpha$ | $\frac{\mu}{4}(1+8\alpha)$ | $4(\mu-1)\alpha+32\mu\alpha^2$ |

where as before we set $\mu := \frac{1}{\theta} + 1$.

We have $C^1$-convergence for a specific value of $(\alpha, \beta)$ provided we can find a $\theta > 0$ so that $\delta > 0$. This is always possible for any point in the open interval (see Figure 3.3). ☐

COROLLARY 3.8. *For $\alpha = \frac{\beta}{4(1-\beta)}$ and $\beta \in [-5/3, 0)$, the scheme $HRC^1$ is $C^1$-convergent.*

PROOF. If $\beta \in [-5/3, 0)$ and $\alpha = \frac{\beta}{4(1-\beta)}$, then $(\alpha, \beta) \in \mathcal{R}$, see Figure 3.1. ☐

]

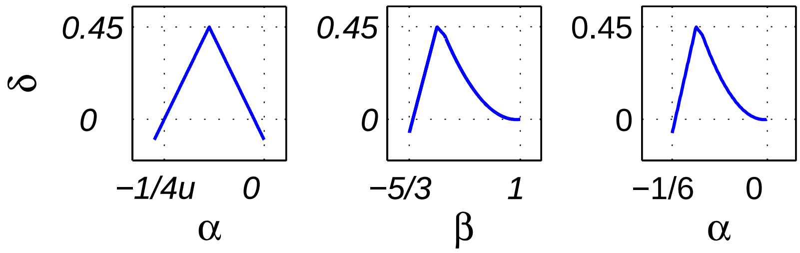


Figure 3.3: The value of $\delta = (1-\gamma_{11})(1-\gamma_{22}) - \gamma_{12}\gamma_{21}$ on the curve segments defined by $\beta = -1$ (left), $\alpha = -\frac{1}{8}$ (center) and $\beta = 16\alpha + 1$ (right) corresponding to $\theta = 10$ or $\mu = 1.1$.

## 4 The control grid

In order to obtain a geometric formulation of the $HRC^1$-algorithm we define *control coefficients* $a_{ij}$ and *control points* $A_{ij}$ relative to the rectangle $R = [a, b] \times [c, d]$ as follows:

$$
\begin{aligned}
A_{00} &= (a, c, a_{00}), && \text{where } a_{00} = f(a, c),\\
A_{10} &= (a + \frac{h}{\lambda}, c, a_{10}), && \text{where } a_{10} = f(a, c) + \frac{hp(a, c)}{\lambda},\\
A_{20} &= (b - \frac{h}{\lambda}, c, a_{20}), && \text{where } a_{20} = f(b, c) - \frac{hp(b, c)}{\lambda},\\
A_{30} &= (b, c, a_{30}), && \text{where } a_{30} = f(b, c),\\
A_{31} &= (b, c + \frac{k}{\lambda}, a_{31}), && \text{where } a_{31} = f(b, c) + \frac{kq(b, c)}{\lambda},\\
A_{32} &= (b, d - \frac{k}{\lambda}, a_{32}), && \text{where } a_{32} = f(b, d) - \frac{kq(b, d)}{\lambda},\\
A_{33} &= (b, d, a_{33}), && \text{where } a_{33} = f(b, d),\\
A_{23} &= (b - \frac{h}{\lambda}, d, a_{23}), && \text{where } a_{23} = f(b, d) - \frac{hp(b, d)}{\lambda},\\
A_{13} &= (a + \frac{h}{\lambda}, d, a_{13}), && \text{where } a_{13} = f(a, d) + \frac{hp(a, d)}{\lambda},\\
A_{03} &= (a, d, a_{03}), && \text{where } a_{03} = f(a, d),\\
A_{02} &= (a, d - \frac{k}{\lambda}, a_{02}), && \text{where } a_{02} = f(a, d) - \frac{kq(a, c)}{\lambda},\\
A_{01} &= (a, c + \frac{k}{\lambda}, a_{01}), && \text{where } a_{01} = f(a, c) + \frac{kq(a, c)}{\lambda}.
\end{aligned} \tag{4.1}
$$

Here $h := b - a$, $k := d - c$ and $\lambda \geq 2$ is a real number to be chosen. The 12 control points are located on the boundary of $R$. We can obtain a control polygon-like structure by adding the four interiour points $A_{11} = A_{10} + A_{01} - A_{00}$, $A_{21} = A_{20} + A_{31} - A_{30}$, $A_{22} = A_{23} + A_{32} - A_{33}$, and $A_{12} = A_{13} + A_{02} - A_{03}$, see Figure 4.1.

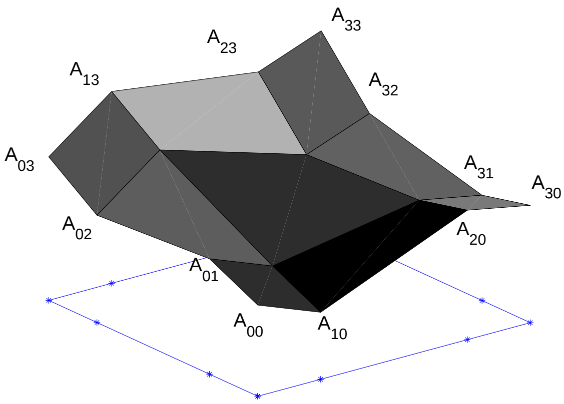


Figure 4.1: Control Grid

If $f$ is the $HRC^1$-interpolant constructed from the given data at the vertices of $R$ then the parametric surface $(x, y, f(x, y))$ with $(x, y) \in R$ interpolates the corner control points $A_{00}, A_{03}, A_{30}, A_{33}$. Moreover each corner rectangle in the control polygon defines a plane which is part of the tangent plane at that vertex. For example the plane containing the four points $A_{00}, A_{10}, A_{01}$, and $A_{11}$ defines the tangent plane to the surface at $A_{00}$.

