RANDOM SERIES OF FUNCTIONS AND APPLICATIONS

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Abstract. In this article we study the continuity properties of trajectories for some random series of functions, $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$ where $(a_k)_{k\geqslant 0}$ is a complex sequence, $(X_k)_{k\geqslant 0}$ is a sequence of real independent random variables, f is a real valued function with period one and summable Fourier coefficients. We obtain almost sure continuity results for these periodic or almost periodic series for a large class of functions f, where the "almost sure" does not depend on the function. The proof relies on gaussian randomization. We show optimality of the results in some cases.

SERIES DE FONCTIONS ALEATOIRES ET APPLICATIONS

Abstract. (Résumé) Dans ce travail, nous étudions des propriétés de continuité de trajectoires de séries de fonctions aléatoires du type $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$ où $(a_k)_{k\geqslant 0}$ est une suite de nombres complexes, $(X_k)_{k\geqslant 0}$ une suite de variables aléatoires réelles et indépendantes, f une fonction 1-périodique à coefficients de Fourier sommables. Nous montrons que, presque sûrement, ces séries de fonctions aléatoires (périodiques ou presque périodiques) sont à trajectoires continues pour une grande classe de fonctions f. Le "presque sûr" est indépendant de f. Les preuves s'appuient sur un procédé de randomisation gaussien. Dans certains cas, nous montrerons l'optimalité des résultats obtenus.

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1. Introduction. Main results

In [1], Berkes studies the almost sure convergence of series defined by :

$$\sum_{k\geqslant 1} a_k f(\alpha n_k)$$

where the sequence (n_k) is lacunary and the function f verifies :

$$f(x+1)=f(x) \qquad \int_0^1 f(x)dx = 0 \qquad \int_0^1 f^2(x)dx=1$$

He shows that the important property of f to ensure the almost sure convergence is $f \in \mathrm{Lip}(\gamma)$ with $\gamma > 1/2$ and $\sum_{k\geqslant 1}|a_k|^2 < +\infty$. In his case, the n_k are strictly lacunary, more precisely, they satisfy the Hadamard gap condition :

$$\frac{n_{k+1}}{n_k} \geqslant q > 1$$

We can naturally adress the question whether the convergence still holds when n_k is polynomial, and for which class of functions. We are going to answer the question when the sequence (n_k) is randomly generated.

Let us mention that the result exists when (n_k) is a deterministic polynomial sequence and (a_k) is randomly distributed (see [5] and [6]).

We want to study the convergence properties of series of functions sampled by a random process. More precisely, consider the torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ and define $A(\mathbb{T})$ as the set of complex valued functions whose Fourier coefficients are absolutely summable :

$$A(\mathbb{T}) = \{f : \mathbb{T}\to\mathbb{C}, f(\alpha)=\sum_{j\in\mathbb{Z}}\hat{f}(j)\exp(2i\pi\alpha j), \sum_{j\in\mathbb{Z}}|\hat{f}(j)| < +\infty\}$$

$(a_k)_{k\geqslant 0}$ will denote a sequence of real numbers and $(X_k)_{k\geqslant 0}$ a sequence of independent real random variables defined on the probabilised space $(\Omega, \mathcal{A}, \mathbb{P})$. Our aim is to study the convergence, when $\omega\in\Omega$ is fixed, of the series of functions

$$\forall\alpha\in\mathbb{R}, \qquad F(\alpha,\omega)=\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$$

Is it possible to give conditions on the sequence $(a_k)_{k\geqslant 0}$ in order to find a $\mathcal{A}-$measurable set Ω_0 independent of the function f, such that $\mathbb{P}(\Omega_0)=1$, on which the series uniformly converges?

Note that when X_k does not take integer values, F is not a periodical function of the torus. For us, α will be real and we will deal with this "almost periodical" case. That is why we have to study the properties of F on a compact $[-M, M]$ and not only $[0,1]$ (see for example [3])

For all $f\in A(\mathbb{T})$, define

$$||f|| := \sum_{j\in\mathbb{Z}}|\hat{f}(j)| < +\infty\,.$$

$$|||f||| := \sum_{j\in\mathbb{Z}} |\hat{f}(j)|\sqrt{\log(|j|+3)} < +\infty\,.$$

and

$$B(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, f(\alpha) = \sum_{j\in\mathbb{Z}} \hat{f}(j)\exp(2i\pi\alpha j), |||f||| < +\infty\}$$

Remark that $B(\mathbb{T}) \subset A(\mathbb{T})$.
In the following, we will give conditions for $\alpha \mapsto F(\alpha,\omega)$ to have continuous trajectories $\mathbb{P}-$almost surely.
We will denote by φ_X the characteristic function of the random variable X

$$\forall t \in \mathbb{R}, \varphi_X(t) = \mathbb{E}(e^{2i\pi tX})$$

Theorem 1.1. *Let $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables and let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers such that, for any compact K which does not contain 0:*

$$(\mathcal{H}) \qquad \forall\varepsilon > 0, \exists N > 0, \sup_{m>n\geqslant N}\ \sup_{\alpha\in K}\ \sup_{j\in\mathbb{Z}-\{0\}} \left|\sum_{k=n}^{m} a_k\varphi_{X_k}(j\alpha)\right| < \varepsilon\,.$$

Assume moreover that:
case 1:(polynomial) *there exists $\beta > 0$ and $d > 0$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(k^d)$ and*

$$(1) \qquad \sum_{n\geqslant 1} \frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n\sqrt{\log n}} < +\infty$$

case 2:(subexponential) *there exists $\beta > 0$ and $\gamma \in]0,1[$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(2^{k^\gamma})$ and*

$$(2) \qquad \sum_{n\geqslant 1} \frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n^{1-\frac{\gamma}{2}}} < +\infty$$

then in both cases, there exists a measurable set Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$, for any $f \in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$: for $\alpha \in \mathbb{R}-\{0\}$, $F(\alpha,\omega)$ is well defined, $\alpha \mapsto F(\alpha,\omega)$ is continuous and the series defining F converges uniformly on every compact which does not contain $\{0\}$.

Remark 1.1.

1. *It is worth noticing that the set Ω_0 does not depend on the class of functions f ($A(\mathbb{T})$ or $B(\mathbb{T})$).*
2. *when $(X_k)_{k\geqslant 0}$ takes integer values, condition $|||f||| < \infty$ becomes $||f|| < \infty$.*
3. *we will give conditions on the law of the process $(X_k)_{k\geqslant 0}$ to fulfill hypothesis $(\mathcal{H})$.*
4. *For example, when $|a_k| = \mathcal{O}(k^{-\delta})$, in case 1, if $\delta > 1/2$, then condition 1 holds and in case 2, if $\delta > \frac{\gamma+1}{2}$, then condition 2 holds.*

(5) *Concerning case 2, if* $\gamma \geqslant 1$ *(*$\mathbb{E}|X_k|^\beta$ *growths exponentially), one can prove using remark 2.1 that the series* $\sum a_k$ *has to converge. The function* F *is then obviously well defined using only Cauchy Schwarz inequality.*

In case condition $(\mathcal{H})$ is hard to check, it is possible to split up the hypothesis on the sequence (a_k) and the characteristic function φ_{X_k} either using Abel's summation method or using Cauchy Schwarz inequality.
Define:

$$c_n = \begin{cases} 1+\sqrt{\log n} & \text{in the polynomial case} \\ n^{\frac{\gamma}{2}} & \text{in the subexponential case} \end{cases}$$

Corollary 1.2. *Let* $(X_k)_{k\geqslant 0}$ *be a sequence of independent real valued random variables*

