Performance evaluation of demodulation with diversity – A combinatorial approach II: Bijective methods

D. Krob * , E.A. Vassilieva †

Abstract

This paper is devoted to the presentation of a combinatorial approach for analyzing the performance of a generic family of demodulation methods used in mobile telecommunications. We show that a fundamental formula in this context is in fact highly connected with a slight modification of a very classical bijection of Knuth between pairs of Young tableaux of conjugate shapes and $\{0,1\}$-matrices. These considerations allowed us to obtain the first explicit expressions for several important specializations of the performance evaluation formula that we studied.

1 Introduction

Modulating a numeric signal means to transform it into a wave form. Modulation is therefore a technique of main interest in a number of engineering domains such as computer networks, mobile communications, satellite transmissions, ... Due to their practical importance, modulation methods were of course widely studied in signal processing. The classical Proakis textbook devotes for instance a full chapter to this subject (cf. Chapter 5 of [13]). One should also point out that one of the most important problems in this area is to be able to evaluate the performance characteristics of the optimum receivers associated with a given modulation method, which reduces to the computation of various probabilities of errors (see again Chapter 5 of [13]).

Among the different families of modulation protocols used in practice, an important class consists in methods where the modulation reference (i.e. a fixed digital sequence) is also modulated and transmitted. In this kind of situation, the demodulation decision needs to take into account several noisy informations (the transmitted signal, the transmitted reference, but also copies of these two signals). It turns out that the probability of errors appearing in such contexts leads very often to the computation of the following type of probability:

$$P(U < V) = P\left(U = \sum_{i=1}^{N} |u_i|^2 \; < \; V = \sum_{i=1}^{N} |v_i|^2 \right) , \qquad (1)$$

where the u_i and v_i's stand for independent centered complex Gaussian random variables with variances denoted by $E[\,|u_i|^2\,] = \chi_i$ and $E[\,|v_i|^2\,] = \delta_i$ for every $i \in [1, N]$ (see also Section 3.2).

* *Corresponding author:* LIX (CNRS) - Ecole Polytechnique - Route de Saclay - 91128 Palaiseau Cedex – France – e-mail: dk@lix.polytechnique.fr

† LIX (CNRS) - Ecole Polytechnique - Route de Saclay - 91128 Palaiseau Cedex – France – e-mail: katya@lix.polytechnique.fr

The problem of computing explicitely this last probability was hence studied by several researchers from signal processing (cf. [2, 9, 13, 16]). The most interesting result in this direction was obtained by Barrett (cf. [2]) who proved that the probability defined by (1) is equal to

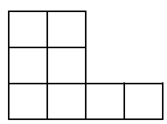
$$P(U < V) = \sum_{k=1}^{N} \left(\prod_{j \neq k} \frac{1}{1 - \delta_k^{-1}\delta_j} \prod_{j=1}^{N} \frac{1}{1 + \delta_k^{-1}\chi_j} \right) . \qquad (2)$$

This last formula can also be described in a purely combinatorial way, using Young tableaux (cf. Section 3.2). This new approach has already led to the first, both algorithmically efficient and numerically stable, practical method for computing the probability $P(U < V)$ (cf. Section 3.2 or [5, 6]). In this paper, we continue the combinatorial study of Barrett's formula by connecting it with a very classical bijection of Knuth (cf. Section A.4.3 of [7] or [10]) between pairs of Young tableaux of conjugate shapes and $\{0, 1\}$-matrices. These considerations allowed us in particular to get the first explicit expressions for several specializations of formula (2) (cf. Section 6).

2 Background

2.1 Partitions and Young tableaux

A *partition* is a finite nondecreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers. The number m of elements of λ is called the *length* of the partition λ. One can represent each such partition λ by a *Ferrers diagram* of *shape* λ, that is to say by a diagram of $\lambda_1 + \ldots + \lambda_m$ boxes whose i-th row contains exactly λ_i boxes for every $1 \leq i \leq m$. The Ferrers diagram associated with the partition $\lambda = (2, 2, 4)$ is for instance given below.



The *conjugate* partition $\tilde{\lambda}$ of a given partition λ is the partition obtained by reading the heights of the columns of the Ferrers diagram associated with λ. For instance, for the partition $\lambda = (2, 2, 4)$ of the above figure, we have $\tilde{\lambda} = (1, 1, 3, 3)$.

When λ is a partition whose Ferrers diagram is contained into the square represented by the partition $N^N = (N, \ldots, N)$ with N rows of length N, one can also define the *complementary* partition $\overline{\lambda}$ of λ which is the conjugate of the partition ν whose Ferrers diagram is the complement (read from bottom to top) of the Ferrers diagram of λ in the square (N^N). Note that this definition is relative to a given size N and that the square does not have to be the smallest one containing λ. For instance, for $N = 6$ and $\lambda = (1, 1, 2, 3)$, we have $\nu = (3, 4, 5, 5, 6, 6)$ and $\overline{\lambda} = (2, 4, 5, 6, 6, 6)$ (see Figure 1).

Let A be a totally ordered alphabet. A *tabloid* of shape λ over A is a filling of the boxes of a Ferrers diagram of shape λ with letters of A. A tabloid is called a *Young tableau* when its rows and its columns consist respectively of non decreasing and strictly increasing sequences of letters of A. One can see below a Young tableau of shape $(2, 2, 4)$ over $A = \{ a_1 < \ldots < a_5 \}$.

a_3	a_5		
a_2	a_2		
a_1	a_1	a_1	a_4

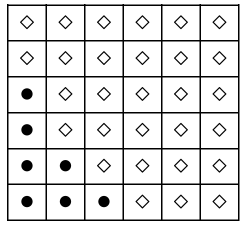


Figure 1: Two complementary partitions : $\lambda = (1, 1, 2, 3)$ and $\overline{\lambda} = (2, 4, 5, 6, 6, 6)$.

One associates with any Young tableau T over A the monomial A^T which is the product of all letters of A that occur in the different boxes of T. One has for instance $A^T = a_1^3\, a_2^2\, a_3\, a_4\, a_5$ for T the Young tableau of the last example. The *Schur function* $s_\lambda(A)$ associated with the partition λ is then defined as the sum of all monomials A^T for T running over all Young tableaux of shape λ. We recall that the Schur functions are symmetric polynomials that form a linear basis of the algebra of symmetric polynomials over A (cf. Section 1.3 of [12]).

2.2 Knuth's bijection

Knuth's bijection is a famous one-to-one correspondence between $\{0, 1\}$-matrices and pairs of Young tableaux of conjugate shapes (cf. [10]). It is based on the *column insertion* process which is a classical combinatorial construction that we present now. Let A be a totally ordered alphabet. The fundamental step of the column insertion process associates with a letter $a \in A$ and a Young tableau T over A a new Young tableau $T(a)$ over A defined as follows.

1. If a is strictly larger than all the entries of the first column of T, the tableau $T(a)$ is obtained by putting a in a new box at the top of the first column of T.

2. Otherwise one can consider the smallest entry b of the first column of T which is greater than or equal to a. The tableau $T(a)$ is then obtained by replacing b by a and by applying recursively our insertion scheme, starting now by trying to insert b in the second column of T. Our process continues until a replaced entry can go at the top of the next column or until it becomes the only entry of a new column.

One can easily check that $T(a)$ is always a Young tableau. Moreover our process can be reverted if one knows which new box it created. Let now $w = a_1 \dots a_N$ be a word over A. The result of the column insertion process applied to w is the Young tableau obtained by column inserting successively $a_1, \dots, a_N$ as described above, starting from the empty Young tableau.

Note 2.1 *The Young tableau which is obtained by applying the column insertion process to a word $w = a_1 \dots a_N$ over A is the same as the tableau obtained by applying the row insertion process (i.e. Schensted's algorithm) to its mirror image $\tilde{w} = a_N \dots a_1$ (see [7] for more details).*

We are now in the position to present Knuth's construction. Let M be a matrix from the set $\mathcal{M}_{N\times N}(\{0, 1\})$ of square $\{0, 1\}$-matrices of order N. Knuth's bijection associates with M a pair (P, Q) of Young tableaux with conjugate shapes over the alphabet $[1, N]$ as described below.

1. Construct the 2-row array A_N which results by listing the N^2 pairs (i, j) of $[1, N] \times [1, N]$ in lexicographic order, i.e.

$$A_N = \begin{pmatrix} 1 & \dots & 1 & 2 & \dots & 2 & \dots\dots & N & \dots & N \\ 1 & \dots & N & 1 & \dots & N & \dots\dots & 1 & \dots & N \end{pmatrix} .$$

2. Take in this array all the entries corresponding to the 1's of M in order to get an array

$$\mathcal{A}(M) = \begin{pmatrix} u_1 & u_2 & \ldots\ldots & u_r \\ v_1 & v_2 & \ldots\ldots & v_r \end{pmatrix}.$$

3. Form the word $w_1(M) = v_1 \ldots v_r$ obtained by reading from left to right the bottom entries (the entries of the second row) of $\mathcal{A}(M)$. The column insertion process applied to $w_1(M)$ gives the Young tableau P.

4. Form finally the second Young tableau Q by placing for every $i \in [1, r]$ the i-th element u_i of the first row of $\mathcal{A}(M)$ in the box which is conjugate to the i-th box created during the column insertion process that led to P.