After one step of subdivision the rectangle $R$ is divided into four subrectangles (Cf. Figure 2.1). On each of the four sub-rectangles we can compute new control points $\bar{A}_{ij}$. To compute these control points we can use (4.1) shifted to each subrectangle. In particular, we replace $h$ and $k$ by $h/2$ and $k/2$ respectively.

By using (2.1), (2.2), (2.3) and (2.4) and inverting the formulas in (4.1) it is possible to express the new control coefficients $\{\bar{a}_{ij}\}$ in terms of the original

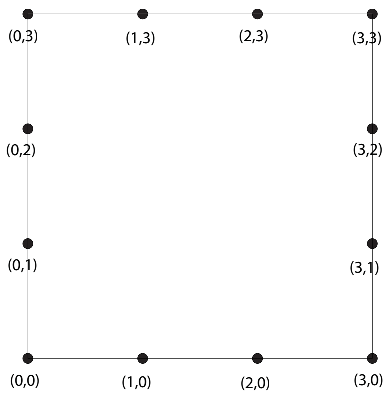


Figure 4.2: The control points projected on the original rectangle

$\xrightarrow{S}$

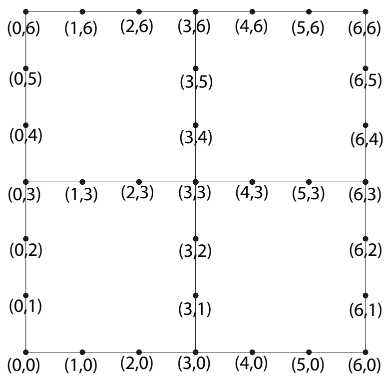


Figure 4.3: The projected control points after one subdivision

control coefficients $a_{ij}$. We restrict attention to the one parameter family given by $\alpha = \beta/(4(1-\beta))$. We also write $\lambda = u(1-\beta)$, where $u$ is a free parameter to be chosen later. With the aid of a computer algebra system it can be proved that:

PROPOSITION 4.1. *Suppose $\alpha = \beta/(4(1-\beta))$ and $\lambda = u(1-\beta)$. With the indexing used in Figures 4.2 and 4.3 we have*

$$
\begin{bmatrix}
\bar{a}_{0,0}\\ \bar{a}_{1,0}\\ \bar{a}_{2,0}\\ \bar{a}_{3,0}\\ \bar{a}_{4,0}\\ \bar{a}_{5,0}\\ \bar{a}_{6,0}\\ \bar{a}_{0,1}\\ \bar{a}_{3,1}\\ \bar{a}_{6,1}\\ \bar{a}_{0,2}\\ \bar{a}_{3,2}\\ \bar{a}_{6,2}\\ \bar{a}_{0,3}\\ \bar{a}_{1,3}\\ \bar{a}_{2,3}\\ \bar{a}_{3,3}\\ \bar{a}_{4,3}\\ \bar{a}_{5,3}\\ \bar{a}_{6,3}\\ \bar{a}_{0,4}\\ \bar{a}_{3,4}\\ \bar{a}_{6,4}\\ \bar{a}_{0,5}\\ \bar{a}_{3,5}\\ \bar{a}_{6,5}\\ \bar{a}_{0,6}\\ \bar{a}_{1,6}\\ \bar{a}_{2,6}\\ \bar{a}_{3,6}\\ \bar{a}_{4,6}\\ \bar{a}_{5,6}\\ \bar{a}_{6,6}
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{vy}{4u} & \frac{1}{4}v\gamma & \frac{1}{4}w\gamma & \frac{wy}{4u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{4}y & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{wy}{4u} & \frac{1}{4}w\gamma & \frac{1}{4}v\gamma & \frac{vy}{4u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{4}x & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}x & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{vy}{4u} & 0 & 0 & 0 & \frac{1}{4}v\gamma & 0 & \frac{1}{4}w\gamma & 0 & \frac{wy}{4u} & 0 & 0 & 0\\
\frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u}\\
0 & 0 & 0 & \frac{vy}{4u} & 0 & \frac{1}{4}v\gamma & 0 & \frac{1}{4}w\gamma & 0 & 0 & 0 & \frac{wy}{4u}\\
\frac{1}{4}y & 0 & 0 & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & \frac{1}{4}y & 0 & 0 & 0\\
\frac{1}{4}x & \frac{1}{4} & 0 & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & \frac{1}{4}x & \frac{1}{4} & 0 & 0\\
\frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u}\\
\frac{1}{4}x & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{1}{4}x & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{1}{4}x & \frac{u\gamma}{8} & \frac{u\gamma}{8} & \frac{1}{4}x\\
\frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u}\\
0 & 0 & \frac{1}{4} & \frac{1}{4}x & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4}x\\
0 & 0 & 0 & \frac{1}{4}y & 0 & \frac{u\gamma}{4} & 0 & \frac{u\gamma}{4} & 0 & 0 & 0 & \frac{1}{4}y\\
\frac{wy}{4u} & 0 & 0 & 0 & \frac{1}{4}w\gamma & 0 & \frac{1}{4}v\gamma & 0 & \frac{vy}{4u} & 0 & 0 & 0\\
\frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{wx}{4u} & \frac{1}{8}w\gamma & \frac{1}{8}w\gamma & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u} & \frac{1}{8}v\gamma & \frac{1}{8}v\gamma & \frac{vx}{4u}\\
0 & 0 & 0 & \frac{wy}{4u} & 0 & \frac{1}{4}w\gamma & 0 & \frac{1}{4}v\gamma & 0 & 0 & 0 & \frac{vy}{4u}\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}x & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}x\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{vy}{4u} & \frac{1}{4}v\gamma & \frac{1}{4}w\gamma & \frac{wy}{4u}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4}y & \frac{u\gamma}{4} & \frac{u\gamma}{4} & \frac{1}{4}y\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{wy}{4u} & \frac{1}{4}w\gamma & \frac{1}{4}v\gamma & \frac{vy}{4u}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a_{0,0}\\ a_{1,0}\\ a_{2,0}\\ a_{3,0}\\ a_{0,1}\\ a_{3,1}\\ a_{0,2}\\ a_{3,2}\\ a_{0,3}\\ a_{1,3}\\ a_{2,3}\\ a_{3,3}
\end{bmatrix},
$$