Assume that, for any compact K *which does not contain* 0 *:*

$$(\mathcal{H}') \qquad \sup_{N\geqslant 1}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}-\{0\}} \Big|\sum_{k=0}^{N}\varphi_{X_k}(j\alpha)\Big| < \infty\,.$$

Let $(a_k)_{k\geqslant 1}$ *be a sequence of complex numbers enjoying the following properties*

(1) $\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{nc_n} < +\infty$
(2) $\sum_{k\geqslant 1}|a_k - a_{k+1}|$ *converges*

then there exists a measurable set Ω_0 *with* $\mathbb{P}(\Omega_0)=1$ *such that for all* $\omega\in\Omega_0$*, for any* $f\in B(\mathbb{T})$ *such that* $\int_{\mathbb{T}} f(t)dt = 0$*: for* $\alpha\in\mathbb{R}-\{0\}$*,* $F(\alpha,\omega)$ *is well defined,* $\alpha\mapsto F(\alpha,\omega)$ *is continuous and the series defining* F *converges uniformly on every compact which does not contain* $\{0\}$*.*

Corollary 1.3. *Let* $(X_k)_{k\geqslant 0}$ *be a sequence of independent real valued random variables*

Assume that, for any compact K *which does not contain* 0 *:*

$$(\mathcal{H}'') \qquad \forall\varepsilon>0, \exists N>0,\ \sup_{m>n\geqslant N}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}-\{0\}}\left(\sum_{k=n}^{m}|\varphi_{X_k}(j\alpha)|^2\right) < \varepsilon\,.$$

Let $(a_k)_{k\geqslant 1}$ *be a sequence of complex numbers enjoying*

$$\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{nc_n} < +\infty$$

then there exists a measurable set Ω_0 *with* $\mathbb{P}(\Omega_0)=1$ *such that for all* $\omega\in\Omega_0$*, for any* $f\in B(\mathbb{T})$ *such that* $\int_{\mathbb{T}} f(t)dt = 0$*: for* $\alpha\in\mathbb{R}-\{0\}$*,* $F(\alpha,\omega)$ *is well defined,* $\alpha\mapsto F(\alpha,\omega)$ *is continuous and the series defining* F *converges uniformly on every compact which does not contain* $\{0\}$*.*

Remark 1.2.
The previous corollaries will be useful for example when the law of X_k *is obtained by convolution product (see corollary 4.2). As the condition*

$\sum_{k\geqslant 1}|a_k-a_{k+1}|$ *is often hard to check, corollary 1.3 is sometimes better to use.*

The proof of theorem 1.1 will start by looking separately at $F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ and $\mathbb{E}(F(\alpha,.))$. It turns out that hypothesis $(\mathcal{H})$ will be used only to deal with the expectation. That is why we think interesting to state the result for:

$$F(\alpha,\omega)-\mathbb{E}(F(\alpha,.)):=\sum_k a_k[f(\alpha X_k(\omega)-\mathbb{E}(f(\alpha X_k))$$

Theorem 1.4. *Let* $(X_k)_{k\geqslant 0}$ *be a sequence of independent real valued random variables such that there exists* $\beta>0$ *and* $d>0$ *with* $\mathbb{E}|X_k|^\beta=\mathcal{O}(k^d)$ *or* $\gamma\in]0,1[$ *with* $\mathbb{E}|X_k|^\beta=\mathcal{O}(2^{k^\gamma})$. *Let* $(a_k)_{k\geqslant 1}$ *be a sequence of complex numbers enjoying the following property*

$$\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{nc_n}<+\infty$$

then there exists a measurable set Ω_0 *with* $\mathbb{P}(\Omega_0)=1$ *such that for all* $\omega\in\Omega_0$, *for any* $f\in B(\mathbb{T})$ *such that* $\int_{\mathbb{T}}f(t)dt=0$ *: for* $\alpha\in\mathbb{R}$, $F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ *is well defined,* $\alpha\mapsto F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ *is continuous, the series defining* $F-\mathbb{E}(F)$ *converges uniformly on every compact and there exists* $C_\omega>0$ *such that for all* $\alpha\in\mathbb{R}$:

$$|F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))|\leqslant C_\omega|||f|||\sqrt{\log(|\alpha|+2)}$$

Remark 1.3.

1. *We also discuss the optimality of hypothesis on* (a_k) *of theorem 1.4 in section 2.*
2. *We also have:*

$$\mathbb{E}\sup_{T>1}\frac{\sqrt{\int_0^T|F(t,\omega)-\mathbb{E}(F(t,.))|^2dt}}{\sqrt{T\log T}}<\infty$$

This result relies on uniform estimations of the size of some trigonometric polynomials, more precisely on the following :
Recall that $\log^+=\max(\log,0)$.

Theorem 1.5. *Let* λ *and* Λ *be two integers with* $\lambda\leqslant\Lambda$, $(X_k)_{k\geqslant 0}$ *be a sequence of independent real valued random variables such that there exists* $\beta>0$ *such that,* $\forall N\geqslant 0$, $\mathbb{E}|X_N^\beta|<\infty$. *Define*

$$\forall N\geqslant 0,\Phi_\beta(N)=2+\max(N,\mathbb{E}|X_N^\beta|)$$

Let $M\geqslant 1$ *and* $I_M=[-M,M]$. *Let* $(a_k)_{k\geqslant 1}$ *be a sequence of real or complex numbers.*

Define

$$A_{\lambda,\Lambda,M}=\sqrt{\log\left(M\Phi_\beta(\Lambda)\right)\sum_{k=\lambda}^{\Lambda}|a_k|^2},$$

then

$$\mathbb{E}\sup_{j\in\mathbb{Z}}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha jX_k(\omega)-\mathbb{E}\exp 2i\pi\alpha jX_k\right]}{\sqrt{A^2_{\lambda,\Lambda,M}\log\left(|j|+3\right)}}\right|<\infty$$

Remark 1.4. *When*$(X_k)_{k\geqslant 0}$ *takes integer values, the proof of theorem 1.5 is easier. Namely, using the fact that* $\alpha\mapsto j\alpha$ *(mod 1) is onto for* $j\neq 0$*, we get*

$$\sup_{j\in\mathbb{Z}^*}\sup_{\alpha\in\mathbb{T}}\left|\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha jX_k(\omega)-\mathbb{E}\exp 2i\pi\alpha jX_k\right]\right|$$

$$=\sup_{\alpha\in\mathbb{T}}\left|\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha X_k(\omega)-\mathbb{E}\exp 2i\pi\alpha X_k\right]\right|$$

the result of theorem 1.5 becomes then :

$$\mathbb{E}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha X_k(\omega)-\mathbb{E}\exp 2i\pi\alpha X_k\right]}{\sqrt{A^2_{\lambda,\Lambda,M}}}\right|<\infty$$

When $(X_k)_{k\geqslant 1}$ *takes real values, the proof is more tedious. It relies on a fine inequality about decoupling gaussian random functions (see section 3.). We can see here why, for integer-valued* X_k*, we can work with the functional space* $A(\mathbb{T})$*, whereas for real-valued* X_k*, we need to introduce the space* $B(\mathbb{T})$*.*

2. Proof of theorem 1.1 and corollary 1.2

First, we split F into two parts as follows :

$$\sum_k a_kf(\alpha X_k(\omega))=\sum_k a_k[f(\alpha X_k(\omega)-\mathbb{E}(f(\alpha X_k))+\sum_k a_k\mathbb{E}(f(\alpha X_k))$$

