May's Theorem for Trees

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Abstract

Kenneth May in 1952 proved a classical theorem characterizing simple majority rule for two alternatives. The present paper generalizes May's theorem to the case of three alternatives, but where the voters' preference relations are required to be trees with the alternatives at the leaves.

1 Introduction

In 1952, Kenneth May gave an elegant characterization of simple majority decision based on a set with exactly two alternatives [9]. This work is a model of the classic voting situation where there is two candidates and the candidate with the most votes is declared the winner. May's theorem is a fundamental result in the area of social choice and it has inspired many extensions. See [2], [3], [4], [5], [8], and [10] for a sample of these results.

The goal of the current paper is to state and prove a version of May's theorem in the context of trees. In what follows, **tree** will mean a rooted tree with labelled leaves and unlabelled interior vertices, and no vertex except possibly the root can have degree 2. In the biological literature, such a tree T might represent the evolutionary history of the set S of species, with interior vertices of T representing ancestors of the species in S. Clearly the simplest nontrivial case is when $|S| = 3$. In this case, there are exactly 4 distinct trees with leaves labelled by the set S. It is within this context that we define a version of simple majority decision for trees and characterize it in terms of three conditions. There is a clear connection between our conditions and those given by May.

This paper is divided into four sections with this introduction being the first section. Section 2 is background material on May's work and includes the statement of May's Theorem. Section 3 contains the definition of majority decision for trees, and the main result of this paper is stated and proved in Section 4.

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2 Background on May's Work

Let $S = \{x, y\}$ be a set with two alternatives. The binary relations $R_{-1} = \{(x, x), (y, y), (y, x)\}$, $R_0 = S \times S$, and $R_1 = \{(x, x), (y, y), (x, y)\}$ are the three weak orders on S. The relation R_{-1} represents the situation where y is strictly preferred to x, R_1 represents the situation where x is strictly preferred to y, and R_0 represents indifference between x and y.

Let $K = \{1, \ldots, k\}$ be a set with $k \geq 2$ individuals and let $\mathcal{W}(S)$ be the set $\{R_{-1}, R_0, R_1\}$. A function of the form

$$f : \mathcal{W}(S)^k \to \mathcal{W}(S)$$

is called a **group decision function** by May.

For any $p = (D_1, \ldots, D_k)$ in $\mathcal{W}(S)^k$ and for any $i \in \{-1, 0, 1\}$ let

$$N_p(i) = |\{D_j : D_j = R_i\}|.$$

That is, $N_p(i)$ is the number of times the relation R_i appears in the k-tuple p. It follows that $N_p(-1) + N_p(0) + N_p(1) = k$ and $N_p(i) \geq 0$ for each $i \in \{-1, 0, 1\}$. The group decision function

$$M : \mathcal{W}(S)^k \to \mathcal{W}(S)$$

defined by

$$M(p) = \begin{cases} R_{-1} & \text{if } N_p(1) - N_p(-1) < 0 \\ R_1 & \text{if } N_p(1) - N_p(-1) > 0 \\ R_0 & \text{if } N_p(1) - N_p(-1) = 0 \end{cases}$$

for any k-tuple p is called, for obvious reasons, **simple majority decision**. The consensus weak order $M(p)$ has y strictly preferred to x if more individuals rank y strictly over x than x strictly over y. There is indifference between x and y if the number of individuals that strictly prefer y over x is the same as the number of individuals that strictly prefer x over y. Finally, $M(p)$ has x strictly preferred to y if the number of individuals that rank x strictly over y is more than the number of individuals that rank y strictly over x.

May simplified the notation used above as follows. The relation R_{-1} is identified with the number -1, the relation R_0 is identified with the number 0, and the relation R_1 is identified with 1. Using this identification we can think of a group decision function as a function with domain $\{-1, 0, 1\}^k$ and range $\{-1, 0, 1\}$.

Let $f : \{-1, 0, 1\}^k \to \{-1, 0, 1\}$ be a group decision function. Then reasonable properties that f may or may not satisfy are the following.

(A) For any k-tuple $p = (D_1, \ldots, D_k)$ and for any permutation α of K,

$$f(D_{\alpha(1)}, \ldots, D_{\alpha(k)}) = f(D_1, \ldots, D_k).$$

(N) For any k-tuple $p = (D_1, \ldots, D_k)$,

$$f(-D_1, \ldots, -D_k) = -f(D_1, \ldots, D_k).$$

(PR) For any k-tuples $p = (D_1, \ldots, D_k)$ and $p' = (D'_1, \ldots, D'_k)$,

$$\text{if } f(D_1, \ldots, D_k) \in \{0, 1\}, D'_i = D_i \text{ for all } i \neq i_0, \text{ and } D'_{i_0} > D_{i_0},$$

then

$$f(D'_1, \ldots, D'_k) = 1.$$

The conditions (A), (N), and (PR) correspond to conditions II, III, and IV given on pages 681 and 682 in [9]. Condition (A) states that f is a symmetric function of its arguments and thus individual voters are anonymous. Condition (N) is called **neutrality**. This axiom is motivated by the idea that the consensus outcome should not depend upon any labelling of the alternatives. Condition (PR) is called **positive responsiveness** since it reflects the notion that a group decision function should respond in a positive way to changes in individual preferences. If the consensus outcome $f(p)$ does not rank y strictly preferred to x and one individual i_0 changes their vote in a favorable way toward x, then the consensus outcome $f(p')$ should strictly prefer x to y.

We now can state May's result.

Theorem 1 *A group decision function is the method of simple majority decision if and only if it satisfies (A), (N), and (PR).*

3 Trees with 3 Leaves

As we have noted, May studied majority decision for two alternatives, which is the simplest non-trivial case for weak orders. Since our goal is to prove a version of May's result for trees, we too restrict our attention to the simplest non-trivial case for trees; namely when $|S| = 3$. For $S = \{x, y, z\}$, and $\{u, v\} \subset S$, let $T_{\{u,v\}}$ denote the tree with one non-root vertex of degree three adjacent to the root, u, and v. Let $T_\emptyset$ be the tree whose only internal vertex is the root.

Let $\mathcal{T}(S)$ be the set $\{T_{\{x,y\}}, T_{\{x,z\}}, T_{\{y,z\}}, T_\emptyset\}$ of all trees with the leaves labelled by the elements of S. We will call a function of the form

$$C : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

a **consensus function** to conform with current useage [6] . An element $P = (T_1, \ldots, T_k)$ in $\mathcal{T}(S)^k$ is called a **profile** and the output $C(P)$ is called a **consensus tree**. For any profile $P = (T_1, \ldots, T_k)$ and for any two element subset $\{u, v\}$ of S, let

$$N_P(uv) = |\{T_i : T_i = T_{\{u,v\}}\}|.$$

Also, let

$$N_P(\emptyset) = |\{T_i : T_i = T_\emptyset\}|.$$

So $N_P(xy) + N_P(xz) + N_P(yz) + N_P(\emptyset) = k$. The consensus function

$$Maj : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

defined by

$$Maj(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_p(uv) > \frac{k}{2} \\ T_\emptyset & \text{otherwise} \end{cases}$$

is called **majority rule** [7]. This consensus function is well known but it is not the best analog of simple majority decision *sensu* May. We feel that a better candidate is the consensus function

$$M : \mathcal{T}(S)^k \to \mathcal{T}(S)$$

defined by

$$M(P) = \begin{cases} T_{\{u,v\}} & \text{if } N_p(uv) > \max\{N_P(uw), N_P(vw)\} \\ T_\emptyset & \text{otherwise} \end{cases}$$

where $\{u, v, w\} = \{x, y, z\}$. It is easy to see that if $Maj(P) = T_{\{u,v\}}$ for some two element subset $\{u, v\}$ of S, then $M(P) = Maj(P)$. The converse is not true. For example, if $P = (T_1, \ldots, T_k)$ such that $T_1 = T_{\{x,y\}}$ and $T_i = T_\emptyset$ for all $i \neq 1$ in K, then $M(P) = T_{\{x,y\}}$ and $Maj(P) = T_\emptyset$. For the remainder of this paper the function M will be called **majority decision**.