*where*

$$\gamma := -\beta,\ v := u+1,\ w := u-1,\ x := 1+u\beta,\ y := 2+u\beta. \tag{4.2}$$

*We denote the transformation matrix by $S$.*

## 5 Local shape constrains

We consider only the function case, where the starting data are values of $f, p$ and $q$ on the vertices of a rectangle $[a,b]\times[c,d]$ in $\mathbb{R}^2$. We consider the one parameter family given by $\alpha = \frac{\beta}{4(1-\beta)}$ with $\beta \in [-1, 0)$. Corollary 3.8 implies $C^1$-convergence for any $\beta \in [-1, 0)$. We let $\lambda = u(1-\beta)$, where $u$ is a free parameter. We also recall that for $\beta = -1$, the interpolant is the Sibson-Thomson element which is piecewise quadratic.

*5.1 Positive interpolants*

We prove that if the control grid is positive, then the interpolant is positive. We use this result to give an algorithm to get a positive interpolant whenever the initial data make it possible.

PROPOSITION 5.1. *Suppose that* $1 \le u \le -1/\beta$. *If the initial control grid is positive, i.e.* $a_{k\ell} \ge 0$ *for all* $k, \ell$, *then the interpolant* $f$ *is positive.*

PROOF. With the hypothesis $1 \le u \le -1/\beta$ and $-1 \le \beta < 0$ the quantities $\gamma, v, w, x, y$ in (4.2) are nonnegative so that all entries in the matrix $S$ in Proposition 4.1 are nonnegative. In the subdivision process we apply the matrix $S$ recursively and it follows that all control coefficients on all levels are nonnegative. But then the values of the function $f$ on $\cup(\mathcal{P}_n \times \mathcal{Q}_n) = \mathcal{P} \times \mathcal{Q}$ are nonnegative. We have the result on $[a, b] \times [c, d]$ by continuous extension. □

We describe an algorithm to build a nonnegative interpolant on $[a, b] \times [c, d]$. Suppose that the initial data satisfy

$$\begin{aligned} &f(a,c) \ge 0 \text{ and } (p(a,c) \ge 0, q(a,c) \ge 0 \text{ if } f(a,c) = 0),\\ &f(b,c) \ge 0 \text{ and } (p(b,c) \le 0, q(b,c) \ge 0 \text{ if } f(b,c) = 0),\\ &f(a,d) \ge 0 \text{ and } (p(a,d) \ge 0, q(a,d) \le 0 \text{ if } f(a,d) = 0),\\ &f(b,d) \ge 0 \text{ and } (p(b,d) \le 0, q(b,d) \le 0 \text{ if } f(b,d) = 0). \end{aligned} \tag{5.1}$$

ALGORITHM 5.1. *Let* $h := b - a$, $k := d - c$ *and choose* $\lambda \ge 2$ *such that*

$$\begin{array}{lcl} a_{10} = f(a,c) + h\frac{p(a,c)}{\lambda} \ge 0 & , & a_{01} = f(a,c) + k\frac{q(a,c)}{\lambda} \ge 0,\\ a_{20} = f(b,c) - h\frac{p(b,c)}{\lambda} \ge 0 & , & a_{31} = f(b,c) + k\frac{q(b,c)}{\lambda} \ge 0,\\ a_{13} = f(a,d) + h\frac{p(a,d)}{\lambda} \ge 0 & , & a_{02} = f(a,d) - k\frac{q(a,d)}{\lambda} \ge 0,\\ a_{23} = f(b,d) - h\frac{p(b,d)}{\lambda} \ge 0 & , & a_{32} = f(b,d) - k\frac{q(b,d)}{\lambda} \ge 0. \end{array} \tag{5.2}$$

*Define* $\beta = \frac{1}{1-\lambda}$ *and* $\alpha = \frac{\beta}{4(1-\beta)}$.

*Perform* $HRC^1$ *defined by* (2.1), (2.2), (2.3) *and* (2.4).

Since $\lambda \ge 2$, we obtain $\beta \in [-1, 0)$ so that the scheme is $C^1$-convergent. In view of (5.1), since $a_{00} = f(a, c) \ge 0$, $a_{30} = f(b, c) \ge 0$, $a_{33} = f(b, d) \ge 0$ and $a_{03} = f(a, d) \ge 0$ it is possible to choose $\lambda \ge 2$ so that the remaining control coefficients are nonnegative. By Proposition 5.1 the interpolant $f$ is nonnegative.

EXAMPLE 5.1.