By reversing the steps of the described construction, we can recover the array $\mathcal{A}(M)$ (and hence our matrix M) from the pair (P, Q). We find the box in which Q has the largest entry; if there are several equal entries, the box that is farthest to the right is selected. Then we perform the reverse column insertion to P starting with the conjugate of the selected box and remove the selected box from Q. We obtain a new pair of Young tableaux with conjugate shapes and perform the same procedure up to the moment when we get two empty Young tableaux.

Example 2.2 *Let us consider the matrix*

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then the arrays $\mathcal{A}_3$ and $\mathcal{A}(M)$ are respectively equal to

$$\mathcal{A}_3 = \begin{pmatrix} 1 & 1 & \boxed{1} & \boxed{2} & 2 & 2 & 3 & \boxed{3} & \boxed{3} \\ 1 & 2 & \boxed{3} & \boxed{1} & 2 & 3 & 1 & \boxed{2} & \boxed{3} \end{pmatrix} \quad \textit{and} \quad \mathcal{A}(M) = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 3 & 1 & 2 & 3 \end{pmatrix}$$

where in $\mathcal{A}_3$ we boxed the entries corresponding to the 1's of M. Thus $w_1(M) = (3, 1, 2, 3)$. Knuth's bijection associates with M the following pair of Young tableaux of conjugate shapes:

$$(P, Q) = \left(\begin{array}{|c|c} \hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|} \hline 2 \\ \hline 1 & 3 & 3 \\ \hline \end{array} \right).$$

We now present a variant of Knuth's bijection that we will need in the sequel (see also Section A.4.3 of [7]). Let M be again a matrix of $\mathcal{M}_{N \times N}(\{0, 1\})$. One can associate with M a new pair (R, S) of Young tableaux with conjugate shapes over the alphabet $[1, N]$ which is constructed as follows.

1. Construct first the 2-row array $\widetilde{\mathcal{A}}_N$ which is equal to the sequence of the N^2 pairs (i, j) of $[1, N] \times [1, N]$ taken in the following order:

$$\widetilde{\mathcal{A}}_N = \begin{pmatrix} N & \ldots & N & \ldots\ldots & 2 & \ldots & 2 & 1 & \ldots & 1 \\ 1 & \ldots & N & \ldots\ldots & 1 & \ldots & N & 1 & \ldots & N \end{pmatrix}.$$

2. Take in this array all the entries corresponding to the 1's of M. We get an array $\widetilde{\mathcal{A}}(M)$.

3. Form the word $\widetilde{w}_1(M)$ obtained by reading from left to right the bottom entries of $\widetilde{\mathcal{A}}(M)$. The column insertion process applied to $\widetilde{w}_1(M)$ gives the Young tableau R.

4. Form finally the second Young tableau S from R and $\widetilde{\mathcal{A}}(M)$ using the conjugate sliding process (see Section A.4.3 of [7] for more details) applied to $\widetilde{w}_1(M)$. Now we will briefly describe this process. To this purpose, let us first set

$$\widetilde{\mathcal{A}}(M) = \begin{pmatrix} \tilde{u_1} & \dots & \tilde{u_r} \\ \tilde{v_1} & \dots & \tilde{v_r} \end{pmatrix} .$$

To construct the second Young tableau S, we apply the following procedure.

- Start with one single box containing $\tilde{u_1}$.
- For each i in $[2, r]$, apply successively the following rules.
  - Add an empty box conjugate to the i-th box that appears during the construction of the Young tableau R and slide there the greater of its two neighbours to the left or below. If the two neighbours have the same entry, the one below is chosen. If there is only one neighbour, it is chosen by abuse of terminology. This creates a new empty box. This sliding process continues until the empty box is the first one of the first column.
  - Put in this box, the following entry $\tilde{u_i}$.

At the end of this procedure, we get the Young tableau S. The tableaux R and S are then of conjugate shape.

The following symmetry result connects then the two previous constructions (cf. Section A.4.3 of [7] for more details).

Theorem 2.3 (Knuth; [10]) *Let M be a matrix of $M_{N\times N}(\{0,1\})$ and let ${}^tM$ be the transpose of M. Let (P,Q) be the result of Knuth's bijection applied to M and let (R,S) be the result of the process described above applied to ${}^tM$. Then one has $(P,Q)=(S,R)$.*

Example 2.4 *Take again the matrix M of Example 2.2. The transpose of M is*

$${}^tM = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} .$$

Then the arrays $\widetilde{\mathcal{A}}_3$ and $\widetilde{\mathcal{A}}({}^tM)$ are respectively given by

$$\widetilde{\mathcal{A}}_3 = \begin{pmatrix} \boxed{3} & 3 & \boxed{3} & 2 & 2 & \boxed{2} & 1 & \boxed{1} & 1 \\ \boxed{1} & 2 & \boxed{3} & 1 & 2 & \boxed{3} & 1 & \boxed{2} & 3 \end{pmatrix} \quad \textit{and} \quad \widetilde{\mathcal{A}}({}^tM) = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{pmatrix}$$

where in $\widetilde{\mathcal{A}}_3$ we boxed the entries coresponding to the 1's of ${}^tM$. Thus $\widetilde{w}_1({}^tM) = (1,3,3,2)$. The above construction associates with ${}^tM$ the following pair (R,S) of Young tableaux:

$$(R,S) = \left(\begin{array}{|c|cc} \hline 2 & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \; , \; \begin{array}{|c|c} \hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array} \right) = (Q,P) .$$

2.3 Plactic relations

The column insertion process can also be described algebraically by the plactic formalism developed by Lascoux and Schützenberger (cf. [11]) that we will now present. Let A be a totally ordered alphabet. The *plactic monoid* is the monoid constructed over A and subject to the following relations (discovered by Knuth (cf. [10])):

$$\begin{cases} aba \equiv baa, \quad bba \equiv bab, \quad \text{for every } a < b \in A \ , \\ acb \equiv cab, \quad bca \equiv bac, \quad \text{for every } a < b < c \in A \ . \end{cases}$$

Two words over A are identified under the plactic relations if and only if the Young tableaux obtained by applying the column insertion process to their mirror images are equal (cf. [7, 11]).

We now present an important property of the plactic monoid that we will use in the sequel. Let T be a Young tableau over A. One can associate with T a word $w(T)$ over A by reading the columns of T from top to bottom and left to right. The words associated with Young tableaux in such a way are called *tableau words*. For instance the tableau word associated with the Young tableau T at the end of Section 2.1 is $w(T) = a_3\, a_2\, a_1\, a_5\, a_2\, a_1\, a_1\, a_4$. Note that applying the column insertion to the mirror image of a tableau word $w(T)$ yields the tableau T. Observe also (see Section 2.1 of [7] or [11]) that a word over A is equivalent with respect to the plactic relations to a unique tableau word (which is therefore associated with the Young tableau given by the column insertion process applied to the mirror image of w).

3 Performance analysis of demodulation protocols

3.1 Demodulation with diversity

Our initial motivation for studying Barrett's formula came from mobile communications. The probability $P(U < V)$ given by formula (1) appears indeed naturally in the performance analysis of demodulation methods based on diversity which are standard in such a context. In order to motivate more strongly our paper, we first present in details this last situation.

We consider a model where one transmits an information $b \in \{-1, +1\}$ on a noisy channel [1]. A reference $r = 1$ is also sent on the noisy channel at the same time as b. We assume that we receive N pairs $(x_i(b), r_i)_{1 \le i \le N} \in (\mathbb{C} \times \mathbb{C})^N$ of data (the $x_i(b)$'s) and references (the r_i's) [2] that have the following form

$$\begin{cases} x_i(b) &=& a_i\, b + \nu_i & \text{for every } 1 \le i \le N, \\ r_i &=& a_i\, \sqrt{\beta_i} + \nu'_i & \text{for every } 1 \le i \le N, \end{cases}$$

where $a_i \in \mathbb{C}$ is a complex number that models the channel fading associated with $x_i(b)$ [3], where $\beta_i \in \mathbb{R}^+$ is a positive real number that represents the excess of signal to noise ratio (SNR) which is available for the reference r_i [4] and where $\nu_i \in \mathbb{C}$ and $\nu'_i \in \mathbb{C}$ denote finally two independent complex white Gaussian noises. We also assume that every a_i is a complex random variable distributed according to a centered Gaussian density of variance α_i for every $i \in [1, N]$.

[1] This situation corresponds to Binary Phase Shift Keying (BPSK).

[2] This situation corresponds to spatial diversity, i.e. when more than one antenna is available, but also to multipath reflexion contexts. These two types of situations typically occur in mobile communications.

[3] Fading is typically the result of the absorption of the signal by buildings. Its complex nature comes from the fact that it models both an attenuation (its modulus) and a dephasing (its argument).

[4] This number β_i is usually greater or equal to 1. In practice however, one often takes $\beta_i = 1$.

According to these assumptions, all observables of our model, i.e. the pairs $(x_i(b), r_i)_{1\leq i\leq N}$, are complex Gaussian random variables. We finally also assume that these N observables are N independent random variables which have their image in $\mathbb{C}^2$. Under these hypotheses it is proved in [4] that

$$\log\left(\frac{P(b=+1|X)}{P(b=-1|X)}\right) = \sum_{i=1}^{N} \frac{4\,\alpha_i\,\sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\,(x_i(b)|r_i) \tag{3}$$

with $X = (x_i(b), r_i)_{1\leq i\leq N}$ and where $(\star|\star)$ denotes the Hermitian scalar product. The demodulation decision is based on the associated Bayesian criterium. One indeed decides that b was equal to 1 (resp. to -1) when the right hand side of Formula (3) is positive (resp. negative). Intuitively this means that one decides that the value $b = 1$ was sent when the $x_i(b)$'s are more or less globally in the same direction than the r_i's. Figure 2 illustrates the case $N = 1$ and one can see that a noisy reference r has a positive (resp. negative) Hermitian scalar product with a noisy information x when x corresponds to a small pertubation of 1 (resp. -1).

