

Low lying twisting and acoustic modes of a rotating Bose-Einstein condensate

F. Chevy

Laboratoire Kastler-Brossel*, ENS, 24 rue Lhomond, 75005 Paris

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We present a calculation of the low lying spectrum of a rotating Bose-Einstein condensate. We show that in a cylindrical geometry, there exist two linear branches, one associated with usual acoustic excitations, the other corresponding to a twisting mode of the vortex lattice. Using a hydrodynamical approach we derive the elasticity coefficient of the vortex lattice and calculate the spectrum of condensate in a three dimensional harmonic trap with cylindrical symmetry.

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By contrast with classical hydrodynamics, a quantum fluid cannot rotate like a solid body. Instead, it carries quantized vortices self organizing along a triangular Abrikosov lattice when the rotation is fast enough. Vortices constitute a universal characteristic of quantum fluids and were directly observed in systems as different as liquid helium [1], superconductors [2], gaseous Bose-Einstein condensates (BEC) [3, 4, 5, 6] and very recently fermionic superfluids [7]. The study of the excitations of a vortex lattice was initiated by the work of Tkachenko [8] that was restricted to the study of the propagation of waves transversely to the rotation axis in an incompressible superfluid. The full understanding of these excitations in the case of dilute gases remains a challenge, due to the non trivial interplay between the elasticity of the vortex lattice and the phonon modes associated with the compressibility of these systems. Partial results were obtained in the case of a single vortex line [9, 10], two-dimensional systems [11, 12, 13], fast rotating systems [14, 15, 16] or using the rotational hydrodynamics formalism [17, 18, 20], but a complete theory remains to be found.

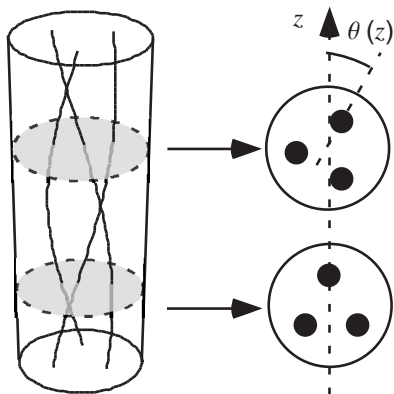


FIG. 1: Left: Structure of the twiston mode for a cylindrical BEC carrying three vortex lines. Right: arrangement of the vortex lattice in the planes defined by the shaded areas. The vortex lattice is rotated by an angle $\theta(z)$ depending on the axial position along the trap. This twisting induces the propagation of an elastic wave along the condensate axis.

In this letter, we present a study of the low lying modes propagating along the axis of an elongated vortex lattice. Up to now, these modes have only been considered for homogeneous and unbounded fluids [16]. In the more realistic case of a trapped gaseous BEC, we show that these modes can be understood as Goldstone modes arising from U(1) and O(2) broken symmetries, associated respectively with the choice of the phase of the macroscopic wave function of the condensate and of the direction of the vortex lattice. These two broken symmetries thus give rise to two low energy excitation branches associated respectively to a modulation of the phase of the wave function and of the direction of the vortex lattice in the z direction. Since the gradient of the phase is proportional to the velocity of the BEC, the first branch is associated with an acoustic wave propagating along the cloud (“phonon” branch). The second mode corresponds to the twisting of the vortices and is related to the elasticity of the lattice (“twiston” branch, Fig. 1). This scheme will be worked out first using a perturbative resolution of the Bogoliubov-de Gennes equations in a cylindrical trap, a method already used in the stability analysis of solitons [19]. We will show that this solution can be interpreted in a macroscopic framework, leading to elasto-hydrodynamical equations for the motion of the BEC in the presence of the vortex lattice. In particular, we propose the first derivation of the elastic response coefficient of the vortex lattice, starting from first principles. Finally, local density approximation will allow for the calculation of the lowest energy modes in the presence of an axial trapping.