-Step 1 : (first part of the sum)
Let $(N_k)_{k\geqslant 1}$ be a strictly increasing sequence of integers and define

$$\forall k\geqslant 1,\qquad P_k(\alpha)=\sum_{l=N_k+1}^{N_{k+1}}a_l\left[f(\alpha X_l(\omega))-\mathbb{E}f(\alpha X_l)\right]$$

where $f\in B(\mathbb{T})$. We want to study the following series, for all $M\geqslant 1$:

$$\sum_k\sup_{\alpha\in[-M,M]}|P_k(\alpha)|$$

We have :

$$|P_k(\alpha)|\leq\sum_{j\in\mathbb{Z}}|\hat{f}(j)|\left|\sum_{l=N_k+1}^{N_{k+1}}a_l[\exp\left(2\pi j\alpha X_l(\omega)\right)-\mathbb{E}\exp\left(2\pi j\alpha X_l\right)]\right|$$

Hence, using theorem 1.5, there exists a positive integrable random variable ξ such that

$$\sup_{\alpha\in[-M,M]}|P_k(\alpha)|\leq \xi|||f|||\sqrt{\log\left(M\Phi_\beta(N_{k+1})\right)\sum_{j=N_k+1}^{N_{k+1}}|a_j|^2} \tag{3}$$

where

$$\xi=\sup_{j\in\mathbb{Z}}\sup_{k\geqslant 1}\sup_{\alpha\in I_M}\left|\frac{1}{\sqrt{A_{k,M}^2\log\left(|j|+3\right)}}\sum_{l=N_k+1}^{N_{k+1}}a_l\left[e^{2i\pi\alpha jX_l(\omega)}-\mathbb{E}e^{2i\pi\alpha jX_l}\right]\right|$$

with

$$A_{k,M}^2=\log\left(M\Phi_\beta(N_{k+1})\right)\sum_{l=N_k+1}^{N_{k+1}}|a_l|^2$$

First, in the polynomial case, that is to say when there exists $d>0$ with

$$\Phi_\beta(N)=\mathcal{O}(N^d)$$

then we choose $N_k=2^{2^k}$ and we need to prove that

$$\sum_k 2^{k/2}\left(\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}|a_l|^2\right)^{1/2}<+\infty$$

now we use the following equivalent:

$$\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}\frac{1}{l(\log(l))^{1/2}}\approx 2^{k/2}$$

which may be computed by comparing series and integral, hence :

$$\begin{aligned}2^{k/2}\left(\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}|a_l|^2\right)^{1/2} &\leqslant C\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}\frac{1}{l(\log(l))^{1/2}}\left(\sum_{l=2^{2^k}+1}^{\infty}|a_l|^2\right)^{1/2}\\ &\leqslant C\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}\frac{\left(\sum_{j=l}^{\infty}|a_j|^2\right)^{1/2}}{l(\log(l))^{1/2}}\end{aligned}$$

and, using condition 1 :

$$\begin{aligned}\sum_k 2^{k/2}\left(\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}|a_l|^2\right)^{1/2} &\leqslant \sum_k C\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}\frac{\left(\sum_{j=l}^{\infty}|a_j|^2\right)^{1/2}}{l(\log(l))^{1/2}}\\ &\leqslant \sum_{n\geqslant 2}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n\sqrt{\log n}}<+\infty\end{aligned}$$

this implies :

$$\sum_{k\geqslant 1}\sup_{\alpha\in[-M,M]}|P_k(\alpha)|<\infty$$

almost everywhere on the measurable set $\Omega_o=\{\omega\in\Omega,\xi(\omega)<\infty\}$. By construction, this set does not depend on the choice of f.
Secondly, in the subexponential case, that is when there exists $\gamma\in]0,1[$ with

$$\Phi_\beta(N)=\mathcal{O}(2^{N^\gamma})$$

we choose $N_k=2^k$ and we need to prove that

$$\sum_k 2^{\gamma k/2}\left(\sum_{l=2^k+1}^{2^{k+1}}|a_l|^2\right)^{1/2}<+\infty$$

Using the following equivalent:

$$\sum_{l=2^k+1}^{2^{k+1}}\frac{1}{l^{1-\frac{\gamma}{2}}}\approx 2^{\gamma k/2}$$

and doing the same kind of computation as before, using condition 2:

$$\sum_{n\geqslant 2}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n^{1-\frac{\gamma}{2}}}<+\infty$$

implies

$$\sum_{k\geqslant 1}\sup_{\alpha\in[-M,M]}|P_k(\alpha)|<\infty$$

We also get from (3), for all $\alpha\in\mathbb{R}$:

$$|P_k(\alpha)|\leqslant\xi|||f|||\sqrt{\log(|\alpha|+1)}\sqrt{\log(\Phi_\beta(N_{k+1}))\sum_{j=N_k+1}^{N_{k+1}}|a_j|^2}$$

summing on k, we get the inequality:

$$|F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))|\leqslant C\xi(\omega)|||f|||\sqrt{\log(|\alpha|+2)}$$

where C only depends on (a_k) and $\mathbb{E}(\xi)<\infty$
This ends the proof of theorem 1.4 which is also the first step of the proof of theorem 1.1

-Step 2 : (second part of the sum)
Let K be a compact which does not contain zero and $\alpha\in K$. Let $n<m$ be

two integers,

$$
\begin{aligned}
&(4) & |\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))| &= |\sum_{j\in\mathbb{Z}^*}\sum_{k=n}^{m} a_k \hat{f}(j)\mathbb{E}(\exp(2i\pi j\alpha X_k))| \\
&(5) & &= |\sum_{j\in\mathbb{Z}^*}\sum_{k=n}^{m} a_k \hat{f}(j)\varphi_{X_k}(j\alpha)| \\
&(6) & &= |\sum_{j\in\mathbb{Z}^*} \hat{f}(j)(\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha))| \\
&(7) & &\leqslant (\sum_{j\in\mathbb{Z}^*} |\hat{f}(j)|) \sup_{j\in\mathbb{Z}^*} |\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha)|
\end{aligned}
$$

At this point, to prove theorem 1.1, we can conclude directly by using hypothesis $(\mathcal{H})$ to get

$$\sup_{n<m}\sup_{\alpha\in K} |\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))| < \varepsilon$$

as long as m and n are large enough.
To prove corollary 1.2, we use Abel's summation. Let $\phi_p = \sum_{k=0}^{p-1}\varphi_{X_k}$, we have :

$$
\begin{aligned}
&(8) & \sum_{k=n}^{m} a_k\varphi_{X_k}(j\alpha) &= \sum_{k=n}^{m} a_k(\phi_{k+1}(j\alpha) - \phi_k(j\alpha)) \\
&(9) & &= \sum_{k=n+1}^{m+1} a_{k-1}\phi_k(j\alpha) - \sum_{k=n}^{m} a_k\phi_k(j\alpha) \\
&(10) & &= -a_n\phi_n(j\alpha) + a_m\phi_{m+1}(j\alpha) \\
&(11) & &+ \sum_{k=n+1}^{m} (a_{k-1} - a_k)\phi_k(j\alpha)
\end{aligned}
$$

and :

$$(7) \leqslant |||f||| \sup_{N\geqslant 1}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}^*} |\phi_N(j\alpha)| \left[|a_m| + |a_n| + \sum_{k=n+1}^{m} |a_k - a_{k-1}|\right]$$

we now conclude using hypothesis $\mathcal{H}'$ and hypothesis (1) on the sequence (a_n) in corollary 1.2 :

$$\sup_{n<m}\sup_{\alpha\in K} |\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))|$$

$$\leqslant |||f||| \left[|a_m| + |a_n| + \sum_{k=n+1}^{m} |a_k - a_{k-1}|\right] \sup_{N\geqslant 1}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}^*} |\phi_N(j\alpha)| < \varepsilon$$

as long as m and n are large enough.
To prove corollary 1.3, we use Cauchy Schwarz inequality in the following way:

$$\left|\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha)\right| \leqslant \sqrt{\sum_{k=n}^{m} |a_k|^2}\sqrt{\sum_{k=n}^{m} |\varphi_{X_k}(j\alpha)|^2}$$

and we conclude using condition H''.

Remark 2.1.

(1) *the most general hypothesis we can put on the sequence $(a_k)_{k\geqslant 1}$ is the following : there exists a strictly increasing sequence $(N_k)_{k\geqslant 1}$ such that*

$$\sum_{k=1}^{\infty} \sqrt{\log \Phi_\beta(N_{k+1}) \sum_{l=N_k+1}^{N_{k+1}} |a_l|^2} < \infty$$

(2) *if the process $(X_k)_{k\geqslant 1}$ takes integer values then $f \in A(\mathbb{T})$ can be assumed without any other hypothesis.*

(3) *If $\sum |a_k|^2$ diverges, then we can construct a stochastic process $(X_k)_{k\geqslant 1}$ verifying the hypothesis of theorem 1.4 and find $f \in B(\mathbb{T})$ such that the convergence of the series is not uniform on any compact. In that sense, the conditions imposed to the sequence $(a_k)_{k\geqslant 1}$ are optimal. Remark that in this case, condition 1 is not fulfilled.*

Namely consider a sequence of independent random variables with disjoint supports. For all $k \geqslant 1$ the support of X_k is the set of integers belonging to $[k^2, (k+1)^2-1]$ and hence, the hypothesis on the moment is verified. We will come back to the law of X_k later. Now choose f in the following way : for all $\alpha \in \mathbb{T}$, $f(\alpha) = \exp(2i\pi\alpha)$. Thus $f \in A(\mathbb{T})$ et $|||f||| < \infty$. As a consequence, if the convergence of the series defining F was uniform in α on $\mathbb{T}$, then we would have :

$$\sqrt{\int_0^1 \left|\sum_{k=1}^{\infty} a_k \left[\exp 2i\pi\alpha X_k(\omega) - \mathbb{E}\exp 2i\pi\alpha X_k\right]\right|^2 d\alpha}$$
$$\leq \sup_{\alpha\in[0,1]} \left|\sum_{k=1}^{\infty} a_k \left[\exp 2i\pi\alpha X_k(\omega) - \mathbb{E}\exp 2i\pi\alpha X_k\right]\right| < \infty$$

By construction, $\mathbb{P}-$ *almost surely :*

$$\int_0^1 \left|\sum_{k=1}^{\infty} a_k \left[\exp 2i\pi\alpha X_k(\omega) - \mathbb{E}\exp 2i\pi\alpha X_k\right]\right|^2 d\alpha$$

$$= \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |\exp\left(2i\pi\alpha X_k(\omega)\right) - \mathbb{E}\exp\left(2i\pi\alpha X_k\right)|^2 d\alpha$$

$$\geqslant \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 \left|1 - |\mathbb{E}\exp\left(2i\pi\alpha X_k\right)|\right|^2 d\alpha$$

Assume now that the law of X_k *is uniform on the* $2k+1$ *integers of* $[k^2, (k+1)^2 - 1]$*. for all* $k \geqslant 1$*.*

$$|\mathbb{E}\exp\left(2i\pi\alpha X_k\right)| = \frac{1}{2k+1}\left|\frac{\sin \pi\alpha(2k+1)}{\sin \pi\alpha}\right|$$

Using Lebesgue convergence theorem, we get

$$\lim_{k\to+\infty} \int_0^1 \left|1 - |\mathbb{E}\exp\left(2i\pi\alpha X_k\right)|\right|^2 d\alpha = 1$$

and we also get the divergence of the series with positive terms

$$\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 \left|1 - |\mathbb{E}\exp\left(2i\pi\alpha X_k\right)|\right|^2 d\alpha = \infty$$

A contradiction with uniform convergence of the centered part.

3. Proof of theorem 1.5

Let us begin by restating some inequalities obtained by Fernique ([4]) which will be useful in the proof of theorem 1.5. For more information on gaussian techniques in this framework, see [7] and [2].

Inequality 3.1. *Let* $(G_k)_{k\geqslant 1}$ *be a sequence of Banach space valued gaussian random variables* $(B, \|\cdot\|)$ *defined on a probabilised space* $(\Omega, \mathcal{A}, \mathbb{P})$*. Then :*

$$\mathbb{E}\sup_{k\geqslant 1} \| G_k \| \leq K_1 \left\{ \sup_{k\geqslant 1} \mathbb{E} \| G_k \| + \mathbb{E}\sup_{k\geqslant 1} | \lambda_k \sigma_k | \right\}$$

where $(\lambda_k)_{k\geqslant 1}$ *is an isonormal sequence,* K_1 *a universal constant and for all* $k \geqslant 1$*,*

$$\sigma_k = \sup_{f\in B', \|f\|\leq 1} \|< G_k, f >_B\|_{2,\mathbb{P}}$$

Inequality 3.2. *Let* g *be a real valued stationary gaussian random variable, separable and continuous in quadratic mean. Let* m *be its associated spectral measure on* $\mathbb{R}^+$ *defined by*

$$\mathbb{E}[| g(s) - g(t) |^2] = 2\int_0^{\infty} [1 - \cos 2\pi u(s-t)] m(du)$$

We have

$$\mathbb{E}\sup_{\alpha\in[0,1]} g(\alpha) \leq K\left\{\sqrt{\int_0^\infty \min(u^2,1)m(du)} + \int_0^\infty \sqrt{m\left(\left]e^{x^2},\infty\right[\right)}dx\right\}$$

where K is a universal constant.

Inequality 3.3. *(***decoupling***) Let $X = \{X(t) : t \in \mathbb{T}\}$ be a gaussian random function defined on a finite or countable set T. Let $\{T_k, k \in [1,n]\}$ be a covering of T. Let $S = T_1 \times \cdots \times T_n$. The following inequality holds :*

$$\mathbb{E}\left\{\sup_{k\in[1,n]}\left[\sup_{t\in T_k} X(t) - \mathbb{E}\sup_{t\in T_k} X(t)\right]\right\} \leq \frac{\pi}{2}\cdot\sup_{s\in S}\left\{\mathbb{E}\left[\sup_{k\in[1,n]} X(s_k)\right]\right\}$$

The following estimation generalizes inequality 3.2 and will be useful in the almost periodic case because it gives estimations on arbitrarily large intervals.

Inequality 3.4. *Let g a real valued stationnary gaussian random function, separable and continuous in quadratic mean. Let m its associated spectral measure on $\mathbb{R}^+$ defined as in inequality 3.2. There exists a universal constant K such that*

$$\mathbb{E}\sup_{\alpha\in[-M,M]} g(\alpha)$$
$$\leq K\left\{\sqrt{\int_0^\infty \min(2Mu^2,1)\ m(du)} + \int_0^\infty \sqrt{m\left(\left]\frac{e^{x^2}}{2M},\infty\right[\right)}dx\right\}$$

Let us now come to the proof

-Step 1: In this part, we replace our problem by a question of regularity of trajectories of random gaussian functions.

