Recognizable Subsets of the Two Letter Plactic Monoid

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Abstract

The plactic monoid $Pla(a,b)$ on two letters will be studied from the point of view of classical language theory. In particular, we will give the fine structure of its recognizable subsets.

1 Introduction

The history of the plactic monoid can be traced back to Schensted's paper of 1961 [**?**] in which an algorithm was given that allows to find the length of the largest increasing subsequence of a sequence of integers. This algorithm sets up a correspondence between words w over some totally ordered alphabet A and Young tableaux. The Young tableau $P(w)$ associated with a word by Schensted's algorithm is called its insertion tableau and the length of its last row is the length of the longest increasing subsequence of the word.

In 1970, Knuth proved that the insertion tableaux of two words u and v coincide if and only if these words are equivalent with respect to the congruence of A^* generated by the relations

$$\begin{cases} aba \equiv baa, \quad bba \equiv bab \quad \text{when } a < b \\ acb \equiv cab, \quad bca \equiv bac \quad \text{when } a < b < c. \end{cases} \tag{1}$$

However, Knuth did not exploit the fact that these relations were the defining relations of a monoid, a feat which was performed by Lascoux and Schützenberger in [**?**]. They decided to call it plactic monoid and put it to good use in one of the

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first proofs of the so called Richardson-Littlewood rule. The continued development of the theory of the plactic monoid turned it into a major tool in a variety of combinatorial contexts, such as the Kosta-Foulkes polynomials, charge and cocharge of Young tableaux etc.

In more recent developments it turned out that the plactic monoid also figures prominently in the theory of quantum groups mainly due to the strong relationship, discovered by Jimbo and Miwa, between the usual Robinson-Schensted correspondence and quantum $U_q(gl_n)$. Leclerc and Thibon even managed to obtain a quantum characterization of the plactic monoid in [**?**].

Thus it becomes clear that the plactic monoid is an object worthy of further attention. In this paper, we study it from the classical point of view of language theory. In particular, we give the fine structure of the recognizable subsets of the plactic monoid on two letters. Due to the fact that it has a cross section of entropy zero, the recognizable subsets are of a particularly simple form.

2 Generalities

2.1 Rational and Recognizable Subsets of a Monoid

Let $\mathcal{M}$ be a monoid. A subset R of $\mathcal{M}$ will be called recognizable if there exists a finite monoid $\mathcal{N}$ and a monoid morphism $\phi : \mathcal{M} \to \mathcal{N}$ such that $\phi^{-1}(\phi(R)) = R$ (cf. [**?**]). A subset R of $\mathcal{M}$ is rational if it can be constructed in a finite number of steps, each of which consists of choosing a one element subset $\{m\} \subseteq \mathcal{M}$ or of forming the union, product or the star of sets obtained in previous steps. The set of rational subsets of $\mathcal{M}$ will be denoted by $Rat(\mathcal{M})$ whereas the set of recognizable subsets will be denoted by $Rec(\mathcal{M})$. In general there is no strong relationship between the two sets. In some cases of practical relevance, however, such a relationship exists.

Proposition 2.1 *Let $\mathcal{M}$ be a monoid.*

1. *If $\mathcal{M}$ is finitely generated then $Rec(\mathcal{M}) \subseteq Rat(\mathcal{M})$.*
2. *If $\mathcal{M}$ is free then $Rec(\mathcal{M}) = Rat(\mathcal{M})$.*

In order to determine whether or not a given subset of a monoid is recognizable, it often pays to consider the residuals of the set.

Definition 2.2 *Let $\mathcal{M}$ be a monoid, $L \subseteq \mathcal{M}$ and $m \in \mathcal{M}$. Then the set*

$$m^{-1}L = \{x \in \mathcal{M} | mx \in L\}$$

is called a (left) residual of L.

Note that $(m_1m_2)^{-1}L = m_2^{-1}(m_1^{-1}L)$. The fact relevant to the determination of the recognizability of a set L is the following.