*In Figure 5.1, we have computed three nonnegative* $HRC^1$*-interpolants choosing the same data on the vertices of* $[0, 1]^2$ *except* $p(0, 1)$ *which values are successively -1, -1.5 and -3. The values of* $\lambda$ *are the smallest one so that* (5.2) *holds. In the first case* $(f_1, p_1, q_1)$, *we have* $\lambda = 2$ *so that* $\alpha = -1/8$ *and* $\beta = -1$ *and we obtain the quadratic spline interpolant with piecewise linear derivatives.*

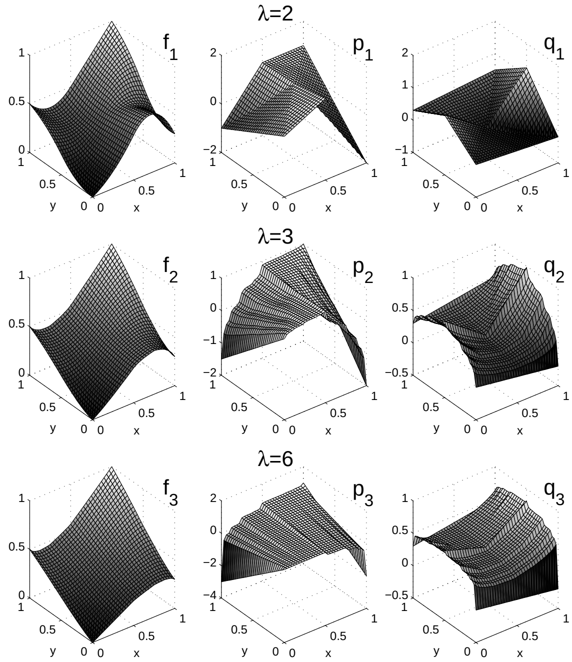


Figure 5.1: Nonnegative interpolants

*5.2 Monotone interpolants*

We prove that if the control polygon is increasing in the variable $x$ then the interpolant is an increasing function in $x$. We use this result to give an algorithm to get an increasing interpolant in $x$ as soon as the data make it possible.

LEMMA 5.2. *Suppose that $u = -1/\beta$ with $\beta \in [-1, 0)$.*

$$\text{If} \left\{ \begin{array}{c} a_{00} \le a_{10} \le a_{20} \le a_{30}, \\ a_{01} \le a_{31}, \\ a_{02} \le a_{32}, \\ a_{03} \le a_{13} \le a_{23} \le a_{33}, \end{array} \right. \quad \text{then} \left\{ \begin{array}{c} \bar a_{00} \le \bar a_{10} \le \bar a_{20} \le \bar a_{30} \le \bar a_{40} \le \bar a_{50} \le \bar a_{60}, \\ \bar a_{01} \le \bar a_{31} \le \bar a_{61}, \\ \bar a_{02} \le \bar a_{32} \le \bar a_{62}, \\ \bar a_{03} \le \bar a_{13} \le \bar a_{23} \le \bar a_{33} \le \bar a_{43} \le \bar a_{53} \le \bar a_{63}, \\ \bar a_{04} \le \bar a_{34} \le \bar a_{64}, \\ \bar a_{05} \le \bar a_{35} \le \bar a_{65}, \\ \bar a_{06} \le \bar a_{16} \le \bar a_{26} \le \bar a_{36} \le \bar a_{46} \le \bar a_{56} \le \bar a_{66}. \end{array} \right.$$

PROOF. We define (Cf. Figures 4.2 and 4.3) horizontal differences $d_{i,j} := a_{i+1,j} - a_{i,j}$ for $i = 0, 1, 2$ and $j = 1, 2$, $d_{0,j} := a_{3,j} - a_{0,j}$ for $j = 1, 2$, $\bar d_{i,j} := \bar a_{i+1,j} - \bar a_{i,j}$ for $i = 0, 1, \ldots, 5$ and $j = 0, 3, 6$, and $\bar d_{i,j} := \bar a_{i+3,j} - \bar a_{i,j}$ for $i = 0, 3$ and $j = 1, 2, 4, 5$. We use the results of Proposition 4.1 and a computer algebra system to obtain:

$$
\begin{bmatrix}
\bar d_{0,0}\\ \bar d_{1,0}\\ \bar d_{2,0}\\ \bar d_{3,0}\\ \bar d_{4,0}\\ \bar d_{5,0}\\ \bar d_{0,1}\\ \bar d_{3,1}\\ \bar d_{0,2}\\ \bar d_{3,2}\\ \bar d_{0,3}\\ \bar d_{1,3}\\ \bar d_{2,3}\\ \bar d_{3,3}\\ \bar d_{4,3}\\ \bar d_{5,3}\\ \bar d_{0,4}\\ \bar d_{3,4}\\ \bar d_{0,5}\\ \bar d_{3,5}\\ \bar d_{0,6}\\ \bar d_{1,6}\\ \bar d_{2,6}\\ \bar d_{3,6}\\ \bar d_{4,6}\\ \bar d_{5,6}
\end{bmatrix}
=
\begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4} & 0 & 0 & 0 & 0 & 0\\
-\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4} & 0 & 0 & 0 & 0 & 0\\
-\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4} & 0 & 0 & 0 & 0 & 0\\
\frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0\\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0\\
\frac{1-\beta}{4} & \frac{1-\beta}{8} & 0 & \frac{1-\beta}{8} & \frac{1+\beta}{8} & \frac{1+\beta}{4} & \frac{1+\beta}{8} & 0\\
0 & \frac{1-\beta}{8} & \frac{1-\beta}{4} & \frac{1-\beta}{8} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1+\beta}{4}\\
\frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0\\
0 & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & 0\\
0 & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & 0\\
0 & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & -\frac{\beta}{8} & 0 & -\frac{\beta}{8} & 0\\
0 & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & 0\\
0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4}\\
\frac{1+\beta}{4} & \frac{1+\beta}{8} & 0 & \frac{1+\beta}{8} & \frac{1-\beta}{8} & \frac{1-\beta}{4} & \frac{1-\beta}{8} & 0\\
0 & \frac{1+\beta}{8} & \frac{1+\beta}{4} & \frac{1+\beta}{8} & \frac{1-\beta}{8} & 0 & \frac{1-\beta}{8} & \frac{1-\beta}{4}\\
0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0\\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2}\\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4}\\
0 & 0 & 0 & 0 & 0 & -\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4}\\
0 & 0 & 0 & 0 & 0 & -\frac{\beta}{4} & -\frac{\beta}{2} & -\frac{\beta}{4}\\
0 & 0 & 0 & 0 & 0 & \frac{1+\beta}{4} & \frac{1+\beta}{2} & \frac{1+\beta}{4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
d_{0,0}\\ d_{1,0}\\ d_{2,0}\\ d_{0,1}\\ d_{0,2}\\ d_{0,3}\\ d_{1,3}\\ d_{2,3}
\end{bmatrix}
$$

