
EFFECTIVE RESUMMATION METHODS FOR AN IMPLICIT RESURGENT FUNCTION

by

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Dedicated to Cécile

Abstract. — Our main aim in this self-contained article is at the same time to detail the relationships between the resurgence and the hyperasymptotic theories, and to demonstrate how these theories can be used for an implicit resurgent function. For this purpose we consider after Stokes the question of the effective Borel-resummation of an exact Bohr-Sommerfeld-like implicit resurgent function whose values on an explicit semi-lattice provide the zeros of the Airy function. The resurgent structure encountered resembles what one usually gets in nonlinear problems, so that the method described here is quite general.

1. Introduction and summary

The problem of computing the sum of a Borel-resummable divergent series expansion has already a long history, which traces back to the Stokes's epoch marking paper [38] where an analogous of the summation to the least term (the “*méthode des astronomes*” of Poincaré [35]) play a key argument. Various methods have been developed since then, most of them being justified in the framework of the Gevrey theory [36, 37, 5, 39, 3], see also [25]. One of them, which will be used in this paper, is the resummation by factorial series [42, 28, 29, 27, 41] and its recent extension [15].

In most applications, the Borel resumable divergent series expansion enjoys the property of being resurgent [18, 19, 20, 6, 13, 9, 7, 8, 34, 1]. In this case the Borel sum can be calculated by the hyperasymptotic theory [4, 30, 31]. The efficiency of this method has been demonstrated in various problems [32, 24, 33, 23, 11].

These problems (multiple integrals, linear and nonlinear ODE's, difference equations, PDE's, ...) have a common feature : the resurgent properties of the divergent series to be resummed, that is, roughly speaking, the Riemann sheet structure of its Borel transform, is basically governed by a the so-called “formal integral” which can be derived directly from the problem. This means that the various series expansions playing a role in the hyperasymptotics are known, up to the Stokes multipliers which have (and can be) computed in the hyperasymptotic scheme. This certainly

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explain why up to now no general links have been written between resurgence and hyperasymptotic theories.

The main goal of this self-contained article is precisely to provide the relationships between the resurgence and hyperasymptotic theories. This will be done by the way of an example. We shall examine the effective resummation of a divergent series defined implicitly, so that no formal integral is computable. Nevertheless, as we shall see, the resurgent structure can be described through general tools from resurgent theory, and this allows hyperasymptotic expansions.

We shall be interested here in a celebrated test problem, the calculation of the zeros of the Airy function

$$(1) \quad \text{Ai}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos\left(ks + \frac{s^3}{3}\right) ds.$$

In [4], Berry and Howls have already shown how the hyperasymptotics can be used to solve this problem, their method being based on the hyperasymptotics of the Airy function itself. Our approach differs from theirs and resembles (and by the way justify) the idea of Stokes [38] who first translates the problem into a simpler implicit resurgent-resummable equation and formally solve it. This can be rigorously justified in the framework of resurgence analysis, leading to the notion of “model equation” through a resurgent-resummable change of variable (this is a key-idea in many problems, see, e.g., [12, 13, 16, 8]). This will be done in §2. What remains to do then is to Borel-sum the change of variable. For that purpose we first use, in §3, the direct method by factorial series as explained in [15]. We then turn to the hyperasymptotics in §5. However, to apply this second method we first have to analyze the resurgent properties of our implicitly defined formal series. This is what we do in §4. Using the alien differential calculus, it can be derived that the Riemann sheet structure of the Borel transform is analogous to that of a solution of a nonlinear differential equation, a Riccati equation for instance [33], with a one-dimensional lattice of singularities. This means that our method can be adapted to a large class of problems, for instance to the so-called exact Bohr-Sommerfeld equations [10, 16, 2] which are nowadays quite common in physics.

2. The zeros of the Airy function: the Stokes-Borel approach

Notation 2.1. — In this article, if $\{\omega_n\}$ is a set of points in \mathbb{C} with no accumulation point, we note:

- $\mathbb{C}_{\{\omega_n\}} = \mathbb{C} \setminus \{\omega_n\}$ and $\begin{array}{c} \mathbb{C}_{\{\omega_n\}}^\infty \\ \pi \downarrow \\ \mathbb{C}_{\{\omega_n\}} \end{array}$ its universal covering.
- For $\rho > 0$ small enough we write $\mathbb{C}_{\{\omega_n\},\rho} = \mathbb{C} \setminus \bigcup_n \overline{D(\omega_n, \rho)}$ where $\overline{D(\omega, \rho)}$ is the close disc centered on ω with radius ρ , and $\begin{array}{c} \mathbb{C}_{\{\omega_n\},\rho}^\infty \\ \pi_\rho \downarrow \\ \mathbb{C}_{\{\omega_n\},\rho} \end{array}$ its universal covering.
- For $\zeta \in \mathbb{C}_{\{\omega_n\}}^\infty$ (resp. $\zeta \in \mathbb{C}_{\{\omega_n\},\rho}^\infty$) we write $|\zeta| := |\pi(\zeta)|$ (resp. $|\zeta| := |\pi_\rho(\zeta)|$).

For a given formal series expansion $\varphi(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{z^n} \in \mathbb{C}[[z^{-1}]]$:

- Its *minor* is defined as $\tilde{\varphi}(\zeta) = \sum_{n=1}^{\infty} \alpha_n \frac{\zeta^{n-1}}{(n-1)!}$, that is its formal Borel transform when forgetting its constant term α_0 .
- One says that φ is a *small* formal series expansion if $\alpha_0 = 0$.
- φ is said to be Gevrey-1, and we note $\varphi \in \mathbb{C}[[z^{-1}]]_1$, if its minor $\tilde{\varphi}$ converges at the origin.

We first recall some well-known fact on the Airy function, but this will help us introducing necessary notations and definitions.

2.1. The Airy function as a Borel sum. — It is known since Stokes that the asymptotics of the Airy function is essentially governed by the series expansion

$$(2) \quad \varphi_{Ai}(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad \text{with} \quad a_n = \left(-\frac{3}{4}\right)^n \frac{\Gamma(n+1/6)\Gamma(n+5/6)}{2\pi\Gamma(n+1)}.$$

This series expansion is divergent but enjoys the following properties (see, e.g., [26]):

Proposition 2.1. — *The series expansion $\varphi_{Ai}(z)$ is Gevrey-1 and its minor $\widetilde{\varphi}_{Ai}(\zeta) \in \mathbb{C}\{\zeta\}$ extends analytically to $\mathbb{C}_{\{0, -4/3\}}^{\infty}$. Moreover, for any $\rho > 0$ and $B > 0$, there exists $A = A(\rho, B)$ such that*

$$\forall \zeta \in \mathbb{C}_{\{0, -4/3\}, \rho}^{\infty}, \quad |\widetilde{\varphi}_{Ai}(\zeta)| \leq Ae^{B|\zeta|}.$$

This proposition implies that for any $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pi\}$, the function

$$(3) \quad s_{\theta}\varphi_{Ai}(z) = a_0 + \int_0^{\infty e^{i\theta}} \widetilde{\varphi}_{Ai}(\zeta) e^{-z\zeta} d\zeta.$$

is a well-defined holomorphic function in $\Re(z e^{i\theta}) > 0$: φ_{Ai} is *Borel-resummable in the direction θ* and $s_{\theta}\varphi_{Ai}$ is its *Borel-sum* in that direction whose asymptotics is given by φ_{Ai} : Proposition 2.1 and a theorem of Nevanlinna [15, 27] induce for instance that, for any $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pi\}$, there exists $r > 0$ such that, for any $B > 0$,

$$\exists A > 0, \quad \forall \Re(z e^{i\theta}) > B, \quad \forall n \geq 0, \quad \left| s_{\theta}\varphi_{Ai}(z) - \sum_{k=0}^n \frac{a_k}{z^k} \right| \leq Ae^{Br} \frac{n!}{r^n |z|^n (\Re(z e^{i\theta}) - B)}.$$

The link between φ_{Ai} and the Airy function is given by the following proposition [26]:

Proposition 2.2. — *For $|\arg(z)| < \pi/2$, $|z| > 0$, resp. $|\arg(k)| < \pi/3$, $|k| > 0$,*

$$2\sqrt{\pi}k^{1/4}Ai(k) = e^{-2z/3}s_0\varphi_{Ai}(z) \quad \text{where} \quad k = z^{2/3}.$$

(Here and in the sequel we use the convention that $t^{\alpha} = |t|e^{i\alpha \arg(t)}$).

To analytically continue $s_0\varphi_{Ai}(z)$ one just has, by Cauchy, to rotate the direction of summation θ in (3). For $\theta \in [0, -\pi[$ one thus gets:

for $|\arg(z) + \theta| < \pi/2$, $|z| > 0$, resp. $|\arg(k) + 2\theta/3| < \pi/3$, $|k| > 0$,

$$2\sqrt{\pi}k^{1/4}Ai(k) = e^{-2z/3}s_{\theta}\varphi_{Ai}(z).$$

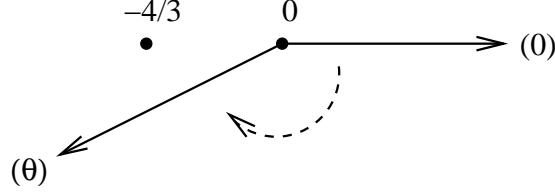


FIGURE 1. Rotating the direction of Borel-resummation.

A Stokes phenomenon occurs for the $-\pi$ direction, since in that direction one meets a singularity for $\widetilde{\varphi}_{Ai}$: φ_{Ai} is not Borel-resummable in that direction, but *right and left Borel-resummable*. For the right-resummation

$$S_{-\pi^+} \varphi_{Ai}(z) = a_0 + \int_0^{\infty e^{-i\pi^+}} \widetilde{\varphi}_{Ai}(\zeta) e^{-z\zeta} d\zeta.$$

one integrates along a path avoiding the singularity as shown on Fig. 2. The left-resummation $S_{-\pi^-} \varphi_{Ai}(z)$ is defined in a similar way.

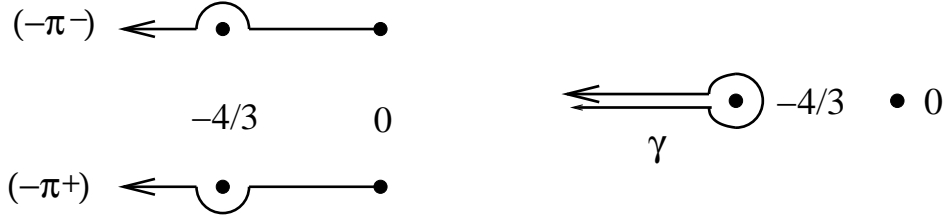


FIGURE 2. Right and left Borel-resummation.

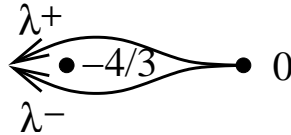
One can compare right and left-resummations, since

$$(4) \quad S_{-\pi^-} \varphi_{Ai}(z) = S_{-\pi^+} \varphi_{Ai}(z) + \int_{\gamma} \widetilde{\varphi}_{Ai}(\zeta) e^{-z\zeta} d\zeta$$

where the path γ is drawn on Fig. 2. It can be shown [26] that, locally near $\zeta = -4/3$, $\widetilde{\varphi}_{Ai}(\zeta)$ reads

$$(5) \quad \widetilde{\varphi}_{Ai}(\zeta) = \frac{b_0}{2i\pi(\zeta + 4/3)} + \tilde{h}(\zeta + 4/3) \frac{\ln(\zeta + 4/3)}{2i\pi} + hol(\zeta + 4/3)$$

where \tilde{h} , hol are holomorphic functions near 0.

FIGURE 3. The paths of analytic continuations λ^\pm .