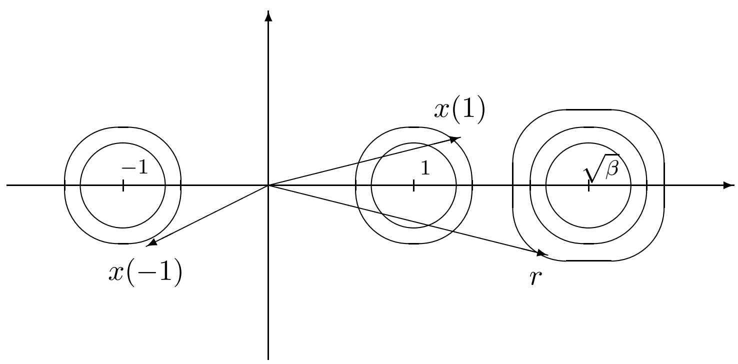


Figure 2: Two possible noisy bits $x(1)$ and $x(-1)$ and a noisy reference r in the case $N = 1$.

The bit error probability (BER) of our model is the probability that the value $b = 1$ was decoded in -1, i.e. the probability that one had

$$\sum_{i=1}^{N} \frac{4\,\alpha_i\,\sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\,(x_i(1)|r_i)\ < 0\ .$$

Using the parallelogram identity, it is now easy to rewrite this last probability as

$$P(\ \sum_{i=1}^{N}\ |u_i|^2 - \sum_{j=1}^{N}\ |v_i|^2 < 0\ )$$

where u_i and v_i denote for every $i \in [1, N]$ the two variables defined by setting

$$u_i = \left(\frac{\alpha_i\,\sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\right)^{1/2}(x_i(1)+r_i)\quad\text{and}\quad v_i = \left(\frac{\alpha_i\,\sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\right)^{1/2}(x_i(1)-r_i)\ .$$

Our various hypotheses imply then immediately that the u_i's and the v_i's are independent complex Gaussian random variables. Hence the performance analysis of our model relies exactly on Barrett's Formula (2) as already indicated in the introduction of our paper (cf. Formula (1)).

It is also interesting to point out the explicit relation between the values of χ_i and δ_i appearing in Barrett's formula and the α_i and β_i which is the following:

$$\chi_i = 2\frac{\alpha_i\,(\beta_i+1)}{1+\alpha_i\,(\beta_i+1)}\left(\sqrt{\Delta_i}+\alpha_i\,\sqrt{\beta_i}\right) \quad \text{and} \quad \delta_i = 2\frac{\alpha_i\,(\beta_i+1)}{1+\alpha_i\,(\beta_i+1)}\left(\sqrt{\Delta_i}-\alpha_i\,\sqrt{\beta_i}\right)$$

with $\Delta_i = (\alpha_i+1)(\alpha_i\beta_i+1)$.

3.2 Barrett's formula

As we saw in the last section, Barrett's formula is connected with the performance analysis of demodulation methods based on diversity. More generally, the performance analysis of many other practical digital transmission systems is based on the computation of the probability that a given Hermitian quadratic form q in complex centered Gaussian variables is negative. Numerous examples of such situations can be found for instance in Proakis's standard textbook (cf. [13]).

The problem of computing such a probability was therefore addressed by several researchers from signal processing. A first formula for this probability was derived by Turin (cf. [16]) and used later by Barrett (cf. [2]) who expressed it as a rational function of the eigenvalues of the covariance matrix associated with q. Formula (2) appears then as a special case of this more general result of Barrett. Alternate methods based either on contour integration or on algebraic manipulations (as in [9] or in annex B of [13]) provide other approaches that involve numerical quadrature of trigonometric functions.

However, all these methods lead to algorithms that are not numerically stable due to the presence of artificial singularities such as the situation $\delta_i = \delta_j$ in Barrett's formula (2) [5]. The first efficient and stable method for computing the probability $P(U<V)$ defined by (1) was obtained by Dornstetter, Krob and Thibon (cf. [5]) using techniques from the theory of symmetric functions (cf. [6]). For the sake of completeness, we recall below their algorithm.

- `Step 1.` Consider the two polynomials $X(z)$ and $\Delta(z)$ of $\mathbb{R}[z]$ defined by setting

$$X(z) = \prod_{i=1}^{N}(1-\chi_i\,z) \quad \text{and} \quad \Delta(z) = \prod_{i=1}^{N}(1+\delta_i\,z)\ .$$

- `Step 2.` Compute the unique polynomial $\pi(z)$ of $\mathbb{R}[z]$ of degree $d(\pi) \le N-1$ such that

$$\pi(z)\,X(z)+\mu(z)\,\Delta(z) = 1$$

  where $\mu(z)$ stands for some polynomial of $\mathbb{R}[z]$ of degree $d(\mu) \le N-1$.

- `Step 3.` Evaluate $\pi(0) = P(U<V)$.

The efficiency and the numerical stability of this algorithm come from the fact that the second step of the above method can be realized by the classical generalized Euclidean algorithm which has the two above mentioned properties.

[5] These last singularities typically create numerical problems in the context of demodulation with diversity described in Section 3.1 where one must deal both with χ_i's and δ_i's that are very close to each other.

3.3 The combinatorial version of Barrett's formula

Using Barrett's formula, it was proved (cf. [6]) that Formula (1) reduces to

$$P(U < V) = \frac{F(\chi, \delta)}{\prod_{1 \leq i,j \leq N} (\chi_i + \delta_j)} \tag{4}$$

where $F(\chi, \delta)$ denotes the symmetric (with respect to the χ_i's and the δ_j's) polynomial

$$F(\chi, \delta) = \sum_{\lambda \subseteq (N^{N-1})} s_{\overline{(\lambda,N)}}(\{\chi_1, \ldots, \chi_N\})\, s_{(\lambda,N)}(\{\delta_1, \ldots, \delta_N\}) \ , \tag{5}$$

$\overline{(\lambda, N)}$ representing the complement of the partition (λ, N) in the square N^N (cf. [6]). Recall that $s_\lambda(X)$ denote the Schur function associated with the partition λ over the alphabet X which is equal by definition to the sum of all monomials X^T, defined as the product of all variables of X involved in T, for T running over all possible Young tableaux of shape λ.

From this combinatorial interpretation of a Schur function, it follows that the monomials involved in the right hand side of equation (5) are exactly the monomials obtained by taking the product of the elements of all square tableaux of shape (N^N) consisting of two Young tableaux of complementary shapes (cf. Figure 1 of Section 2.1) that respect the two constraints:

- **Condition B1**: the first Young tableau is only filled by variables that belong to the ordered alphabet $\delta = \{\, \delta_1 < \ldots < \delta_N \,\}$ and *the length of its first row is equal to* N,
- **Condition B2**: the second Young tableau is only filled by variables that belong to the ordered alphabet $\chi = \{\, \chi_1 < \ldots < \chi_N \,\}$.

A typical example of such a combinatorial structure is given in Figure 3. The first tableau is written there in the usual way. On the other hand, the second tableau is organized differently: its rows (resp. its columns) are placed from top to bottom (resp. from right to left) in the space corresponding to the complement of the first tableau within the square (N^N). Note finally that the tableaux formed out in such a way are examples of the so-called $(k, l)-$semi-standard tableaux in the sense of Remmel (see [14] or [15]).

χ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_5	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_4	χ_3	χ_2
δ_3	δ_3	δ_5	χ_5	χ_4	χ_3
δ_2	δ_2	δ_3	δ_4	χ_4	χ_3
δ_1	δ_1	δ_2	δ_2	δ_2	δ_3

Figure 3: A typical example of complementary fillings of a square tableau.

Example 3.1 *For $N = 2$, Barrett's formula reduces to*

$$P(U < V) = \frac{\chi_1\, \chi_2\, (\delta_1^2 + \delta_1\, \delta_2 + \delta_2^2) + (\chi_1 + \chi_2)\, (\delta_1^2\, \delta_2 + \delta_1\, \delta_2^2) + \delta_1^2\, \delta_2^2}{(\chi_1 + \delta_1)\, (\chi_1 + \delta_2)\, (\chi_2 + \delta_1)\, (\chi_2 + \delta_2)}$$

and one can check that the eight monomials occurring in the numerator of this last expression are exactly the products of the entries of the following eight combinatorial structures:

$$\begin{array}{|c|c|}\hline \chi_1 & \chi_2 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \chi_1 & \chi_2 \\ \hline \delta_1 & \delta_2 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \chi_1 & \chi_2 \\ \hline \delta_2 & \delta_2 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \delta_2 & \chi_1 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \delta_2 & \chi_2 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \delta_2 & \chi_1 \\ \hline \delta_1 & \delta_2 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \delta_2 & \chi_2 \\ \hline \delta_1 & \delta_2 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline \delta_2 & \delta_2 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\ .$$

The complexity of Formula (4) is $O(N^2\,\gamma_N)$ where γ_N denotes the number of monomials involved in its numerator or equivalently the number of square tableaux of shape (N^N) filled as in the typical example of Figure 3. Unfortunately $\gamma_N = 2^{N^2-1}$ (see below) from which it follows that Formula (4) is impracticable when N grows. This combinatorial formula is however not useless since it leads to efficient expressions for several interesting specializations of Barrett's formula (cf. Section 6). Formula (4) can also be reformulated in terms of the algorithm given at the end of Section 3.2, which is both practically efficient (its complexity is quadratic as Barrett's formula) and numerically stable as already stated (cf. [5, 6]).

Proposition 3.2 *The number γ_N of square tableaux of shape (N^N) filled by two complementary Young tableaux satisfying conditions* **B1** *and* **B2** *is equal to $\gamma_N = 2^{N^2-1}$.*

Proof – Let us denote by $P(t)$ the polynomial of $\mathbb{N}[t]$ that results from the substitution in the numerator $F(\chi,\delta)$ (of the righthand side of (4)) of χ_i and δ_i by t^i for every $i \in [1,N]$. Then a combination of Formulas (2) and (4) yields:

$$P(t) = \prod_{1\le i,j\le N} \left(t^i + t^j\right) \left(\sum_{k=1}^{N} \left(\prod_{1\le j\ne k\le N} \frac{1}{1-t^{j-k}} \prod_{j=1}^{N} \frac{1}{1+t^{j-k}} \right)\right) . \qquad (6)$$

Note now that the special case $z = 0$ in the following partial fraction expansion

$$\frac{1}{\displaystyle\prod_{i=1}^{N} (1-t^i\,z) \prod_{i=1}^{N} (1+t^i\,z)}$$

$$= \sum_{k=1}^{N} \frac{1}{1-t^k\,z} \left(\prod_{\substack{j=1\\ j\ne k}}^{N} \frac{1}{1-t^{j-k}} \prod_{j=1}^{N} \frac{1}{1+t^{j-k}} \right) + \sum_{k=1}^{N} \frac{1}{1+t^k\,z} \left(\prod_{j=1}^{N} \frac{1}{1+t^{j-k}} \prod_{\substack{j=1\\ j\ne k}}^{N} \frac{1}{1-t^{j-k}} \right)$$

leads immediately to the identity

$$\sum_{k=1}^{N} \left(\prod_{1\le j\ne k\le N} \frac{1}{1-t^{j-k}} \prod_{j=1}^{N} \frac{1}{1+t^{j-k}} \right) = \frac{1}{2} \qquad (7)$$

from which we deduce that one has

$$P(t) = \frac{1}{2} \left(\prod_{1\le i,j\le N} (t^i + t^j) \right) .$$

One can now immediately conclude that $\gamma_N = P(1) = 2^{N^2-1}$ which completes our proof. ■

We will add finally that the proof of Proposition 3.2 is purely analytic. A bijective proof that explains better this result is now coming (see Section 5).

4 Column words and their complements

This section is devoted to the presentation of several combinatorial results of independent interest concerning column words that will be used in the sequel in our paper.

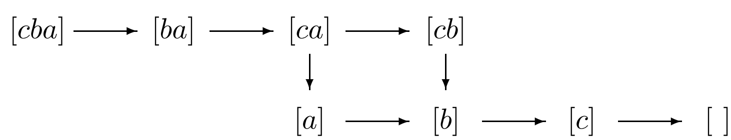
4.1 Column words

Let A be a totally ordered alphabet. We recall that a *column* (of length k) over A is just a Young tableau of shape $1^k = (1, \ldots, 1)$ over A. If $P = \{ p_1 < \ldots < p_r \}$ is a subset of A, we denote by $[P]$ or by $[p_r, \ldots, p_1]$ the unique column of length r filled with all letters of P. We also denote by $\mathcal{C}(A) = \{ [P],\ P \subset A \}$ the set of all columns over A (including the empty column denoted by $[\,]$). A word over the alphabet $\mathcal{C}(A)$ is then said to be a *column word*.

It is important to note that one can associate with every Young tableau T over A a column word $[T]$ which is just the concatenation of the columns of T (considered now as letters of $\mathcal{C}(A)$) read from left to right. The column word associated with the Young tableau given at the end of Section 2.1 is for instance equal to $[a_3\, a_2\, a_1]\, [a_5\, a_2\, a_1]\, [a_1]\, [a_4]$.

The column words associated with Young tableaux can be characterized using the partial order $\preceq$ on the alphabet $\mathcal{C}(A)$ which is defined as follows. Let $P = \{ p_1 < \ldots < p_r \}$ and $Q = \{ q_1 < \ldots < q_s \}$ be two subsets of A. Then one says that $[P] \preceq [Q]$ if one has $s \le r$ and $p_i \le q_i$ for every $1 \le i \le s$. In other words, the column $[P]$ is less or equal to the column $[Q]$ if and only if one gets a Young tableau when putting $[Q]$ at the right of $[P]$. It is then easy to see that each column word associated with a Young tableau is a non decreasing column word (with respect to $\preceq$) and that conversely each non decreasing column word encodes a Young tableau [6].

Example 4.1 *Let $A = \{ a < b < c \}$. Then one has $\mathcal{C}(A) = \{ [\,], [a], [b], [c], [ba], [ca], [cb], [cba] \}$ and the associated partial order $\preceq$ is given by the Hasse diagram below.*



We need one further notation. If $P = \{ p_1 < \ldots < p_r \}$ and $Q = \{ q_1 < \ldots < q_s \}$ are two subsets of A such that $p_r < q_1$, then we denote by $[Q, P]$ the unique column of length $r + s$ which is filled by all the elements of P and Q.

4.2 Complement of a column word

Let P and Q be two subsets of a totally ordered alphabet A. The column $[P]$ is said to be the *complement* (within A) of the column $[Q]$, denoted by $\overline{[Q]}$, if and only if one has $P = A \setminus Q$. More generally the *complement* of a column word $[w] = [P_1] \ldots [P_n] \in \mathcal{C}(A)^*$ is the column word $\overline{[w]}$ defined by setting $\overline{[w]} = \overline{[P_n]} \ldots \overline{[P_1]}$.

We can now give the following important result that will be used in the next section when we will deal with complementation of Young tableaux.

Proposition 4.2 *The complement of a non decreasing column word is a non decreasing column word.*

[6] This last encoding is however not one-to-one due to the fact that the empty column is allowed in the context of column words.

Proof – The proof of our result is based on the two following lemmas, whose proofs can be easily made by suitable inductions and that are left to the reader.

Lemma 4.3 *Let P and Q be two subsets of A such that each element of P is strictly less than each element of Q. Then one has $\overline{[P]} \preceq \overline{[Q,P]}$.*

Lemma 4.4 *Let P and Q be two subsets of A of the same cardinality such that $[P] \preceq [Q]$. Then for the complements of the columns constructed on P and Q, one has $\overline{[Q]} \preceq \overline{[P]}$.*

Let P and Q be two subsets of A such that each element of P is strictly less than each element of Q. Let also R be another subset of A of the same cardinality than P. Suppose finally that the inequality $[Q,P] \preceq [R]$ holds. Then as an immediate consequence of the two last lemmas, one has $\overline{[R]} \preceq \overline{[Q,P]}$. This ends the proof of our result. ∎

Note 4.5 *Observe that Proposition 4.2 shows that the complement $\overline{[T]}$ of the column word $[T]$ associated with a Young tableau T over A also naturally encodes a Young tableau called the* complement *(within A) of T (see [11] p. 140).*

4.3 Complementation and plactic equivalence

Let A be a totally ordered alphabet. Let us then denote by π the *natural projection* of $\mathcal{C}(A)^*$ onto A^*, i.e. the morphism defined by setting $\pi([P]) = p_r \, \ldots \, p_1$ for every subset $P = \{ p_1 < \ldots < p_r \}$ of A. The following result (that can be seen as an extension of Property 3.4 of [11]) gives a simple condition for the plactic equivalence $\equiv$ to be preserved under complementation. Remind that the length of a column word $[u]$ is its number of columns (not the number of letters in the word $\pi([u])$!).

Theorem 4.6 *Let $[u]$ and $[v]$ be two column words of $\mathcal{C}(A)^*$ that have the same length. Then one has $\pi([u]) \equiv \pi([v])$ if and only if one has $\pi(\overline{[u]}) \equiv \pi(\overline{[v]})$.*

Proof – Let us first prove two lemmas that correspond to two special cases of our theorem (i.e. the situations where $[u] = [\ ]\,[Q,p]$ and $[v] = [Q]\,[p]$ for the first lemma, and where all columns involved in $[u]$ and $[v]$ are reduced to letters of A for the second lemma).