The hypothesis implies that $d_{k\ell} \geq 0$. Since $-1 \leq \beta < 0$, we obtain $\bar d_{ij} \geq 0$. $\square$

We can extend the result recursively on each sub rectangle of $\mathcal{P}_n \times \mathcal{Q}_n$. At each step the control grid is increasing in the direction $x$ so that the function $p$ is nonnegative on $\cup(\mathcal{P}_n \times \mathcal{Q}_n) = \mathcal{P} \times \mathcal{Q}$ By continuous extension $p$ is nonnegative on $[a,b] \times [c,d]$ and $f$ is increasing in $x$.

PROPOSITION 5.3. *Suppose that $u = -1/\beta$ with $\beta \in [-1,0)$. If the initial grid is increasing in $x$, i.e.* $\begin{cases} a_{00} \leq a_{10} \leq a_{20} \leq a_{30},\\ a_{01} \leq a_{31},\\ a_{02} \leq a_{32},\\ a_{03} \leq a_{13} \leq a_{23} \leq a_{33}, \end{cases}$ *then the interpolant $f$ is increasing in $x$.*

We define an algorithm to construct an interpolant on $[a,b] \times [c,d]$ that is increasing in $x$. Suppose that the initial data satisfy
$p(a,c) \geq 0$, $p(b,c) \geq 0$, $f(a,c) < f(b,c)$ and
$p(a,d) \geq 0$, $p(b,d) \geq 0$, $f(a,d) < f(b,d)$.

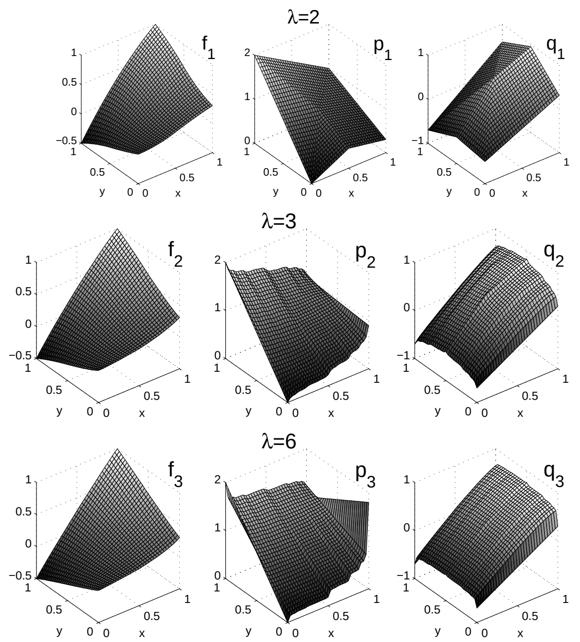


Figure 5.2: Increasing interpolants in the variable $x$

Algorithm 5.2. *Let $h := b-a, k =: d-c$ and choose $\lambda \geq 2$ such that*

$$
\begin{array}{rcl}
a_{10} = f(a,c) + h\frac{p(a,c)}{\lambda} & \leq & a_{20} = f(b,c) - h\frac{p(b,c)}{\lambda}, \\
a_{01} = f(a,c) + k\frac{q(a,c)}{\lambda} & \leq & a_{31} = f(b,c) + k\frac{q(b,c)}{\lambda}, \\
a_{02} = f(a,d) - k\frac{q(a,d)}{\lambda} & \leq & a_{32} = f(b,d) - k\frac{q(b,d)}{\lambda}, \\
a_{13} = f(a,d) + h\frac{p(a,d)}{\lambda} & \leq & a_{23} = f(b,d) - h\frac{p(b,d)}{\lambda}.
\end{array}
\tag{5.3}
$$

*Define $\beta = \frac{1}{1-\lambda}$ and $\alpha = \frac{\beta}{4(1-\beta)}$.*

*Perform $HRC^1$ defined by* (2.1), (2.2), (2.3), *and* (2.4).

Since the $\beta$ used in this algorithm always belongs to the interval $[-1,0)$ the interpolating scheme is $C^1$-convergent. Morover, since $a_{00} \leq a_{10}$, $a_{20} \leq a_{30}$, $a_{03} \leq a_{13}$, $a_{23} \leq a_{33}$, the control grid and the interpolant $f$ are increasing in the variable $x$.

Example 5.2. *In the 3 following pictures (Figure 5.2), we have computed three interpolants which are monotone in the $x$-direction. We choose the same data on the vertices of $[0,1]^2$ except $p(1,0)$ which values are successively 0.3, 0.9 and 1.8. The values of $\lambda$ are the smallest one satisfying* (5.3). *In the first case $(f_1, p_1, q_1)$, we have $\lambda = 2$ so that $\alpha = -1/8$ and $\beta = -1$ and we obtain the quadratic spline interpolant with piecewise linear derivatives.*

## 6 Examples of global constrains

### *6.1 The First Example*

We start with the grid defined by

$$\{(x_i, y_j)\} = \{0, 0.25, 0.7, 0.92, 1\} \times \{0, 0.2, 0.6, 1\}$$

and sub-rectangles $R_{i,j}, i = 1, \ldots, 4, j = 1, \ldots, 3$. The initial data for the function $f$ at the vertices of the grid are

| $x \backslash y$ | 0 | 0.2 | 0.6 | 1 |
|---|---|---|---|---|
| 0 | 0 | −0.9511 | 0.5878 | 0.0000 |
| 0.25 | 0.0625 | −0.8886 | 0.6503 | 0.0625 |
| 0.7 | 0.4900 | −0.4611 | 1.0778 | 0.4900 |
| 0.92 | 0.8464 | −0.1047 | 1.7000 | 1.7000 |
| 1 | 1.0000 | 0.0489 | 1.7000 | 1.7000 |

They are stricly increasing along $x$ except that $f(0.92, 0.6) = f(1, 0.6) = f(0.92, 1) = f(1, 1)$. Since the initial data where sampled from the function $f(x, y) = x^2 - \sin(2\pi y)$, we compute the exact derivatives $p$ and $q$ and we add a random number in $[0, 0.2]$ for $p$. We have an exception for $R_{4,3}$. We choose the example:

$p$

| $x \backslash y$ | 0 | 0.2 | 0.6 | 1 |
|---|---|---|---|---|
| 0 | 0.0388 | 0.1098 | 0.1255 | 0.1675 |
| 0.25 | 0.6810 | 0.6863 | 0.6398 | 0.5743 |
| 0.7 | 1.5138 | 1.4670 | 1.4794 | 1.4851 |
| 0.92 | 1.9664 | 1.9711 | 0 | 0 |
| 1 | 2.0469 | 2.0784 | 0 | 0 |

$q$

| $x \backslash y$ | 0 | 0.2 | 0.6 | 1 |
|---|---|---|---|---|
| 0 | −6.2832 | −5.0832 | 1.9416 | 6.2832 |
| 0.25 | −6.2832 | −5.0832 | 1.9416 | 6.2832 |
| 0.70 | −6.2832 | −5.0832 | 1.9416 | 6.2832 |
| 0.92 | −6.2832 | −5.0832 | 0 | 0 |
| 1 | −6.2832 | −5.0832 | 0 | 0 |

All the derivatives $p = f_x$ are nonnegative except $p(0.92, 0.6) = p(1, 0.6) = p(0.92, 1) = p(1, 1) = 0$. We add $q(0.92, 0.6) = q(1, 0.6) = q(0.92, 1) = q(1, 1) = 0$.

On each subrectangle $R_{i,j}$, we compute the smallest $\lambda_{i,j} \geq 2$ which gives an increasing control grid in the variable $x$. For the rectangle $R_{4,3}$, we can built a constant interpolant with any $\lambda_{4,3}$. Let us choose $\lambda_{4,3} = 2$. Then we compute $\lambda = \max \lambda_{i,j}$, $\beta = \frac{1}{1-\lambda}$ and $\alpha = \frac{\beta}{4(1-\beta)}$. On each subrectangle, we perform $HRC^1$ defined by (2.1), (2.2), (2.3) and (2.4). See Figure 6.1.

### *6.2 The Second Example*

This example was proposed in [2]. The initial grid is $\{-0.07, 0.33, 0.55, 0.69, 0.84, 0.93, 0.98, 1.02, 1.08, 1.13\} \times \{-2.3, -1.61, -0.92, -0.51, -0.22, 0.0\}$, and we will use the sub-rectangles $R_{i,j}, i = 1, \ldots, 9, j = 1, \ldots, 5$. The given data $f^0_{i,j}$ of the

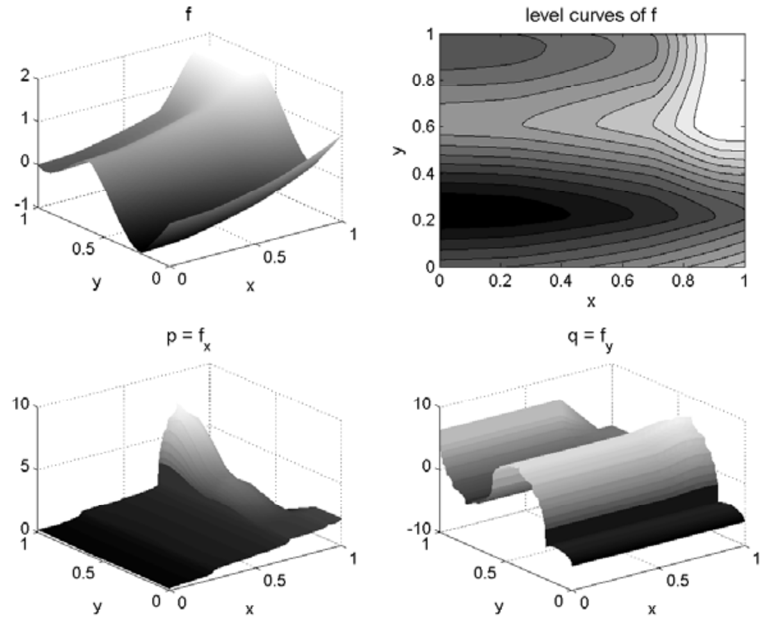


Figure 6.1: Increasing interpolant in the variable $x$ on a mesh with , $\lambda = 3.1843$