Note that b_0 is the residue at $-4/3$ of the analytic continuation of $\widetilde{\varphi}_{Ai}$, while the \tilde{h} may be defined as

$$\tilde{h}(\zeta + 4/3) = \lambda^+ \widetilde{\varphi}_{Ai}(\zeta) - \lambda^- \widetilde{\varphi}_{Ai}(\zeta)$$

where λ^\pm are the paths of analytic continuation in the direction $(-\pi)$ drawn on Fig. 3.

Since \tilde{h} is holomorphic near the origin, it can be considered as the minor of a (Gevrey-1) series expansion $h \in \mathbb{C}[[z^{-1}]]$. Fixing its constant term to be b_0 , we translate these informations into a single formula, namely

$$\Delta_{-4/3}^z \varphi_{Ai}(z) = h(z).$$

The operator $\Delta_{-4/3}^z$ is the so-called *alien derivation* at $-4/3$ (the superscript z is just added here to remind the name of the variable). The following proposition precise the effect of the action of $\Delta_{-4/3}^z$ on φ_{Ai} [26]:

Proposition 2.3. — *One has*

$$\Delta_{-4/3}^z \varphi_{Ai} = -i\varphi_{Bi} \quad \text{where} \quad \varphi_{Bi}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{z^n} = \varphi_{Ai}(-z).$$

Note that $\varphi_{Bi}(z) = \varphi_{Ai}(-z)$ implies that $\widetilde{\varphi_{Bi}}(\zeta) = -\widetilde{\varphi_{Ai}}(-\zeta)$. From propositions 2.1 and 2.3 one easily gets:

Proposition 2.4. — *The series expansion φ_{Bi} is Gevrey-1 and its minor $\widetilde{\varphi_{Bi}}$ extends analytically to $\mathbb{C}_{\{0,+4/3\}}^\infty$. For any $\rho > 0$ and $B > 0$, there exists $A = A(\rho, B)$ such that*

$$\forall \zeta \in \mathbb{C}_{\{0,4/3\},\rho}^\infty, \quad |\widetilde{\varphi_{Ai}}(\zeta)| \leq Ae^{B|\zeta|}.$$

Moreover,

$$\Delta_{4/3}^z \varphi_{Bi} = -i\varphi_{Ai}.$$

Remark 2.1. — Propositions 2.1, 2.3 and 2.4 imply that φ_{Ai} and φ_{Bi} belong to the algebra RES of simply ramified resurgent functions (see [21]).

Now a direct consequence of (4) is that

$$\int_{\gamma} \widetilde{\varphi_{Ai}}(\zeta) e^{-z\zeta} d\zeta = e^{+4z/3} \left(b_0 + \int_0^{\infty e^{-i\pi}} \tilde{h}(\zeta) e^{-z\zeta} d\zeta \right) = e^{+4z/3} \mathbf{s}_{-\pi}(\Delta_{-4/3}^z \varphi_{Ai})(z).$$

Returning to the Airy function, what we have got so far is that, for $|\arg(z) - \pi| < \pi/2$, $|z| > 0$, resp. $|\arg(k) - 2\pi/3| < \pi/3$, $|k| > 0$,

$$\begin{aligned} 2\sqrt{\pi}k^{1/4} Ai(k) &= e^{-2z/3} \mathbf{s}_{-\pi} \varphi_{Ai}(z) \\ &= e^{-2z/3} \mathbf{s}_{-\pi} \varphi_{Ai}(z) - e^{+2z/3} \mathbf{s}_{-\pi}(\Delta_{-4/3}^z \varphi_{Ai})(z) \\ &= e^{-2z/3} \mathbf{s}_{-\pi} \varphi_{Ai}(z) + ie^{+2z/3} \mathbf{s}_{-\pi} \varphi_{Bi}(z). \end{aligned}$$

The Stokes phenomenon being analyzed, one can go on rotating the direction of Borel-resummation. We finally arrive to the following result:

Lemma 2.1. — *For $|\arg(z) - 3\pi/2| < \pi/2$, $|z| > 0$, resp. $|\arg(k) - \pi| < \pi/3$, $|k| > 0$,*

$$(6) \quad 2\sqrt{\pi}k^{1/4} Ai(k) = e^{-2z/3} \mathbf{s}_{-3\pi/2} \varphi_{Ai}(z) + ie^{+2z/3} \mathbf{s}_{-3\pi/2} \varphi_{Bi}(z).$$

We are now in position to analyze the Airy function in a sector of the complex plan bisected by the negative real axis, where the zeros we are looking for lay. For this purpose we are going to reduce our problem to a model equation, in the spirit of Stokes [38].

2.2. The zeros of the Airy function. — One sees from lemma 2.1 that, for $|\arg(k) - \pi| < \pi/3$, $|k| > 0$, the zeros of the Airy function are those of the function given by the right-hand side of the equality (6).

We introduce

$$(7) \quad \psi_{Ai}(x) = \varphi_{Ai}(z), \quad \psi_{Bi}(x) = \varphi_{Bi}(z), \quad \text{where} \quad z = xe^{3i\pi/2}.$$

For the resurgence and Borel-resummation viewpoint, the change of variable $z = xe^{3i\pi/2}$ is quite innocent. In effect, a Borel-resummation in z in the direction $-3\pi/2$ is transformed into a Borel-resummation in x in the direction 0, meanwhile propositions 2.1, 2.3 and 2.4 translate into:

Proposition 2.5. — *The series expansion $\psi_{Ai}(x) = \sum_{n=0}^{\infty} (i)^n \frac{a_n}{x^n}$ (resp. $\psi_{Bi}(x) = \sum_{n=0}^{\infty} (-i)^n \frac{a_n}{x^n}$) is Gevrey-1 and its minor $\widetilde{\psi}_{Ai}(\xi)$ (resp. $\widetilde{\psi}_{Bi}(\xi)$) extends analytically to $\mathbb{C}_{\{0, +\frac{4i}{3}\}}^{\infty}$ (resp. $\mathbb{C}_{\{0, -\frac{4i}{3}\}}^{\infty}$). For any $\rho > 0$ and $B > 0$, there exists $A = A(\rho, B)$ such that*

$$\forall \xi \in \mathbb{C}_{\{0, \frac{4i}{3}\}, \rho}^{\infty}, \quad |\widetilde{\psi}_{Ai}(\xi)| \leq Ae^{B|\xi|} \quad \text{resp.} \quad \forall \xi \in \mathbb{C}_{\{0, -\frac{4i}{3}\}, \rho}^{\infty}, \quad |\widetilde{\psi}_{Bi}(\xi)| \leq Ae^{B|\xi|}.$$

Moreover,

$$(8) \quad \Delta_{\frac{4i}{3}}^x \psi_{Ai} = -i\psi_{Bi}, \quad \Delta_{-\frac{4i}{3}}^x \psi_{Bi} = -i\psi_{Ai}.$$

The right-hand side of the equality (6) becomes the function

$$G(x) = e^{\frac{2}{3}ix} s_0 \psi_{Ai}(x) + ie^{-\frac{2}{3}ix} s_0 \psi_{Bi}(x), \quad |\arg(x)| < \pi/2, \quad |x| > 0.$$

Using the linearity of the Borel-resummation, one can write this function as follows:

$$(9) \quad G(x) = \cos\left(\frac{2}{3}x - \frac{\pi}{4}\right)R(x) + \sin\left(\frac{2}{3}x - \frac{\pi}{4}\right)S(x)$$

with

$$(10) \quad \begin{aligned} R(x) &= s_0 r(x), \quad S(x) = s_0 s(x) \\ r(x) &= e^{i\pi/4} \psi_{Ai}(x) + e^{i\pi/4} \psi_{Bi}(x) = 2e^{i\pi/4} \sum_{m=0}^{\infty} (-1)^m \frac{a_{2m}}{x^{2m}} \\ s(x) &= e^{3i\pi/4} \psi_{Ai}(x) - e^{3i\pi/4} \psi_{Bi}(x) = 2e^{i\pi/4} \sum_{m=1}^{\infty} (-1)^m \frac{a_{2m-1}}{x^{2m-1}}. \end{aligned}$$

To proceed it will be convenient to use the following definition throughout the rest of this article.

Definition 2.1. — We shall say that a formal series expansion $\varphi(x) \in \mathbb{C}[[x^{-1}]]$ is resurgent if φ is Gevrey-1 and if its minor extends analytically on $\mathbb{C}_{\{\frac{4i}{3}\mathbb{Z}\}}^\infty$.

In fact, Definition 2.1 just defines a special class of resurgent functions, see [18, 19, 20, 6].

Note that, from proposition 2.5, the formal series expansions ψ_{Ai} , ψ_{Bi} , and their linear combinations r and s are resurgent, and Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$.

We shall use the following theorem:

Theorem 2.1. — We assume that $\varphi_1, \varphi_2 \in \mathbb{C}[[x^{-1}]]_1$ are resurgent formal series expansions.

1. The product $\varphi_1 \cdot \varphi_2$ is a resurgent formal series expansion. If φ_1, φ_2 are Borel-resummable in a direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, then the product $\varphi_1 \cdot \varphi_2$ is Borel-resummable in that direction and $S_\theta(\varphi_1 \cdot \varphi_2)(x) = S_\theta\varphi_1(x) \cdot S_\theta\varphi_2(x)$.

2. If φ_1 is small and if $\Psi(\varepsilon) = \sum_{n=0}^{\infty} \alpha_n \varepsilon^n \in \mathbb{C}\{\varepsilon\}$ is a holomorphic function near

$\varepsilon = 0$, then the composition $\Psi \circ \varphi_1(x) = \sum_{n=0}^{\infty} \alpha_n \varphi_1^n(x)$ is a resurgent formal

series expansion. Moreover, if φ_1 is Borel-resummable in a direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, then $\Psi \circ \varphi_1$ is Borel-resummable in that direction and $S_\theta(\Psi \circ \varphi_1)(x) = \Psi \circ (S_\theta\varphi_1)(x)$.

3. If φ_1 is small and if $\varphi(x) = x + \varphi_1(x)$, then the composition $\varphi_2 \circ \varphi$ defined by

$$\varphi_2 \circ \varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi_1^n(x)}{n!} \frac{d^n \varphi_2}{dx^n}(x)$$

is resurgent. If φ_1, φ_2 are Borel-resummable in a direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, then $\varphi_2 \circ \varphi$ is Borel-resummable in that direction and $S_\theta(\varphi_2 \circ \varphi)(x) = S_\theta\varphi_2(x + S_\theta\varphi_1(x))$.

Apart from the Borel-resummability, this theorem is just a specialization of more general theorems in resurgence theory [6]. In our case the proof can be done in a simpler way by the methods used in [21, 14, 40], including the Borel-resummability properties (which are also a consequence of the theorem of Ramis-Sibuya [27]).

This theorem 2.1 will allow us to write (as Stokes do [38]) the functions R and S in a polar form:

$$(11) \quad R(x) = M(x) \cos(\Phi(x)) \quad S(x) = M(x) \sin(\Phi(x)).$$

One first defines $M(x)$. Theorem 2.1 implies that the formal series expansion

$$m(x) = (r^2(x) + s^2(x))^{1/2} = 2e^{i\pi/4} \psi_{Ai}^{1/2}(x) \psi_{Bi}^{1/2}(x)$$

is resurgent and Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$. We define

$M(x)$ as the Borel-sum of $m(x)$ in the direction 0.

Note that, since m is not small, it is invertible (this is again a consequence of theorem 2.1). This ensures that, for $\Re(x)$ large enough, $M(x)$ does not vanish.

We now define $\Phi(x)$: r being invertible and s being small, s/r is a small resurgent formal series expansion, Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$. Applying again theorem 2.1, the same properties will be true for the formal series expansion $\phi(x) = \arctan\left(\frac{s(x)}{r(x)}\right)$, and one can define

$$(12) \quad \Phi(x) = \arctan \frac{S(x)}{R(x)} = \arctan \circ s_0 \left(\frac{s}{r} \right) (x) = s_0 \left(\arctan \circ \frac{s}{r} \right) (x).$$

The polar form (11) being justified, one deduces from (9) that:

$$G(x) = M(x) \cos \left(\frac{2}{3}x - \frac{\pi}{4} - \Phi(x) \right).$$

We summarize what we have obtained:

Theorem 2.2. — *For $\Re(x)$ large enough, the zeros of the Airy function $Ai(k)$ with $k = e^{i\pi}x^{2/3}$ are the solutions of the equation*

$$(13) \quad \cos \left(\frac{2}{3}x - \frac{\pi}{4} - \Phi(x) \right) = 0,$$

with $\Phi(x) = s_0\phi(x)$, where $\phi(x) = \arctan\left(\frac{s(x)}{r(x)}\right)$ is a small resurgent formal series expansion, Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$.

2.3. The model equation. — The *model equation* for our problem is the equation

$$(14) \quad \cos\left(\frac{2}{3}t - \frac{\pi}{4}\right) = 0, \quad \Re(t) > 0,$$

deduced from (13) through the change of variable

$$(15) \quad t = x - \frac{3}{2}\Phi(x).$$

The model equation (14) is obviously exactly solvable, the solutions being

$$(16) \quad t = \frac{3}{2}\left(l - \frac{1}{4}\right)\pi, \quad l \in \mathbb{N}^*.$$

To translate this result in term of the zeros of the Airy function, one has to justify the change of variable (15) and to calculate the inverse function.

We know that $\Phi(x) = s_0\phi(x)$ where $\phi(x)$ is a small resurgent formal series expansion. Let us look at (15) at a formal level, that is

$$t = x - \frac{3}{2}\phi(x), \quad \text{or equivalently} \quad x = t + \frac{3}{2}\phi(x).$$

This fixed point problem has a unique formal solution $X(t)$, $X(t) - t \in \mathbb{C}[[t^{-1}]]$, which can be constructed by the formal successive approximation method: if

$$(17) \quad \begin{cases} X_0(t) = t \\ X_{n+1}(t) = t + \frac{3}{2}\phi \circ X_n(t), \quad n \geq 0 \end{cases}$$

then one easily checks that the sequence $({}^bX_n) = (X_n - t)$ converges in the algebra of formal series expansions $\mathbb{C}[[t^{-1}]]$ to a unique small series expansion $f \in \mathbb{C}[[t^{-1}]]$, since for all $n \in \mathbb{N}$, $X_{n+1} - X_n \in t^{-n-1}\mathbb{C}[[t^{-1}]]$. Note that, from theorem 2.1, every bX_n

is a small resurgent formal series expansion, and Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$ (by iteration, since this is true for ϕ). Applying the resurgent implicit function theorem [6], one shows that the limit ${}^bX(t)$ is a small resurgent formal series expansions, Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$ (this can be obtained by direct estimates or by the Ramis-Sibuya theorem). Also, by construction, writing $X(t) = t + {}^bX(t)$:

Proposition 2.6. — *There exist two sectorial neighbourhood of infinity Σ_x and Σ_t of aperture $I =] - \pi/2, \pi/2[$ such that*

$$x \in \Sigma_x, t = x - \frac{3}{2}\Phi(x) \Leftrightarrow t \in \Sigma_t, x = s_0X(t).$$

In this proposition $s_0X(t) = t + s_0{}^bX(t)$ and:

Definition 2.2. — A sectorial neighbourhood of infinity of aperture $] - \pi/2, \pi/2[$ is an open set Σ of \mathbb{C} such that for any open interval $J \subset I$, there is $z \in \Sigma$ such that $zJ \subset \Sigma$, where $zJ := \{z + re^{i\theta}, r > 0, \theta \in J\}$.

Returning to theorem (2.2) what we have obtained is summarized in the following theorem.

Theorem 2.3. — *For $\Re(x)$ large enough, the zeros of the Airy function $Ai(k)$ with $k = e^{i\pi}x^{2/3}$ are given by*

$$(18) \quad x_l = s_0X\left(\frac{3}{2}\left(l - \frac{1}{4}\right)\pi\right), \quad l \in \mathbb{N}^*, l \text{ large enough.}$$

where $X(t)$ is the unique formal solution of the implicit equation

$$(19) \quad X(t) = t + \frac{3}{2}\phi \circ X(t), \quad \text{with} \quad \phi(x) = \arctan\left(\frac{s(x)}{r(x)}\right).$$

This formal solution $X(t)$ reads $X(t) = t + {}^bX(t)$ where ${}^bX(t)$ is a small resurgent formal series expansion, Borel-resummable in any direction $\theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}} \setminus \{\pm\pi/2\}$.

3. The zeros of the Airy function: calculation with the factorial series method

To compute the zeros of the Airy function, we first have now to calculate the series expansion $X(t)$. Following theorem 2.3, this first means to expand $\phi(x)$ which is straightforward. We then apply the formal successive approximation method (17). With Maple V Release 5.1 one gets the result to any fixed order:

$$(20) \quad X(t) = t + \frac{5}{32}t^{-1} - \frac{1255}{6144}t^{-3} + \frac{272075}{196608}t^{-5} + \dots = t + \sum_{n=0}^{\infty} \frac{c_n}{t^n}$$

As we shall see, this sole information on $X(t)$ is quite enough to calculate the zeros of the Airy function to any order, using the factorial series method.

We introduce some notations:

Notation 3.1. — For $r > 0$ we note $\mathcal{B}_r = \{\tau \in \mathbb{C} / d(\tau, \mathbb{R}^+) < r\}$, where d is the euclidian distance measure.

- We note Δ the image of the open disc $D(1, 1)$ centered on 1 with radius 1 under the biholomorphic mapping $s \in D(1, 1) \mapsto \tau = -\ln(s) \in \Delta$.
The open set Δ satisfy :

$$\mathcal{B}_{\ln(2)} \subset \Delta \subset \mathcal{B}_{\frac{\pi}{2}} \quad (\text{cf. Fig. 4}).$$

- For $\lambda > 0$, Δ_λ is the homothetic set of Δ defined by: $\Delta_\lambda = \{\lambda\tau / \tau \in \Delta\}$.

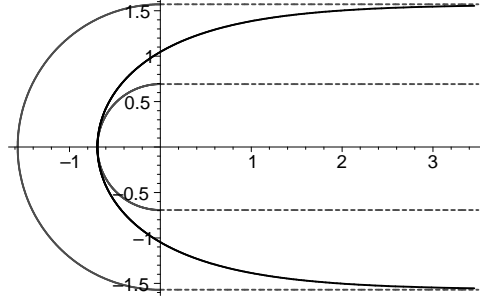


FIGURE 4. The open sets $\mathcal{B}_{\ln(2)} \subset \Delta \subset \mathcal{B}_{\frac{\pi}{2}}$.

We shall use the following theorem whose proof is detailed in [15]:

Theorem 3.1. — Assume that the minor $\tilde{h}(\tau)$ of the formal series expansion $h(t) = \sum_{n=0}^{+\infty} \frac{\alpha_n}{t^n} \in \mathbb{C}[[z^{-1}]]_1$ extends analytically to the open set Δ_λ , $\lambda > 0$. Assume furthermore that there exist $A > 0$ and $B > 0$ such that for every $\tau \in \Delta_\lambda$, $|\tilde{f}(\tau)| \leq Ae^{B|\tau|}$. Then:

- the factorial series $\alpha_0 + \lambda \sum_{n=0}^N \frac{\Gamma(\lambda t)\Gamma(n+1)\beta_n^{(\lambda)}}{\Gamma(\lambda t + n + 1)}$ converges absolutely for $\Re(t) > \max(B, 1/\lambda)$, its sum being the Borel-sum $s_0h(t)$ of h , the $\beta_n^{(\lambda)}$ being deduced from the $\alpha_n^{(\lambda)} = \lambda^{n-1}\alpha_n$ by the Stirling algorithm (proposition 3.1).
- For any $N \geq 0$ and $\Re(t) > B$,

$$(21) \quad \left| s_0h(z) - \left(\alpha_0 + \lambda \sum_{n=0}^N \frac{\Gamma(\lambda t)\Gamma(n+1)\beta_n^{(\lambda)}}{\Gamma(\lambda t + n + 1)} \right) \right| \leq R_{fact}(\lambda, A, B, N, t)$$

$$R_{fact}(\lambda, A, B, N, t) = \frac{A}{(\lambda B)^{\lambda B}} \frac{(N + \lambda B + 1)^{N + \lambda B + 1}}{(N + 1)^N} \left| \frac{\Gamma(\lambda t)\Gamma(N + 1)}{\Gamma(\lambda t + N + 1)(\Re(t) - B)} \right|,$$

Proposition 3.1 (Stirling algorithm). —

$$\forall n \geq 0, \quad \beta_n = \frac{1}{n!} \sum_{k=1}^{n+1} (-1)^{n-k+1} \mathfrak{s}(n, k-1) \alpha_k,$$

where $\mathfrak{s}(n, k)$ are the Stirling cycle numbers (or Stirling numbers of the first kind):

$$\prod_{k=0}^{n-1} (x - k) = \sum_{k=0}^n \mathfrak{s}(n, k) x^k.$$

From theorem 2.3 we see that theorem 3.1 can be applied to the formal series expansion ${}^bX(t)$ as soon as one chooses $\lambda \in]0, 4/\pi[$, so that the open set Δ_λ is included in the cut plane $\mathbb{C} \setminus [\pm i4/3, \pm i\infty[$. Therefore:
for any $\lambda \in]0, 4/\pi[$, there exist $A > 0$ and $B > 0$ such that, for $\Re(t) > B$ and $N \geq 0$,

$$\left| s_0 X(t) - \left(t + \lambda \sum_{n=0}^N \frac{\Gamma(\lambda t) \Gamma(n+1) \beta_n^{(\lambda)}}{\Gamma(\lambda t + n + 1)} \right) \right| \leq R_{fact}(\lambda, A, B, N, t)$$

where the $\beta_n^{(\lambda)}$ are deduced from the $c_n^{(\lambda)} = \lambda^{n-1} c_n$ (cf. (20)) by the Stirling algorithm. As explained in [15], in practice for $\Re(t)$ and N large enough, one can evaluate the remainder term by:

$$(22) \quad R_{fact}(\lambda, A, B, N, t) \sim |\beta_{l, N+1}^{(\lambda)}| \frac{|\Gamma(\lambda t) \Gamma(N+1)|}{|\Re(t) \Gamma(\lambda t + N + 1)|}.$$

Following theorem 2.3, we now evaluate the $x_l = s_0 X\left(\frac{3}{2}\left(l - \frac{1}{4}\right)\pi\right)$, $l \in \mathbb{N}^*$ by the factorial series methods, and then translates the results (and the error estimates) to the values $k_l = -x_l^{2/3}$ which give the zeros of the Airy function. Numerical calculations give very good results, even for $l = 1$. The comparison *Exact* – *Est.* with the “exact zeros” have been made taken for granted that the *AiryAiZeros* function of Maple V Release 5.1 gives the correct answer.

Value of N	Estimates for k_1	Real error
21	-2.338107342	0.68×10^{-7}
41	-2.3381074010	0.95×10^{-8}
61	-2.338107410494	0.34×10^{-10}

TABLE 1. Calculation for the first zero k_1 of the Airy function. We have applied the factorial series method with $\lambda = 1.2$. For the value of t used here, the error estimates (22) do not apply here.

Value of N	Estimates for k_3	Error estimates	Real error
21	-5.5205598280955580	0.13×10^{-12}	0.69×10^{-14}
41	-5.520559828095551049	0.26×10^{-16}	-0.10×10^{-16}
61	-5.5205598280955510591283	0.73×10^{-19}	-0.16×10^{-20}

TABLE 2. Calculation for the third zero k_3 of the Airy function. We have applied the factorial series method with $\lambda = 1.2$ and used (22) for the error estimates.

4. Resurgent structure for the implicit function

We would like to turn to hyperasymptotics. This requires first to determine the resurgent structure of $X(t)$. Since $X(t)$ is defined implicitly by the equation (19), we first have to precise the resurgent structure of the small resurgent formal series expansion $\phi(x) = \arctan\left(\frac{s(x)}{r(x)}\right)$. For that purpose, we shall freely use the alien derivations.

4.1. Resurgent structure for $\phi(x)$. — In proposition (2.5) we have seen that the resurgent structure of ψ_{Ai} and ψ_{Bi} , that is the singularity structure of their minors, was governed by the alien derivatives (8), namely

$$\begin{cases} \Delta_{\frac{4i}{3}}^x \psi_{Ai} = -i\psi_{Bi} \\ \text{otherwise, } \Delta_{\omega}^x \psi_{Ai} = 0 \end{cases} \quad \begin{cases} \Delta_{-\frac{4i}{3}}^x \psi_{Bi} = -i\psi_{Ai} \\ \text{otherwise, } \Delta_{\omega}^x \psi_{Bi} = 0 \end{cases}$$

The alien derivations are derivations in the algebraic sense; in particular they are linear operators. From the very definitions (10) of r and s , one easily gets:

$$\Delta_{\frac{4i}{3}}^x r(x) = e^{-i\pi/4} \psi_{Bi}(x), \quad \Delta_{-\frac{4i}{3}}^x r(x) = e^{-i\pi/4} \psi_{Ai}(x)$$

$$\Delta_{\frac{4i}{3}}^x s(x) = e^{i\pi/4} \psi_{Bi}(x), \quad \Delta_{-\frac{4i}{3}}^x s(x) = -e^{i\pi/4} \psi_{Ai}(x)$$

Applying the Leibniz chain rule, one thus gets:

$$\Delta_{\frac{4i}{3}}^x \frac{s}{r} = \frac{r \Delta_{\frac{4i}{3}}^x s - s \Delta_{\frac{4i}{3}}^x r}{r^2} = \frac{2\psi_{Bi}^2}{(\psi_{Ai} + \psi_{Bi})^2}$$

and

$$\Delta_{-\frac{4i}{3}}^x \frac{s}{r} = -\frac{2\psi_{Ai}^2}{(\psi_{Ai} + \psi_{Bi})^2}.$$

We now need the following general result from resurgence theory [6, 18]:

Theorem 4.1. — *If φ is a small resurgent formal series expansion and if $\Psi(\varepsilon)$ is a holomorphic function near $\varepsilon = 0$, then for any $\omega \in \mathbb{C}^*$,*

$$\Delta_{\omega} \Psi \circ \varphi = \left(\Delta_{\omega} \varphi \right) \cdot \Psi' \circ \varphi$$

where $'$ is the usual derivation.

Thus, for any $\omega \in \mathbb{C}^*$, $\Delta_{\omega}^x \phi = \left(\Delta_{\omega}^x \frac{s}{r} \right) \cdot \arctan' \left(\frac{s}{r} \right)$. Applying this for $\omega = \pm \frac{4i}{3}$, one obtains:

Proposition 4.1. — *The resurgent structure of $\phi(x)$ is governed by:*

$$(23) \quad \begin{cases} \Delta_{\frac{4i}{3}}^x \phi = \frac{1}{2} \frac{\psi_{Bi}}{\psi_{Ai}} \\ \Delta_{-\frac{4i}{3}}^x \phi = -\frac{1}{2} \frac{\psi_{Ai}}{\psi_{Bi}} \\ \text{otherwise, } \Delta_{\omega}^x \phi = 0 \end{cases}$$

4.2. Translation in term of analytic structure. — It is certainly time to pose and to say more about alien derivations (in a way suited to our purpose), so as to explain the meaning and the consequences of (23).

4.2.1. Alien derivations and analytic continuations. — Since ϕ is resurgent (theorem 2.3) its minor $\tilde{\phi}$ is an holomorphic function near 0 which can be analytically prolonged in the direction $\pi/2$ (say), provided to avoid the singular semi-lattice $\frac{4i}{3}\mathbb{N}^*$.

For $n \in \mathbb{N}^*$ and $\xi \in]\frac{4i}{3}n, \frac{4i}{3}(n+1)[$ we note $\tilde{\phi}^{\epsilon_1, \epsilon_2, \dots, \epsilon_n}(\xi)$ the analytic continuation of $\tilde{\phi}(\xi)$ along a path which circumvents each $\frac{4i}{3}k$ to the left ($\epsilon_k = +$) or to the right ($\epsilon_k = -$), see Fig. 5.

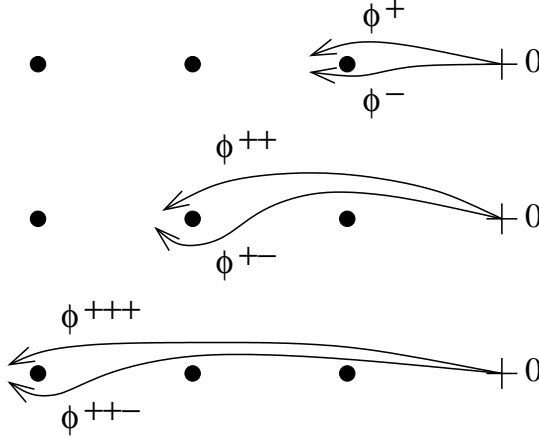


FIGURE 5. The paths of analytic continuations for the direction $\pi/2$ (we have rotated the picture to save place). The bullets are the points $4in/3$, $n \in \mathbb{N}^*$.

We fix a $N \in \mathbb{N}^*$ and consider, for $\xi \in]\frac{4i}{3}(N-1), \frac{4i}{3}N[$ the mean

$$(24) \quad \tilde{\phi}_N(\xi) = \sum_{\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1})} \frac{p(\epsilon)!q(\epsilon)!}{N!} \tilde{\phi}^{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}}(\xi)$$

where the sum is made over the 2^{N-1} $(N-1)$ -lists $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}) = (\pm, \pm, \dots, \pm)$ whereas $p(\epsilon)$ is the number of $+$ and $q(\epsilon) = N-1-p(\epsilon)$ the number of $-$.

In addition to theorem 2.3 it can be shown that the $\tilde{\phi}_N$'s inherit the type of singularities (5) of $\widetilde{\varphi_{A_i}}$ and $\widetilde{\varphi_{B_i}}$, that is

$$(25) \quad \tilde{\phi}_N(\xi) = \frac{b}{2i\pi(\xi - \frac{4i}{3}N)} + \tilde{h}(\xi - \frac{4i}{3}N) \frac{\ln(\xi - \frac{4i}{3}N)}{2i\pi} + \text{hol}(\xi - \frac{4i}{3}N)$$

where \tilde{h} and hol are holomorphic functions near 0. The coefficient b is just the residue of $\tilde{\phi}_N$ at $\frac{4i}{3}N$ while, for $\xi \in]\frac{4i}{3}N, \frac{4i}{3}(N+1)[$,

$$(26) \quad \tilde{h}\left(\xi - \frac{4i}{3}N\right) = \sum_{\epsilon=(\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1})} \frac{p(\epsilon)!q(\epsilon)!}{N!} \left(\tilde{\phi}^{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}, +}(\xi) - \tilde{\phi}^{\epsilon_1, \epsilon_2, \dots, \epsilon_{N-1}, -}(\xi) \right)$$

Denoting by h the inverse formal Borel transform of \tilde{h} with constant term b , one gets the alien derivative of ϕ at $\frac{4i}{3}N$,

$$\Delta_{\frac{4i}{3}N}^x \phi(x) = h(x).$$

The alien derivatives $\Delta_{-\frac{4i}{3}N}^x \phi(x)$ are defined in a similar way. Of course for $\omega \notin \frac{4i}{3}\mathbb{Z}^*$, $\Delta_\omega^x \phi = 0$.

From the fact that $\Delta_\omega^x \phi = 0$ when $\omega \neq \pm \frac{4i}{3}$ (proposition 4.1), it would be wrong to deduce that $\tilde{\phi}$ has no other “glimpsed” singularities in the $\pi/2$ direction than the “seen” (“adjacent” in hyperasymptotic theory) singularity $\frac{4i}{3}$.

Let us consider what happens at $\frac{8i}{3}$. On the one hand, the equality $\Delta_{\frac{8i}{3}}^x \phi = 0$ translates into the fact that the mean function

$$\frac{1}{2} \left(\tilde{\phi}^+(\xi) + \tilde{\phi}^-(\xi) \right), \quad \xi \in]\frac{4i}{3}, \frac{8i}{3}[$$

extends holomorphically near $\xi = \frac{8i}{3}$. In particular concerning the residues,

$$(27) \quad \text{res}_{\frac{8i}{3}} \tilde{\phi}^+(\xi) = -\text{res}_{\frac{8i}{3}} \tilde{\phi}^-(\xi),$$

whereas concerning the variations,

$$\left(\tilde{\phi}^{++}(\xi) + \tilde{\phi}^{-+}(\xi) \right) - \left(\tilde{\phi}^{+-}(\xi) + \tilde{\phi}^{--}(\xi) \right) = 0, \quad \xi \in]\frac{8i}{3}, \frac{12i}{3}[$$

which we writes as

$$(28) \quad \left(\tilde{\phi}^{++}(\xi) - \tilde{\phi}^{-+}(\xi) \right) + \left(\tilde{\phi}^{+-}(\xi) - \tilde{\phi}^{--}(\xi) \right) = 0, \quad \xi \in]\frac{8i}{3}, \frac{12i}{3}[.$$

On the other hand, from (23), (8) and the Leibniz chain rule,

$$\Delta_{\frac{4i}{3}}^x \left(\Delta_{\frac{4i}{3}}^x \phi \right) = \Delta_{\frac{4i}{3}}^x \left(\frac{\psi_{Bi}}{2\psi_{Ai}} \right) = \frac{i}{2} \frac{\psi_{Bi}^2}{\psi_{Ai}^2}.$$

Therefore the analytic function

$$\tilde{\phi}^+(\xi) - \tilde{\phi}^-(\xi),$$

defined for $\xi \in]\frac{4i}{3}, \frac{8i}{3}[$ (which up to a translation in the variable space corresponds to the minor of $\Delta_{\frac{4i}{3}}^x \phi$) is singular at $\xi = \frac{8i}{3}$ with a residue equal to $\frac{i}{2}$ (the constant

term of $\frac{i \psi_{Bi}^2}{2 \psi_{Ai}^2}$ so that

$$(29) \quad \operatorname{res}_{\frac{8i}{3}} \tilde{\phi}^+(\xi) - \operatorname{res}_{\frac{8i}{3}} \tilde{\phi}^-(\xi) = \frac{i}{2},$$

while

$$\left(\tilde{\phi}^{++}(\xi) - \tilde{\phi}^{-+}(\xi) \right) - \left(\tilde{\phi}^{+-}(\xi) - \tilde{\phi}^{--}(\xi) \right) = \frac{i}{2} \widetilde{\left(\frac{\psi_{Bi}^2}{\psi_{Ai}^2} \right)} \left(\xi - \frac{8i}{3} \right)$$

for $\xi \in]\frac{8i}{3}, \frac{12i}{3}[$. This reads also:

$$(30) \quad \left(\tilde{\phi}^{++}(\xi) - \tilde{\phi}^{+-}(\xi) \right) - \left(\tilde{\phi}^{-+}(\xi) - \tilde{\phi}^{--}(\xi) \right) = \frac{i}{2} \widetilde{\left(\frac{\psi_{Bi}^2}{\psi_{Ai}^2} \right)} \left(\xi - \frac{8i}{3} \right), \quad \xi \in]\frac{8i}{3}, \frac{12i}{3}[.$$

Comparing (28) and (30) we get that, for $\xi \in]\frac{8i}{3}, \frac{12i}{3}[$,

$$(31) \quad \begin{cases} \tilde{\phi}^{++}(\xi) - \tilde{\phi}^{+-}(\xi) = \frac{i}{4} \widetilde{\left(\frac{\psi_{Bi}^2}{\psi_{Ai}^2} \right)} \left(\xi - \frac{8i}{3} \right) \\ \tilde{\phi}^{-+}(\xi) - \tilde{\phi}^{--}(\xi) = - \left(\tilde{\phi}^{++}(\xi) - \tilde{\phi}^{+-}(\xi) \right), \end{cases}$$

while the comparison of (27) with (29) gives

$$(32) \quad \operatorname{res}_{\frac{8i}{3}} \tilde{\phi}^+(\xi) = \frac{i}{4}, \quad \operatorname{res}_{\frac{8i}{3}} \tilde{\phi}^-(\xi) = -\frac{i}{4}.$$

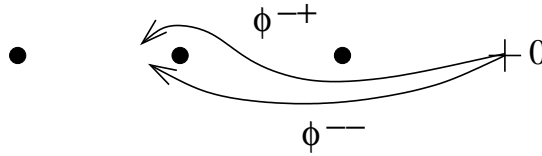


FIGURE 6.

4.2.2. Stokes automorphism. — The above analysis allows us to understand the Stokes phenomenon in the singular direction $\pi/2$. We can write

$$(33) \quad s_{\frac{\pi}{2}-} \phi(x) = s_{\frac{\pi}{2}+} \phi(x) + \sum_{n \geq 1} \int_{\gamma_n} \tilde{\phi}(\xi) e^{-x\xi} d\xi$$

where the paths γ_n are drawn on Fig. 7. From what precedes,

$$\int_{\gamma_1} \tilde{\phi}(\xi) e^{-x\xi} d\xi = e^{-\frac{4i}{3}x} s_{\frac{\pi}{2}+} \frac{1}{2} \frac{\psi_{Bi}}{\psi_{Ai}}(x), \quad \int_{\gamma_2} \tilde{\phi}(\xi) e^{-x\xi} d\xi = e^{-\frac{8i}{3}x} s_{\frac{\pi}{2}+} \frac{i}{4} \frac{\psi_{Bi}^2}{\psi_{Ai}^2}(x).$$

To get the whole picture, there is no need to do things by hand as we have previously done. We introduce a definition:

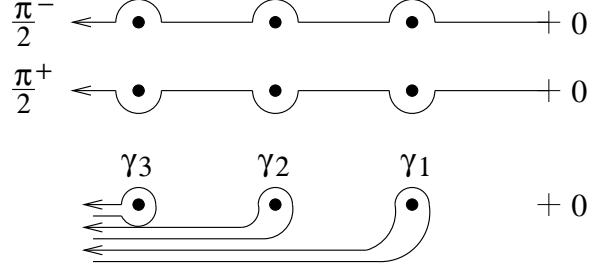


FIGURE 7. Comparing left and right Borel-resummation in the direction $\pi/2$ (we have rotated the picture). The bullets are the points $4in/3$, $n \in \mathbb{N}^*$.

Definition 4.1. — The alien derivation in the direction $\pi/2$ is

$$(34) \quad \underline{\Delta}_{\pi/2}^x = \sum_{n \in \mathbb{N}^*} \dot{\Delta}_{\frac{4i}{3}n}^x,$$

where $\dot{\Delta}_{\omega}^x = e^{-\omega x} \Delta_{\omega}^x$ is the pointed alien derivation at $\omega \in \mathbb{C}^*$.

It can be shown [18, 6] that:

Theorem 4.2. — One has

$$(35) \quad \mathfrak{S}_{\pi/2}^- = \mathfrak{S}_{\pi/2}^+ \circ \mathfrak{S}_{\pi/2} \quad \text{where} \quad \mathfrak{S}_{\pi/2} = \exp(\underline{\Delta}_{\pi/2}^x) = \sum_{n=0}^{\infty} \frac{(\underline{\Delta}_{\pi/2}^x)^n}{n!}.$$

The operator $\mathfrak{S}_{\pi/2}$ is the Stokes automorphism in the direction $\pi/2$.

In this theorem, by automorphism we mean automorphism of resurgent algebras (we do not precise here these algebras, see, e.g., [6]).

From (23) we know that:

$$\underline{\Delta}_{(\pi/2)}^x \phi = \frac{1}{2} \frac{\psi_{Bi}}{\psi_{Ai}} e^{-\frac{4i}{3}x}.$$

Using (8) one easily sees that for every $k \geq 2$, $\Delta_{\frac{4ki}{3}}^x \left(\Delta_{\frac{4i}{3}}^x \phi \right) = 0$, so that $(\underline{\Delta}_{(\pi/2)}^x)^2 \phi$ reduces to $e^{-\frac{8i}{3}x} \Delta_{\frac{4i}{3}}^x \left(\Delta_{\frac{4i}{3}}^x \phi \right)$:

$$(\underline{\Delta}_{(\pi/2)}^x)^2 \phi = \frac{i}{2} \left(\frac{\psi_{Bi}}{\psi_{Ai}} e^{-\frac{4i}{3}x} \right)^2.$$

More generally, for $n \geq 1$,

$$(\underline{\Delta}_{(\pi/2)}^x)^n \phi = -\frac{i}{2} \Gamma(n) \left(i \frac{\psi_{Bi}}{\psi_{Ai}} e^{-\frac{4i}{3}x} \right)^n.$$

Thus $\mathfrak{S}_{(\pi/2)} \phi$ is the following *transseries*:

$$(36) \quad \mathfrak{S}_{(\pi/2)} \phi = \phi - \frac{i}{2} \sum_{n \geq 1} \frac{1}{n} \left(i \frac{\psi_{Bi}}{\psi_{Ai}} \right)^n e^{-\frac{4in}{3}x} = \phi + \frac{i}{2} \ln \left(1 - i \frac{\psi_{Bi}}{\psi_{Ai}} e^{-\frac{4i}{3}x} \right).$$

Of course one defines the *Stokes automorphism in the direction $-\pi/2$* (the other singular direction) in a similar way. The calculation gives

$$(37) \quad \mathfrak{S}_{(-\pi/2)} \phi = \phi + \frac{i}{2} \sum_{n \geq 1} \frac{1}{n} \left(i \frac{\psi_{Ai}}{\psi_{Bi}} \right)^n e^{\frac{4in}{3}x} = \phi - \frac{i}{2} \ln \left(1 - i \frac{\psi_{Ai}}{\psi_{Bi}} e^{\frac{4i}{3}x} \right).$$

4.3. Resurgent structure for the implicit function. — We now return to the resurgent formal series expansion $X(t)$ implicitly defined by equation (19). We would like to analyze its resurgent structure from what we now know concerning ϕ . This requires the following theorem, in complement with theorem 2.1:

Theorem 4.3. — *We assume that $\varphi_1(t)$ and $\varphi_2(x)$ are resurgent formal series expansions. We assume furthermore that φ_1 is small and we note $\varphi(t) = t + \varphi_1(t)$. Then for any $\omega \in \mathbb{C}^*$,*

$$(38) \quad \dot{\Delta}_\omega^t (\varphi_2 \circ \varphi)(t) = (\dot{\Delta}_\omega^x \varphi_2) \circ \varphi(t) + \left(\dot{\Delta}_\omega^t \varphi(t) \right) \left(\varphi_2' \circ \varphi(t) \right).$$

In this theorem ' means the usual derivation, while $\Delta_\omega^t \varphi(t) = \Delta_\omega^t \varphi_1(t)$. One can also translate (38) in term of usual alien derivatives:

$$(39) \quad \Delta_\omega^t (\varphi_2 \circ \varphi)(t) = e^{-\omega \varphi_1(t)} \left[(\Delta_\omega^x \varphi_2) \circ \varphi(t) \right] + \left(\Delta_\omega^t \varphi(t) \right) \left(\varphi_2' \circ \varphi(t) \right).$$

Up to the knowledge of the author, theorem 2.1 only appears in [13] (with a misprint) with no explanation, so that it is perhaps worth to give here at least a sketch of proof.

Proof. — We recall that the composition function $\varphi_2 \circ \varphi$ is defined by

$$\varphi_2 \circ \varphi(t) = \sum_{n=0}^{\infty} \frac{\varphi_1^n(t)}{n!} \frac{d^n \varphi_2}{dx^n}(t).$$

We recall also the following result [18, 6]:

Theorem 4.4. — *The pointed alien derivation $\dot{\Delta}_\omega^t = e^{-\omega t} \Delta_\omega^t$ commutes with the usual derivation $\frac{d}{dt}$.*

At a formal level this theorem and the Leibniz chain rule imply that, for $\omega \in \mathbb{C}^*$,

$$\dot{\Delta}_\omega^t (\varphi_2 \circ \varphi)(t) = \sum_{n=1}^{\infty} \left(\dot{\Delta}_\omega^t \varphi_1(t) \right) \frac{\varphi_1^{n-1}(t)}{(n-1)!} \frac{d^n \varphi_2}{dx^n}(t) + \sum_{n=0}^{\infty} \frac{\varphi_1^n(t)}{n!} \left(\frac{d^n}{dx^n} (\dot{\Delta}_\omega^x \varphi_2) \right)(t),$$

which is just

$$\dot{\Delta}_\omega^t (\varphi_2 \circ \varphi)(t) = \left(\dot{\Delta}_\omega^t \varphi(t) \right) \left(\varphi_2' \circ \varphi(t) \right) + (\dot{\Delta}_\omega^x \varphi_2) \circ \varphi(t).$$

To justify this formal reasoning, one just need to go back to the proof of the resurgence of the composition function $\varphi_2 \circ \varphi$ (see [6]) and to translate the alien derivation in the Borel plane. \square

We apply theorem 2.1 to the implicit equation defining $X(t) = t + {}^bX(t)$ (see theorem 2.3): for $\omega \in \mathbb{C}^*$,

$$\begin{aligned}\Delta_\omega^t X &= \frac{3}{2} \Delta_\omega^t (\phi \circ X) \\ &= \frac{3}{2} \left[e^{-\omega {}^bX(t)} (\Delta_\omega^x \phi) \circ X + \Delta_\omega^t X (\phi' \circ X) \right].\end{aligned}$$

Since ϕ' is small, $2 - 3\phi' \circ X$ is invertible, its inverse being a resurgent formal series expansions. We thus obtain:

$$\Delta_\omega^t X = \frac{3}{2 - 3\phi' \circ X} e^{-\omega {}^bX(t)} (\Delta_\omega^x \phi) \circ X.$$

Meanwhile (19) implies also $X' = 1 + \frac{3}{2} X' (\phi' \circ X)$ so that

$$X' = \frac{2}{2 - 3\phi' \circ X}$$

This finally gives:

Lemma 4.1. — For $\omega \in \mathbb{C}^*$,

$$\Delta_\omega^t X = \frac{3}{2} X' \cdot e^{-\omega {}^bX(t)} (\Delta_\omega^x \phi) \circ X.$$

With proposition 4.1 this implies that:

Proposition 4.2. — The resurgent structure of $X(t)$ is governed by:

$$(40) \quad \begin{cases} \Delta_{\frac{4i}{3}}^t X = \frac{3}{4} X' \cdot e^{-(4i/3) {}^bX(t)} \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right) \circ X \\ \Delta_{-\frac{4i}{3}}^t X = -\frac{3}{4} X' \cdot e^{(4i/3) {}^bX(t)} \left(\frac{\psi_{Ai}}{\psi_{Bi}} \right) \circ X \\ \text{otherwise, } \Delta_\omega^t X = 0 \end{cases}$$

The calculation gives:

$$(41) \quad \begin{aligned}\Delta_{\frac{4i}{3}}^t X &= \frac{3}{4} - \frac{15}{128} t^{-2} + \frac{3765}{8192} t^{-4} - \frac{1360375}{262144} t^{-6} + \dots = \sum_{m=0}^{\infty} \frac{c_{(m, \frac{4i}{3})}}{t^m} \\ \Delta_{-\frac{4i}{3}}^t X &= -\frac{3}{4} + \frac{15}{128} t^{-2} - \frac{3765}{8192} t^{-4} + \frac{1360375}{262144} t^{-6} - \dots = \sum_{m=0}^{\infty} \frac{c_{(m, -\frac{4i}{3})}}{t^m}\end{aligned}$$

where

$$c_{(m, \frac{4i}{3})} = -c_{(m, -\frac{4i}{3})} \in \mathbb{R}.$$

(This relies to the fact that $\psi_{Bi}(x) = \psi_{Ai}(-x)$ and that $\psi_{Ai}(ix) \in \mathbb{R}[[x^{-1}]]$).

Note that (40) implies that the alien derivative $\underline{\Delta}_{\frac{t}{2}}^t X$ (resp. $\underline{\Delta}_{-\frac{t}{2}}^t X$) reduces to $e^{-\frac{4i}{3}t} \Delta_{\frac{4i}{3}}^t X$ (resp. $e^{\frac{4i}{3}t} \Delta_{-\frac{4i}{3}}^t X$).

To go further we remark that theorem 4.4 translates into the fact that

$$(42) \quad \Delta_\omega^t \frac{d}{dt} = \left(-\omega + \frac{d}{dt} \right) \Delta_\omega^t.$$

This being said, one uses again theorem 4.3 to calculate $(\Delta_{\frac{4i}{3}}^t)^2 X$:

$$\begin{aligned} (\Delta_{\frac{4i}{3}}^t)^2 X &= \frac{3}{4} \Delta_{\frac{4i}{3}}^t \left(X' e^{-(4i/3)^b X(t)} \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right) \circ X \right) \\ &= \frac{3}{4} (\Delta_{\frac{4i}{3}}^t X') e^{-(4i/3)^b X(t)} \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right) \circ X - i X' (\Delta_{\frac{4i}{3}}^t X) e^{-(4i/3)^b X(t)} \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right) \circ X \\ &\quad + \frac{3}{4} X' e^{-(4i/3)^b X(t)} \left(e^{-(4i/3)^b X(t)} \left(\Delta_{\frac{4i}{3}}^t \frac{\psi_{Bi}}{\psi_{Ai}} \right) \circ X + \Delta_{\frac{4i}{3}}^t X \cdot \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right)' \circ X \right). \end{aligned}$$

Since $\Delta_{\frac{4i}{3}}^x \frac{\psi_{Bi}}{\psi_{Ai}} = i \frac{\psi_{Bi}^2}{\psi_{Ai}^2}$, and taking into account (40) and (42), one gets:

$$\begin{aligned} (\Delta_{\frac{4i}{3}}^t)^2 X &= \\ e^{-(8i/3)^b X(t)} &\left[\left(\frac{9}{16} X'' - \frac{3}{2} i (X')^2 + \frac{3}{4} i X' \right) \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right)^2 \circ X + \frac{9}{8} (X')^2 \cdot \left[\left(\frac{\psi_{Bi}}{\psi_{Ai}} \right)' \frac{\psi_{Bi}}{\psi_{Ai}} \right] \circ X \right]. \end{aligned}$$

The calculation gives

$$(43) \quad (\Delta_{\frac{4i}{3}}^t)^2 X = -\frac{3}{4}i + \frac{15}{128}i t^{-2} + \frac{45}{256}t^{-3} - \frac{3765}{8192}i t^{-4} + \dots = \sum_{m=0}^{\infty} \frac{c(m, \frac{4i}{3}, \frac{4i}{3})}{t^m}.$$

In a similar way,

$$\begin{aligned} (\Delta_{-\frac{4i}{3}}^t)^2 X &= \\ e^{(8i/3)^b X(t)} &\left[\left(\frac{9}{16} X'' + \frac{3}{2} i (X')^2 - \frac{3}{4} i X' \right) \left(\frac{\psi_{Bi}}{\psi_{Ai}} \right)^2 \circ X + \frac{9}{8} (X')^2 \cdot \left[\left(\frac{\psi_{Bi}}{\psi_{Ai}} \right)' \frac{\psi_{Bi}}{\psi_{Ai}} \right] \circ X \right], \\ (44) \quad (\Delta_{-\frac{4i}{3}}^t)^2 X &= \frac{3}{4}i - \frac{15}{128}i t^{-2} + \frac{45}{256}t^{-3} + \frac{3765}{8192}i t^{-4} + \dots = \sum_{m=0}^{\infty} \frac{c(m, -\frac{4i}{3}, -\frac{4i}{3})}{t^m}. \end{aligned}$$

Here again $(\Delta_{\frac{x}{2}}^t)^2 X$ (*resp.* $(\Delta_{-\frac{x}{2}}^t)^2 X$) reduces to $e^{-\frac{8i}{3}t} (\Delta_{\frac{4i}{3}}^t)^2 X$ (*resp.* $e^{\frac{8i}{3}t} (\Delta_{-\frac{4i}{3}}^t)^2 X$).

More generally we have

$$\begin{aligned} \mathfrak{S}_{(\pi/2)} X &= X + \sum_{n \geq 1} e^{-\frac{4i}{3}nt} \frac{1}{n!} X_n(t) \quad X_n = (\Delta_{\frac{4i}{3}}^t)^n X \\ (45) \quad \mathfrak{S}_{(-\pi/2)} X &= X + \sum_{n \geq 1} e^{\frac{4i}{3}nt} \frac{1}{n!} X_{-n}(t) \quad X_{-n} = (\Delta_{-\frac{4i}{3}}^t)^n X \end{aligned}$$

For latter purpose (§5.3) it is also necessary to calculate the alien derivatives $\Delta_{-\frac{4i}{3}}^t (\Delta_{\frac{4i}{3}}^t X)$ and $\Delta_{+\frac{4i}{3}}^t (\Delta_{-\frac{4i}{3}}^t X)$. The reasoning is quite analogous to what we have done previously. One obtains:

$$\begin{aligned} \Delta_{-\frac{4i}{3}}^t (\Delta_{\frac{4i}{3}}^t X) &= -\frac{3}{4}i X' - \frac{9}{16}X'' \\ (46) \quad \Delta_{-\frac{4i}{3}}^t (\Delta_{\frac{4i}{3}}^t X) &= -\frac{3}{4}i + \frac{15}{128}i t^{-2} - \frac{45}{256}t^{-3} - \frac{3765}{8192}i t^{-4} + \dots = \sum_{m=0}^{\infty} \frac{c(m, \frac{4i}{3}, -\frac{4i}{3})}{t^m}. \end{aligned}$$

while

$$\Delta_{\frac{4i}{3}}^t (\Delta_{-\frac{4i}{3}}^t X) = \frac{3}{4}i X' - \frac{9}{16}X'',$$

$$(47) \quad \Delta_{\frac{4i}{3}}^t (\Delta_{-\frac{4i}{3}}^t X) = \frac{3}{4}i - \frac{15}{128}i t^{-2} - \frac{45}{256}t^{-3} + \frac{3765}{8192}i t^{-4} + \dots = \sum_{m=0}^{\infty} \frac{c_{(m, -\frac{4i}{3}, \frac{4i}{3})}}{t^m}.$$

Note that

(48)

$$\mathfrak{S}_{(\pi/2)} X_1 = X_1 + \sum_{n \geq 1} e^{-\frac{4i}{3}nt} \frac{1}{n!} X_{(1,n)}(t) \quad \text{with} \quad X_{(1,1)} = \Delta_{\frac{4i}{3}}^t X_1 = (\Delta_{\frac{4i}{3}}^t)^2 X$$

$$\mathfrak{S}_{(-\pi/2)} X_1 = X_1 + \sum_{n \geq 1} e^{\frac{4i}{3}nt} \frac{1}{n!} X_{(1,-n)}(t) \quad \text{with} \quad X_{(1,-1)} = \Delta_{-\frac{4i}{3}}^t X_1 = \Delta_{-\frac{4i}{3}}^t (\Delta_{\frac{4i}{3}}^t X)$$

while

(49)

$$\mathfrak{S}_{(\pi/2)} X_{-1} = X_{-1} + \sum_{n \geq 1} e^{-\frac{4i}{3}nt} \frac{1}{n!} X_{(-1,n)}(t) \quad \text{with} \quad X_{(-1,1)} = \Delta_{\frac{4i}{3}}^t X_{-1} = \Delta_{\frac{4i}{3}}^t (\Delta_{-\frac{4i}{3}}^t X)$$

$$\mathfrak{S}_{(-\pi/2)} X_{-1} = X_{-1} + \sum_{n \geq 1} e^{\frac{4i}{3}nt} \frac{1}{n!} X_{(-1,-n)}(t) \quad \text{with} \quad X_{(-1,-1)} = \Delta_{-\frac{4i}{3}}^t X_{-1} = (\Delta_{-\frac{4i}{3}}^t)^2 X$$

5. The zeros of the Airy function: hyperasymptotic method

We now show how the informations we have got about X can be used in hyperasymptotic calculations. Since this is the first paper where the relationships between resurgent theory and hyperasymptotics is done, we shall detail the constructions. However we refer to [32] for questions related to optimal truncations at each hyperasymptotic level, and for remainder estimates.

5.1. Level-0. — We start at the 0 level, which is just the summation to the least term. From theorem 2.3 and referring for instance to [15], one sees that for $r \in]0, 4/3[$, there exist $A > 0$, $B > 0$ such that, for $\Re(t) > B$ et $n \geq 0$,

$$(50) \quad \left| s_0 X(t) - t - \sum_{k=0}^n \frac{c_k}{t^k} \right| \leq R_{as}(r, A, B, n, t)$$

$$R_{as}(r, A, B, n, t) = A e^{Br} \frac{n!}{r^n |t|^n (\Re(t) - B)}.$$

To minimize $R_{as}(r, A, B, n, xt)$, one is brought to choose $n = [r|z|]$ as optimal truncation ($[.]$ is the integer part). For such a n one evaluates in practice $R_{as}(r, A, B, n, t)$ as

$$R_{as}(r, A, B, n, t) \sim \frac{|c_{n+1}|}{|t|^n \Re(t)}$$

for n and $|x|$ large enough. However, since $c_n = 0$ for n even, we shall use these estimates only for $n + 1$ odd. The calculations made with $n = \sup_{0 < r < 4/3} [r|t|]$ provide tables 3 and 4.

Optimal n	Estimates for k_1	Error estimates	Real error
5	-2.33863	0.147×10^{-2}	0.52×10^{-3}

TABLE 3. Calculation for the first zero k_1 of the Airy function by level-0 hyperasymptotics.

Optimal n	Estimates for k_3	Error estimates	Real error
17	-5.52055982870	0.133×10^{-8}	0.60×10^{-9}

TABLE 4. Calculation for the first zero k_3 of the Airy function by level-0 hyperasymptotics.

5.2. Level-1. — For $\Re(t) > B$ (B large enough), the Borel sum of $X(t) = t + \sum_{n=0}^{\infty} \frac{c_n}{t^n}$ reads

$$s_0 X(t) = t + \int_0^{+\infty} e^{-t\tau} \tilde{X}(\tau) d\tau,$$

where $\tilde{X}(\tau)$ is the minor of $X(t)$ (since $c_0 = 0$, see (20)).

For practical calculation one can introduce a cut-off $b > 0$ large enough in the integral so that, instead of working with the Borel-sum $s_0 X(t)$ one considers the function

$$(51) \quad s_0^b X(t) = t + \int_0^b e^{-t\tau} \tilde{X}(\tau) d\tau.$$

This can be justified as follows. Since X is Borel-resummable (in the direction 0) there exist $A > 0$ and $B > 0$ such that for $\tau \in B_r$ (see Notation 3.1), $r > 0$ small enough (here $0 < r < 4/3$), $|\tilde{X}(\tau)| \leq Ae^{B|\tau|}$.

For $0 < \delta < \pi/2$ and $\mu > 1$ we note

$$P_{\delta, \mu}(B) = \left\{ t \in \mathbb{C} / |\arg(t)| \leq \frac{\pi}{2} - \delta, |z| \geq \frac{\mu B}{\sin(\delta)} \right\}.$$

For $t \in P_{\delta, \mu}(B)$ we have $\Re(t) - B \geq (1 - \frac{1}{\mu}) \sin(\delta)|t| \geq (\mu - 1)B$, therefore

$$\left| \int_b^{+\infty} \tilde{X}(\tau) e^{-t\tau} d\tau \right| \leq \frac{e^{(B - \Re(t))b}}{\Re(t) - B} \leq \frac{e^{-(1 - \frac{1}{\mu})b \sin(\delta)|t|}}{(\mu - 1)B}.$$

This ensures that, for $t \in P_{\delta, \mu}(B)$ and for $b > 0$ large enough, $|s_0 X(t) - s_0^b X(t)|$ is numerically negligible.

In what follows, it is worth to work with a family of such $s_0^b X(t)$, b large enough. For $t \in P_{\delta, \mu}(B)$, we shall note $s_0^{\square} X(t)$,

$$(52) \quad s_0^{\square} X(t) = t + \int_0^{[0]} e^{-t\tau} \tilde{X}(\tau) d\tau, \quad [0] = be^{i\theta}, b > 0 \text{ large enough.}$$

a member of this family. One defines s_θ^\square (*resp.* $s_{\theta+}^\square, s_{\theta-}^\square$) in a similar way, calling these operators *pre-Borel-resummation* (*resp.* *right, left pre-Borel-resummation*) in the direction θ . One of the main advantage of working with pre-Borel-resummation is that a pre-Borel-sum extends analytically as an entire function.

In (52) we represent the function $\tilde{X}(\tau)$ in term of a Cauchy integral representation

$$(53) \quad \tilde{X}(\tau) = \frac{1}{2i\pi} \oint du \frac{\tilde{X}(u)}{u - \tau}.$$

By a binomial expansion to the order of truncation N_0 one gets the well-known Hermite formula for $\tilde{X}(\tau)$ which, used in (51), gives

$$(54) \quad s_0^\square X(t) = t + \sum_{n=0}^{N_0-1} \frac{c_n}{t^n} + R(t, N_0)$$

where

$$(55) \quad R(t, N_0) = \frac{1}{2i\pi t^{N_0}} \int_0^{[\arg(t)]} dwe^{-w} w^{N_0-1} \int_\gamma du \frac{\tilde{X}(u)}{(1 - w/(tu))u^{N_0}}.$$

The contour γ encircles the line segment $[0, b]$ as in Fig 8. We then deform γ as shown in Fig. 8. By the Cauchy theorem, on can write

$$(56) \quad \begin{aligned} R(t, N_0) = & \frac{1}{2i\pi t^{N_0}} \sum_{k=1}^l \int_0^{[\arg(t)]} dwe^{-w} w^{N_0-1} \int_{\gamma_k} du \frac{\tilde{X}(u)}{(1 - w/(tu))u^{N_0}} \\ & + \frac{1}{2i\pi t^{N_0}} \sum_{k=1}^l \int_0^{[\arg(t)]} dwe^{-w} w^{N_0-1} \int_{\gamma_k^*} du \frac{\tilde{X}(u)}{(1 - w/(tu))u^{N_0}} \\ & + \frac{1}{2i\pi t^{N_0}} \int_0^{[\arg(t)]} dwe^{-w} w^{N_0-1} \int_C du \frac{\tilde{X}(u)}{(1 - w/(tu))u^{N_0}}. \end{aligned}$$

In (56) the γ_k are the bounded paths drawn on Fig. 8 and the γ_k^* are their complex conjugates. The number of terms l in each sum depends on the chosen cut-off. The path C consists in the remaining arcs (in dotted lines on Fig. 8). As shown in [32], the path C gives a contribution to $R(t, N_0)$ which can be bounded away to an exponential level smaller than the one to which the hyperasymptotics is eventually taken, i.e., less than $\exp(-M|t|)$ for any chosen $M > 0$, so that we shall forget this term in what follows.

For each γ_i (*resp.* γ_i^*) we make the change of variable $w = vu/\omega_k$ (*resp.* $w = vu/\omega_k^*$) where $\omega_k = 4ik/3$. Equality (56) becomes (forgetting the contribution of the path C as we explained):

(57)

$$(57) \quad \begin{aligned} R(t, N_0) = & \frac{1}{2i\pi t^{N_0}} \sum_{k=1}^l \frac{1}{\omega_k^{N_0}} \int_0^{[\arg(t)]} dv \int_{\gamma_k} du e^{-vu/\omega_k} v^{N_0-1} \frac{v^{N_0-1}}{1 - v/(t\omega_k)} \tilde{X}(u). \\ & + \frac{1}{2i\pi t^{N_0}} \sum_{k=1}^l \frac{1}{\omega_k^{*N_0}} \int_0^{[\arg(t)]} dv \int_{\gamma_k^*} du e^{-vu/\omega_k} \frac{v^{N_0-1}}{1 - v/(t\omega_k^*)} \tilde{X}(u). \end{aligned}$$

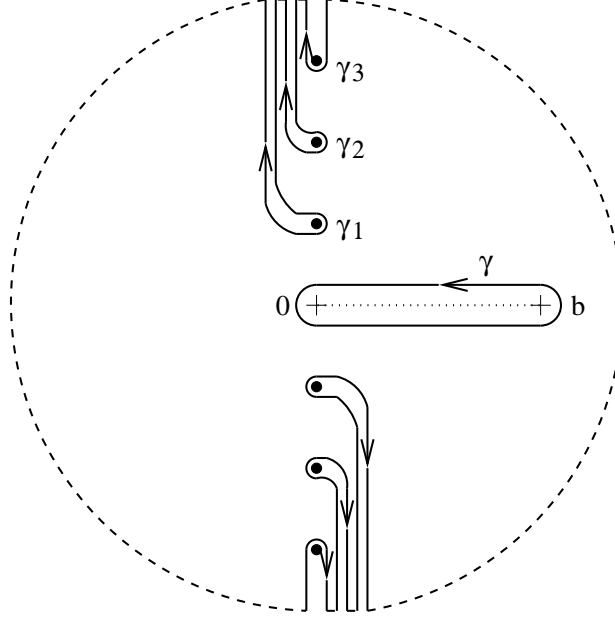


FIGURE 8. The contour γ and its homotopic deformation. The bullets are the singular points $\omega_k = 4ik/3$ and $\omega_k^* = -4ik/3$, $k \in \mathbb{N}^*$.

Up to their orientations (and the cut-off), the contours γ_k (*resp.* γ_k^*) are those used when one compares left and right Borel-resummation in the direction $\pi/2$ (see Fig. 7) (*resp.* $-\pi/2$). Using the known action of the Stokes automorphisms $\mathfrak{S}_{(\pi/2)}$ and $\mathfrak{S}_{(-\pi/2)}$ on X (see (45)) and making the convenient change of variable $\tau = u - \omega_k$ (*resp.* $\tau = u - \omega_k^*$) in each term, one obtains

$$(58) \quad R(t, N_0) = -\frac{1}{2i\pi t^{N_0}} \sum_{k=1}^l \frac{1}{\omega_k^{N_0}} \int_0^{[\arg(t)]} dv e^{-v} \frac{v^{N_0-1}}{1-v/(t\omega_k)} S_{(\pi/2)+}^{\square} \frac{1}{k!} X_k(v/\omega_k) \\ - \frac{1}{2i\pi t^{N_0}} \sum_{k=1}^l \frac{1}{\omega_k^{*N_0}} \int_0^{[\arg(t)]} dv e^{-v} \frac{v^{N_0-1}}{1-v/(t\omega_k^*)} S_{(-\pi/2)+}^{\square} \frac{1}{k!} X_{-k}(v/\omega_k^*).$$

Note that, since one works with pre-Borel-resummation, $S_{(\pi/2)+}^{\square} X_k$ and $S_{(-\pi/2)+}^{\square} X_{-k}$ extend as entire functions, so that (58) is well-defined.

Equation (58) may be written in the following way by suitable changes of variables $\tau = v/\omega_k$ and $\tau = v/\omega_k^*$.

$$(59) \quad R(t, N_0) = -\frac{1}{2i\pi t^{N_0-1}} \sum_{k=1}^l \int_0^{[\arg(t)-\arg(\omega_k)]} d\tau e^{-\omega_k \tau} \frac{\tau^{N_0-1}}{t-\tau} S_{(\pi/2)+}^{\square} \frac{1}{k!} X_k(\tau) \\ - \frac{1}{2i\pi t^{N_0-1}} \sum_{k=1}^l \int_0^{[\arg(t)-\arg(\omega_k^*)]} d\tau e^{-\omega_k^* \tau} \frac{\tau^{N_0-1}}{t-\tau} S_{(-\pi/2)+}^{\square} \frac{1}{k!} X_{-k}(\tau).$$

The resurgence formula (58) (in the sense of Dingle-Berry-Howls [4, 17]) is the key-point in hyperasymptotic theory.

The algorithm for the level-1 hyperasymptotics is now as follows (see [32]):

- Only the seen (adjacent) singularities $\omega_1 = 4i/3$ and $\omega_1^* = -4i/3$ play a role. This means that we write

$$(60) \quad R(t, N_0) = -\frac{1}{2i\pi t^{N_0-1}} \int_0^{[\arg(t)-\arg(\omega_1)]} d\tau e^{-\omega_1 \tau} \frac{\tau^{N_0-1}}{t-\tau} S_{(\pi/2)^+}^{\square} X_1(\tau) \\ -\frac{1}{2i\pi t^{N_0-1}} \int_0^{[\arg(t)-\arg(\omega_1^*)]} d\tau e^{-\omega_1^* \tau} \frac{\tau^{N_0-1}}{t-\tau} S_{(-\pi/2)^+}^{\square} X_{-1}(\tau).$$

modulo a remainder which will be negligible at this level 1.

- We replace each right Borel-sum $s_{(\pi/2)^+}^{\square} X_1(\tau)$ and $s_{(-\pi/2)^+}^{\square} X_{-1}(\tau)$ by their truncated asymptotic expansions. For reasons of symmetries, the order of truncation $N_1 \leq N_0$ will be the same for each of these expansions. One then extends the bounded contours of integration up to infinity. Putting all pieces together, using (45) and (41), one thus obtains

$$(61) \quad s_0 X(t) = t + \sum_{n=0}^{N_0-1} \frac{c_n}{t^n} \\ -\frac{1}{2i\pi t^{N_0-1}} \sum_{n=0}^{N_1-1} c_{(n, \frac{4i}{3})} \int_0^{\infty e^{-i \arg(\omega_1)}} d\tau e^{-\omega_1 \tau} \frac{\tau^{N_0-n-1}}{t-\tau} \\ -\frac{1}{2i\pi t^{N_0-1}} \sum_{n=0}^{N_1-1} c_{(n, -\frac{4i}{3})} \int_0^{\infty e^{-i \arg(\omega_1^*)}} d\tau e^{-\omega_1^* \tau} \frac{\tau^{N_0-n-1}}{t-\tau} + R(N_0, N_1, t)$$

- Taking into account that the seen singularities from ω_1 (*resp.* ω_1^*) are at a distance $|\omega_1|$, and for reasons explained in [32], the choices $N_0 = 2|\omega_1||t| = 8/3|t|$ and $N_1 = |\omega_1||t| = 4/3|t|$ give optimal truncations, for which the remainder term behaves like $R(N_0, N_1, t) = \exp(-N_0|t|)O(1)$, for $t \in P_{\delta, \mu}(B)$.

Note that one can write (61) using the the canonical hyperterminants [4, 22, 23, 32], defined by:

$$(62) \quad \left\{ \begin{array}{l} F^{(0)}(t) = 1 \\ F^{(1)} \left(t; \begin{array}{c} M_0 \\ \sigma_0 \end{array} \right) = \int_0^{\infty e^{-i\theta_0}} d\tau_0 e^{-\sigma_0 \tau_0} \frac{\tau_0^{M_0-1}}{t-\tau_0}, \quad \theta_0 = \arg(\sigma_0) \\ F^{(l+1)} \left(t; \begin{array}{ccc} M_0, & \cdots, & M_l \\ \sigma_0, & \cdots, & \sigma_l \end{array} \right) = \\ \int_0^{\infty e^{-i\theta_0}} d\tau_0 \cdots \int_0^{\infty e^{-i\theta_l}} d\tau_l e^{-(\sigma_0 \tau_0 + \cdots + \sigma_l \tau_l)} \frac{\tau_0^{M_0-1} \cdots \tau_l^{M_l-1}}{(t-\tau_0)(\tau_0-\tau_1) \cdots (\tau_{l-1}-\tau_l)} \\ (\theta_i = \arg(\sigma_i)). \end{array} \right.$$

Remark 5.1. — With formula (41) we have noted that

$$c_{(m, \frac{4i}{3})} = -c_{(m, -\frac{4i}{3})} \in \mathbb{R}.$$

Since formula (61) reads also as
(63)

$$\begin{aligned} s_0 X(t) = & t + \sum_{n=0}^{N_0-1} \frac{c_n}{t^n} \\ & + \frac{1}{2\pi t^{N_0-1}} \sum_{n=0}^{N_1-1} (-i)^{N_0-n-1} c_{(n, \frac{4i}{3})} \int_0^{+\infty} dx e^{-4x/3} \frac{x^{N_0-n-1}}{t+ix} \\ & - \frac{1}{2\pi t^{N_0-1}} \sum_{n=0}^{N_1-1} i^{N_0-n-1} c_{(n, -\frac{4i}{3})} \int_0^{+\infty} dx e^{-4x/3} \frac{x^{N_0-n-1}}{t-ix} + R(N_0, N_1, t), \end{aligned}$$

we see that the realness of $s_0 X(t)$ is well preserved by the level-1 hyperasymptotics.

We turn now to numerical experiments. Formula (61) give tables 5 and 6.

Optimal N	Estimates for k_1	Real error
$N_0 = 9, N_1 = 5$	-2.33810834	0.93×10^{-6}

TABLE 5. Calculation for the first zero k_1 of the Airy function by level-1 hyperasymptotics.

Optimal N	Estimates for k_3	Real error
$N_0 = 35, N_1 = 17$	-5.520559828095551059172	0.42×10^{-19}

TABLE 6. Calculation for the first zero k_3 of the Airy function by level-1 hyperasymptotics.

5.3. Level-2. — What we have done at the level-1 can be repeated. We just detail here how the informations got from the resurgence analysis can be used in this context, referring to [32] for what concerns the questions of optimal truncations and remainder estimates.

We go back to (59) with $l = 2$. For $k = 2$, one just replace the pre-Borel-sums $s_{(\pi/2)+}^{\square} \frac{1}{2!} X_2(\tau)$ and $s_{(-\pi/2)+}^{\square} \frac{1}{2!} X_{-2}(\tau)$ by their truncated asymptotic expansions. With the notations (45), (43), (44), (62), this gives:

(64)

$$\begin{aligned} s_0 X(t) = & t + \sum_{n=0}^{N_0-1} \frac{c_n}{t^n} \\ & - \frac{1}{2i\pi t^{N_0-1}} \sum_{n=0}^{N_2-1} \frac{c_{(n, \frac{4i}{3}, \frac{4i}{3})}}{2} F^{(1)} \left(t; \begin{matrix} N_0 - n \\ \omega_2 \end{matrix} \right) + \frac{c_{(n, -\frac{4i}{3}, -\frac{4i}{3})}}{2} F^{(1)} \left(t; \begin{matrix} N_0 - n \\ \omega_2^* \end{matrix} \right) \\ & - \frac{1}{2i\pi t^{N_0-1}} \int_0^{[\arg(t) - \arg(\omega_1)]} d\tau e^{-\omega_1 \tau} \frac{\tau^{N_0-1}}{t - \tau} s_{(\pi/2)+}^{\square} X_1(\tau) \\ & - \frac{1}{2i\pi t^{N_0-1}} \sum_{k=1}^l \int_0^{[\arg(t) - \arg(\omega_1^*)]} d\tau e^{-\omega_1^* \tau} \frac{\tau^{N_0-1}}{t - \tau} s_{(-\pi/2)+}^{\square} X_{-1}(\tau) + R(N_0, N_2, t). \end{aligned}$$

For $s_{(\pi/2)^+}^{\square} X_1(\tau)$ and $s_{(-\pi/2)^+}^{\square} X_{-1}(\tau)$ we copy what we have done at the level-1 hyperasymptotics: using (48) and (49) we get:

$$(65) \quad \begin{aligned} s_{(\pi/2)^+}^{\square} X_1(\tau) &= \sum_{n=0}^{N_1-1} \frac{c_{(n,4i/3)}}{\tau^n} \\ &- \frac{1}{2i\pi\tau^{N_1-1}} \sum_{n=0}^{N_2-1} c_{(n, \frac{4i}{3}, \frac{4i}{3})} \int_0^{\infty} e^{-i \arg(\omega_1)} d\tau_1 e^{-\omega_1 \tau_1} \frac{\tau_1^{N_1-n-1}}{\tau - \tau_1} \\ &- \frac{1}{2i\pi\tau^{N_1-1}} \sum_{n=0}^{N_2-1} c_{(n, \frac{4i}{3}, -\frac{4i}{3})} \int_0^{\infty} e^{-i \arg(\omega_1^*)} d\tau_1 e^{-\omega_1^* \tau_1} \frac{\tau_1^{N_1-n-1}}{\tau - \tau_1} + R(N_1, N_2, \tau) \end{aligned}$$

and

$$(66) \quad \begin{aligned} s_{(-\pi/2)^+}^{\square} X_{-1}(\tau) &= \sum_{n=0}^{N_1-1} \frac{c_{(n,-4i/3)}}{\tau^n} \\ &- \frac{1}{2i\pi\tau^{N_1-1}} \sum_{n=0}^{N_2-1} c_{(n, -\frac{4i}{3}, \frac{4i}{3})} \int_0^{\infty} e^{-i \arg(\omega_1)} d\tau_1 e^{-\omega_1 \tau_1} \frac{\tau_1^{N_1-n-1}}{\tau - \tau_1} \\ &- \frac{1}{2i\pi\tau^{N_1-1}} \sum_{n=0}^{N_2-1} c_{(n, -\frac{4i}{3}, -\frac{4i}{3})} \int_0^{\infty} e^{-i \arg(\omega_1^*)} d\tau_1 e^{-\omega_1^* \tau_1} \frac{\tau_1^{N_1-n-1}}{\tau - \tau_1} + R(N_1, N_2, \tau) \end{aligned}$$

Plugging (65) and (66) in (64) we obtain:

$$(67) \quad \begin{aligned} s_0 X(t) &= t + \sum_{n=0}^{N_0-1} \frac{c_n}{t^n} \\ &- \frac{1}{2i\pi t^{N_0-1}} \sum_{n=0}^{N_1-1} c_{(n, \frac{4i}{3})} F^{(1)} \left(t; \begin{matrix} N_0 - n \\ \omega_1 \end{matrix} \right) + c_{(n, -\frac{4i}{3})} F^{(1)} \left(t; \begin{matrix} N_0 - n \\ \omega_1^* \end{matrix} \right) \\ &- \frac{1}{2i\pi t^{N_0-1}} \sum_{n=0}^{N_2-1} \frac{c_{(n, \frac{4i}{3}, \frac{4i}{3})}}{2} F^{(1)} \left(t; \begin{matrix} N_0 - n \\ \omega_2 \end{matrix} \right) + \frac{c_{(n, -\frac{4i}{3}, -\frac{4i}{3})}}{2} F^{(1)} \left(t; \begin{matrix} N_0 - n \\ \omega_2^* \end{matrix} \right) \\ &+ \left(-\frac{1}{2i\pi} \right)^2 \frac{1}{t^{N_0-1}} \sum_{n=0}^{N_2-1} \left[c_{(n, \frac{4i}{3}, \frac{4i}{3})} F^{(2)} \left(t; \begin{matrix} N_0 - N_1 + 1, & N_1 - n \\ \omega_1, & \omega_1 \end{matrix} \right) \right. \\ &+ c_{(n, \frac{4i}{3}, -\frac{4i}{3})} F^{(2)} \left(t; \begin{matrix} N_0 - N_1 + 1, & N_1 - n \\ \omega_1, & \omega_1^* \end{matrix} \right) + c_{(n, -\frac{4i}{3}, \frac{4i}{3})} F^{(2)} \left(t; \begin{matrix} N_0 - N_1 + 1, & N_1 - n \\ \omega_1^*, & \omega_1 \end{matrix} \right) \\ &\left. + c_{(n, -\frac{4i}{3}, -\frac{4i}{3})} F^{(2)} \left(t; \begin{matrix} N_0 - N_1 + 1, & N_1 - n \\ \omega_1^*, & \omega_1^* \end{matrix} \right) \right] + R(N_0, N_1, N_2, t). \end{aligned}$$

Note that, since we always use right-Borel-resummations, this induces the following convention for the hyperterminants, when $\arg \sigma_j = \arg \sigma_{j+1} \pmod{2\pi}$ for some j :

$$(68) \quad F^{(l+1)} \left(t; \begin{matrix} M_0, & \cdots, & M_l \\ \sigma_0, & \cdots, & \sigma_l \end{matrix} \right) = \lim_{\epsilon \downarrow 0} F^{(l+1)} \left(t; \begin{matrix} M_0, & & M_1, & \cdots, & M_l \\ \sigma_0 e^{-i\epsilon}, & & \sigma_1 e^{-i(l-1)\epsilon} & \cdots, & \sigma_l \end{matrix} \right).$$

For reasons explained in [32], the choices $N_0 = 3|\omega_1||t| = 12/3|t|$ and $N_1 = 2|\omega_1||t| = 8/3|t|$ and $N_2 = |\omega_1||t| = 4/3|t|$ give optimal truncations, for which the remainder term behaves like $R(N_0, N_1, N_2, t) = \exp(-N_0|t|)O(1)$, for $t \in P_{\delta, \mu}(B)$. Numerical experiments are illustrated by tables 7 and 8.

Optimal N	Estimates for k_1	Real error
$N_0 = 14, N_1 = 9, N_2 = 5$	-2.33810741077	0.31×10^{-9}

TABLE 7. Calculation for the first zero k_1 of the Airy function by level-2 hyperasymptotics.

Optimal N	Estimates for k_3	Real error
$N_0 = 52, N_1 = 35, N_2 = 17$	-5.520559828095551059129855522	0.87×10^{-26}

TABLE 8. Calculation for the first zero k_3 of the Airy function by level-2 hyperasymptotics.

5.4. Higher level. — From a theoretical viewpoint it is possible to perform higher hyperasymptotics. This requires to go deeper in the resurgent structure of $X(t)$. The method is just like what we have done in §4.3, even if the alien calculus becomes somewhat complicated. For the hyperasymptotics part, the existence of collinear singularities induce the need of great care in choosing the correct branches of the hyperterminants (in relation with (68)). But, according to the specialists, this problem can be mastered (see [33]).

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