4 Main Result

Following are translations of the conditions (A), (N), and (PR) to the context of trees. Let $C : \mathcal{T}(S)^k \to \mathcal{T}(S)$ be a consensus function, and consider the following conditions.

(A)$^+$ For any profile $P = (T_1, ..., T_k)$ and any permutation α of K,

$$C(P_\alpha) = C(P).$$

where $P_\alpha = (T_{\alpha(1)}, ..., T_{\alpha(k)})$.

Let $\beta : S \rightarrow S$ be a permutation. Then β induces a map on $\mathcal{T}(S)$ as follows: $\beta T_{\emptyset} = T_{\emptyset}$ and $\beta T_{\{u,v\}} = T_{\{\beta(u),\beta(v)\}}$ for any two element subset $\{u, v\}$ of S. If $P = (T_1, ..., T_k)$ is a profile, then set $\beta P = (\beta T_1, ..., \beta T_k)$.

(N)$^+$ For any profile $P = (T_1, ..., T_k)$ and any permutation β of S,

$$C(\beta P) = \beta C(P).$$

(PR)$^+$ This condition has three parts.
(1) For any profiles $P = (T_1, ..., T_k)$ and $P' = (T'_1, ..., T'_k)$, if $T'_i = T_i$ for all $i \neq i_0$ and $T'_{i_0} = T_{\{x,y\}}$, then $C(P) = T_{\{x,y\}}$ implies $C(P') = T_{\{x,y\}}$.

(2) For any profiles $P = (T_1, ..., T_k)$ and $P' = (T'_1, ..., T'_k)$, if $T'_i = T_i$ for all $i \neq i_0$, $T_{i_0} \notin \{T_{\emptyset}, T_{\{x,y\}}\}$, and $T'_{i_0} = T_{\emptyset}$, then $C(P) = T_{\{x,y\}}$ implies $C(P') = T_{\{x,y\}}$.

(3) Let $P = (T_1, ..., T_k)$ be a profile such that $C(P) = T_{\emptyset}$ and $T_{i_0} \in \{T_{\emptyset}, T_{\{x,y\}}\}$. Then there exists a profile $P' = (T'_1, ..., T'_k)$ such that $T'_i = T_i$ for all $i \neq i_0$, $T'_{i_0} \neq T_{i_0}$, and $C(P') \notin \{T_{\emptyset}, T_{\{x,y\}}\}$.

It is easy to make direct comparisons between conditions (A) and (N) for group decision functions and conditions (A)$^+$ and (N)$^+$ for consensus functions. A comparison between conditions (PR) and (PR)$^+$ requires a bit more thought. The hypotheses of condition (PR) allow for different possibilities. One possibility, for example, is when $f(p) = 1$, $D'_{i_0} = 1$, and $D_{i_0} \in \{-1, 0\}$. Another possibility is $f(p) = 1$, $D'_{i_0} = 0$, and $D_{i_0} = -1$. These two possibilities translate into items (1) and (2) in (PR)$^+$. The tree $T_{\{x,y\}}$ is identified with 1 and the tree $T_{\emptyset}$ is identified with 0.

The final item (3) in (PR)$^+$ corresponds to the case when $f(p) = 0$ in (PR). Now $D'_{i_0} > D_{i_0}$ implies that $D_{i_0} \in \{0, -1\}$. This in turn is motivation for the hypothesis $T_{i_0} \in \{T_{\emptyset}, T_{\{x,y\}}\}$. Notice the change in identification with the tree $T_{\{x,y\}}$ now corresponding to -1. The conclusion in (PR) can be written as $f(p') \notin \{-1, 0\}$ which corresponds to the conclusion $C(P') \notin \{T_{\emptyset}, T_{\{x,y\}}\}$ in (PR)$^+$.