Lemma 4.7 *Let p be an element of A and let Q be a subset of A such that p is strictly less than each element of Q. Then one has $\pi(\overline{[p]}\ \overline{[Q]}) \equiv \pi(\overline{[Q,p]}\,[A])$.*

Proof – Let $\omega(A)$ be the maximal element of the totally ordered alphabet A and let q^+ denote the immediate successor in A of any element $q \in A$ distinct of $\omega(A)$. The interpretation of the plactic equivalence in terms of the column insertion process allows then to show that one has $q' \, \pi(\overline{[q]}) \equiv \pi(\overline{[q]}) \, q'$ when $q' \neq q$ and $q \, \pi(\overline{[q]}) \equiv \pi(\overline{[q^+]}) \, q^+$ when $q \neq \omega(A)$. We are now in position to prove that one has

$$\pi(\overline{[p]}\ \overline{[Q]}) \equiv \pi(\overline{[Q,p]}\,[A])\,.$$

We will start from its lefthand side and use the first of the two identities established above to bring all the elements $\{ q \in \overline{[Q]},\, q > p \}$ to the left with respect to $\overline{[p]}$. In the similar way, we will use the second identity to move all the elements $\{ q \in \overline{[Q]},\, q \leq p \}$ to the left with respect to $\overline{[p]}$. This leads to the desired relation and ends the proof. ∎

Lemma 4.8 *Let $a_1 \ldots a_n \in A^*$ and $b_1 \ldots b_n \in A^*$ be two equivalent (with respect to the plactic relations) words over A. Then one has $\pi(\overline{[a_n]} \ldots \overline{[a_1]}) \equiv \pi(\overline{[b_n]} \ldots \overline{[b_1]})$.*

Proof – This lemma follows immediately from the fact the plactic relations are stable under complementation as shown in the proof of Property 3.4 of [11]. ■

Let now $[u] = [P_1] \ldots [P_N]$ and $[v] = [Q_1] \ldots [Q_N]$ be two column words of $\mathcal{C}(A)^*$ of the same length N such that $\pi([u]) = a_1 \ldots a_n$ and $\pi([v]) = b_1 \ldots b_n$ are plactically equivalent words of A^*. Let us denote by p_i the number of letters of A involved in the column P_i for every $1 \le i \le N$. Using the fact that the identity $\pi([\ ]^{p_1-1}\,[P_1] \ldots [\ ]^{p_N-1}\,[P_N]) \equiv \pi([a_1] \ldots [a_n])$ can be obtained by using only relations of the type $[\ ]\,[Q,p] \equiv [Q]\,[p]$, one can now deduce from Lemma 4.7 and from the fact that $\pi([A])$ commutes with every letter of A that one has

$$\pi(\overline{[u]}\,[A]^{n-N}) \equiv \pi(\overline{[P_N]}\,[A]^{p_N-1} \ldots \overline{[P_1]}\,[A]^{p_1-1}) \equiv \pi(\overline{[a_n]} \ldots \overline{[a_1]}) \ .$$

Symmetrically one can also prove that $\pi(\overline{[v]}\,[A]^{n-N}) \equiv \pi(\overline{[b_n]} \ldots \overline{[b_1]})$. Using Lemma 4.8, one deduces from these relations that $\pi(\overline{[u]}\,[A]^{n-N}) \equiv \pi(\overline{[v]}\,[A]^{n-N})$, from which it is immediate to obtain that $\pi(\overline{[u]}) \equiv \pi(\overline{[v]})$ according to the interpretation of the plactic equivalence in terms of the column insertion process. ■

Let us denote by $\widetilde{\pi}$ the *mirror projection* of $\mathcal{C}(A)^*$ onto A^*, i.e. the anti-morphism [7] defined by setting $\widetilde{\pi}([P]) = p_1 \ldots p_r$ for every subset $P = \{\, p_1 < \ldots < p_r \,\}$ of A. We are now in the position to show how the column insertion process acts with respect to complementation of column words.

Corollary 4.9 *Let $[w]$ be a column word of $\mathcal{C}(A)^*$ of length n, let T be the Young tableau obtained by applying the column insertion process to $\widetilde{\pi}([w])$ and let m be the number of columns of T. Then the Young tableau obtained by applying the column insertion process to $\widetilde{\pi}(\overline{[w]})$ is the Young tableau naturally associated with the non decreasing column word $[A]^{n-m}\,\overline{[T]}$.*

Proof – Observe first that one must have $m \le n$ due to the structure of the column insertion process. Our result follows immediately from Theorem 4.6 applied to the two column words $[w]$ and $[T]\,[\ ]^{n-m}$ and from the basic properties of the plactic equivalence (cf. Section 2.3). ■

Example 4.10 *Let us take $A = \{1, 2, 3\}$ and $[w] = [3][21][3]$. Then we get $\pi([w]) = 3213$ and $\widetilde{\pi}([w]) = 3123$. Applying the column insertion process to $\widetilde{\pi}([w])$, we get the Young tableau*

$$T = \begin{array}{|c|c} \hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \ .$$

Hence we have $m = 2$ and $n - m = 1$. Observe that $\overline{[w]} = [21][3][21]$ and $\widetilde{\pi}(\overline{[w]}) = 12312$. Applying the column bumping process to $\widetilde{\pi}(\overline{[w]})$, we get the Young tableau

$$[A]\,\overline{[T]} = \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} \ .$$

[7] That is to say a mapping $\widetilde{\pi}$ satisfying $\widetilde{\pi}([P_1] \ldots [P_n]) = \widetilde{\pi}([P_n]) \ldots \widetilde{\pi}([P_1])$ for every $[P_1] \ldots [P_n] \in \mathcal{C}(A)^*$.

5 A bijective proof of Proposition 3.2

Proposition 3.2 gave us the number γ_N of monomials involved in $F(\chi, \delta)$ in a purely analytic way. In particular, its proof did not provide any insight, neither on the structure of $F(\chi, \delta)$, nor on the simplicity of the fact that one has $\gamma_N = 2^{N^2-1}$ which is indeed remarkable. This section will be devoted to the construction of a bijective proof that explains this result more deeply. This bijection will also help us for studying a number of specializations of Barrett's formula of practical interest (see Section 6).

5.1 A more general combinatorial structure

Let us first introduce a natural generalization of the combinatorial structures that appeared in Section 3.3, that is to say the set $\mathcal{T}_N$ of all square tableaux of shape (N^N) divided as in this last section into two complementary Young tableaux (but without any constraint on them) filled by elements of the alphabets δ and χ, respectively. The two Young tableaux that form an element of $\mathcal{T}_N$ will again be organized as already depicted in Section 3.3. The following picture shows two typical examples of elements of $\mathcal{T}_6$.

χ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_5	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_5	χ_2	χ_1
δ_3	δ_3	δ_4	χ_6	χ_2	χ_1
δ_2	δ_2	δ_2	δ_2	χ_2	χ_1
δ_1	δ_1	δ_1	δ_1	δ_1	δ_1

δ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_5	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_4	χ_3	χ_2
δ_3	δ_3	δ_5	χ_5	χ_4	χ_3
δ_2	δ_2	δ_3	δ_4	χ_4	χ_3
δ_1	δ_1	δ_2	δ_2	δ_2	χ_4

Figure 4: Two typical elements of $\mathcal{T}_6$.

As we will see in the sequel, it is in fact possible to construct a bijection between $\mathcal{T}_N$ and the set $\mathcal{M}_{N\times N}(\{0,1\})$ of all square $\{0,1\}$-matrices of size N, which implies that the cardinality of $\mathcal{T}_N$ is equal to 2^{N^2}. It follows then from this last result that $\gamma_N = 2^{N^2-1}$ due to the fact that the number of elements of $\mathcal{T}_N$ whose first tableau has a first row of length N is obviously (use the symmetry with respect to the main diagonal of the square (N^N) and exchange the role of the alphabets χ and δ in order to pass from one case to the other) equal to the number of elements of $\mathcal{T}_N$ whose second tableau has a first row of length N (which means equivalently that the first tableau has a first row of length strictly less than N).

5.2 Description of the bijection

We now present our bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and $\mathcal{T}_N$. Our construction is based on a slight variation of the well known Knuth correspondence (cf. Section 2.2) that has an interesting symmetry property which is used to derive some practically important specializations of Barrett's formula.

Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$. We apply first Knuth's bijection (as described in Section 2.2) to M in order to get a pair (P, Q) of Young tableaux of conjugate shapes λ and $\tilde{\lambda}$. We then associate with Q a new Young tableau $\overline{Q}$ of shape $\overline{\lambda}$ (the complementary partition of λ within the square (N^N)) which is defined as follows.

- We denote first the length of λ by m (or equivalently the number of columns of Q). We then decide (by abuse of terminology) that Q also has columns indexed by integers strictly greater than m which are all empty.

- We can now define a unique tabloid $\overline{Q}$ of shape $\overline{\lambda}$ by requiring that for every $i \in [1, N]$ the i-th column of $\overline{Q}$ consists exactly of all the letters of the alphabet $\{1, \ldots, N\}$, sorted in increasing order from bottom to top, that do not appear in the $(N{-}i{+}1)$-th column of Q.

Observe that the column word obtained by reading from left to right the columns of $\overline{Q}$ (considered here as letters of $\mathcal{C}(\{1, \ldots N\})$) is equal to $[A]^{N-m}\,\overline{[Q]}$. It follows then immediately from Proposition 4.2 that the tabloid $\overline{Q}$ is also a Young tableau.

Hence $\Psi(M) = (P, \overline{Q})$ is a pair of complementary Young tableaux within the square (N^N). To obtain from it an element of $\mathcal{T}_N$, it suffices to associate with each entry i of P (resp. $\overline{Q}$) the letter δ_i (resp. χ_i) of the alphabet δ (resp. χ). We denote by $\Phi(M)$ the element of $\mathcal{T}_N$ that corresponds in such a way to the initial matrix M. Since the mapping $Q \to \overline{Q}$ is one to one, Ψ is clearly a bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and pairs of Young tableaux of complementary shapes over the alphabet $[1, N]$ while Φ is a bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and $\mathcal{T}_N$.