Let us consider an independent copy of $X = (X_k)_{k\geqslant 1}$ denoted by $X' = (X'_k)_{k\geqslant 1}$ defined on another probabilized space $(\Omega', \mathcal{A}', \mathbb{P}')$. We call $\mathbb{E}_*$ the integration symbol whose index refers to the space of integration.

Using classical convexity properties, to prove (1), it is enough to show

$$\text{(12)}\quad \mathbb{E}_{X,X'}\sup_{j\in\mathbb{Z}}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda} a_k\left[e^{2i\pi\alpha jX_k} - e^{2i\pi\alpha jX'_k}\right]}{\sqrt{A^2_{\Lambda,\lambda,M}\log(|j|+3)}}\right| < \infty$$

Let us now symmetrize the problem: consider the following separable family of random functions, with continuous trajectories

$$(f_k)_{k\geqslant 1} = \{f_k(\alpha,j) = a_k(\exp 2i\pi\alpha jX_k - \exp 2i\pi\alpha jX'_k), \alpha\in I_M, j\in\mathbb{Z}\}_{k\geqslant 1}$$

By construction f is a symmetric family of random functions, that is to say their law is sign-invariant. More precisely, call $\{\varepsilon_k, k \geqslant 1\}$ a sequence of independent Rademacher random variables (taking the values $+1$ and -1 with probability 1/2), defined on a third space $(\Omega'', \mathcal{A}'', \mathbb{P}'')$, independent of X and X'. $\{f_k, k \geqslant 1\}$ and $\{\varepsilon_k f_k, k \geqslant 1\}$ have the same law. Thus for all integers (Λ, λ) such that $\Lambda \geqslant \lambda$, $\sum_{k=\lambda}^{\Lambda} f_k$ and $\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k$ also have the same law.

That is why (12) can be written on a larger space of integration in the following way:

$$\mathbb{E}_{X,X',\varepsilon} \sup_{j\in\mathbb{Z}} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k(\alpha, j)}{\sqrt{A^2_{\Lambda,\lambda,M} \log\left(|j|+3\right)}} \right|$$

We deduce a sufficient condition for (12) to be realised

$$\mathbb{E}_{X,\varepsilon} \sup_{j\in\mathbb{Z}} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k a_k \exp 2i\pi\alpha j X_k}{\sqrt{A^2_{\Lambda,\lambda,M} \log\left(|j|+3\right)}} \right| \tag{13}$$

We then use a precious tool in the theory of gaussian random functions: the contraction principle. This tool is built on a quite simple idea: replace the choice of signs by a sequence of gaussian random variables with mean zero and variance one. This idea can be explained by the following property: given g a gaussian random variable with mean zero and variance 1 and ε a Rademacher random variable, if g and ε are independent, then g and $\varepsilon|g|$ have the same law.

As a consequence, in order to prove (13), we show

$$\mathbb{E}_{X,g,g'} \sup_{j\in\mathbb{Z}} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} a_k (g_k \cos 2\pi\alpha j X_k + g'_k \sin 2\pi\alpha j X_k)}{\sqrt{A^2_{\lambda,\Lambda,M} \log\left(|j|+3\right)}} \right| < +\infty$$

where $\{g_k, k \geqslant 1\}$ et $\{g'_k, k \geqslant 1\}$ are two sequences of independent identically distributed random variables with law $\mathcal{N}(0, 1)$, independent of X and ε, defined on two other probabilised spaces.

Conditionally to X, the problem is reduced to studying the regularity of the trajectories of stationary gaussian random variables. This concludes the first step of the proof.

-Step 2 : In this part, we use the gaussian tools introduced in the beginning.

Conditionally to X, call $G(\lambda, \Lambda, j, \alpha)$ the following quantity

$$\frac{1}{\sqrt{A^2_{\lambda,\Lambda,M} \log\left(|j|+3\right)}} \sum_{k=\lambda}^{\Lambda} a_k \left[g_k \cos\left(2\pi\alpha j X_k\right) + g'_k \sin\left(2\pi\alpha j X_k\right) \right]$$

If j, λ and Λ are fixed, $G(\alpha) := G(\lambda, \Lambda, j, \alpha)$ is a random function with almost surely continuous trajectories (up to a modification of trajectories). That is why it is enough to show that G is bounded on $I_M \cap \mathbb{Q}$. Moreover, we will assume that $|j| \leq J$ where J is a large fixed integer.

Let us begin by finding an upper bound for

$$\mathbb{E}_{g,g'} \sup_{|j|\leq J} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M\cap\mathbb{Q}} |G(\lambda, \Lambda, j, \alpha)|$$

First remark that if Y_t $(t \in T)$ is a gaussian random function defined on T, then for all $t_0 \in T$ we have (see [4] (480))

$$\mathbb{E} \sup_{t \in T} |Y_t| \leq \mathbb{E}|Y_{t_0}| + \mathbb{E} \sup_{t \in T} Y_t$$

In this way, we get rid of the absolute value. Apply this remark to $G(\lambda, \Lambda, j, \alpha)$ with $\alpha = 0$, $\lambda = \Lambda = 1$ and $j = 0$ and let us find an upper bound for

$$\mathbb{E}_{g,g'} \sup_{|j| \leq J} \sup_{\lambda \geqslant 1} \sup_{\Lambda \geqslant \lambda} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(\lambda, \Lambda, j, \alpha) \tag{14}$$

In order to apply the decoupling inequality 3.3, define

$$T = \{-J, \cdots, J\} \times H \times (I_M \cap \mathbb{Q})$$

where J is a large enough integer and H is the upper triangle of dimension 2 in $\mathbb{N} \times \mathbb{N}$ (see figure 1 below) ($\lambda \in \mathbb{N}$ and $\Lambda \geqslant \lambda$.) A point in $t \in T$ will be written $t = (j, \lambda, \Lambda, \alpha)$.

This set T is at most countable and we will find an upper bound for $\mathbb{E}_{g,g'} \sup_{t \in T} G(t)$ independently of J and then conclude by taking the supremum on $j \in J$. Define

$$T_j = \{j\} \times H \times (I_M \cap \mathbb{Q})$$

It is obvious that $\{T_j\}_{j=-J,\cdots,J}$ is a covering of T. Define

$$S = T_{-J} \times \cdots \times T_J$$

Using inequality 3.3, we have

$$\mathbb{E}_{g,g'} \sup_{t \in T} G(t) \leq \frac{\pi}{2} \sup_{s \in S} \mathbb{E}_{g,g'} \sup_{-J \leqslant j \leqslant J} G(s_j) + \sup_{-J \leqslant j \leqslant J} \mathbb{E}_{g,g'} \sup_{t \in T_j} G(t)$$

where s_j is a point in T_j.

-Step 3 : We study now

$$\sup_{s \in S} \mathbb{E} \sup_{-J \leqslant j \leqslant J} G(s_j).$$

We can rewrite this in the following way :

$$\sup \mathbb{E}_{g,g'} \sup_{-J \leqslant j \leqslant J} \frac{\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[g_k \cos(2\pi j \alpha_j X_k) + g_k^{'} \sin(2\pi j \alpha_j X_k)\right]}{\sqrt{A^2_{\lambda_j,\Lambda_j,M} \log(|j|+3)}} \tag{15}$$

where the first supremum is taken on

$$\{(\alpha_{-J}, \cdots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}, (\,(\lambda_{-J}, \Lambda_{-J})\,, \cdots, (\lambda_J, \Lambda_J)\,) \in H^{2J+1}\,\}$$

Fix $\{(\alpha_{-J}, \cdots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}$ and $((\lambda_{-J}, \Lambda_{-J})\,, \cdots, (\lambda_J, \Lambda_J)\,) \in H^{2J+1}\,\}$. Define the gaussian process

$$G_j := \frac{\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[g_k \cos(2\pi j \alpha_j X_k) + g_k^{'} \sin(2\pi j \alpha_j X_k)\right]}{\sqrt{A^2_{\lambda_j,\Lambda_j,M} \log(|j|+3)}}$$

In order to get an upper bound for 15, we remark that

$$\sup \mathbb{E}_{g,g'} \sup_{-J\leqslant j\leqslant J} G_j$$

$$\leq \sup_{((\lambda_{-J},\lambda_{-J}),\cdots,(\lambda_J,\lambda_J))\in H^{2J+1}} \mathbb{E}_{g,g'} \sup_{-J\leqslant j\leqslant J} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)| \tag{16}$$

We will get an upper bound for the right hand side of 16 independently of J, which will give us an upper bound for 15 by taking the supremum on $j \in J$.

Applying inequality 3.1 to the finite sequence of random gaussian functions

$$(G(j,\lambda_j,\Lambda_j,\alpha))_{-J\leq j\leq J}$$

we prove that $\mathbb{E}_{g,g'} \sup_{-J\leq j\leq J} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)|$ is less than

$$C\left\{ \sup_{-J\leq j\leq J} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)| + \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| \right\} \tag{17}$$

where C is a universal constant, $(\xi_j)_{-J\leq j\leq J}$ is an isonormal sequence and

$$q_j \leq \sup_{\alpha\in I_M} ||G(j,\lambda_j\Lambda_j,\alpha)||_{2,g,g'}$$

This gives us the following upper bound

$$q_j \leq \frac{1}{\sqrt{\log(M\Phi_\beta(\Lambda_j))\log(|j|+3)}}$$

As for all j we have $\Lambda_j \geqslant 1$ and $M \geqslant 1$ we easily get

$$\mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| \leq \mathbb{E}_\xi \sup_{-J\leq j\leq J} \left| \frac{1}{\sqrt{\log(|j|+3)}} \xi_j \right| \tag{18}$$

$$\leq C\mathbb{E}_\xi \sup_{j\in\mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log(|j|+3)}} \right| \tag{19}$$

The exponential integrability of gaussian vectors gives

$$\mathbb{E}_\xi \sup_{j\in\mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log|j|+3}} \right| = C < \infty$$

and hence

$$\sup_{(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1}} \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| = C < \infty$$

independently of J and of the sequence $(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1})$.

We then get independently of J on the whole integration space

$$\mathbb{E}_X \sup_{(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1}} \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| \leq C < \infty$$

where C is a constant.

Let us now try to find an upper bound for

$$\sup_{((\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1}} \sup_{-J\leq j\leq J} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)| \tag{20}$$

independently of J.

We will use inequality 3.4. Let us choose a finite sequence

$$((\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J)$$

and an integer $|j|\leq J$. The gaussian random function $G_j(\alpha):=G(j,\lambda_j,\Lambda_j,\alpha)$ is stationary. Its associated spectral measure on $\mathbb{R}^+$ is defined by

$$m_j = \frac{1}{\log\left(|j|+3\right)\log\left(M\Phi_\beta(\Lambda_j)\right)\left(\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\right)} \sum_{k=\lambda_j}^{\Lambda_j} |a_k|^2 \delta_{j|X_k|}$$

where δ_u is the Dirac measure in the point u.

We get

$$\mathbb{E}_{g,g'} \sup_{\alpha\in I_M} |G_j(\alpha)|$$

$$\leq C\left\{\sqrt{\int_0^\infty \min(\frac{u^2}{2M},1)m_j(du)} + \int_0^\infty \sqrt{m_j\left(]\frac{e^{x^2}}{2M},\infty[\right)}dx\right\}$$

It is obvious that the first term is less than

$$\sqrt{m_j(\mathbb{R}^+)} \leq C\frac{1}{\sqrt{\log\left(2M\right)\log\left(|j|+3\right)}}$$

where C is a universal constant because for all j we have $\Phi_\beta(\Lambda_j)\geqslant 2$.

For the second term, it can be rewritten in the following way

$$\int_0^\infty \sqrt{m_j\left(]\frac{\exp x^2}{2M},\infty[\right)}dx$$

$$= \left[\frac{1}{\log\left(|j|+3\right)\log\left(M\Phi_\beta(\Lambda_j)\right)\left(\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\right)}\right]^{\frac{1}{2}}$$

$$\times \int_0^\infty \sqrt{\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2 \mathbf{1}_{\{2M|jX_k|>e^{x^2}\}}}dx$$

Using

$$\forall k\geqslant 1, \qquad \mathbf{1}_{\{2M|jX_k|>e^{x^2}\}} \leq \mathbf{1}_{\{2M|j|\sup_{l\leq k}|X_l|>e^{x^2}\}}$$

we cut $\mathbb{R}^+$ in the integral according to the increasing subdivision

$$\{0\}\bigcup\{\sqrt{\log^+\left(2M|j|\sup_{l\leq k}\mid X_l\mid\right)}, \lambda_j\leq k\leq\Lambda_j\}$$

We get thus an upper bound for the previous integral

$$\begin{aligned}
&\sum_{k=\lambda_j}^{\Lambda_j} \left[\sqrt{\log^+(2M|j|\sup_{l\leq k} \mid X_l \mid)} - \sqrt{\log^+(2M|j|\sup_{l\leq k-1} \mid X_l \mid)}\right]\left(\sum_{l=\lambda_j}^{\Lambda_j}|a_l|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{l=\lambda_j}^{\Lambda_j}|a_l|^2\right)^{\frac{1}{2}} \sum_{k=\lambda_j}^{\Lambda_j}\left[\sqrt{\log^+(2M|j|\sup_{l\leq k} \mid X_l \mid)}\right. \\
&- \left.\sqrt{\log^+(2M|j|\sup_{l\leq k-1} \mid X_l \mid)}\right] \\
&\leq 2\left(\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\right)^{\frac{1}{2}} \sqrt{\log^+(2M|j|\sup_{l\leq \Lambda_j} \mid X_l \mid)}
\end{aligned}$$

Consequently, $\forall |j|\leq J$,

$$\begin{aligned}
\int_0^\infty \sqrt{m_j\left(\left]\frac{\exp x^2}{M},\infty\right[\right)}dx &\leq 2\sqrt{\frac{\log^+(2M|j|\sup_{l\leq\Lambda_j} \mid X_l \mid)}{\log(|j|+3)\log(M\Phi_\beta(\Lambda_j))}} \\
&\leq 4\sqrt{\frac{\log^+(2M\sup_{1\leq l\leq\Lambda_j} \mid X_l \mid)}{\log(M\Phi_\beta(\Lambda_j))}} \\
&\leq 4\sup_{1\leq l\leq\Lambda_j}\sqrt{\frac{\log^+(2M \mid X_l \mid)}{\log(M\Phi_\beta(l))}} \\
&\leq 4\sup_{N\geqslant 1}\sqrt{\frac{\log^+(2M \mid X_N \mid)}{\log(M\Phi_\beta(N))}} \\
&\leq 4\left(1+\sup_{N\geqslant 1}\sqrt{\frac{\log^+(\mid X_N \mid)}{\log\Phi_\beta(N)}}\right)
\end{aligned}$$