We consider a dilute Bose-Einstein condensate rotating at an angular velocity Ω_0 along the z axis. At rest, the system is described in the mean-field approximation by a macroscopic wave-function ψ_0 solution of the Gross-Pitaevskii equation

$$\left(\hat{h}_0 + g|\psi_0|^2\right) \psi_0 = 0. \quad (1)$$

where the single particle hamiltonian \hat{h}_0 is given by $\hat{h}_0 = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y) - \Omega_0\hat{L}_z - \mu_0$. V is the transverse trapping potential, \hat{L}_z the angular momentum in the z

direction, μ_0 the chemical potential and g the coupling constant characterizing the 2-body interactions. We assume for the moment there is no confining potential in the z direction. Nevertheless, we impose periodic boundary conditions in this direction, with period L , to allow for the normalisation of the wave-function.

In the linear regime, we write function ψ of the condensate is given by $\psi = \psi_0 + \delta\psi$. $\delta\psi$ is then solution of the Bogoliubov de-Gennes equations $i\hbar\partial_t\Phi = \widehat{\mathcal{L}}\Phi$, with $\Phi = (\delta\psi, \delta\psi^*)$ and

$$\widehat{\mathcal{L}} = \begin{pmatrix} \widehat{h}_0 + 2g|\psi_0|^2 & g\psi_0^2 \\ -g\psi_0^{*2} & -\widehat{h}_0 - 2g|\psi_0|^2 \end{pmatrix}. \quad (2)$$

We recall a few properties of $\widehat{\mathcal{L}}$ relevant for the discussion [21, 22]. First, although $\widehat{\mathcal{L}}$ is not hermitian, it is orthogonal for the quadratic form with signature (1,-1) defined by $(\Phi_1|\Phi_2) = \int d^3r \Phi_1^\dagger \widehat{\sigma}_3 \Phi_2$, where $\widehat{\sigma}_3$ is the diagonal Pauli matrix. Second, the eigenvectors Φ_α of $\widehat{\mathcal{L}}$ do not constitute a complete basis. Indeed, one can show that each zero energy mode is associated with an anomalous Jordan mode Φ'_α such that $\widehat{\mathcal{L}}\Phi'_\alpha = \Phi_\alpha$. In the case under study here, we assume there are only two zero energy modes associated respectively with the U(1) and O(2) freedom of choice of the phase of the wave function and the orientation of the vortex lattice. The associated eigenvectors are respectively denoted by Φ_p (phase symmetry) and Φ_r (rotational symmetry) and are given by [21, 22]

$$\begin{aligned} \Phi_p &= \begin{pmatrix} \psi_0 \\ -\psi_0^* \end{pmatrix} & \Phi'_p &= \begin{pmatrix} \partial_\mu \psi_0 \\ \partial_\mu \psi_0^* \end{pmatrix} \\ \Phi_r &= \begin{pmatrix} \widehat{L}_z \psi_0 \\ -(\widehat{L}_z \psi_0)^* \end{pmatrix} & \Phi'_r &= \begin{pmatrix} \partial_\Omega \psi_0 \\ \partial_\Omega \psi_0^* \end{pmatrix}, \end{aligned} \quad (3)$$

Let us now proceed with the calculation of the long wavelength modes of $\widehat{\mathcal{L}}$. Using the translational invariance of the system we can write $\Phi_\alpha(x, y, z) = \bar{\Phi}_\alpha(x, y)e^{ik_\alpha z}$. We therefore see that $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_0 + \delta\widehat{\mathcal{L}}$, where $\widehat{\mathcal{L}}_0$ acts on the transverse degrees of freedom only, $\delta\widehat{\mathcal{L}} = \epsilon_{k_\alpha} \widehat{\sigma}_3$, with $\epsilon_{k_\alpha} = \hbar^2 k_\alpha^2 / 2m$.

Since we only care for long wavelength eigenmodes, we have ϵ_{k_α} vanishingly small and we can treat $\delta\widehat{\mathcal{L}}$ as a perturbation of $\widehat{\mathcal{L}}_0$. However, starting the perturbation expansion from the zero energy modes Φ_r and Φ_p of \mathcal{L}_0 associated with the rotational and phase symmetries gives rise to *linear* excited branches. Indeed, due to the non diagonalizability of $\widehat{\mathcal{L}}_0$ the first order term of the low momentum expansion scales like $\epsilon_{k_\alpha}^{1/2}$ rather than ϵ_{k_α} [23]. This singularity can be proven rigorously by diagonalizing the projection of $\delta\widehat{\mathcal{L}}$ on the vector space spanned by the zero energy and anomalous modes. Physically, this result corresponds to the fact that one should

recover a linear phonon dispersion law $E_\alpha \propto k_\alpha \propto \epsilon_{k_\alpha}^{1/2}$ associated with usual acoustic waves.