Example 5.1 *Let us continue Example 2.2. Knuth's bijection applied to the matrix M introduced in this example gives a pair (P, Q) of tableaux of conjugate shapes $\lambda = (1, 1, 2)$ and $\tilde{\lambda} = (1, 3)$. The shape $\overline{\lambda} = (2, 3)$, complementary to the shape λ within the square (3^3), provides the shape of the tableau $\overline{Q}$. Filling in its entries by taking (in the reverse order) the complements in $\{1, 2, 3\}$ of the entries of the columns of Q, we obtain the tableau*

$$\overline{Q} = \begin{array}{|c|c|c}\hline 2 & 2 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad .$$

The element $\Phi(M)$ of $\mathcal{T}_3$ associated with M is then the following rewriting of the pair $(P, \overline{Q})$:

$$\Phi(M) \quad = \quad \begin{array}{|c|c|c|}\hline \delta_3 & \chi_2 & \chi_1 \\ \hline \delta_2 & \chi_2 & \chi_1 \\ \hline \delta_1 & \delta_3 & \chi_3 \\ \hline \end{array} \quad .$$

5.3 Symmetry properties of the bijection

In this section we present of a strong symmetry property of the bijection Φ. We start by giving first a new method for constructing the second Young tableau $\overline{Q}$ associated by Φ with a given $\{0, 1\}$-matrix M.

1. Construct the 2-row array B_N which results by listing the N^2 pairs (i, j) of $[1, N] \times [1, N]$ in lexicographic order with respect to the second entry, i.e.

$$B_N = \left(\begin{array}{cccccccccccc} 1 & \ldots & N & 1 & \ldots & N & \ldots\ldots & 1 & \ldots & N \\ 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots\ldots & N & \ldots & N \end{array} \right) .$$

2. Select in this array all the entries corresponding to the 0's of M. We obtain then a word $w_2(M)$ by reading the top components of the selected entries. The result of the column insertion process applied to $w_2(M)$ is a Young tableau Q'.

It turns out that the Young tableau Q' obtained in this way is exactly the second Young tableau $\overline{Q}$ constructed by the bijection Ψ, presented in Section 5.2, when applied to the matrix M.

Proposition 5.2 *Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$, let $\overline{Q}$ be the second Young tableau constructed by the bijection Ψ applied to M and let Q' be the Young tableau constructed as above. Then one has $Q' = \overline{Q}$.*

Proof – Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$ and let ${}^tM$ be its transpose matrix. Let $\widetilde{A}_N$ be the 2-row array associated with ${}^tM$ as defined in Section 2.2 and let B_N be the 2-row array associated with M as defined above. Let us then associate with these two 2-row arrays the two following column words $[u(M)]$ and $[v(M)]$ of length N defined by setting:

- $[u(M)] = [I_1]\dots[I_N]$ where I_i denotes the sequence (possibly empty) of the entries, written from right to left, of the second row of $\widetilde{A}_N$ corresponding to the 1's of the i-th row of ${}^tM$;
- $[v(M)] = [J_N]\dots[J_1]$ where J_i denotes the sequence (possibly empty) of the entries, written from right to left, of the first row of B_N corresponding to the 0's of the i-th column of M.

For instance, if we take the matrix M of Example 2.2, we have $[u(M)] = [2]\,[3]\,[31]$ (cf. Example 2.4) and $[v(M)] = [2]\,[21]\,[31]$ (cf. Example 5.3 that follows).

The reader can now check that one always has $[v(M)] = \overline{[u(M)]}$ (as can be observed in the previous example). Our proposition follows then from Corollary 4.9 due to the fact that Q is the result of the column insertion process applied to $\widetilde{w}_1(M) = \widetilde{\pi}([u(M)])$ according to Theorem 2.3 and that Q' is the result of the column insertion process applied to $w_2(M) = \widetilde{\pi}([v(M)])$ according to the construction presented above. ∎

Example 5.3 *This example continues Example 2.2 and Example 5.1. In this case, we have:*

$$B_3 = \begin{pmatrix} \boxed{1} & 2 & \boxed{3} & \boxed{1} & \boxed{2} & 3 & 1 & \boxed{2} & 3 \\ \boxed{1} & 1 & \boxed{1} & \boxed{2} & \boxed{2} & 2 & 3 & \boxed{3} & 3 \end{pmatrix}$$

where we boxed the entries that correspond to the 0's of the associated matrix M. Hence $w_2(M) = (1,3,1,2,2)$. Then the column insertion process applied to $w_2(M)$ gives the Young tableau:

$$Q' = \begin{array}{|c|c|c} \hline 2 & 2 & \multicolumn{1}{|c}{} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} = \overline{Q}\ .$$

The following symmetry result is now an immediate consequence of the new intepretation of the bijection Ψ that follows from the construction given above and Theorem 2.3.

Corollary 5.4 *Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$ and let (P,Q') be the result of the bijection Ψ applied to M. Then the result of the bijection Ψ applied to the matrix $s_{0,1}({}^tM)$ obtained by exchanging the 0's and the 1's in the transpose matrix ${}^tM$ of M is equal to (Q',P).*

Example 5.5 *Let us consider again the matrix M of Example 2.2. Then one has:*

$$s_{0,1}({}^tM) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

The reader can then easily check that $w_1(s_{0,1}({}^tM)) = (1,3,1,2,2)$ and $w_2(s_{0,1}({}^tM)) = (3,1,2,3)$ from which it follows that (taking here again all the notations of the previous examples):

$$\Psi(s_{0,1}({}^tM)) = \left(\begin{array}{|c|c|c} \hline 2 & 2 & \multicolumn{1}{|c}{} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\ ,\ \begin{array}{|c|c} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right) = (Q',P)\ .$$

6 Some specializations of Barrett's formula

In this section we show how the bijection constructed in Section 5 can be effectively used to find explicit expressions for several specializations of Barrett's formula.

6.1 Matrices involved in the combinatorial version of Barrett's formula

Let us denote by $\mathcal{N}_N$ the set of all square matrices M of $\mathcal{M}_{N\times N}(\{0,1\})$ such that the length of the first row of the first Young tableau P associated with M by the bijection Ψ (constructed in Section 5.2) is exactly equal to N. Furthermore, let $\mu(t)$ stand for the monomial obtained by taking the product of all entries of an element t of $\mathcal{T}_N$. According to the results of Section 3.3, the symmetric polynomial $F(\chi,\delta)$ defined by relation (5), i.e. the nominator of the combinatorial expression (4) of the probability of error (1), can be expressed as

$$F(\chi,\delta) = \sum_{M\in\mathcal{N}_N} \mu(\Phi(M)), \tag{8}$$

where Φ stands for the second bijection constructed in Section 5.2.

In order to better understand the combinatorial version of Barrett's formula, we will explore the fine structure of $\mathcal{N}_N$. Let again M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$. Observe that the length of the first row of the Young tableau P associated by Ψ with M is exactly the length of the longest non increasing subsequence in $w_1(M)$ according to Greene's theorem (cf. [8] or Chapter 3 of [7]) and to the construction of P (cf. Section 2.2). Since a non increasing subsequence in $w_1(M)$ corresponds to a strictly increasing subsequence, for the North-East order [8], in the set of the entries of M associated with 1's, we get the following characterization of $\mathcal{N}_N$.

Proposition 6.1 *A matrix $M \in \mathcal{M}_{N\times N}(\{0,1\})$ belongs to $\mathcal{N}_N$ if and only if there exists a sequence of 1's of length N in M such that the corresponding entries form a strictly increasing sequence (of length N) in the North-East order.*

Example 6.2 *Let us consider again the matrix of Example 5.5, denoted here by M', i.e.*

$$M' = \begin{pmatrix} 1 & 0 & \boxed{1} \\ 1 & \boxed{1} & 0 \\ 0 & \boxed{1} & 0 \end{pmatrix}.$$

The entries associated with the three 1's of M' boxed on the above picture correspond to the strictly increasing sequence $(3,2) \prec_{NE} (2,2) \prec_{NE} (1,3)$ in the North-East order. According to Proposition 6.1, M' belongs therefore to $\mathcal{N}_3$, which just means that the length of the first row of the first tableau associated by Ψ to M' is equal to 3 as it can be directly checked in Example 5.5.

Note 6.3 *Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$. Then for every $k \in [1,N]$, let us consider:*

- *the largest number $L_0(M,k)$ that can be realized as the sum of the lengths of k disjoint sequences (possibly empty) of 0's in M such that the corresponding sequences of entries are strictly increasing for the South-East order;* [9]

[8] We define the North-East order $\prec_{NE}$ over $[1,N]^2$ by setting $(i,j) \prec_{NE} (k,l)$ if and only if $i > k$ and $j \leq l$.
[9] We define the South-East order $\prec_{SE}$ over $[1,N]^2$ by setting $(i,j) \prec_{SE} (k,l)$ if and only if $i < k$ and $j \leq l$.