Proposition 2.3 *Let $\mathcal{M}$ be a monoid and $L \subseteq \mathcal{M}$. Then L is recognizable if and only if $\{m^{-1}L | m \in \mathcal{M}\}$ is a finite set.*

2.2 Cross Sections and Entropy

If A is an alphabet, R a family of pairs of elements of A^* and $\equiv$ is the congruence defined by R on A, then a language $S \subseteq A^*$ is said to be a *cross section* of the quotient monoid $M = A^*/\equiv$ if and only if for each $w \in A^*$ there is a unique $s \in S$ such that $w \equiv s$. In particular, two elements $s, t \in S$ are equivalent with respect to $\equiv$ iff and only if they are equal. Furthermore, a cross section S will be called rational if S is a rational subset of A^*. Cross sections are important tools for the description of quotient monoids since they give a representative for each equivalence class of $\equiv$ and thus can be viewed as elements of the quotient monoid $A^*/\equiv$. Thus, it is often necessary to decide whether or not a given set constitutes a cross section of a certain monoid. This can be achieved by defining an action $S \times A^* \to S$ and verifying certain properties of this action.

Lemma 2.4 *If there exists an action* $. : S \times A^* \to S$ *of* A^* *on* S *such that*

1. $s.u_i = s.v_i$ *for all* $(u_i, v_i) \in R$,
2. $s.1_{A^*} = s$ *for all* $s \in S$,
3. $(s.u).v = s.uv$ *for all* $s \in S$ *and* $u, v \in A^*$,

then S *is a cross section of* $A^*/\equiv$, *where* $\equiv$ *is the congruence on* A^* *generated by* R.

The idea behind this action is that for a cross section S, $s \in S$ and $w \in A^*$ there is a unique element $s' \in S$ such that $s' \equiv sw$. This element s' is then the result of the action, that is $s' = s.w$.

If the action from $S \times A^*$ into S is extended into a mapping from $A^* \times A^*$ into A^* by defining $u.v$ to be the element in S equivalent to uv then the following is true, where, for a word $u \in A^*$, nor necessarily in S, and a set $X \subseteq S$, $u^1 X = \{v \in S \mid u.v \in X\}$.

Lemma 2.5 *Let* $X \subseteq S$ *and* u, $v \in A^*$. *If* $u \equiv v$ *then* $u^{-1}X = v^{-1}X$.

Rational cross sections of slow growth have a particular impact on the form of the recognizable sets of a monoid.

Definition 2.6 *Let* P *be a subset of* A^*. *The entropy of* P *is*

$$e(P) = \limsup_{n \to \infty} \frac{\log |P \cap A^n|}{n}.$$

Zero entropy of the star of a language L has a marked effect since it prohibits the existence of free monoids on more than one generator inside L^*. More specifically:

Lemma 2.7 *Let* L *be a non-empty subset of* A^*. *Then the following are equivalent.*

1. *$e(L^*) = 0$.*
2. *There exists $t \in A^*$ and $I \subseteq \mathbb{N}$ such that $L^* = \{t^i | i \in I\}$.*
3. *There exists $t \in A^*$, n, $m \in \mathbb{N}$ and a finite subset I of $\mathbb{N}$ such that*

$$L^* = \left(\bigcup_{i \in I} t^i \right) \cup t^m (t^n)^*.$$

Proof: The equivalence between 2 and 3 is the classical characterization of one letter rational languages. Part 3 implies part 1 quite obviously, whereas part 2 follows from part 1 by the observation that an entropy 0 subset of A^* cannot contain a free monoid on more than one generator. □

Proposition 2.8 *If $M = A^*/\equiv$ has a rational cross section of entropy 0 then the recognizable subsets of M are finite unions of finite products of words and stars of words.*

Proof: Let $X \subseteq M$ be recognizable and suppose $\pi : A^* \to M$ is the natural projection. If S is a rational cross section, then $X = \pi\left(\pi^{-1}(X)\right) = \pi\left(\pi^{-1}(X) \cap S\right)$. