Noncommutative Ribbons and Quasi-differential Operators

Daniel Krob – Roberto Mantaci
LIAFA – Université Paris VII
2, Place Jussieu – 75251 Paris Cedex 05 – France
e-mail: dk, mantaci@liafa.jussieu.fr

Abstract

This paper is devoted to a noncommutative generalisation of a classical result occuring in the context of the modular representation theory of the symmetric group. We prove that a noncommutative Schur ribbon function R_I is annihilated by the quasi-differential operator D_{P_k} if and only if the composition I is the external border of a k-core.

1 Introduction

Ordinary Schur functions can be interpreted as the Frobenius characteristics of the irreducible representations of the symmetric group in characteristic 0. When one wants to work with modular representations of the symmetric group, things become much more complicated since the algebra of the symmetric group does not remain semisimple in this new context. One can however show that the two Grothendieck rings naturally associated with irreducible and projective indecomposable modules of the symmetric group in characteristic k are respectively isomorphic to

$$Sym/\mathcal{I}_k \quad \text{and} \quad Sym_{-k}$$

where $\mathcal{I}_k$ denotes the ideal of the algebra of symmetric functions Sym which is generated by the power sums indexed by a multiple of k and where Sym_{-k} denotes the subalgebra of Sym generated by the power sums which are indexed by a non-multiple of k. The Schur functions that belong to Sym_{-k} are of special interest since they are exactly the Frobenius characteristics of the Specht modules. One can show that these Schur functions are characterized by the fact that they are indexed by k-cores.

This paper is intented as a first step towards the generalization to noncommutative symmetric functions (introduced in [G-T]) of the previous framework. We indeed show that the good noncommutative analogues of the Schur functions, i.e. the so called noncommutative ribbon Schur functions, belong to a noncommutative analogue of Sym_{-k} (coming from Lazard's elimination theorem) if and only if they are indexed by a composition which is the border of a k-core.

The main problem which remains clearly open would be to find a good representation theoretic interpretation of such a result. Since the representation theoretic interpretation of noncommutative symmetric functions is given by the 0-Hecke algebra (see [KT1, KT2, KT3]), there is certainly some two parameter (q and t) deformation of the Hecke algebra where one should both consider degeneracies at $q = 0$ and at t a k-root of unity that would give us the required representation theoretic interpretation of our work. This question is unfortunately still open!

2 Preliminaries

2.1 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* defined in [G-T] is the free associative algebra $\mathbf{Sym} = \mathbb{C}\langle S_1, S_2, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called the *complete* symmetric functions. It is convenient to set $S_0 = 1$. Let then t be an indeterminate commuting with the S_k. If one introduces the generating series

$$\sigma(t) := \sum_{k \geq 0} S_k \, t^k \ ,$$

it is possible to define other families of noncommutative symmetric functions by setting

$$\begin{cases} \lambda(t) = \sigma(-t)^{-1}, \\ \dfrac{d}{dt}\sigma(t) = \sigma(t)\psi(t) \ , \quad \sigma(t) = \exp(\phi(t)), \end{cases}$$

where $\lambda(t)$, $\psi(t)$ and $\phi(t)$ are the generating series defined by

$$\lambda(t) := \sum_{k=0}^{+\infty} \Lambda_k \, t^k \, , \qquad \psi(t) := \sum_{k=1}^{+\infty} \Psi_k \, t^{k-1} \, , \quad \phi(t) := \sum_{k=1}^{+\infty} \frac{\Phi_k}{k} \, t^k \ .$$

The noncommutative symmetric functions Λ_k are called *elementary* symmetric functions and Ψ_k and Φ_k are respectively called *power sums* of *first* and *second kind.*

The algebra **Sym** can also be endowed with a Hopf algebra structure. Its comultiplication Δ is defined by any of the following equivalent formulas:

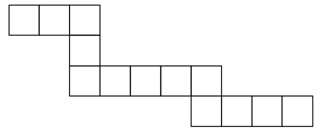
$$\Delta(S_n) = \sum_{k=0}^{n} S_k \otimes S_{n-k} \, , \qquad \Delta(\Lambda_n) = \sum_{k=0}^{n} \Lambda_k \otimes \Lambda_{n-k} \, ,$$

$$\Delta(\Psi_n) = 1 \otimes \Psi_n + \Psi_n \otimes 1 \, , \qquad \Delta(\Phi_n) = 1 \otimes \Phi_n + \Phi_n \otimes 1 \, .$$

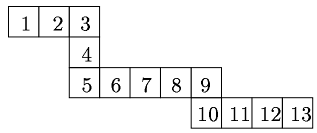
The free Lie algebra generated by the family $(\Phi_n)_{n \geq 1}$ or equivalently by $(\Psi_n)_{n \geq 1}$ is then exactly the Lie algebra of the primitive elements of **Sym** with respect to Δ.

A *composition* is just a sequence $I = (i_1, i_2, \ldots, i_r)$ of strictly positive integers. If m denotes the sum $i_1 + \cdots + i_j$ of the parts of such a composition I, we will say that I is a composition of m. If I is a composition of m, we will write $I \models m$ and call m the *weight* of I. The integer r will be called the *length* of I and denoted by $\ell(I)$. Compositions can be represented by ribbon diagrams, i.e. by connected skew Ferrers diagrams that do not contain 2 by 2 squares. One associates with the composition $I = (i_1, i_2, \ldots, i_r)$ the ribbon diagram whose j-th row has exactly i_j boxes.

Example 2.1 The composition $I = (3, 1, 5, 4)$ is a composition of 13. It can be represented by the ribbon diagram given below.



It is often useful to number the boxes of a ribbon diagram by starting from the top left and finishing at the bottom right. The previous ribbon diagram can then be numbered as follows:



Definition 2.2 *If $I = (i_1, i_2, \ldots, i_r)$ is a composition, we shall denote by $\overline{I}$ the mirror image of I defined by $\overline{I} = (i_r, i_{r-1}, \ldots, i_1)$.*

We can also equip the set of all compositions of a given integer m with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1, 2, 1)$ are exactly $(1, 2, 1)$, $(3, 1)$, $(1, 3)$ and (4).

Let us now observe that **Sym** can be naturally graded by the weight function w defined by setting $w(S_n) = n$ for every $n \geq 1$. The homogeneous component of weight n of **Sym** in this grading will then be denoted by $\mathbf{Sym}_n$. If $(F_n)_{n \geq 1}$ is now a family of homogeneous noncommutative symmetric functions, i.e. a family of noncommutative symmetric functions such that $F_n \in \mathbf{Sym}_n$ for every $n \geq 1$, we set then

$$F^I = F_{i_1} F_{i_2} \ldots F_{i_r}$$

for every composition $I = (i_1, i_2, \ldots, i_r)$. The families (S^I), (Λ^I), (Φ^I) and (Ψ^I) are then homogeneous linear bases of **Sym** that can be constructed in such a way.

There is also another very important basis of **Sym** that is indexed by compositions. If I is a composition, we can indeed define the ribbon Schur function R_I by setting:

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} S^J$$

(see [G-T] for more details). One can then show that the family (R_I) is a basis of **Sym**. We will use extensively the following rule for multiplying two noncommutative ribbon Schur functions (cf. [G-T]).

Proposition 2.3 *Let $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_s)$ be compositions. Then one has:*

$$R_I R_J = R_{I \cdot J} + R_{I \triangleright J}$$

where we set $I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s)$ and $I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s)$.

2.2 Quasi symmetric functions

Malvenuto and Reutenauer (cf. [MvR]) showed that the dual bialgebra **Sym*** of the noncommutative Hopf algebra **Sym** can be identified with the algebra $Qsym$ of quasi-symmetric functions, introduced by Gessel (see [Ge]). This last algebra is defined as follows. Let X be an infinite alphabet totally ordered by some total order $<$. A commutative polynomial $P \in \mathbb{C}[X]$ is then said to be quasi symmetric if one has

$$(P, x_1^{i_1} \dots x_n^{i_n}) = (P, y_1^{i_1} \dots y_n^{i_n})$$

for every strictly increasing sequences $(x_1 < x_2 < \dots < x_n)$ and $(y_1 < y_2 < \dots < y_n)$ of letters of X and every sequence $(i_1, i_2, \dots, i_n) \in \mathbb{N}^n$. The set of all quasi symmetric polynomials of $\mathbb{C}[X]$ form an algebra denoted by $Qsym$ which is called the algebra of *quasi symmetric functions*.

A natural basis of $Qsym$ is formed by the *quasi-monomial functions* defined by setting

$$M_I = \sum_{y_1 < y_2 < \dots < y_p} y_1^{i_1} y_2^{i_2} \dots y_p^{i_p} ,$$

for every $I = (i_1, \dots, i_p)$. Another convenient basis is constituted by the *quasi-ribbon functions* defined by setting

$$F_I = \sum_{J \succeq I} M_J$$

for every I. One can then introduce a pairing between $Qsym$ and **Sym** by setting equivalently

$$\langle R_I, F_J \rangle = \delta_{IJ} \quad \text{or} \quad \langle S^I, M_J \rangle = \delta_{IJ} \ .$$

With respect to this pairing, $Qsym$ becomes exactly the Hopf dual of **Sym** (cf [G-T] for more details). The graded dual basis of (Ψ^I) will then be denoted by (P_I).

2.3 Commutative symmetric functions

The usual algebra of commutative symmetric functions will be denoted here by Sym. We refer the reader to [Macd] or to [LS] for any details concerning the classical theory of symmetric functions. The Schur functions (s_λ) form in particular an important basis of Sym indexed by partitions.

One can define a morphism c from the algebra of noncommutative symmetric functions into the algera of commutative symmetric functions by asking that $c(S_n) = h_n$ (using here Macdonald's notations) for every $n \geq 1$. The image of a noncommutative symmetric function F under this morphism will be called the *commutative image* of F. This terminology is justified by the fact that $c(\Lambda_n) = e_n$ and $c(\Psi_n) = c(\Phi_n) = p_n$ for every $n \geq 1$, using again Macdonald's notations.

One can also show that the commutative image of a noncommutative ribbon function R_I is the so called ribbon Schur function r_I. This last commutative symmetric function is defined in the following way. A composition I can be represented as a skew Ferrers diagram λ/μ and the ribbon Schur function r_I is then just the skew Schur Function $s_{\lambda/\mu}$ (cf [Macd] for details). Figure 1 gives the example of the composition $I = (3, 1, 5, 4)$ interpreted as the skew Ferrers diagram $(10\,7\,3\,3/6\,2\,2)$.

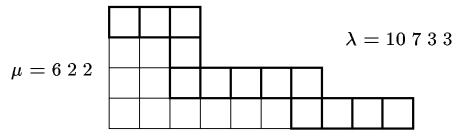


Figure 1: The ribbon Schur function $r_{3\,1\,5\,4}$.

Remark 2.4 The two symmetric functions r_I and $r_{\overline{I}}$ are always equal.

Several properties of noncommutative Schur ribbons are inherited by their commutative images. In particular, the multiplication rule stated in Proposition 2.3 holds clearly true for commutative ribbon Schur functions.

2.4 Some combinatorial results for ribbon diagrams

Definition 2.5 *Let I be a composition interpreted as a skew Ferrers diagram λ/μ. We say that a composition J of weight k is removable from I at position i if and only if one has:*

1. *the boxes numbered from i to $i+k-1$ in the ribbon diagram associated with I form a ribbon diagram of shape J;*
2. *one still gets a skew Ferrers diagram by removing from the Ferrers diagram λ/μ the boxes numbered from i to $1+k-1$.*

Example 2.6 The next figure shows that a composition of weight 4 is removable from the composition $(3, 1, 5, 4)$ at position 6, but not at position 4. By removing a ribbon from another ribbon, one obtains two disconnected ribbons (possibly empty).

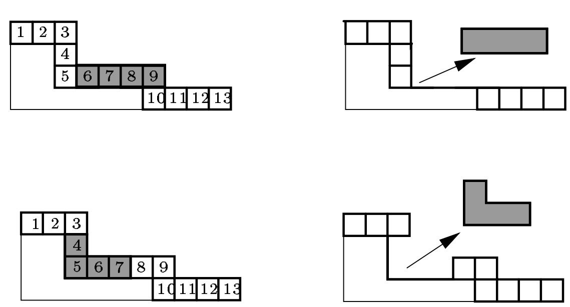


Figure 2: Removable (top) and non removable (bottom) compositions.

Remark 2.7 *Notice that saying that a composition of* weight k *is removable from a composition I is equivalent to saying that a ribbon diagram of* length k *is removable from the ribbon diagram associated with I.*

Well known objects in the theory of representations are the so called k-cores.

Definition 2.8 *Let λ be a partition. We say that λ is a k-core if it is not possible to remove any composition of weight k at the border of the Ferrers diagram λ in order to obtain another Ferrers diagram.*

Definition 2.9 *Let n be a positive integer, let $I = (i_1, i_2, \ldots, i_r)$ be a composition of n and let i be an integer in $\{1, 2, \ldots, n\}$. We say that the composition I passes through i if there exists an integer $k \in \{1, 2, \ldots, r\}$ such that $i_1 + \cdots + i_k = i$.*

For instance the composition $I = (3, 1, 5, 4)$ of 13 passes exactly through 3, 4, 9 and 13.

Proposition 2.10 *Let I be a composition. Then a composition of weight k is removable from I at position i if and only if I passes through $i+k-1$ and I does not pass through $i-1$.*

Proof. A composition J of lenght k is removable from I at the first position ($i = 1$) if and only if the last box of J (which is the k-th box of I) is the last box of its row, i.e. if and only if I passes through k.

A composition J of weight k is clearly removable from the composition I at position $i > 1$ if and only if the two following conditions are satisfied:

- the $(i-1)$-th box of I (which preceeds the first box of J) is not the last box of its row, i.e. I does not pass through $i-1$;
- the $(i+k-1)$-th box of I (which is the last box of J) is the last box of its row, i.e. I passes through $i+k-1$.

Since all the situations were considered, the proof is now complete. $\diamondsuit$

Definition 2.11 *We will say that a composition I is k-solid if no composition of weight k is removable from I.*

Remark 2.12 *It is straightforward from the definitions that a composition I is k-solid if and only if it is the border of a k-core.*

Lemma 2.13 *Let I be a composition of n. Then I is k-solid if and only if either $n < k$, or $n > k$ and I satisfies the following conditions:*

i) I passes through $n-k$,

ii) I does not pass through k,

iii) for every i, the composition I does not pass through $i+k$ if it does not pass through i.

Proof. A ribbon of length k is certainly not k-solid as it is possible to remove it entirely. Of course, ribbons of length smaller than k are k-solid. Let us then supppose that n is greater than k. Condition *i)* insures that no composition of weight k is removable at the end of I. Condition *ii)* insures that no composition of weight k is removable at the beginning of I. Condition *iii)* insures that no composition of weight k is removable at any other position i of I (according to Proposition 2.10). $\diamondsuit$

2.5 Differential and quasi differential operators

The algebra of commutative symmetric functions Sym is equipped with a canonical scalar product (,) which is defined by requiring that the basis of Schur functions forms an orthonormal basis for it, i.e. that one has $(s_\lambda, s_\mu) = \delta_{\lambda\mu}$ for all partitions λ and μ (see [Macd] or [LS]). It is worth noticing that the algebra of $Qsym$ of quasi symmetric functions contains the algebra Sym of symmetric functions.

This scalar product defined in Sym is related to the pairing between **Sym** and $Qsym$ since one can prove that one has

$$\langle F, f \rangle = (c(F), f) \qquad (1)$$

for every noncommutative symmetric function F and every quasi symmetric function f which is a symmetric function of Sym (see [Ge] or [G-T]).

Let f be a symmetric function. The differential operator D_f is defined as the adjoint of the multiplication operator $M_f : g \longrightarrow fg$, i.e. by requiring

$$(D_f(g), h) = (g, M_f(h)) = (g, fh)$$

for every $g, h \in Sym$. One can prove in particular that $D_{p_k} = k\,\partial/\partial p_k$ (cf [Macd] or [LS]). The Murnaghan-Nakayama rule (cf [Macd] or [LS]) explicitly describes the actions of these last differential operators on commutative ribbon Schur functions.

Proposition 2.14 (Murnaghan-Nakayama rule) *Let I be a composition. Then one has*

$$D_{p_k}(r_I) = \sum_{\substack{J \text{ removable from } I \\ J \models k}} (-1)^{\ell(J)-1}\, r_{I_1}\, r_{I_2}$$

where I_1 and I_2 are the ribbons obtained by removing J from I in all possible ways.

We will use extensively the following corollary of the previous proposition.

Corollary 2.15 *Let I be a composition of the integer k. In the expansion of the commutative ribbon Schur function r_I over the basis of power sums, the term p_k appears with coefficient $(-1)^{\ell(I)-1}/k$. Moreover no other monomial of this expansion contains the term p_k.*

Proof. By Proposition 2.14, one has $D_{p_k}(r_I) = (-1)^{\ell(I)-1}$. Since $D_{p_k} = k\,\partial/\partial p_k$, one has

$$\frac{\partial}{\partial p_k}(r_I) = \frac{(-1)^{\ell(I)-1}}{k}$$

and the conclusion is now immediate. ◇

It is also interesting to introduce a notion of quasi differential operator in the context of noncommutative symmetric functions. Let f be a quasi-symmetric function. The quasi differential operator D_f is then defined by setting

$$\langle D_f(F), g \rangle = \langle F, gf \rangle$$

for every noncommutative symmetic function F and every quasi symmetric function g. The previous property of the pairing given at the beginning of this section shows that

$$c(D_f(F)) = D_f(c(F))$$

when f is a quasi symmetric function which is already a symmetric function. This justifies the fact that we used the same notation for differential operators and quasi differential operators.

3 Commutative ribbons and differential operators

3.1 A tensor decomposition of Sym

One can clearly decompose Sym as follows

$$Sym = Sym_k \otimes Sym_{-k} \ ,$$

where $Sym_k = \mathbb{C}[p_k, p_{2k}, p_{3k}, \ldots]$ is the algebra generated by the power sums indexed by multiples of k and where Sym_{-k} is the algebra generated by the remaining power sums. This decomposition is of interest in the theory of modular representations of the symmetric group. The Schur functions that are in Sym_{-k} are indeed exactly the Frobenius characteristics of the Specht k-modular representations of the symmetric group (cf. [CR] and [R]). These Schur functions can be characterized exactly as follows.