Working out the perturbative expansion in power of $\sqrt{\epsilon_{k_\alpha}}$ yields

$$\bar{\Phi}_\alpha = (a_r \Phi_r + a_p \Phi_p) + E_\alpha (a_r \Phi'_r + a_p \Phi'_p) + \dots, \quad (4)$$

where E_α is solution of the eigenequation

$$\sum_{\gamma=r,p} (\epsilon_{k_\alpha} A_{\beta\gamma} - E_\alpha^2 B_{\beta\gamma}) a_\gamma = 0, \quad (5)$$

with $\beta \in \{r, p\}$, $A_{\beta\gamma} = (\Phi_\beta | \widehat{\sigma}_z \Phi_\gamma) / L$, and $B_{\beta\gamma} = (\Phi_\beta | \Phi'_\gamma) / L$. L is introduced here so that A and B are defined as 1D linear quantities that we can use in a local density approach as demonstrated later in this paper. Using Eq. 3 yields the simple expressions

$$\begin{aligned} (\Phi_p | \widehat{\sigma}_z \Phi_p) &= 2N & (\Phi_p | \Phi'_p) &= \partial_{\mu_0} N \\ (\Phi_r | \widehat{\sigma}_z \Phi_r) &= 2\langle \widehat{L}_z^2 \rangle & (\Phi_r | \Phi'_r) &= \partial_{\Omega_0} \langle \widehat{L}_z \rangle \\ (\Phi_p | \widehat{\sigma}_z \Phi_r) &= 2\langle \widehat{L}_z \rangle & (\Phi_p | \Phi'_r) &= \partial_{\Omega_0} N \\ (\Phi_r | \widehat{\sigma}_z \Phi_p) &= 2\langle \widehat{L}_z \rangle & (\Phi_r | \Phi'_p) &= \partial_{\mu_0} \langle \widehat{L}_z \rangle, \end{aligned} \quad (6)$$

where N is the total atom number [24]. The set of two equations (5) yields two different linear excitation branches. The highest branch has a non-zero velocity for vanishing Ω_0 and can therefore be identified with a phonon mode. The other branch has a vanishing velocity at small Ω and is therefore the twiston mode.

It is striking that the coefficients of Eq. (5) can be expressed as simple combinations of macroscopic quantities such as the atom number or the angular momentum. Similarly to superfluid hydrodynamics, this suggests that the formalism developed here can be expressed in term of an elasto-hydrodynamical theory. In order to clarify this link, we first note that the frequency of the long wavelength modes we are interested in is much smaller than the frequencies characterizing the evolution of the transverse degrees of freedom. This means in particular that the dynamics in the (x, y) plane is frozen and that we can therefore define local chemical potential $\mu(z, t)$, angular velocity $\Omega(z, t)$, phase of the wavefunction $\chi(z, t)$ and angle of the vortex lattice $\theta(z, t)$. In other word, the wave function can be written as

$$\psi(x, y, z, t) = \psi_0(\mu(z, t), \Omega(z, t), \chi(z, t), \theta(z, t), x, y) \quad (7)$$

where ψ_0 is the solution of the Gross Pitaevskii equation (1). Expanding the macroscopic quantities around their equilibrium values yields

$$\delta\psi = \delta\mu \partial_{\mu_0} \psi_0 + \delta\Omega \partial_{\Omega_0} \psi_0 + \delta\chi \partial_{\chi_0} \psi_0 + \delta\theta \partial_{\theta_0} \psi_0, \quad (8)$$

where $\mu = \mu_0 + \delta\mu$, $\Omega = \Omega_0 + \delta\Omega$, etc. Noting further that $\delta\psi$ is the first component of Φ_α and that $\partial_\chi\psi_0 = i\psi_0$ and $\partial_\theta\psi_0 = -i\hat{L}_z\psi_0/\hbar$, we see that Eqn. (8) is equivalent to

$$\Phi = i\delta\chi\Phi_p - i\frac{\delta\theta}{\hbar}\Phi_r + \delta\mu\Phi'_p + \delta\Omega\Phi'_r. \quad (9)$$

Comparing to Eq. (4) this leads to the identification

$$a_r = -i\delta\theta/\hbar = \delta\Omega/E_\alpha \quad (10)$$

$$a_p = i\delta\chi = \delta\mu/E_\alpha, \quad (11)$$

which gives, using the identity $\partial_t = -iE_\alpha/\hbar$,

$$\partial_t\delta\theta = \delta\Omega \quad (12)$$

$$\hbar\partial_t\delta\chi = -\delta\mu \quad (13)$$

The interpretation of these two relations is straightforward. Indeed, eq. 12 is the analogue of the Kelvin-Helmholtz theorem [25] and implies that the vortex lattice rotates at the same speed as the local velocity flow. As for eq. 13, it is a version of the Bernoulli theorem, with $\hbar\delta\chi/m$ playing the role of the velocity potential.

Starting from the identification of the $a_{r,p}$ coefficient, we now show that the eigensystem (5) can be interpreted as conservation laws for the system. Indeed, the conservation of particle number, momentum and angular momentum in the z direction yield at first order in perturbation the very general set of equations

$$\partial_t\delta n + \partial_z(n_0v) = 0 \quad (14)$$

$$mn_0\partial_tv + \partial_zf = 0 \quad (15)$$

$$\partial_t\delta\ell_z + \partial_z\Gamma = 0 \quad (16)$$

where n_0 is the linear particle density, v is the local velocity in the z direction, ℓ_z is the angular momentum per unit length. f and Γ are respectively the 1D momentum and angular momentum currents, which can be identified with a force and a torque and will be calculated from the microscopic formalism presented above.

Combining Eq. (14) and (15) and using a generalized Gibbs-Duhem relation [26] for a rotating system $df = n_0d\mu + \ell_z^0d\Omega$ yields

$$\partial_t^2 [\delta\mu\partial_{\mu_0}n_0 + \delta\Omega\partial_{\Omega_0}n_0] = \frac{1}{m}\partial_z [n_0\partial_z\delta\mu + \ell_z^0\partial_z\delta\Omega] \quad (17)$$

where we have used the adiabatic following of the transverse degrees of freedom to set $\delta n = \delta\mu\partial_{\mu_0}n_0 + \delta\Omega\partial_{\Omega_0}n_0$.

After making the replacement $\hbar\partial_t = -iE_\alpha$ and $\partial_z = ik_\alpha$, we see that Eq. (17) is strictly equivalent to the $\alpha = p$ component of Eq. (5).

A similar analysis shows that the $\alpha = r$ component of Eq. (5) is equivalent to the angular momentum conservation if the current Γ is taken to be

$$\Gamma = \ell_z^0v - \kappa\partial_z\delta\theta, \quad (18)$$

where $\kappa = \Delta L_z^2/Lm$, and $\Delta L_z^2 = \langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2/N$ is the fluctuation of the angular momentum. The term of Γ proportional to the local velocity v corresponds the convective part of the angular momentum current. The second term is associated with the elastic response of the vortex lattice to a torsion and κ is therefore the elastic modulus of the lattice.

The hydrodynamical formalism developed above can be used to extend our calculation to the case of a weak trapping in the z direction. In this case, local density approximation can be applied. At equilibrium, the angular velocity is still uniform along the cloud. By contrast, the local chemical potential is now given by $\mu_0(z) = \mu_c - m\omega_z^2z^2/2$, where μ_c is the chemical potential at the center of the trap. This permits to define local matrices $A(z)$ and $B(z)$ and yields the equation for the vector $\mathbf{X}(z, t) = (\delta\mu(z, t), \delta\Omega(z, t))$

$$2m\partial_t^2 [B(z) \cdot \mathbf{X}(z, t)] = \partial_z [A(z) \cdot \partial_z \mathbf{X}(z, t)] \quad (19)$$

In general, A and B must be calculated using a numerical resolution of the Gross-Pitaevskii equation. However, in the regime of fast rotation, quantities appearing in Eq. (19) can be evaluated assuming a classical rotational flow $\mathbf{v} = \mathbf{\Omega}_0 \times \mathbf{r}$. We have first checked that at zero angular velocity we could recover the spectrum of an elongated condensate studied in [27]. We have also verified that the center of mass was indeed evolving at a frequency ω_z , in accordance with Kohn's theorem. The other eigenfrequencies ω_α are evaluated using a variational method. We indeed note that the ω_α^2 are the eigenvalues of the operator $\hat{T} = B^{-1} \cdot \partial_z(A \cdot \partial_z \cdot)/2m$ which is hermitian for the scalar product defined by $\langle \mathbf{Y} | \mathbf{X} \rangle = \int \mathbf{Y}^\dagger(z) \cdot B(z) \cdot \mathbf{X}(z) dz$. We have calculated the ω_α using polynomial trial wave functions of order 5. The variation of ω_α for the two lowest twiston modes – corresponding respectively to odd and even symmetries [28] – is displayed in Fig. 2.

In conclusion, we have demonstrated the existence of two low energy excitation branches of a rotating elongated Bose-Einstein condensate. The validity of our calculation is limited by two conditions. First the applicability of the hydrodynamical formalism to evaluate the matrix elements of A and B requires a dense vortex lattice, hence a large rotation frequency. Using an imaginary time evolution of 2D Gross-Pitaevskii equation [29], we have checked it was true for $\Omega_0 \gtrsim 0.8\omega_\perp$. Second the breakdown of the perturbation expansion will happen when the ω_α become comparable with the frequency

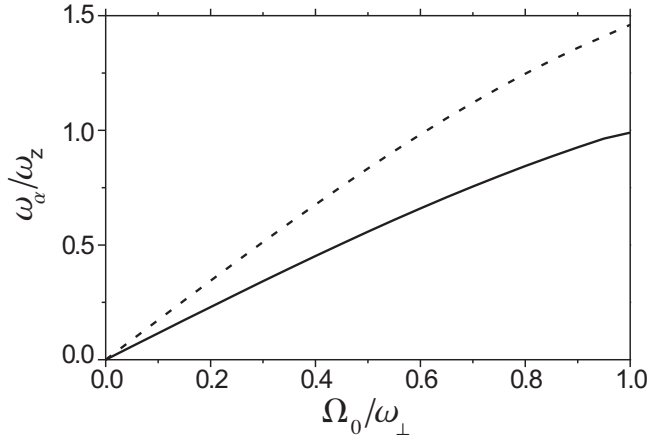


FIG. 2: Low lying twiston spectrum of a trapped rotating Bose-Einstein condensate. The two branches correspond to the frequency of the lowest odd (full line) and even (dashed line) twiston modes and were calculated using polynomials trial wave functions.

ω_T of the first transverse excited mode, *i.e.* the lowest Tkachenko mode. Using the expression of ω_T found in [15], we find that for typical values $\mu/\hbar\omega_\perp = 20$ and $\omega_\perp/\omega_z = 20$, the perturbation expansion fails for $\Omega_0/\omega_\perp \lesssim 0.96$. Although the width of validity of our approximation might seem narrow at first sight, it must be noted that experimentally regular vortex lattices are only observed on a relatively narrow range of rotation frequencies, comparable to our validity domain ($\Omega_0/\omega_\perp \in [0.7, 0.96]$ for [30]). A last issue concerns the experimental observation of twistons. It must be noted that, in practice, it is difficult to excite directly the vortex degrees of freedom of a gaseous BEC and it is therefore more convenient to excite an acoustic mode by perturbing the trapping potential and to use the linear [31, 32] or non linear [33] coupling to vortical modes. In principle, the coupling existing between acoustic and torsional degrees of freedom should permit an excitation of the twiston mode using an external potential. However, one can show using perturbation theory that a simple modulation of the trap center or frequency is only weakly coupled to the twiston mode. To obtain a significant coupling to torsional modes, one therefore needs to impart cubic or quartic perturbation to the gas.

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