- *the largest number $L_1(M,k)$ that can be realized as the sum of the lengths of k disjoint sequences (possibly empty) of 1's in M such that the corresponding sequences of entries are strictly increasing for the North-East order.*

We also define by convention $L_0(M,0) = L_1(M,0) = 0$. Greene's theorem (cf. [8] or Chapter 3 of [7]) used in connection with the constructions of Sections 2.2 and 5.3 shows that

- $L_0(M,k) - L_0(M,k-1)$ *is equal to the length of the k-th column of Q',*

- $L_1(M,k) - L_1(M,k-1)$ *is equal to the length of the k-th row of P,*

for every $k \in [1,N]$, if we set $\Psi(M) = (P,Q')$. Proposition 5.2 then implies that the following simple, but surprising, identity always holds for every $k \in [1,N]$:

$$L_0(M,k) - L_0(M,k-1) + L_1(M,N-k+1) - L_1(M,N-k) = N \ .$$

As an illustration of these results, let us again consider the matrix M of Example 2.2, i.e.

$$M = \begin{pmatrix} \boxed{0} & \textcircled{0} & \boxed{\mathbf{1}} \\ \boxed{\mathbf{1}} & \textcircled{0} & \boxed{0} \\ \triangle\, 0 & \textcircled{\mathbf{1}} & \triangle\, \mathbf{1} \end{pmatrix} .$$

Then one has $L_0(M,1) = 2$, $L_0(M,2) = 4$, $L_0(M,3) = 5$, $L_1(M,1) = 2$, $L_1(M,2) = 3$, $L_1(M,3) = 4$ (the corresponding subsequences of 0's and 1's are boxed, circled and triangled in the above picture) from which it is easy to check all the results of this note.

Let M be a matrix of $\mathcal{N}_N$. According to Proposition 6.1 and to the definition of the North-East order, there exists a sequence σ of length N of 1's in M such that the corresponding sequence of entries has the form $\sigma' = ((N-k+1, j_k))_{1 \le k \le N}$ where $(j_k)_{1 \le k \le N}$ stands for an increasing sequence of integers of $[1,N]$. One can obviously encode such a sequence of 1's by the pseudo-composition [10] $p(\sigma) = (p_k)_{1 \le k \le N}$ of N defined by letting p_k to be the number (possibly equal to zero) of 1's of σ that belong to the k-th column of M [11]. We denote by $p(M)$ the greatest (in the lexicographic order on $\mathbb{N}^N$) pseudo-composition that can be associated in such a way with M. The set $\mathcal{N}_N$ can then be partitioned as

$$\mathcal{N}_N = \bigcup_{p \in \mathcal{P}_N} \mathcal{N}_{p,N} \tag{9}$$

where $\mathcal{P}_N$ denotes the set of all pseudo-compositions of length N of N and where $\mathcal{N}_{p,N}$ stands for the set of all matrices $M \in \mathcal{N}_N$ whose associated pseudo-permutation $p(M)$ is equal to p.

Let us now associate with every pseudo-composition $p = (p_1, \ldots, p_N)$ of $\mathcal{P}_N$ the integer $\mu(p)$ defined as the smallest element μ of $[1,N]$ such that $p_1 + \ldots + p_\mu = N$. The following result gives a fine characterization of the matrices of $\mathcal{N}_{p,N}$.

Proposition 6.4 *Let $p = (p_1, \ldots, p_N)$ be a pseudo-composition of $\mathcal{P}_N$. Furthermore, let also $(j_k)_{1 \le k \le N}$ denote the unique increasing sequence of integers defined by demanding every k in $[1,N]$ to be repeated p_k times. A matrix M belongs to $\mathcal{N}_{p,N}$ if and only if it satisfies the two following properties:*

[10] A pseudo-composition of an integer N is a sequence of nonnegative integers (including 0) whose sum is N.

[11] The sequence $(j_k)_{1 \le k \le N}$ that characterizes σ' (or equivalently σ) as described above, is indeed the unique increasing sequence of N elements of $[1,N]$ obtained by repeating each integer $k \in [1,N]$ exactly p_k times.

- **Condition C1:** *for every* $k \in [1, N]$*, the* $(N-k+1, j_k)$*-entry of* M *is equal to 1;*
- **Condition C2:** *for every* $k \in [1, \mu(p)-1]$*, the* $(N-(p_1+\ldots+p_k), k)$*-entry of* M *is equal to* 0*.*

Proof – Condition **C1** is equivalent to the existence of N values 1 in M whose associated entries form a strictly increasing sequence for the North-East order encoded by the pseudo-permutation p. On the other hand, condition **C2** expresses that no greater pseudo-permutation can be associated with a strictly increasing (for the North-East order) sequence of N entries corresponding to 1's of M. ■

Example 6.5 *Let us consider the matrix* $M \in \mathcal{M}_{3\times 3}(\{0,1\})$ *defined by setting*

$$M = \begin{pmatrix} 0 & \textcircled{0} & \boxed{1} \\ \textcircled{0} & \boxed{1} & 1 \\ \boxed{1} & 0 & 0 \end{pmatrix} .$$

The sequences σ_1 *and* σ_2 *of* 1*'s of* M *given by the associated sequences of entries*

$$\sigma'_1 = (\,(3,1) \prec_{NE} (2,2) \prec_{NE} (1,3)\,) \quad \text{and} \quad \sigma'_2 = (\,(3,1) \prec_{NE} (2,3) \prec_{NE} (1,3)\,)$$

are the unique sequences of length 3 *of* 1*'s in* M *whose corresponding sequences of entries are strictly increasing for the North-East order. Since* $p(\sigma_1) = (1,1,1)$ *and* $p(\sigma_2) = (1,0,2)$*, we get* $p(M) = (1,1,1)$*. One can also check that Proposition 6.4 holds: we boxed (resp. circled) here the entries of* M *that are constrained by condition* **C1** *(resp.* **C2***) as expected.*

6.2 A first specialization : $\chi_i = \chi$ and $\delta_i = \delta$ for every i

Let us consider the situation where all χ_i's are equal to some fixed value χ and all δ_i's to some fixed value δ. Then according to relation (8), the symmetric polynomial $F(\chi, \delta)$ defined by relation (5) reduces to the two variable polynomial

$$F_1(\chi, \delta) = \sum_{i=N}^{N^2} \alpha_i \, \chi^{N^2-i} \, \delta^i \tag{10}$$

where α_i denotes the number of matrices of $\mathcal{N}_N$ with i 1's and $N^2 - i$ 0's (the above expression comes from the fact that $\alpha_i = 0$ for every $0 \leq i \leq N-1$ since every matrix of $\mathcal{N}_N$ has at least N 1's). It now follows from relation (9) and from Proposition 6.4 that one has

$$\alpha_i = \sum_{\mu=1}^{N} \sum_{\substack{p \in \mathcal{P}_N \\ \mu(p)=\mu}} \binom{N^2 - (N+\mu-1)}{i-N} \tag{11}$$

since having i 1's in a matrix of $\mathcal{N}_{p,N}$ means placing $i-N$ 1's (N 1's are already constrained by condition **C1**) in the $N^2 - (N+\mu(p)-1)$ positions not taken both by the N 1's fixed by Condition **C1** and by the $\mu(p)-1$ 0's fixed by Condition **C2**. Now note that the number of pseudo-compositions p of $\mathcal{P}_N$ such that $\mu(p) = \mu$ is just the number of integer solutions of the equation $i_1 + \ldots + i_\mu = N$ with $i_\mu \geq 1$ or equivalently of the equation $i'_1 + \ldots + i'_\mu = N-1$

(without any constraint), which is classically known to be equal to the binomial coefficient of order $(N-1, N-2+\mu)$ (cf. [3]). It follows then from relation (11) that one has

$$\alpha_i = \sum_{\mu=1}^{N} \binom{N-2+\mu}{N-1} \binom{N^2-N-\mu+1}{i-N} .$$

Substituting this last value in relation (10), we obtain

$$\begin{aligned} F_1(\chi,\delta) &= \sum_{i=N}^{N^2} \left(\sum_{\mu=1}^{N} \binom{N-2+\mu}{N-1} \binom{N^2-N-\mu+1}{i-N} \right) \chi^{N^2-i}\,\delta^i \\ &= \sum_{\mu=0}^{N-1} \left(\sum_{i=0}^{N^2-N} \binom{N^2-N-\mu}{i} \chi^{N^2-N-\mu-i}\,\delta^i \right) \binom{N-1+\mu}{N-1} \chi^{\mu}\,\delta^N \\ &= \delta^N \left(\sum_{\mu=0}^{N-1} \binom{N-1+\mu}{N-1} (\chi+\delta)^{N^2-N-\mu}\,\chi^{\mu} \right) \end{aligned}$$

from which the following simple formula for the specialization of the probability of error (12) that we are presently studying can now easily be deduced:

$$P_1(U<V) = \left(\frac{\delta}{\chi}\right)^N \left(\sum_{\mu=0}^{N-1} \binom{N-1+\mu}{N-1} \left(\frac{\chi}{\chi+\delta}\right)^{N+\mu} \right). \tag{12}$$

Note that Formula (12) was already obtained in [5] by purely analytic methods.

6.3 Two other specializations

In order to illustrate the genericity of our bijective method, we now show how it can be applied in two special cases that can also occur in practice (cf. Section 3.1). Our first example (cf. Section 6.3.1) was discussed asymptotically in [5]. We deal below with it in full generality. In our second example (cf. Section 6.3.2) we give an explicit formula (different from Barrett's formula) for a specialization that has not been considered before.