Theorem 3.1 *A Schur function s_λ is in Sym_{-k} if and only if λ is a k-core.*

We want to generalize this result to the noncommutative case where the ribbon Schur functions R_I play the role of the usual Schur functions. Our first step in this direction will be to characterize the commutative ribbon Schur functions that belongs to Sym_{-k}. When one expands a ribbon Schur function r_I of Sym_{-k} into the basis of power sums, no p_{jk} appears in the expansion. Therefore a ribbon Schur function r_I of Sym_{-k} satisfies

$$D_{p_{jk}}(r_I) = jk\, \frac{\partial (r_I)}{\partial p_{jk}} = 0$$

for every $j \geq 1$. In the main theorem of this section (Theorem 3.2), we characterize the compositions I such that $D_{p_k}(r_I) = 0$. In Corollary 3.10, we show that this condition is equivalent to the fact that $D_{p_{jk}}(r_I) = 0$ for every $j \geq 1$. Hence a ribbon Schur function r_I will belong to Sym_{-k} if and only if it is annihilated by the differential operator D_{p_k}. It follows from this remark that Theorem 3.2 will give in fact an exact characterization of the ribbon Schur functions that belongs to Sym_{-k}.

3.2 Ribbon Schur functions in Sym_{-k}

The following theorem is the main result of this section.

Theorem 3.2 *Let I be a composition. Then the two following assertions are equivalent:*

1. $D_{p_k}(r_I) = 0$;
2. *I or $\overline{I}$ is k-solid.*

Remark 3.3 Before entering into the proof of our theorem, it is worth noticing that the two conditions "I is k-solid" and "$\overline{I}$ is k-solid" are not equivalent. Let us consider for instance the composition $I = (1, 3, 3)$ which is 3-solid. It is indeed easy to check that no composition of weight 3 (in other terms, no ribbon of length 3) can be removed from it. On the other hand, its mirror image $\overline{I} = (3, 3, 1)$ is not 3-solid. Indeed, it is possible to remove two ribbons of length 3 from it, one at the beginning (at position 1) and one at the end (at position 5). However, $D_{p_3}(r_{3,3,1})$ is equal to zero. Indeed, using Murnaghan-Nakayama rule, one immediately gets $D_{p_3}(r_{3,3,1}) = r_{3,1} - r_{3,1} = 0$.

Proof. Let us first show that condition 2 implies condition 1. If I is k-solid, then no ribbon of length k is removable from it and hence, by Proposition 2.14, we get $D_{p_k}(r_I) = 0$. If $\overline{I}$ is k-solid, one has of course $D_{p_k}(r_I) = D_{p_k}(r_{\overline{I}}) = 0$. This ends the first part of our proof.

The following lemmas will now prove the converse of our theorem, i.e. that condition 1 implies condition 2.

Lemma 3.4 *Let I be a composition of $n \geq k$ such that $D_{p_k}(r_I) = 0$. Then either I or $\overline{I}$ satisfies the conditions i) and ii) of Lemma 2.13.*

Proof. Let I be a composition of $n \geq k$ such that $D_{p_k}(r_I) = 0$. Suppose first that I satisfies condition *i*) of Lemma 2.13. Then it is not possible to remove from I a ribbon of length k at the position $n - k + 1$ (at the end of I). This implies that it is not possible to remove a ribbon of length k at the beginning of I either. Indeed, if this was possible, by the Murghanam-Nakayama Rule, we would have

$$D_{p_k}(r_I) = \pm r_{I_k} + \sum_{I_1, I_2 \neq \emptyset} \pm r_{I_1} r_{I_2}$$

where I_k denotes the composition of weight $n - k$ obtained by removing a ribbon of length k at the beginning of I and where all the compositions that appear in the remaining sum of the above expression are not empty. According to Corollary 2.15, this implies that the coefficient of p_{n-k} in the previous expression cannot be zero. Hence, $D_{p_k}(r_C)$ is not equal to 0 which contradicts our hypothesis. So it is not possible to remove a ribbon of length k at the beginning of I. Hence I also satisfies required condition *ii*) of Lemma 2.13.

Suppose now that I does not satisfy condition *i*) of Lemma 2.13. If I does not pass through $n - k$, then $\overline{I}$ does not pass through k. This means that it is not possible to remove a ribbon of length k at the beginning of $\overline{I}$. For the same reason as in the previous case, it is not possible to remove a ribbon of length k at the end of $\overline{I}$ either. This implies that $\overline{I}$ passes through $n - k$ and hence that $\overline{I}$ satisfies both conditions *i*) and *ii*) of Lemma 2.13. ♢

We are only left with proving that if I is a composition of $n \geq k$ that satisfies the conditions *i*) and *ii*) of Lemma 2.13 and such that $D_{p_k}(r_I) = 0$, then I also satisfies condition *iii*) of Lemma 2.13. We will prove this property in the following lemma.

Lemma 3.5 *Let I be a composition of $n \geq k$ that satisfies conditions i) and ii) of Lemma 2.13 and such that $D_{p_k}(r_I) = 0$. Then, if I does not pass through some integer i (with $n \geq i + k$), the composition I does not pass through $i + k$ either.*

Proof. Let I be a composition that satisfies the hypotheses of our lemma. Suppose that there exists an integer i such that I does not pass through i and such that I passes through $i + k$ and choose the minimal integer i with this property. Hence, it is possible to remove a ribbon of length k at the position $i + 1$ according to Proposition 2.10. The Murnaghan-Nakayama rule implies then that

$$D_{p_k}(r_I) = \pm r_{I_i} r_{I_{n-k-i}} + \sum_{(I_1, I_2) \neq (I_i, I_{n-k-i})} \pm r_{I_1} r_{I_2} \qquad (2)$$

where I_i and I_{n-k-i} denote respectively the two ribbons of length i and $n-k-i$ obtained by removing from I a ribbon of length k at position $i+1$ and where the other terms of the sum correspond to the products of ribbons obtained by removing from I a ribbon of length k at any other position where this is allowed. According to Corollary 2.15, the coefficient of $p_i\, p_{n-k-i}$ in the product $r_{I_i}\, r_{I_{n-i-k}}$ is nonzero. This monomial must clearly be annihilated by another such monomial in the above sum. However, such a monomial can be found in a product $r_{I_1}\, r_{I_2}$ of the second part of the sum only if this product is obtained by one of the three following ways:

- by removing a ribbon of length k at the beginning of I,
- by removing a ribbon of length k at the end of I,
- by removing a ribbon of length k at the position $n-i-k+1$ of I.

The first two possibilities cannot occur according to our hypotheses. Hence it must be possible to remove a ribbon of length k at the position $n-i-k+1$ of I. This implies that I passes through $n-i-k$ and does not pass through $n-i$ according to Proposition 2.10. Moreover, we have $i \neq n-i-k$. Otherwise, the second part of the sum above would then not contain any monomial p_i^2. This last monomial would then only appear once with nonzero coefficient in the sum, which is not possible.

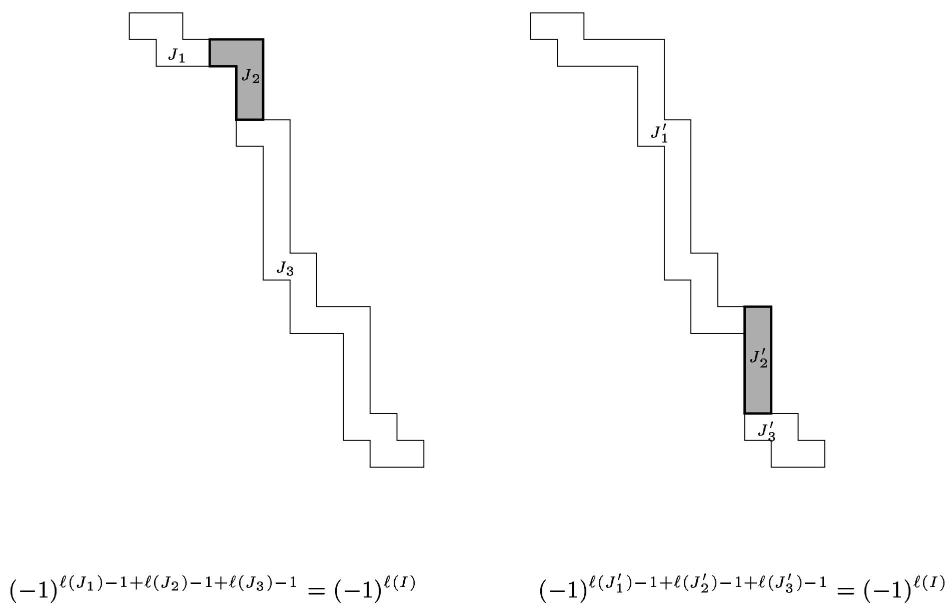


Figure 3: The two ways to obtain $p_i\, p_{n-i-k}$: the coefficients are equal.

Let us look now at the exact coefficient of $p_i\, p_{n-i-k}$ in the expansion (2) (that is given by the Murnaghan-Nakayama rule and its Corollary 2.15). As observed above, terms $p_i\, p_{n-i-k}$ correspond exactly to the removing of a ribbon of length k at position $i+1$ and at position $n-i-k+1$. We immediately find (see Figure 3) that the coefficient of $p_i\, p_{n-i-k}$ in both cases would be $(-1)^{l(I)-1}/(i\,(n-i-k))$. Hence their sum would be different from 0, which is not possible. This contradiction ends the proof. $\diamondsuit$

The proof of Theorem 3.2 beeing achieved, we can now give some consequences of our result. Let us start with the following remark.

Remark 3.6 Let I be a k-solid composition. Then if I passes through i, I passes also through $i-k$. This condition is indeed clearly equivalent to condition *iii*) of Lemma 2.13.

We deduce from this remark the following corollaries.

Corollary 3.7 *Let I be a k-solid composition of $n > k$. Let us consider the euclidean division decomposition $n = qk + r$ (with $0 \le r < k$) of n. Then I passes through $r + ik$ for every $i \in [0, q]$ and does not pass through $i\,k$ for every $i \in [1, q]$.*

Proof. By condition *i*) of Lemma 2.13, I passes through $n - k = r + qk - k = r + (q-i)k$. Hence by the previous remark, I passes through $r + (q-1)k - jk$ for all $j \in [1, q-1]$. On the other hand, by condition *ii*) of Lemma 2.13, I does not pass through k. Condition *iii*) of Lemma 2.13 implies then that I does not pass through $k + jk$ for all $j \in [1, q-1]$. ♢

Corollary 3.8 *Let I be a composition of n such that $D_{p_k}(r_I) = 0$. Then $n \bmod k \neq 0$.*

Proof. Suppose that $n \bmod k = 0$. There exists therefore an integer q such that $n - k = (q-1)\,k$. Up to exchanging I with $\overline{I}$, we can suppose that I is k-solid. Using the previous corollary, we see that I does not pass through $(q-1)k$. On the other hand, according to condition *i*) of Lemma 2.13, I passes through $n - k$. Since $n - k = (q-1)\,k$, we get now a contradiction. ♢

Corollary 3.9 *Let I be a composition of n and let $n = qk + r$ be the euclidean division decomposition of n. Then I is k-solid if and only if one can decompose it as $I = I_0 \cdot I_1 \cdot \ldots \cdot I_q$ where I_j are compositions with the following properties:*

- *I_0 is a composition of r,*
- *I_j is a composition of k for every $j \in [1, q]$,*
- *for every $j \in [1, q-1]$, the composition I_{j+1} is less fine than or equal to I_j,*
- *none of the compositions $I_1, \ldots, I_q$ passes through $k - r$,*
- *if I_0 does not pass through an integer i, then none of the compositions $I_1, \ldots, I_q$ passes through $k - r + i$.*

Hence the compositions I such that $D_{p_k}(r_I) = 0$ are the compositions described in the previous corollary or their mirror images.

Corollary 3.10 *Let I be a composition. Then one has $D_{p_{jk}}(r_I) = 0$ for every $j \geq 1$ if and only if I or $\overline{I}$ is k-solid.*

Proof. If one has $D_{p_{jk}}(r_I) = 0$ for every $j \geq 1$, then in particular one has $D_{p_k}(r_I) = 0$ and Theorem 3.2 allows to conclude.

Conversely, first notice that up to exchanging I with $\overline{I}$ we can assume that I is k-solid. Then I satisfies all conditions of Corollary 3.9. Using the characterization given by this corollary, we easily get then that I is also jk-solid for all $j \geq 1$. Theorem 3.2 allows then to conclude that $D_{p_{jk}}(r_I) = 0$ for every $j \geq 1$. ♢

Example 3.11 The composition $I = (1,1,1,1,2,1,2,2,1,2,2,3)$ is 5-solid since one can decompose it according to Corollary 3.9 in the following way:

$$I = \underbrace{1\,1\,1\,1}_{I_0}\ \underbrace{2\,1\,2}_{I_1}\ \underbrace{2\,1\,2}_{I_2}\ \underbrace{2\,3}_{I_3}\ \underbrace{5}_{I_4}\ .$$

The composition $I' = (6,6,4,2,3,1,2,3,1,1,1,2)$ is annihilated by D_{p_6} because its mirror image is 6-solid. One can indeed decompose this last composition according to Corollary 3.9 as follows:

$$\overline{I'} = \underbrace{2}_{I_0}\ \underbrace{1\,1\,1\,3}_{I_1}\ \underbrace{2\,1\,3}_{I_2}\ \underbrace{2\,4}_{I_3}\ \underbrace{6}_{I_4}\ \underbrace{6}_{I_5}\ .$$

4 Noncommutative ribbons and quasi differential operators

4.1 A tensor decomposition of Sym

As in the commutative case, the algebra **Sym** has an analogous interesting decomposition. Let $\Pi = (\Pi_n)_{n\geq 1}$ be a sequence of Lie idempotents with $w(\Pi_n) = n$. For any positive integer k, we can then define the subalgebra $\mathbf{Sym}_k(\Pi)$ of **Sym** generated by the elements of the sequence Π which are indexed by some multiple of k, i.e.

$$\mathbf{Sym}_k(\Pi) = \mathbb{C}\langle \Pi_k, \Pi_{2k}, \Pi_{3k}, \dots \rangle\ .$$

We must now introduce another important subalgebra of **Sym**. Let us denote by T_k the set of all compositions t of the form

$$t = (i_1k, i2k, \dots, i_rk, i_{r+1})$$

with $i_{r+1} \not\equiv 0 \pmod{k}$. We associate with every element $t = (i_1k, i2k, \dots, i_rk, i_{r+1}) \in T_k$ the noncommutative symmetric function $\Pi[t]$ defined by setting

$$\Pi[t] = [\Pi_{i_1k}, [\Pi_{i_2k}, [\dots, [\Pi_{i_rk}, \Pi_{i_{r+1}}] \dots]]]\ .$$

The subalgebra $\mathbf{Sym}_{-k}(\Pi)$ is then defined by setting

$$\mathbf{Sym}_{-k}(\Pi) = \mathbb{C}\,\langle\, \Pi[t],\ t \in T_k\,\rangle\ .$$

According to Lazard's elimination theorem (see [B]), the subalgebra $\mathbf{Sym}_{-k}(\Pi)$ is freely generated by the family $\{\,\Pi[t],\ t \in T\,\}$. Moreover this last classical result of Lazard also shows that one has the following tensor decomposition

$$\mathbf{Sym} = \mathbf{Sym}_k(\Pi) \otimes \mathbf{Sym}_{-k}(\Pi)$$

of **Sym**. An important fact is that the algebra $\mathbf{Sym}_{-k}(\Pi)$ does not depend on the sequence Π of Lie idempotents that was used to define it.

Proposition 4.1 *Let $\Pi = (\Pi_n)_{n\geq 1}$ and $\Pi' = (\Pi'_n)_{n\geq 1}$ be two sequences of homogeneous Lie idempotents. Then one has*

$$\mathbf{Sym}_{-k}(\Pi) = \mathbf{Sym}_{-k}(\Pi')\ .$$

Proof. Let m be a positive integer such that $m \not\equiv 0 \bmod k$. Since Π_m and Π'_m are homogeneous Lie idempotents of weight m, one has

$$\Pi_m \in \Pi'_m + [L(\Pi'), L(\Pi')] \ .$$

By the free Lie algebra version of the Lazard's elimination theorem (cf [B]), one has

$$L(\Pi') = L(\Pi'[t],\ t \in T_k) \oplus L(\Pi'_k, \Pi'_{2k}, \Pi'_{3k}, \dots) \ .$$

Hence one can write $\Pi_m = \ell_1 + \ell_2$ where $\ell_1 \in L(\Pi'[t],\ t \in T_k)$ and $\ell_2 \in L(\Pi'_k, \Pi'_{2k}, \dots)$. The element ℓ_2 is homogeneous of weight a multiple of k. But Π_m is by definition homogeneous of weight $m \not\equiv 0 \bmod k$. Hence we must have $\ell_2 = 0$, which implies $\Pi_m \in L(\Pi'[t],\ t \in T_k)$. Thus, for every compositions $(i_1, i_2, \dots i_r)$ and for all $m \not\equiv 0 \bmod k$, one has

$$[\Pi_{i_1 k}, [\Pi_{i_2 k}, \dots [\Pi_{i_r k}, \Pi_m] \dots]] \in L(\Pi'[t],\ t \in T_k)$$

since $L(\Pi'[t],\ t \in T_k)$ is a Lie ideal according to Lazard's elimination theorem (see again [B]). It follows now easily that $\Pi[t] \in \mathbf{Sym}_{-k}(\Pi')$ for all $t \in T_k$. Thus we have

$$\mathbf{Sym}_{-k}(\Pi) \subset \mathbf{Sym}_{-k}(\Pi') \ .$$

The final desired equality follows by symmetry. ◇

Therefore the algebra $\mathbf{Sym}_{-k}(\Pi)$ can be denoted more simply by $\mathbf{Sym}_{-k}$. In particular, one has $\mathbf{Sym}_{-k} = \mathbf{Sym}_{-k}(\Psi)$, where $\Psi = (\Psi_n)_{n \geq 1}$. It is also interesting to observe that the proof of the previous proposition shows in fact the following stronger result.

Corollary 4.2 *Let $\Pi = (\Pi_n)_{n \geq 1}$ and $\Pi' = (\Pi'_n)_{n \geq 1}$ be two sequences of homogeneous Lie idempotents. Then one has*

$$L(\Pi[t],\ t \in T_k) = L(\Pi'[t],\ t \in T_k) \ .$$

Let us now state the following technical lemma.