We need a lemma before we can state and prove our main result.

Lemma 2 *Suppose* $C : \mathcal{T}(S)^k \rightarrow \mathcal{T}(S)$ *satisfies (A)*$^+$*, (N)*$^+$*, and (PR)*$^+$*. If* $P = (T_1, \ldots, T_k)$ *is a profile where* $C(P) = H_{\{x,y\}}$*, then* $N_P(xy) > \max\{N_P(xz), N_P(yz)\}$*.*

Proof. Assume that $N_P(xy) = N_P(xz)$. Then $|K_1| = |K_2|$ where $K_1 = \{i \in K : T_i = T_{\{x,y\}}\}$ and $K_2 = \{i \in K : T_i = T_{\{x,z\}}\}$. Choose a permutation α of K such that α maps K_1 onto K_2, K_2 onto K_1, and $\alpha(i) = i$ for all $i \in K \setminus (K_1 \cup K_2)$. Define $\beta : S \rightarrow S$

by $\beta(x) = x, \beta(y) = z$, and $\beta(z) = y$ and note that $P = \beta P_\alpha$. It follows from (A)$^+$ and (N)$^+$ that

$$C(P) = C(\beta P_\alpha) = \beta C(P_\alpha) = \beta C(P).$$

But $C(P) = T_{\{x,y\}}$ and $T_{\{x,y\}} \neq \beta T_{\{x,y\}}$. This contradiction implies that $N_P(xy) \neq N_P(xz)$.

A similar argument shows that $N_P(xy) \neq N_P(yz)$.

Let $r = N_P(xz) - N_P(xy)$ and assume $r > 0$. Choose $i_0 \in \{i \in K : T_i = T_{\{x,z\}}\}$ and define $P' = (T'_1, \ldots, T'_k)$ by $T'_i = T_i$ for all $i \neq i_0$ and

$$T'_{i_0} = \begin{cases} T_\emptyset & \text{if } r = 1 \\ T_{\{x,y\}} & \text{if } r \geq 2 \end{cases}$$

It follows from (PR)$^+$ that $C(P') = T_{\{x,y\}}$. Note that

$$N_P(xy) \leq N_{P'}(xy) \leq N_{P'}(xz) < N_P(xz).$$

Since K is finite this process can be continued (if necessary) until we find a profile P^* such that $C(P^*) = T_{\{x,y\}}$ and $N_{P^*}(xy) = N_{P^*}(xz)$. This contradicts the first part of the proof. Therefore, $N_P(xy) > N_P(xz)$.

A similar argument shows that $N_P(xy) > N_P(yz)$ and the proof is complete. □

Theorem 3 *The consensus function $C : \mathcal{T}(S)^k \to \mathcal{T}(S)$ is the majority decision function if and only if C satisfies (A)$^+$, (N)$^+$, and (PR)$^+$.*

Proof. First, it is straightforward to verify that the consensus function M satisfies (A)$^+$, (N)$^+$, and (PR)$^+$.

Suppose $C : \mathcal{T}(S)^K \to \mathcal{T}(S)$ satisfies (A)$^+$, (N)$^+$, and (PR)$^+$. Let $P = (T_1, \ldots, T_k)$ be an arbitrary profile. The goal is to show that $C(P) = M(P)$.

If $C(P) = T_{\{x,y\}}$, then, by Lemma 2, $N_P(xy) > \max\{N_P(xz), N_P(yz)\}$. By the definition of M, $M(P) = T_{\{x,y\}}$ and so $C(P) = M(P)$.