Finally, on the whole integration space, we get the following upper bound :

$$\mathbb{E}_X \sup_{(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J)\in H^{2J+1}} \sup_{-J\leq j\leq J} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M}|G_j(\alpha)| \tag{21}$$

$$\leq C\left(1+\mathbb{E}_X\sup_{N\geqslant 1}\sqrt{\frac{\log^+(\mid X_N \mid)}{\log\Phi_\beta(N)}}\right)<\infty \tag{22}$$

-Step 4 : To end the proof of theorem 1.5, it remains to deal with

$$\sup_{-J\leq j\leq J}\mathbb{E}_{g,g'}\sup_{(\lambda,\Lambda)\in H}\sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,\lambda,\Lambda,\alpha)$$

Let us fix j. In order to apply inequality 3.1, we need to replace the supremum on $(\lambda,\Lambda)\in H$ by a supremum on only one variable, in other words,

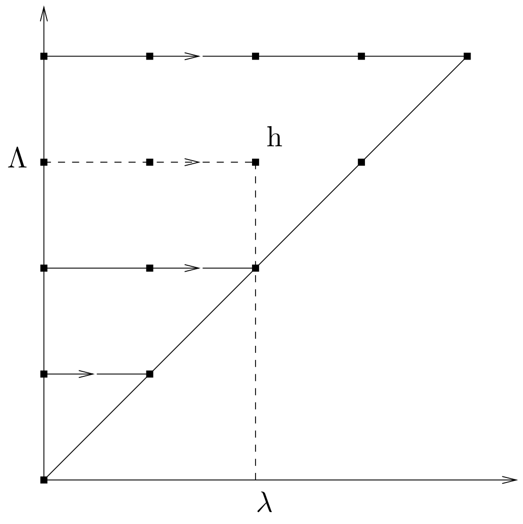

FIGURE 1. renumbering the upper triangle in $\mathbb{N}^2$

we need to renumber H using one variable $h \in \mathbb{N}^2$ given by the formula :

$$h = \lambda + \frac{\Lambda(\Lambda-1)}{2}$$

(see figure 1).

Let us fix j. Inequality 3.1 gives us :

$$\mathbb{E}_{g,g'} \sup_{h\geqslant 1} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,h,\alpha) \leqslant K_1 (\sup_{h\geqslant 1} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,h,\alpha) + \mathbb{E}_{g,g'} \sup_{h\geqslant 1} |\lambda_h \sigma_h|) \tag{23}$$

where $(\lambda_h)_{h\geqslant 1}$ is an isonormal sequence. The inequality :

$$\frac{\Lambda(\Lambda-1)}{2} \leqslant h \leqslant \frac{\Lambda(\Lambda+1)}{2}$$

gives a polynomial dependence between Λ and h, hence :

$$\sigma_h = \mathcal{O}(\frac{1}{\sqrt{\log h}})$$

and the first term in the right hand side of inequality (23) is dealt with in the same way as before. Finally, we get :

$$\mathbb{E}_{g,g'} \sup_{(\lambda,\Lambda)\in H} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,\lambda,\Lambda,\alpha) \leq C \left(1 + \sup_{N\geqslant 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}}\right)$$

That is to say, by integrating on the whole space,

$$\mathbb{E}_X \sup_{1\leq j\leq n} \mathbb{E}_{g,g'} \sup_{(\lambda,\Lambda)\in H} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,\lambda,\Lambda,\alpha) \leq C \left(1 + \mathbb{E}_X \sup_{N\geqslant 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}}\right) \tag{24}$$

Let us prove that

$$\mathbb{E}_X \sup_{N\geqslant 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}} < \infty$$

Using Jensen inequality, we can get rid of the square root. Let $\delta > 0$. For all $N \geqslant 1$, we have

$$\beta \log^+ |X_N| \leq \beta \log^+ \left[\frac{|X_N|}{\Phi_\beta^\delta(N)} \right] + \beta \log^+ [\Phi_\beta^\delta(N)]$$

noticing that $\Phi_\beta^\delta(N)) \geqslant 2$ it is sufficient to show

$$\mathbb{E} \sup_{N\geqslant 1} \log^+ \left[\frac{|X_N|^\beta}{\Phi_\beta^{\delta\beta}(N)} \right] < \infty$$

Using now the inequality $\log^+(x) \leq x$ for any $x \geqslant 0$, it is sufficient to prove

$$\sum_{N\geqslant 1} \frac{\mathbb{E}|X_N|^\beta}{\Phi_\beta^{\delta\beta}(N)} < \infty$$

And as $\mathbb{E}|X_N|^\beta \leqslant \Phi_\beta(N)$ and $\Phi_\beta(N) \geqslant N$, if we chose $\delta = \frac{3}{\beta}$ we get the conclusion. Steps 3 (see 21), step 4 (see 24) lead us to the announced result of theorem 1.5.

4. Applications

Let us begin by giving an example where the (X_k) are uniformly distributed :

Example 4.1. *Suppose that* $\mathcal{L}(X_k) = \mathcal{U}([\mu_k - \sigma_k/2, \mu_k + \sigma_k/2])$ *with* $\sigma_k > 0$ *et* $\mathbb{E}(X_k) = \mu_k$ *with* $\mu_k = \mathcal{O}(k^d)$ *for some* $d > 0$. *The characteristic function of* X_k *can easily be computed :*

$$\varphi_{X_k}(t) = \frac{e^{2i\pi t\mu_k}}{\pi t\sigma_k} \sin(\pi t\sigma_k)$$

Using condition $\mathcal{H}$ *of theorem 1.1, the following condition*

$$\sum_{n\geqslant 1} \frac{|a_n|}{\sigma_n} \text{ converges} \tag{25}$$

and

$$\sum_{n\geqslant 1} \frac{\sqrt{\sum_{k\geqslant n} |a_k|^2}}{n\sqrt{\log n}} < +\infty$$

are sufficient to get the desired convergence.
Notice that using corollary 1.3, condition (25) is replaced by

$$\sum_{n\geqslant 1} \frac{1}{\sigma_n^2} < +\infty$$

The subexponential case could be dealt with in the same way.
If we consider the border case $a_k = \mathcal{O}(k^{-1/2-\varepsilon})$, *it is sufficient that :*

$$\exists \eta > 0, \sigma_k \geqslant k^{\frac{1}{2}+\eta}$$

in this case :

$$\mathbb{P}\{\forall k, X_k \in [\mu_k - \frac{k^{\frac{1}{2}+\eta}}{2}, \mu_k + \frac{k^{\frac{1}{2}+\eta}}{2}]\} = 1$$

which gives an information on the possible dispersion of the variables X_k.

Here are other examples where the conditions of our theorems can be quite easily verified.

Corollary 4.1. *Let* $(X_k)_{k\geqslant 1}$ *be a sequence of real independent random variables whose law can be written in the following way for all* $k \geqslant 1$: $\mathcal{L}(X_k) = \mathcal{L}(\sigma_k \cdot X + \mu_k)$ *where* X *verifies* $\mathbb{E}|X|^\beta < \infty$ *for some* $\beta > 0$. *Moreover, we assume that there exist* $d > 0$ *and* $\delta > 0$ *such that* $|\sigma_k| = \mathcal{O}(k^d)$, $|\mu_k| = \mathcal{O}(k^d)$, *the application* $t \mapsto t^\delta \mathbb{E}\exp(2i\pi tX)$ *is bounded on* $\mathbb{R}$, *and let* $(a_k)_{k\geqslant 1}$ *be a sequence of real or complex numbers satisfying the following two conditions*

1. $|a_k| = O(k^{-\beta})$ *with* $\beta > 1/2$
2. $\sum_{k=1}^{\infty} \frac{1}{|\sigma_k|^{2\delta}} < \infty$

Then there exists a measurable set Ω_o *with full measure* ($\mathbb{P}(\Omega_o) = 1$) *such that for any* $\omega \in \Omega_o$ *for all* $f \in B(\mathbb{T})$ *such that* $\int_{\mathbb{T}} f(t)dt = 0$: *for any compact* K *which does not contain* 0, *the application* $t \in K \mapsto F(t) = \sum_{k\geqslant 1} a_k f(tX_k(\omega))$ *is continuous and the series defining* F *converges uniformly on* K.