Lemma 4.3 *Let $\Pi = (\Pi_n)_{n \geq 1}$ be a sequence of homogeneous Lie idempotents such that $w(\Pi_n) = n$ for every $n \geq 1$ and let (Π^*_I) be the graded dual basis of $QSym$ associated with the basis (Π_I) of* **Sym**. *Then one has $\Pi^*_n = n\, P_n$ for every $n \geq 1$.*

Proof. It suffices to show that $\langle\, k\, P_k, \Pi_k \,\rangle = 1$ for every $k \geq 1$ and that $\langle\, P_k, \Pi^I \,\rangle = 0$ for every $k \geq 1$ and for all compositions I of length greater or equal to 2. It is a known result (see [G-T]) that an element Π_k is a Lie idempotent if and only if one has

$$\Pi_k \in \frac{\Psi_k}{k} + [L(\Psi), L(\Psi)] \tag{3}$$

where $L(\Psi)$ denotes the free Lie algebra generated the noncommutative power sums of first kind. Therefore one can write

$$\Pi_k = \frac{\Psi_k}{k} + \sum_J a_J\, \Psi^J$$

where the compositions J involved in the above sum have all length ≥ 2. Thus we get

$$\langle\, k\,P_k, \Pi_k\,\rangle = \langle\, k\,P_k, \frac{\Psi_k}{k} + \sum_J a_J\,\Psi^J\,\rangle = \langle\, P_k, \psi_k\,\rangle + \sum_J k\,a_J\,\langle\, P_k, \Psi^J\,\rangle = 1$$

since all the terms involved in the second part of the above last sum are equal to 0 due to the fact that all involved J have here length at least 2. On the other hand, for all compositions I of length ≥ 2, one has

$$\langle\, k\,P_k, \Pi^I\,\rangle = \langle\, k\,P_k, \sum_J a_J\,\Psi^J\,\rangle = \sum_J k\,a_J\,\langle\, P_k, \Psi^J\,\rangle = 0$$

since all the J involved in this last sum have length at least 2 according to Equation (3). $\diamondsuit$

Let us now give the following important description of $\mathbf{Sym}_{-k}$ in terms of quasi-differential operators.

Proposition 4.4 *Let $\Pi = (\Pi_n)_{n\geq 1}$ a sequence of Lie homogeneous idempotents such that $w(\Pi_n) = n$ for every $n \geq 1$. Then one has*

$$\mathbf{Sym}_{-k} = \bigcap_{\substack{r\geq 1\\ m_1,\ldots,m_r\in\mathbb{N}^*}} \ker D_{P_{m_1 k,\ldots,m_r k}} \ .$$

Proof. Note first that according to Proposition 4.1, it suffices to work with $\Pi = \Psi$, i.e. suppose that $\Pi_n = \Psi_n$ for every $n \geq 0$. Observe also that our claim says equivalently that the two following assertions are equivalent:

1. $F \in \mathbf{Sym}_{-k}(\Psi)$;
2. $D_{P_{m_1 k,\ldots,m_r k}}(F) = 0$ for all $m_1, \ldots, m_r \in \mathbb{N}^*$.

Let us consider a noncommutative symmetric function F such that $D_{P_{m_1 k,\ldots,m_r k}}(F) = 0$ for all $m_1, \ldots, m_r \in \mathbb{N}^*$. Such a property is clearly equivalent to the fact that

$$\langle\, D_{P_{m_1 k,\ldots,m_r k}}(F), P_I\,\rangle = 0$$

for every $m_1, \ldots, m_r \in \mathbb{N}^*$ and every composition I. By definition of the quasi-differential operator $D_{P_{m_1 k,\ldots,m_r k}}$, one however has

$$\langle\, D_{P_{m_1 k,\ldots,m_r k}}(F), P_I\,\rangle = \langle\, F, P_{m_1 k,\ldots,m_r k}\,P_I\,\rangle = \langle\, \Delta(F), P_{m_1 k,\ldots,m_r k} \otimes P_I\,\rangle\ .$$

Hence F satisfies condition 2 if and only if one has

$$\langle\, \Delta(F), P_{m_1 k,\ldots,m_r k} \otimes P_I\,\rangle = 0$$

for all integers $m_1, \ldots, m_r \in \mathbb{N}^*$ and for every composition I.

Let us now take some noncommutative symmetric function F in $\mathbf{Sym}_{-k}(\Psi)$. Then F is a sum of products of elements $\Psi[t]$ for certain compositions $t \in T_k$, i.e.

$$F = \sum_{i\in\mathcal{I}} a_i\,\Psi[t_1^{(i)}]\,\ldots\,\Psi[t_{r_i}^{(i)}]\ .$$

Since every $\Psi[t]$ is primitive for Δ, a straightforward computation leads us to

$$\Delta(F) = \sum_{u^{(i)} \sqcup\!\sqcup v^{(i)} \in t_1^{(i)} \ldots t_{r_i}^{(i)}} a_i \, \Psi[u^{(i)}] \otimes \Psi[v^{(i)}] \, .$$

This implies that $\langle \, \Delta(F), P_{m_1 k,\ldots,m_r k} \otimes P_I \, \rangle = 0$ because $\langle \, \Psi[u^{(i)}], P_{m_1 k,\ldots,m_r k} \, \rangle$ is necessarily always equal to 0 since the decomposition of the noncommutative symmetric function $\Psi[u^{(i)}]$ over the (Ψ^I) basis does not involve any term only formed of products of homogeneous noncommutative symmetric functions of weight a multiple of k.

Conversely, suppose that $\langle \, \Delta(F), P_{m_1 k,\ldots,m_r k} \otimes P_I \, \rangle = 0$ for all $m_1, \ldots, m_r \in \mathbb{N}^*$ and for every composition I. According to the Lazard's tensor decomposition of the algebra **Sym**, we can decompose F as

$$F = \sum_{i_1,\ldots,i_r \geq 1} \alpha_{(i_1,\ldots,i_r)} \, \Psi_{i_1 k} \, \ldots \, \Psi_{i_r k} \, F^{(i_1,\ldots,i_r)}$$

where every $F^{(i_1,\ldots,i_r)}$ stands for a non commutative symmetric function in $\mathbf{Sym}_{-k}$.

Suppose now that F does not belong to $\mathbf{Sym}_{-k}$. Let then $(p_1, \ldots, p_s)$ be a composition of maximal length such that a term of the form $\Psi_{p_1 k} \, \ldots \, \Psi_{p_s k} \, F^{(p_1,\ldots,p_r)}$ appears in the previous decomposition with $r > 0$. By hypothesis, one has $\langle \, \Delta(F), P_{p_1 k,\ldots,p_s k} \otimes P_I \, \rangle = 0$ for all compositions I. Note now that one has

$$\langle \, \Delta(F), P_{p_1 k,\ldots,p_s k} \otimes P_I \, \rangle = \alpha_{(p_1,\ldots,p_s)} \, \langle \, \Delta(\Psi_{p_1 k} \, \ldots \, \Psi_{p_s k}) \, \Delta(F^{(p_1,\ldots,p_s)}), P_{p_1 k,\ldots,p_s k} \otimes P_I \, \rangle$$

by maximality of $(p_1, \ldots, p_s)$. Since $\alpha_{(p_1,\ldots,p_s)}$ is different form zero, we get

$$\langle \, \Delta(\Psi_{p_1 k} \, \ldots \, \Psi_{p_s k}) \, \Delta(F^{(p_1,\ldots,p_s)}), P_{p_1 k,\ldots,p_s k} \otimes P_I \, \rangle = 0$$

for all compositions I. But the left hand side of the above identity is equal to

$$\langle \, \Psi^{p_1 k,\, \ldots,\, p_s k} \otimes F^{(p_1,\ldots,p_s)}, P_{p_1 k,\ldots,p_s k} \otimes P_I \, \rangle = \langle \, F^{(p_1,\ldots,p_s)}, P_I \, \rangle$$

since all the other terms that may appear in the image by Δ of $\Psi^{p_1 k,\, \ldots,\, p_s k} \, F^{(p_1,\ldots,p_s)}$ are canceled in the scalar product. It follows that

$$\langle \, F^{(p_1,\ldots,p_s)}, P_I \, \rangle = 0$$

for all compositions I. Hence $F^{(p_1,\ldots,p_s)} = 0$ which is impossible. This contradiction shows that F belongs to $\mathbf{Sym}_{-k}$. ♢

Remark 4.5 In the commutative case, Sym_{-k} is the intersection of the kernels of all differential operators $D_{p_{ik}}$ for all $i \geq 1$. This last result does however not hold anymore in the noncommutative case. Following the lines of the proof of our last result, one can in fact easily prove with the help of Lemma 4.3 and of Appendix 1.6.5 of [Re] that one here has

$$\bigcap_{i=1}^{+\infty} \ker D_{P_{ik}} = \mathbf{Sym}_k(\Psi, 1) \otimes \mathbf{Sym}_{-k}$$

where $\mathbf{Sym}_k(\Psi, 1)$ stands for the subalgebra of **Sym** generated by the Lie polynomials in the elements of the family $(\Psi_{ik})_{i \geq 1}$ that do not have any linear part.

4.2 The ribbon functions in $\mathbf{Sym}_{-k}$

The set of all noncommutative ribbon Schur functions form a basis of the algebra **Sym**. We want to characterize the noncommutative ribbons that belong to $\mathbf{Sym}_{-k}$.