6.3.1 A first situation : $\delta_i = \delta$ for every i

Let us consider the situation where δ_i is equal to some fixed value δ for every $i \in [1, N]$, but no restriction is imposed on the χ_i's. According to relation (8) and to the results of Sections 5.2 and 5.3, the symmetric polynomial $F(\chi,\delta)$ reduces to the multivariable polynomial

$$F_2(\chi_1,\ldots,\chi_N,\delta) = \sum_{\substack{i_1,\ldots,i_N \geq 0 \\ 0 \leq i_1+\ldots+i_N \leq N^2-N}} \beta_{i_1,\ldots,i_N}\, \chi_1^{i_1} \,\ldots\, \chi_N^{i_N}\, \delta^{N^2-(i_1+\ldots+i_N)} \tag{13}$$

where $\beta_{i_1,\ldots,i_N}$ stands for the number of matrices of $\mathcal{N}_N$ that have exactly i_k 0's in their k-th row for every $k \in [1, N]$. Let us now associate with every $p = (p_1,\ldots,p_N)$ of $\mathcal{P}_N$ its complementary pseudo-composition $\widetilde{p} = (\widetilde{p}_1,\ldots,\widetilde{p}_N)$ which is defined by setting $\widetilde{p}_k$ to be equal to the number of indices $i \in [1, N]$ such that $p_1+\ldots+p_i = N-k$. It should be observed that $\widetilde{p}_k$ is the number of 0's that any matrix of $\mathcal{N}_{p,N}$ is forced to contain in its k-th row because of condition **C2**. It then follows from relation (9) and from Proposition 6.1 that one has

$$\beta_{i_1,\ldots,i_N} = \sum_{p=(p_1,\ldots,p_N)\in\mathcal{P}_N} \prod_{k=1}^{N} \binom{N-1-\widetilde{p}_k}{i_k-\widetilde{p}_k} \tag{14}$$

since having i_k 0's in the k-th row of a matrix of $\mathcal{N}_{p,N}$ means placing $i_k-\widetilde{p}_k$ 0's ($\widetilde{p}_k$ 0's are justified by condition **C2**) in the $N-1-\widetilde{p}_k$ possible positions of the k-th row not taken both by the unique 1 forced by condition **C1** and by the $\widetilde{p}_k$ 0's forced by condition **C2**. A combination of relations (13) and (14) gives now immediately the formula

$$P_2(U<V) = \frac{1}{\prod_{i=1}^{N}(\chi_i+\delta)^N} \left(\sum_{\substack{p\in\mathcal{P}_N \\ i_1,\ldots,i_N\geq 0 \\ 0\leq i_1+\ldots+i_N\leq N^2}} \prod_{k=1}^{N} \binom{N-1-\widetilde{p}_k}{i_k-\widetilde{p}_k} \chi_1^{i_1}\ldots\chi_N^{i_N}\,\delta^{N^2-(i_1+\ldots+i_N)} \right) \tag{15}$$

for the current specialization of the probability of error (1) that we are studying here. From Formula (15), the reader can also easily get the asymptotic evaluation obtained in [5] that corresponds to the situation $\delta\to 0$.

6.3.2 A second situation : $\chi_i=\chi$ and $\delta_i=\delta$ for $i\leq m$

Let us fix $m\in[1,N]$. We now consider the situation where for $1\leq i\leq m$ the variable χ_i is equal to some fixed value χ and the variable δ_i is equal to some fixed value δ, while for $m+1\leq i\leq N$ both χ_i and δ_i are equal to 1. Then, according to relation (8) and to the results of Sections 5.2 and 5.3, the symmetric polynomial $F(\chi,\delta)$ reduces to the two variable polynomial

$$F_3(\chi,\delta) = \sum_{i,j=0}^{m\,N} \gamma_{i,j}\,\chi^i\,\delta^j \tag{16}$$

where $\gamma_{i,j}$ denotes the number of matrices of $\mathcal{N}_N$ with exactly i 0's in their m first rows and j 1's in their first m columns. Let $p=(p_1,\ldots,p_N)$ be an element of $\mathcal{P}_N$ and let $\widetilde{p}=(\widetilde{p}_1,\ldots,\widetilde{p}_N)$ be its complementary pseudo-composition. Going back to Proposition 6.1, one can easily check that the following properties hold for every matrix $M\in\mathcal{N}_{p,N}$:

- the number of 0's whose entries are enforced by condition **C2** to belong to the $[1,m]\times[1,m]$ square is equal to $\alpha_0(m,p)=\min(m,\widetilde{p}_1+\ldots+\widetilde{p}_N)-\min(m,\widetilde{p}_{m+1}+\ldots+\widetilde{p}_N)$,
- the number of 1's whose entries are enforced by condition **C1** to belong to the $[1,m]\times[1,m]$ square is equal to $\alpha_1(m,p)=\max(0,p_1+\ldots+p_m-(N-m))$,
- the number of 0's whose entries are enforced by condition **C2** to belong to the $[m+1,N]\times[1,m]$ rectangle is equal to $\beta_0(m,p)=\min(\widetilde{p}_{m+1}+\ldots+\widetilde{p}_N,m)$,
- the number of 1's whose entries are enforced by condition **C1** to belong to the $[m+1,N]\times[1,m]$ rectangle is equal to $\beta_1(m,p)=p_1+\ldots+p_m-\alpha_1(m,p)=\min(N-m,p_1+\ldots+p_m)$,
- the number of 0's whose entries are enforced by condition **C2** to belong to the $[1,m]\times[m+1,N]$ rectangle is equal to $\gamma_0(m,p)=\widetilde{p}_1+\ldots+\widetilde{p}_m-\alpha_0(m,p)$,
- the number of 1's whose entries are enforced by condition **C1** to belong to the $[1,m]\times[m+1,N]$ rectangle is equal to $\gamma_1(m,p)=m-\alpha_1(m,p)=\min(m,N-(p_1+\ldots+p_l))$.

Due to the fact that having j 1's in the first m columns and i 0's in the first m rows of M means that there exist exactly k 1's (for some $k \in [0, j]$) whose entries belong to $[1, m] \times [1, m]$, $j - k$ 1's whose entries belong to $[m+1, N] \times [1, m]$ and $i - (m^2 - k)$ 0's whose entries belong to $[m+1, N] \times [1, N]$, one can now easily check that from relation (9), from Proposition 6.1 and from our last considerations it follows that one has

$$\gamma_{i,j} = \sum_{p \in \mathcal{P}_N} \sum_{k=0}^{j} \binom{m^2 - \alpha(m,p)}{k - \alpha_1(m,p)} \binom{m\,(N-m) - \beta(m,p)}{j - k - \beta_1(m,p)} \binom{m\,(N-m) - \gamma(m,p)}{i - (m^2 - k) - \gamma_0(m,p)} \quad (17)$$

where we set $\alpha(m,p) = \alpha_0(m,p) + \alpha_0(m,p)$, $\beta(m,p) = \beta_0(m,p) + \beta_1(m,p)$ and $\gamma(m,p) = \gamma_0(m,p) + \gamma_1(m,p)$. A combination of relations (16) and (17) leads immediately to an explicit formula (that we will not write down) for the current specialization of the probability of error (1) that we are studying here. We leave it as an exercise to the reader to deduce from this (non written) formula the asymptotic evaluations of our probability of error corresponding to the two situations $\chi \to 0$ and $\delta \to 0$.

Acknowledgements

We would like to thank C. Krattenthaler who suggested us a valuable simplification of our initial proof (based on a reduction to the q-Newton formula) of relation (7). We also sincerely thank the anonymous referees for their interesting comments that allowed us greatly to improve the paper.

References

[1] G.E. ANDREWS, *The Theory of Partitions*, Addison Wesley, 1974.

[2] M.J. BARRETT, *Error probability for optimal and suboptimal quadratic receivers in rapid Rayleigh fading channels*, IEEE J. Select. Areas in Commun., **5**, Nr.2, 302–304, February 1987.

[3] L. COMTET, *Analyse combinatoire*, Presses Universitaires de France, 1970.

[4] J.L. DORNSTETTER, Personal communication.

[5] J.L. DORNSTETTER, D. KROB, J.Y. THIBON, *Fast and Stable Computation of Error Probability in Rapid Rayleigh Fading Channels*, LIAFA Technical Report, Paris, 2000.

[6] J.L. DORNSTETTER, D. KROB, J.Y. THIBON, E.A. VASSILIEVA, *Performance evaluation of demodulation with diversity – A combinatorial approach I: Symmetric function theoretical methods*, Discrete Mathematics and Theoretical Computer Science, (5), 191-204, 2002 (URL : `http://www.dmtcs.org/volumes/abstracts/dm050113.abs.html`).

[7] W. FULTON, *Young Tableaux*, Cambridge University Press, 1997.

[8] C. GREENE, *Some partitions associated with a partially ordered set*, J. of Combin. Theory, Ser. A, **20**, 69–79, 1976.

[9] J.P. IMHOF, *Computing the distribution of quadratic forms in normal variables*, Biometrika, **48**, 419–426, 1961.

[10] D.E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math., **34**, 709–727, 1970.

[11] A. Lascoux, M.P. Schützenberger, *Le monoïde plaxique*, Quaderni de la Ricerca Scientifica, A. De Luca Ed., **109**, 129–156, 1981.

[12] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd Edition, Clarendon Press, 1993.

[13] J. Proakis, *Digital Communications*, 3rd Edition, McGraw-Hill, 1995.

[14] J. Remmel, *The combinatorics of* (k, l)*-hook Schur functions*, Contemp. Math., **34**, Amer. Math. Soc., Providence, RI, 1984.

[15] J. Remmel, *A bijective proof of a factorization theorem for* (k, l)*-hook Schur functions*, Linear and Multilinear Algebra, **28**, 119–154, 1990.

[16] G.L. Turin, *The characteristic function of Hermitian quadratic forms in complex normal variables*, Biometrika, **47**, 199–201, 1960.