The proof of corollary 4.1 relies on corollary 1.3.

Example 4.2. *The random variable* X *may have a gaussian law with mean zero and variance one, a Cauchy law, the first Laplace law, an exponential law with parameter* $\lambda > 0$. *Let us precise the gaussian case.*
Here $\mathcal{L}(X) = \mathcal{N}(0,1)$, *we have* $\mathbb{E}\exp 2i\pi tX = e^{-t^2/2}$. *Hence we can use the fact that* $t \mapsto e^{t^2/2}\mathbb{E}\exp 2i\pi tX$ *is bounded on* $\mathbb{R}$. *In this case, the sufficient condition to obtain convergence is*

$$\forall \varepsilon > 0, \exists N > 0, \sup_{m>n\geqslant N} \sup_{\alpha\in K} \sup_{j\in\mathbb{Z}-\{0\}} |\sum_{k=n}^{m} a_k e^{2i\pi\alpha\mu_k j} e^{-j^2\alpha^2\sigma_k^2/2}| < \varepsilon$$

Let $d(0,K)$ *be the distance between* 0 *and the compact* K. *Using the fact that* $|a_k| = O(k^{-\beta})$ *with* $\beta > 1/2$, *the previous condition will be satisfied as soon as :*

$$\exists \varepsilon > 0, \sigma_k \geqslant \frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0,K)}\sqrt{\log k}$$

which, in terms of dispersion of the variables X_k means that infinitely often:

$$X_k \in [\mu_k - 3\frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0,K)}\sqrt{\log k}, \mu_k + 3\frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0,K)}\sqrt{\log k}]$$

We discuss now the case when the laws of X_k are generated by a convolution product of a given law μ. We distinguish two cases : on one hand when the support of μ contains non integer values, on the other hand when the support of μ is contained in $\mathbb{Z}$. The first case is discribed by the following corollary :

Corollary 4.2. *Let $(X_k)_{k\geqslant 1}$ be an sequence of real valued independent random variables such that for all integer $k \geqslant 1$, $\mathcal{L}(X_k) = \mu^{*k}$ where μ is a probability measure on $\mathbb{R}$ with $\mathbb{E}|X_1|^\delta < \infty$ for some δ. Assume the following :*

$$(a) \qquad \varphi_{X_1}(t) = 1 \Longleftrightarrow t = 0 \qquad (X_1 \text{ aperiodic})$$

$$(b) \quad \exists\delta > 0, \qquad \sup_{t\in\mathbb{R}} |t^\delta \mathbb{E}\exp(2i\pi t X_1)| = q < \infty$$

Let $(a_k)_{k\geqslant 1}$ be a sequence of real or complex numbers such that the sequence $|a_k|$ is decreasing and fulfills the two following conditions :

1. *$|a_k| = O(k^{-\beta})$ avec $\beta > 1/2$*
2. *$\sum_{k=1}^\infty |a_k - a_{k+1}| < \infty$*

Then there exists a measurable set Ω_o with full measure ($\mathbb{P}(\Omega_o) = 1$) such that for any $\omega \in \Omega_o$, for all $f \in B(\mathbb{T})$ such that $\int_\mathbb{T} f(t)dt = 0$, for any compact K which does not contain 0, the application $t \in K \mapsto F(t,\omega) = \sum_{k\geqslant 1} a_k f(t X_k(\omega))$ is continuous and the series defining F converges uniformly on K.

Remark 4.1. *If X_1 is striclty aperiodic ($|\varphi_{X_1}(t)| = 1 \Longleftrightarrow t = 0$), then the condition on the differences $|a_k - a_{k+1}|$ may be removed, using corollary 1.3 and the same kind of conputation as in the following proof.*

The random variable X_1 being real valued, its characteristic function is not periodic. Take for example a gaussian law with mean zero and variance one.

Proof : Let K be a compact which does not contain 0. Using Abel 's summation, it is sufficient to prove

$$\sup_{t\in K}\sup_{|j|\geqslant 1}\left|\sum_{k=1}^N (\mathbb{E}\exp 2i\pi j t X_1)^k\right| < \infty$$

independently of N. Let us split the supremum on j respectively into the supremum on the indexes $J(q)$ and $\bar{J}(q)$ where $J(q) = \{j \in \mathbb{Z}^* : |j|^\delta \le \left[\frac{2q}{\varepsilon^\delta}\right]\}$ and 2ε is the distance between 0 and the fixed compact K.
On one hand, using (a), it can be proved that :

$$\forall\varepsilon > 0 \qquad \inf_{|t|>\varepsilon} |t|^\delta |1 - \mathbb{E}\exp(2i\pi t X_1)| > 0$$

this implies

$$\sup_{t\in K}\sup_{j\in J(q)}\left|\sum_{k=1}^{N}(\mathbb{E}\exp 2i\pi jtX_1)^k\right| \leq \sup_{t\in K}\sup_{j\in J(q)} C(\varepsilon)|jt|^\delta \leq C(K)\left\lceil\frac{2q}{\varepsilon^\delta}\right\rceil$$

On the other hand, using (b),

$$\begin{aligned}\sup_{t\in K}\sup_{j\in \bar{J}(q)}\left|\sum_{k=1}^{N}(\mathbb{E}\exp 2i\pi jtX_1)^k\right| &\leq \sup_{t\in K}\sup_{j\in \bar{J}(q)}\sum_{k=1}^{N}\left(\frac{q}{|tj|^\delta}\right)^k\\ &\leq C\sum_{k=1}^{N}\frac{1}{2^k}\leq 2C\end{aligned}$$

where C is a universal constant.
□

As for the integer valued case, we have :

Corollary 4.3. *Let $(X_k)_{k\geqslant 1}$ be an sequence of integer valued independent random variables such that for all integer $k\geqslant 1$, $\mathcal{L}(X_k)=\mu^{*k}$ where μ is a probability measure on $\mathbb{R}$ with $\mathbb{E}|X_1|^\beta<\infty$ for some $\beta>0$. Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers such that $|a_k|=O(k^{-\beta})$ with $\beta>1/2$. Assume either :*

$$\varphi_{X_1}(t)=1 \Longleftrightarrow t=0 \quad \textit{and} \quad \sum_{k\geqslant 1}|a_k-a_{k+1}| \quad \textit{converges}$$

or :

$$|\varphi_{X_1}(t)|=1 \Longleftrightarrow t=0$$

Then there exists a measurable set Ω_o with full measure ($\mathbb{P}(\Omega_o)=1$) such that for any $\omega\in\Omega_o$, for all $f\in A(\mathbb{T})$ such that $\int_{\mathbb{T}}f(t)dt=0$, for any compact K of the torus which does not contain 0, the application $t\in K\mapsto F(t,\omega)=\sum_{k\geqslant 1}a_k f(t\,X_k(\omega))$ is continuous and the series defining F converges uniformly on K.

Example 4.3. *If the law of X_1 is a Poisson law with parameter 1, we use :*

$$\forall t\in\mathbb{T}, |\varphi_{X_1}(t)|\leqslant e^{cos(2\pi t)-1}$$

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