Theorem 4.6 *Let I be a composition. Then the following conditions are equivalent:*

i) $R_I \in \mathbf{Sym}_{-k}$,

ii) $D_{P_k}(R_I) = 0$,

iii) Either I or $\overline{I}$ is k-solid,

iv) $D_{P_{m_1 k,\ldots,m_r k}}(R_I) = 0$ *for all integers* $m_1, \ldots, m_r \geq 1$.

Note first that conditions *i)* and *iv)* are equivalent according to Proposition 4.4 and that *iv)* obviously implies *ii)*. In each of the following subsections, we show the remaining implications.

4.2.1 Condition *ii)* implies condition *iii)*

Lemma 4.7 *Let I be a composition such that $D_{P_k}(R_I) = 0$. Then either I or $\overline{I}$ is k-solid.*

Proof. Suppose that $D_{P_k}(R_I) = 0$. This means that one has

$$\langle\, R_I, P_k F \,\rangle = 0$$

for all quasi-symmetric functions F. Since every symmetric function is a quasi-symmetric function, we get in particular that

$$\langle\, R_I, P_k\, f \,\rangle = 0$$

for every symmetric function f. Using now property (1) of Section 2.5, we get

$$(\, r_I, p_k\, f \,) = 0$$

for every symmetric function f (notice that $P_k = p_k$ as one can also easily deduce it from Lemma 4.3). Hence we have $D_{p_k}(r_I) = 0$ and we can conclude to our Lemma by using the results of the previous section. ♢

4.2.2 Condition *iii)* implies condition *iv)*

Since the family (P_K) form a linear basis of the algebra of the quasi-symmetric functions, condition *iv)* is equivalent to the fact that

$$\langle\, R_I, P_{m_1 k,\ldots,m_r k}\, P_K \,\rangle = 0\ ,$$

for all integers $m_1, \ldots, m_r \geq 1$ and for every composition K. This last condition is itself clearly equivalent to the fact that one has

$$\langle\, \Delta(R_I), P_{m_1 k,\ldots,m_r k} \otimes P_K \,\rangle = 0$$

for all integers $m_1, \ldots, m_r \geq 1$ and for every composition K. Hence, to prove that condition *iii)* implies condition *iv)*, it suffices to prove the following lemma.

Lemma 4.8 *Let I be a composition such that either I or $\overline{I}$ is k-solid. Then one has*

$$\langle\, \Delta(R_I), P_{m_1 k,\dots,m_r k} \otimes P_K \,\rangle = 0$$

for all integers $m_1, \dots, m_r \geq 1$ and for every composition K.

Proof. Let I be a composition such that either I or $\overline{I}$ is k-solid. By definition of a noncommutative ribbon Schur function, note first that one has

$$\Delta(R_I) = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)}\, \Delta(S^J)\ .$$

Hence it suffices to prove that one has

$$\sum_{J \preceq I} (-1)^{\ell(J)}\, \langle\, \Delta(S^J), P_{m_1 k,\dots,m_r k} \otimes P_K \,\rangle = 0 \qquad (4)$$

for every composition K. Recall now that for all composition $J = (j_1, \dots, j_s)$, one has

$$\Delta(S^J) = \Delta(S_{j_1}) \,\dots\, \Delta(S_{j_s})\ .$$

By definition of Δ, one gets

$$\Delta(S^J) = \sum_{l_1=0}^{j_1} \dots \sum_{l_s=0}^{j_s} S_{l_1} \dots S_{l_s} \otimes S_{j_1-l_1} \dots S_{j_s-l_s}\ .$$

By substituting this last expression in (4), we see that we have to show that

$$\sum_{J \preceq I} (-1)^{\ell(J)} \,\langle\, \sum_{l_1=0}^{j_1} \dots \sum_{l_s=0}^{j_s} S_{l_1} \dots S_{l_s} \otimes S_{j_1-l_1} \dots S_{j_s-l_s}, P_{m_1 k,\dots,m_r k} \otimes P_K \,\rangle$$

$$= \sum_{J \preceq I} (-1)^{\ell(J)} \sum_{l_1=0}^{j_1} \dots \sum_{l_s=0}^{j_s} \langle\, S_{l_1} \dots S_{l_s}, P_{m_1 k,\dots,m_r k} \,\rangle\, \langle\, S_{j_1-l_1} \dots S_{j_s-l_s}, P_K \,\rangle = 0$$

for every composition K. Using the expansion of the noncommutative complete symmetric functions on the basis of noncommutative power sums of the first kind (cf [G-T]), one gets

$$S^L = \sum_{C \geq L} \alpha_{C,L}\, \Psi^C$$

for every composition L where $\alpha_{C,L}$ is some rational number that depends on the composition L and its refinement C. Therefore the only terms of the last previous sum that are different from zero are those in which the l_j's that are not equal to zero form a composition which is an anti-refinement of the composition $M = (m_1 k, \dots, m_r k)$. Hence we have to show that

$$\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{L \leq M} \alpha_{M,L} \left(\sum_{J'} \langle\, S_{J'}, P_K \,\rangle \right)\right) = 0$$

for every composition K, where the most internal sum in the above expression is taken over all compositions J' of the integer $n - (m_1 + \dots + m_r)k$ which can be obtained from J in the following way:

1. take a sequence of non negative integers $l_1, \ldots, l_s$ such that the l_j's that are not equal to zero form a composition L which is an anti-refinement of the composition $M = (m_1k, \ldots, m_rk)$ and such that $l_t \leq j_t$ for all $t \in [1, s]$,

2. form then J' by subtracting in J the integer l_t from j_t for all $t \in [1, s]$ (if a part j_t of J is equal to l_t, then this part must "disappears" in J' when l_t is subtracted).

Our last identity is therefore equivalent to the fact that

$$\sum_{L \leq M} \alpha_{M,L} \left(\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{J'} \langle S_{J'}, P_K \rangle \right) \right) = 0$$

for every composition K. Of course it suffices to prove that one has

$$\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{J'} \langle S_{J'}, P_K \rangle \right) = 0$$

for every composition K. Now any composition L which is an anti-refinement of $M = (m_1k, \ldots, m_rk)$ is still a composition of the form $(n_1k, \ldots, n_qk)$. Therefore it suffices to prove that if I is a k-solid composition or the mirror image of a k-solid composition and if $(m_1k, \ldots, m_rk)$ is a sequence of multiples of k, then the identity

$$\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{J'} \langle S_{J'}, P_K \rangle \right) = 0 \qquad (5)$$

holds for every composition K. Let us now recall that J' is here any composition obtained by taking any subsequence $(j_{l_1}, \ldots, j_{l_r})$ of r parts of J such that $j_{l_t} \geq m_tk$ for all $t \in [1, r]$ and by subtracting the integer m_tk from j_{l_t} for all $t \in [1, r]$ (if a part j_{l_t} of J is equal to m_tk, then this part "disappears" in J' after the subtraction).

The fact that Equation (5) holds for every composition K is clearly equivalent to the following identity

$$\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{J'} S_{J'} \right) = 0 \qquad (6)$$

where the internal sum is taken over all compositions J' of $n - (m_1 + \ldots + m_r)k$ obtained using the process described above. This last property is then an immediate consequence of the next lemma 4.13 which says that

$$\sum_{J \preceq I} (-1)^{\ell(J)} \left(\sum_{J'} J' \right)$$

is identically equal to zero when I is either a k-solid or the mirror image a of k-solid composition. This result might be clearer with the following example.

Example 4.9 Let us consider the 3-solid composition $I = (1, 1, 2, 3, 3)$. The anti-refinements of I form then the following set

$$\{ (1,1,2,3,3), (2,2,3,3), (1,3,3,3), (1,1,5,3), (1,1,2,6), (4,3,3), (2,5,3), (2,2,6),$$

$$(1,6,3),(1,3,6),(1,1,8),(7,3),(4,6),(2,8),(1,9),(10)\,\} \ .$$

Consider the sequence $M = (3,3)$. By subtracting the integers of this sequence from the anti-refinements of I (where this is possible), we obtain the followed signed compositions:

$$\begin{array}{lll}
(1,1,2,3,3) & \longrightarrow & -(1,1,2) \\
(2,2,3,3) & \longrightarrow & (2,2) \\
(1,3,3,3) & \longrightarrow & (1,3)+(1,3)+(1,3) \quad \text{(3 times!)} \\
(1,1,5,3) & \longrightarrow & (1,1,2) \\
(4,3,3) & \longrightarrow & -(1,3)-(1,3)-(4) \\
(2,5,3) & \longrightarrow & -(2,2) \\
(1,6,3) & \longrightarrow & -(1,3) \\
(1,3,6) & \longrightarrow & -(1,3) \\
(7,3) & \longrightarrow & (4) \\
(4,6) & \longrightarrow & (1,3)
\end{array}$$

It is straightforward to check that the sum of these signed compositions is identically equal to zero as required.

Before stating the next lemma, we need to introduce some notations.

Definition 4.10 *Let $J=(j_1,\ldots,j_s)$ be a composition, let k and r be two positive integers with $r \le s$. Let $m_1,\ldots,m_r$ be another sequence of positive integers and let $l_1 < \cdots < l_r$ be elements of $[1,s]$ such that $j_{l_t} \ge m_t k$ for all $t \in [1,r]$. We note then by*

$$(J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk}$$

the composition obtained from J by subtracting the integer m_tk from its l_t-th part for all $t \in [1,r]$ (if $j_{l_t} = m_tk$, then the l_t-th part is erased from J).