If $C(P) = T_{\{x,z\}}$, then define $\beta : S \to S$ by $\beta(x) = x, \beta(y) = z$, and $\beta(z) = y$. It follows from (N)$^+$ that

$$C(\beta P) = \beta C(P) = \beta T_{\{x,z\}} = T_{\{x,y\}}.$$

Since $C(\beta P) = T_{\{x,y\}}$ it follows from above that $M(\beta P) = T_{\{x,y\}}$. A second application of (N)$^+$ yields

$$\beta C(P) = C(\beta P) = T_{\{x,y\}} = M(\beta P) = \beta M(P).$$

Since β induces a bijection on $\mathcal{T}(S)$ it follows that $C(P) = M(P)$.

If $C(P) = T_{\{y,z\}}$, then use a variation of the previous argument to establish that $C(P) = M(P)$.

The final case is when $C(P) = T_\emptyset$. Assume that $M(P) \neq T_\emptyset$. By using condition (N)$^+$ if necessary we may assume $M(P) = T_{\{x,y\}}$. By the definition of M, $N_P(xy) > \max\{N_P(xz), N_P(yz)\}$. Let $i_0 \in \{i \in K : T_i = T_{\{x,y\}}\}$. It follows from (PR)$^+$ that there exists a profile $P' = (T'_1, \ldots, T'_k)$ such that $T'_i = T_i$ for all $i \neq i_0$, $T'_{i_0} \neq T_{i_0}$, and $C(P') \notin \{T_\emptyset, T_{\{x,y\}}\}$. Then $C(P') = T_{\{x,z\}}$ or $T_{\{y,z\}}$. Assume without loss of generality that $C(P') = T_{\{x,z\}}$. By Lemma 2 and (N)$^+$, $N_{P'}(xz) > \max\{N_{P'}(xy), N_{P'}(yz)\}$. Thus $T'_{i_0} = T_{\{x,z\}}$ and $T_{i_0} = T_{\{x,y\}}$. In fact, $N_{P'}(xz) = N_{P'}(xy) + 1$. If $K_1 = \{i \in K : T'_i = T_{\{x,z\}}\}$ and $K_2 = \{i \in K : T'_i = T_{\{x,y\}}\}$, then $|K_1| = |K_2| + 1$. Choose a permutation α of K such that α maps K_2 onto $K_1 \setminus \{i_0\}$, $K_1 \setminus \{i_0\}$ onto K_2, and $\alpha(i) = i$ for all $i \in K \setminus (K_2 \cup K_1 \setminus \{i_0\})$. In particular, $\alpha(i_0) = i_0$. Note that $P'_\alpha = \beta P$. It follows from (A)$^+$ and (N)$^+$ that

$$C(P') = C(P'_\alpha) = C(\beta P) = \beta C(P) = \beta T_\emptyset = T_\emptyset,$$

contrary to $C(P') = T_{\{x,z\}}$. This last contradiction completes the proof of our main result. $\square$

It is not possible to drop any one of (A)$^+$, (N)$^+$, (PR)$^+$ and still uniquely determine the consensus function M. The projection function $C_1 : \mathcal{T}(S)^k \to \mathcal{T}(S)$ defined by $C_1(P) = T_1$ for any profile $P = (T_1, \ldots, T_k)$ satisfies (N)$^+$ and (PR)$^+$ but it does not satisfy (A)$^+$. The constant function $C_2 : \mathcal{T}(S)^k \to \mathcal{T}(S)$ defined by $C_2(P) = T_{\{x,y\}}$ for any profile P satisfies (A)$^+$ and (PR)$^+$ but not (N)$^+$. The majority consensus rule Maj satisfies (A)$^+$, (N)$^+$, and items (1) and (2) in (PR)$^+$ but does not satisfy item (3) in (PR)$^+$.

It would be interesting to extend the consensus function M to trees with more than 3 leaves. However, it turns out that there is not a unique extension and in some cases the consensus outcome is not even a tree. The details of this work will be given in a future paper.

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