function $f$ are

| $x\backslash y$ | $-2.3$ | $-1.61$ | $-0.92$ | $-0.51$ | $-0.22$ | $0.0$ |
|---|---|---|---|---|---|---|
| $-0.07$ | $-34.5400$ | $-13.8200$ | $-10.1000$ | $-7.2600$ | $-5.6600$ | $-4.5300$ |
| $0.33$ | $-34.5400$ | $-13.8200$ | $-10.1000$ | $-7.2600$ | $-5.6600$ | $-4.1300$ |
| $0.55$ | $-34.5400$ | $-13.8200$ | $-10.1000$ | $-7.2600$ | $-4.8800$ | $-3.3500$ |
| $0.69$ | $-34.5400$ | $-13.8200$ | $-10.1000$ | $-4.8200$ | $-3.3400$ | $-2.7300$ |
| $0.84$ | $-34.5400$ | $-13.8200$ | $-2.5200$ | $-2.2200$ | $-1.9800$ | $-1.7800$ |
| $0.93$ | $-34.5400$ | $-2.6800$ | $-1.8800$ | $-1.5600$ | $-1.4100$ | $-1.2800$ |
| $0.98$ | $-3.0600$ | $-2.2800$ | $-1.6300$ | $-1.3200$ | $-1.1500$ | $-1.0500$ |
| $1.02$ | $-2.8600$ | $-1.9200$ | $-1.3900$ | $-1.1000$ | $-0.9200$ | $-0.8100$ |
| $1.08$ | $-2.3700$ | $-1.6000$ | $-1.1700$ | $-0.9000$ | $-0.7200$ | $-0.6000$ |
| $1.13$ | $-1.8900$ | $-1.3000$ | $-0.9500$ | $-0.7100$ | $-0.5400$ | $-0.4100$ |

The data are increasing in the directions $x$ and $y$ (see Figure 6.2) so that we will choose non negative derivatives $p$ and $q$ to get an increasing interpolant in both directions. Notice that if $f^0_{i,j} = f^0_{i+1,j}$, we must choose $p^0_{i,j} = p^0_{i+1,j} = 0$ and $q^0_{i,j} = q^0_{i+1,j}$ and similarly on the other direction. With this exception, we can choose any non negative derivatives $p^0_{i,j}$ and $q^0_{i,j}$ to get an increasing interpolant in both directions. Again on each sub-rectangle $R_{i,j}$, we compute the smallest $\lambda_{i,j} \geq 2$ which gives an increasing control grid in the variable $x$ and in the variable $y$. Then we compute $\lambda = \max \lambda_{i,j}$, $\beta = \frac{1}{1-\lambda}$ and $\alpha = \frac{\beta}{4(1-\beta)}$. On each sub-rectangle, we perform $HRC^1$ defined by (2.1), (2.2), (2.3) and (2.4).

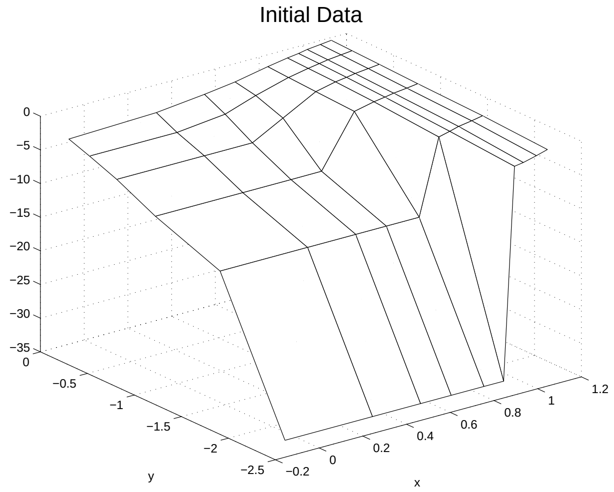


Figure 6.2: Initial Mesh and Function.

*Case 1:* We have computed the initial derivatives $p^0_{i,j}$ and $q^0_{i,j}$ using the standard two point forward differences. The computed value is $\lambda = 4.5455$. See Figure 6.3.

*Case 2:* We took random positive derivatives (between 0 and 2). The computed value is $\lambda = 4.9861$. See Figure 6.4.

## 7 Final Remarks

1. In the shape preserving algorithms the subdivision was carried out using the $HRC^1$-algorithm. The control coefficients were used only to choose parameters to ensure a final interpolant with the desired shape.

2. By applying Proposition 4.1 it is possible to reformulate the $HRC^1$ scheme as a stationary subdivision scheme working on points in $\mathbb{R}^s$. We start with 12 control coefficients $a_{0,0}, a_{1,0}, a_{2,0}, a_{3,0}, a_{0,1}, a_{3,1}, a_{0,2}, a_{3,2}, a_{0,3}, a_{1,3}, a_{2,3}, a_{3,3}$ in $\mathbb{R}^s$, $s \geq 1$, $(\alpha, \beta)$ in the convergence region in Figure 3.1, and $\lambda \geq 2$. Under suitable restrictions on the "rectangular structure" of the initial control coefficients we could then define an algorithm $SRC^1$ based on recursively using the matrix $S$. However we will not consider this any further here.

3. We note that $S$ has negative minors and thus is not a totally positive matrix. For example the $2 \times 2$ minor constructed from the entries in rows 2 and 8 and columns 1 and 2 has the value -1/4 for all values of $\alpha, \beta$ and $\lambda$.