Example 4.11 One has for instance

$$((1,3,8,6,7);2,4,5)^{\frown 2,4,4} = (1,1,8,2,3) \quad \text{and} \quad ((1,3,8,6,7);2,4,5)^{\frown 2,6,4} = (1,1,8,3) \ .$$

Definition 4.12 *Let I be a composition and let $m_1,\ldots,m_r$ be another fixed sequence of positive integers. We note then by $\mathcal{E}_I$ the set which is equal to*

$$\{\,(J;l_1,\ldots,l_r),\ J=(j_1,\ldots,j_s) \preceq I,\ 1 \le l_1 < \cdots < l_r \le s \text{ and } j_{l_t} \ge m_tk \text{ for all } t \in [1,r]\,\} \ .$$

The elements of the set $\mathcal{E}_I$ defined above are pairs made of a composition and a sequence of indices corresponding to parts of the composition from which it is possible to subtract the integers m_tk. Notice that the above set also depends on the sequence $(m_1,\ldots,m_r)$. We however decided to omit this dependence in the notation $\mathcal{E}_I$ for clearness.

Lemma 4.13 *Let I be a composition of n such that either I or $\overline{I}$ is k-solid and let $m_1,\ldots,m_r$ be a fixed sequence of positive integers. Then the formal sum*

$$\sum_{(J;l_1,\ldots,l_r)\in\mathcal{E}_I} (-1)^{\ell(J)}\ (J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk} \qquad (7)$$

is identically equal to 0.

Proof. Up to taking the mirror images of all the compositions that appear in (7), we can suppose that I is a composition such that $\overline{I}$ satisfies the conditions of Corollary 3.9 (i.e. such that $\overline{I}$ is k-solid). Let us now introduce the two sets $\mathcal{E}_{I=}$ and $\mathcal{E}_{I>}$ defined by

$$\begin{cases} \mathcal{E}_{I=} &= \{ (J; l_1, l_2, \dots, l_r) \in \mathcal{E}_I \textit{ such that } j_{l_r} = m_r k \}, \\ \mathcal{E}_{I>} &= \{ (J; l_1, l_2, \dots, l_r) \in \mathcal{E}_I \textit{ such that } j_{l_r} > m_r k \}, \end{cases}$$

These two sets are subsets of $\mathcal{E}_I$ and they are clearly the complement of each other. Consider then the function f from $\mathcal{E}_{I=}$ into $\mathcal{E}_{I>}$ defined by setting

$$f(J; l_1, \dots, l_r) = ((j_1, \dots, j_{l_r-1}, j_{l_r} + j_{l_r+1}, j_{l_r+2}, \dots, j_s); l_1, \dots, l_r)$$

for every composition $J = (j_1, j_2, \dots, j_s)$ of $\mathcal{E}_{I=}$. The function f is well defined. Indeed one has always $l_r \neq s$ since if $l_r = s$ and $j_s = m_r k$, then J would pass through $n - m_r k$. This is not possible because, by the characterization of k-solid and mirror images of k-solid compositions, I does not pass through $n - m_r k$ and I is finer than J by definition.

The function f is injective. Indeed let $(J; l_1, \dots, l_r)$ and $(J'; l'_1, \dots, l'_r)$ be two elements of $\mathcal{E}_{I=}$ such that $f(J; l_1, \dots, l_r) = f(J'; l'_1, \dots, l'_r)$. Then one has

$$((j_1, \dots, j_{l_r} + j_{l_r+1}, \dots, j_s), l_1, \dots, l_r) = ((j'_1, \dots, j'_{l'_r} + j'_{l'_r+1}, \dots, j'_q), l'_1, \dots, l'_r)$$

if we set $J = (j_1, \dots, j_s)$ and $J' = (j'_1, \dots, j'_q)$. This implies that $s = q$ and that $l_t = l'_t$ for all $t \in [1, r]$. Moreover one has $j_t = j'_t$ for all $t \neq l_r, l_r + 1$. Note now that we have $j_{l_r} = j'_{l_r} = m_r k$ (since we are working with elements of $\mathcal{E}_{I=}$) which allows us to conclude that $j_{l_r+1} = j'_{l_r+1}$ since $j_{l_r} + j_{l_r+1} = j'_{l_r} + j'_{l_r+1}$. Thus $J = J'$ which ends proving the injectivity of the function f.

The function f is also surjective. Let $(J; l_1, \dots, l_r)$ be an element of $\mathcal{E}_{I>}$. Suppose that $J = (j_1, \dots, j_{l_r-1}, j_{l_r}, \dots, j_s)$ and let J' be defined as

$$J' = (j_1, \dots, j_{l_r-1}, m_r k, j_{l_r} - m_r k, \dots, j_s) \ .$$

The composition J' has $s + 1$ parts. Let us first show that it is a composition less fine than I. Two cases are now to be considered:

- if $l_r > 1$, the composition J is less fine than I and passes through $j_1 + \ldots + j_{l_r-1}$. Hence I also passes through $j_1 + \ldots + j_{l_r-1}$. By Corollary 3.9, if I passes through an integer m, it also passes through the integer $m + lk$ for all l such that $m + lk \leq n$. Thus I passes through $j_1 + \ldots + j_{l_r-1} + m_r k$ which implies that J' is less fine than I.

- if $l_r = 1$, the composition J' is also less fine than I because I is the mirror image of a k solid composition and hence passes through all the multiples of k that are smaller than n. In particular, I passes through $m_r k$.

This shows that $J' \preceq I$. Hence $(J'; l_1, \dots, l_r)$ belongs to $\mathcal{E}_{I=}$. The fact that f is a surjection follows then immediately since $f(J'; l_1, \dots, l_r) = (J; l_1, \dots, l_r)$.

Thus f is a bijection between $\mathcal{E}_{I=}$ and $\mathcal{E}_{I>}$. This allows us to write

$$\begin{aligned}
&\sum_{(J;l_1,\ldots,l_r)\in\mathcal{E}_I} (-1)^{\ell(J)}\ (J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk} \\
= &\sum_{(J;l_1,\ldots,l_r)\in\mathcal{E}_{I=}} (-1)^{\ell(J)}\,(J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk} \\
+ &\sum_{(J;l_1,\ldots,l_r)\in\mathcal{E}_{I>}} (-1)^{\ell(J)}\,(J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk} \\
= &\sum_{(J;l_1,\ldots,l_r)\in\mathcal{E}_{I=}} (-1)^{\ell(J)}\,(J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk} \\
+ &\sum_{(J;l_1,\ldots,l_r)\in\mathcal{E}_{I=}} (-1)^{\ell(J')}\,(f(J;l_1,\ldots,l_r))^{\frown m_1k,\ldots,m_rk}\ .
\end{aligned}$$

Let us set $(J';l_1,\ldots,l_r) = f(J;l_1,\ldots,l_r)$. It suffices now to notice that $\ell(J') = \ell(J)-1$ and that

$$(J;l_1,\ldots,l_r)^{\frown m_1k,\ldots,m_rk} = (f(J;l_1,\ldots,l_r))^{\frown m_1k,\ldots,m_rk}$$

in order to conclude to our lemma. ◊

Using the last lemma, we immediately get equation (6). This ends therefore the proof of Lemma 4.8 and by way of consequence the proof of Theorem 4.6. ◊

4.3 2-solid ribbons

Theorem 3.2 characterizes the k-solid ribbons. In the particular case $k=2$, one deduces that the only 2-solid ribbons are the compositions of the form $1\,2^m$ for all m in $\mathbb{N}$. The following theorem is an interesting result about the generating series of the 2-solid ribbons.

Theorem 4.14 *The formal series*

$$\mathcal{R} = \sum_{m=0}^{+\infty} R_{1\,2^m}$$

is the tangent of a Lie series in the elements of the family $\Psi = (\Psi_n)_{n\geq 1}$.

Proof. Let us denote by $\mathcal{L}$ the series defined by

$$\mathcal{L} = \arctan(\mathcal{R}) = \sum_{j\geq 0} \frac{(-1)^j}{2j+1}\,\mathcal{R}^{2j+1}\ . \tag{8}$$

The assertion to prove is equivalent to show that $\mathcal{L}$ is a Lie series in the elements of the family Ψ. Note first that using a classical expression of arctan, we immediately get

$$\mathcal{L} = \frac{1}{2i}\log\frac{1+i\mathcal{R}}{1-i\mathcal{R}}\ .$$

It is a well known fact (see for instance [Re], Theorem 3.2) that $\mathcal{L}$ is a Lie series in the elements of the family Ψ if and only if the series

$$\frac{1+i\mathcal{R}}{1-i\mathcal{R}}$$

is a group like element for the comultiplication Δ, i.e. if and only if one has

$$\Delta\left(\frac{1+i\mathcal{R}}{1-i\mathcal{R}}\right) = \frac{1+i\mathcal{R}}{1-i\mathcal{R}} \otimes \frac{1+i\mathcal{R}}{1-i\mathcal{R}} \ .$$

According to Proposition 5.23 of [G-T], we have

$$\mathcal{R} = \left(\sum_{k\geq 0} (-1)^k S_{2k+1} \right) \left(\sum_{k\geq 0} (-1)^k S_{2k} \right)^{-1} ,$$

i.e. $\mathcal{R} = \mathcal{I}\mathcal{P}^{-1}$ where we set

$$\mathcal{I} = \sum_{k\geq 0} (-1)^k \, S_{2k+1} \quad \text{and} \quad \mathcal{P} = \sum_{k\geq 0} (-1)^k \, S_{2k} \ .$$

We are then left with proving that

$$\Delta\left(\frac{1+i\mathcal{I}\mathcal{P}^{-1}}{1-i\mathcal{I}\mathcal{P}^{-1}}\right) = \frac{1+i\mathcal{I}\mathcal{P}^{-1}}{1-i\mathcal{I}\mathcal{P}^{-1}} \otimes \frac{1+i\mathcal{I}\mathcal{P}^{-1}}{1-i\mathcal{I}\mathcal{P}^{-1}} \ ,$$

i.e. that the series

$$\frac{1+i\mathcal{I}\mathcal{P}^{-1}}{1-i\mathcal{I}\mathcal{P}^{-1}}$$

is group like for Δ. Note now that this last expression is equal to

$$(1+i\mathcal{I}\mathcal{P}^{-1})(1-i\mathcal{I}\mathcal{P}^{-1})^{-1} = (\mathcal{P}+i\mathcal{I})\mathcal{P}^{-1}((\mathcal{P}-i\mathcal{I})\mathcal{P}^{-1})^{-1} = (\mathcal{P}+i\mathcal{I})(\mathcal{P}-i\mathcal{I})^{-1} \ .$$

But we can write

$$\begin{aligned} \mathcal{P}+i\mathcal{I} &= \sum_{k\geq 0} (-1)^k \, S_{2k} + i \sum_{k\geq 0} (-1)^k \, S_{2k+1} \\ &= \sum_{k\geq 0} i^{2k} \, S_{2k} + \sum_{k\geq 0} i^{2k+1} \, S_{2k+1} \\ &= \sum_{k\geq 0} i^k \, S_k. \end{aligned}$$

We can also show in the same way that one has

$$\mathcal{P}-i\mathcal{I} = \sum_{k\geq 0} (-i)^k \, S_k \ .$$

Since the series

$$\sigma(t) = \sum_{k\geq 0} S_k \, t^k$$

is a group like element for Δ, we get that

$$(\mathcal{P}+i\mathcal{I})\,(\mathcal{P}-i\mathcal{I})^{-1} = \sigma(i)\,\sigma(-i)^{-1}$$

is also a group like element for Δ, which ends our proof. ◊

Remark 4.15 Let us take again the notations of the proof of our last result. We saw that

$$\mathcal{P} + i\mathcal{I} = \sigma(i) \quad \text{and} \quad (\mathcal{P} - i\mathcal{I})^{-1} = (\sigma(-i))^{-1} \ .$$

Since $\sigma(t) = \exp(\Phi(t))$, we can then write

$$\mathcal{L} = \frac{1}{2i} \log[\exp(\Phi(i)) \exp(-\Phi(-i))] = \frac{1}{2i} H(\Phi(i), -\Phi(-i))$$

where $H(a,b) = \log[\exp(a)\exp(b)]$ is the Hausdorff series in two variables. This gives us an interesting Hausdorff-type expression for the Lie series $\mathcal{L}$ defined by Equation (8).

5 Enumeration

The main result of this section will allow us to count the compositions I of weight n such that $D_{P_k}(R_I) = 0$ or equivalently such that I or $\overline{I}$ is k-solid according to Theorem 4.6. Note first that I and $\overline{I}$ are certainly different in such a case since Lemma 2.13 shows that I and $\overline{I}$ cannot be both k-solid compositions. Note also that Corollary 3.8 shows that n is necessarily not a multiple of k in the situation that we are studying.

Theorem 5.1 *Let n and k be two integers such that $n \bmod k \neq 0$ and let $a_{n,k}$ be the number of compositions I of n which are either k-solid or mirror images of k-solid compositions (i.e. such that $D_{P_k}(R_I) = 0$). Then one has*

$$a_{n,k} = 2\,(q+2)^{r-1}(q+1)^{k-r-1}$$

where q and r are the quotient and the remaining part of the euclidean division of n by k.

Proof. According to the preliminary remarks, one has $a_{n,k} = 2\,b_{n,k}$ where $b_{n,k}$ stands for the number of mirror images of k-solid compositions of n. Suppose now that $n > k$. Then these last compositions are exactly the compositions I of n that satisfy the following properties:

- I passes through k,
- I does not pass through $n - k$,
- if I passes through i, then I passes through $i + k$ if $i + k \leq n$.

Using the usual bijection between compositions of n and subsets of $\{1, \ldots, n\}$ which contain the element n, we see that $b_{n,k}$ is exactly the number of subsets E of $\{1, \ldots, n\}$ which have the following properties:

- $k \in E$,
- $n - k \notin E$,
- if $i \in E$ and $i + k \leq n$, then $i + k \in E$.

Notice that the first and third conditions imply that $2k, \ldots, qk \in E$ whereas the second and the third conditions imply that $n-2k, \ldots, n-qk = r \notin E$. Moreover, if an integer j belongs to E, all the integers $j + lk$ are in E, as long as $j + lk \leq n$. Hence such a set E is uniquely determined by the set

$$\varphi(E) = \{\ j,\ j \neq k,\ j \in E \text{ and } (j-k \notin E \text{ or } j-k \leq 0)\ \}\ .$$

Indeed an integer i belongs to E if and only if there exists $j \in \varphi(E)$ and $l \in [0, q]$ such that $i = j + lk$.

Let us introduce some more new notations. We can decompose every element $j \in \varphi(X)$ as $j = q_j k + r_j$ where one has $0 \leq r_j < k$ and

$$\begin{cases} 0 \leq q_j \leq q & \text{if } 0 \leq r_j \leq r, \\ 0 \leq q_j \leq q-1 & \text{if } r < r_j < k. \end{cases}$$

Note that the integers r_j are never 0 because otherwise j would be a multiple of k, say lk, with $l > 1$ and we would have $lk - k \notin E$. Note also that the integers r_j are never equal to r since otherwise j would be equal to $n - kl$ for some l which is impossible according to a previous remark. Finally observe that, if t and s are in $\varphi(E)$ with $t \neq s$, then r_t and r_s are distinct. Indeed if we had $r_t = r_s$ and $r > s$, then $t = s + lk$ for some $l \geq 1$. But $t - k \notin E$ because $r \in \varphi(E)$. Hence $s = t - lk \notin X$, which is not possible. This contradiction proves the assertion.

It follows that $\varphi(E)$ is uniquely determined by the pair of partial functions

$$f_1 : \{1, \ldots, r-1\} \to \{0, \ldots, q\}\ ,$$

$$f_2 : \{r+1, \ldots, k-1\} \to \{0, \ldots, q-1\}$$

defined by setting

$$f_1(r_j) = q_j,\ \text{for every } r_j \in \{1, \ldots, r-1\},$$

$$f_2(r_j) = q_j,\ \text{for every } r_j \in \{r+1, \ldots, k-1\}\ .$$

These two functions are only partial since there might not exist any $j \in \varphi(E)$ such that $r_j = i$ for some $i \in \{1, \ldots, r-1\} \cup \{r+1, \ldots, k-1\}$. Note now that there are exactly $(q+2)^{r-1}$ such functions f_1 and $(q+1)^{k-r-1}$ such functions f_2. Hence, $b_{n,k} = (q+2)^{r-1}(q+1)^{k-r-1}$ when one has $n > k$.

To complete the proof note that if $n < k$, all the compositions of weight n are k-solid. Hence $b_{n,k} = 2^{n-1}$ in this case. This is however exactly the formula to prove here since $q = 0$ and $r = n$ in this case. ♢

Acknowledgments

The authors would like to thank Michel Morvan for his helpful hints and collaboration in the enumeration of k-solid ribbons.

References

[B] N. Bourbaki, *Algèbres et Groupes de Lie*, Chapters 2 and 3, CCLS, 1972.

[CR] C. W. Curtis, I. Reiner, *Representation Theory of finite groups and associative algebras*, Wiley, 1988.

[G-T] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. **112**, (1995), 218-348.

[Ge] I. Gessel, *Multipartite P-partitions and inner product of skew Schur functions*, Contemp. Math. **34**, (1984), 289-301.

[KT1] D. Krob, J.Y. Thibon, *Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at* $q = 0$, J. of Alg. Comb., **6**, (4), 339–376, 1997.

[KT2] D. Krob, J.Y. Thibon, *A crystalizable version of* $U_q(Gl_n)$, in "Proceedings of the 9-th Formal Power Series and Algebraic Combinatorics conference" (C. Krattenthaler, Ed.), Universität Wien, (2), 326-338, 1997.

[KT3] D. Krob, J.Y. Thibon, *Noncommutative symmetric functions V: A crystalizable version of* $U_q(Gl_n)$, Int. J. of Alg. and Comput. (to appear).

[LS] A. Lascoux, M. P. Schützenberger, *Formulaire raisonné des fonctions symétriques*, Publication L.I.T.P., Université Paris 7, Oct. 1985.

[Macd] I. G. Macdonald, *Symmetric functions and Hall Polynomials*, Oxford Math. Monographs, Oxford University Press, 2nd Ed., 1994.

[MvR] C. Malvenuto, C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177**, (1995), 967-982.

[R] G. de B. Robinson, *Representation Theory of the Symmetric Group*, University of Toronto Press, 1961.

[Re] C. Reutenauer, *Lie Algebras*, Oxford University Press, 1993.