4. Unfortunately, the algorithm $HRC^1$ is in general not able to give a convex interpolant when starting with convex data. To see this we consider the

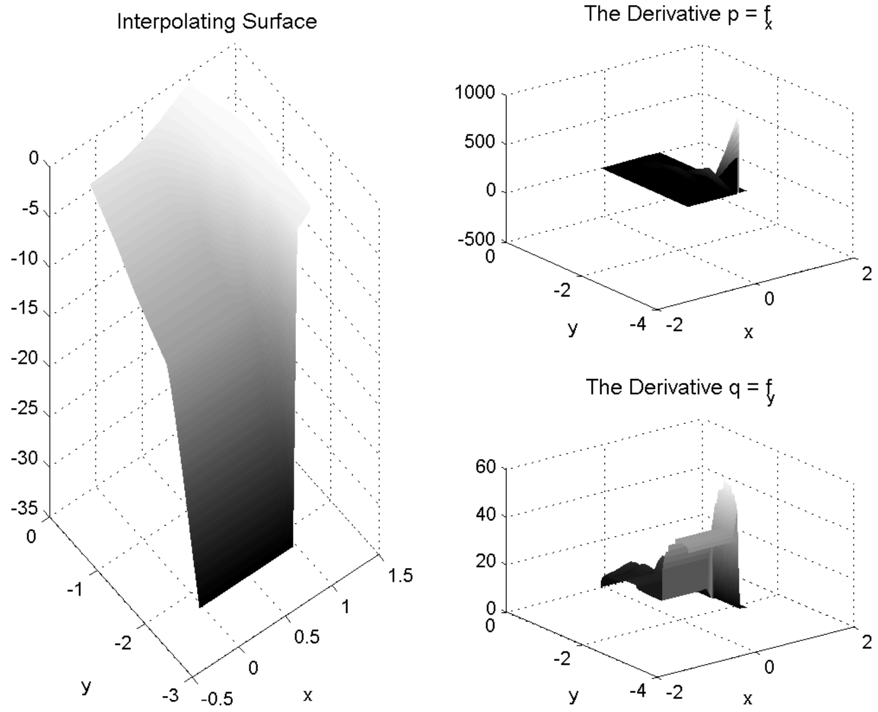


Figure 6.3: Increasing interpolant using forward differences to estimate derivatives.

function given by

$$\phi(x,y) := \begin{cases} (1-x-y)^3 \text{ if } 1-x-y \geq 0 \\ 0 \text{ if } 1-x-y \leq 0 \end{cases}$$

This function is $C^2$ and convex on $[0,1]^2$. To construct a convex $HRC^1$-interpolant $f$ we note that $f$ must be convex along the diagonal $\delta := \{(x,y) \in [0,1]^2 : x+y=1\}$. Since $\phi$ and its partial derivatives vanish at the two corners $(1,0)$ and $(0,1)$ the same holds true for $f$. This means that $f$ must vanish identically on $\delta$. We now show that this is not possible regardless of how we choose $\alpha$ and $\beta$.

At step 0, we sample the function and its derivatives on the vertices of the square $[0,1]^2$ and we obtain $f^0_{i,j} = p^0_{i,j} = q^0_{i,j} = 0$ for $(i,j) \neq (0,0)$ and $f^0_{0,0} = 1$, $p^0_{0,0} = q^0_{0,0} = -3$. Using (2.4) we compute the values at the midpoint $(1/2, 1/2)$. We find $f^1_{11} = \frac{1}{4} + 3\alpha$ and $p^1_{11} = q^1_{11} = -\frac{1}{2} - \beta$. Convexity on $\delta$ implies that $\alpha = -1/12$ and $\beta = -1/2$. Moreover for these values of the parameters we must have $f^n_{ij} = 0$ for all points on $\delta$. But already the value $f^2_{31}$ at the point $(3/4, 1/4)$ on $\delta$ is nonzero. To see this

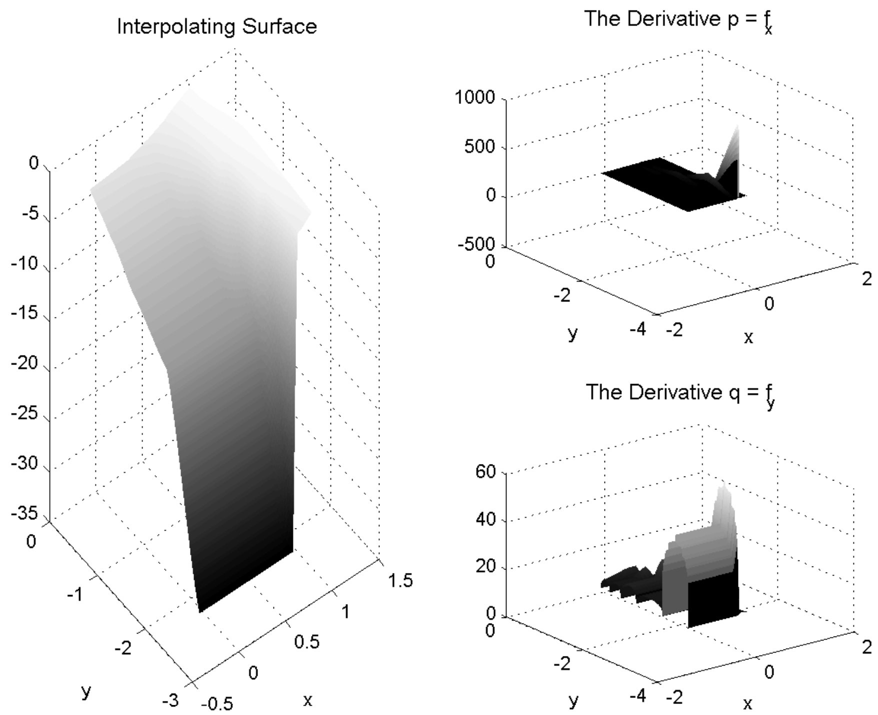


Figure 6.4: Increasing interpolant with random derivatives.

we first compute $f^1_{1,0} = 1/4$, $p^1_{1,0} = -3/4$, $q^1_{1,0} = -3/2$ by (2.2), and then $f^1_{2,1} = p^1_{2,1} = q^1_{2,1} = 0$ by (2.3). We now find $f^2_{3,1} = 1/64 \neq 0$ using (2.4). Thus the $HRC^1$-interpolant is not convex.

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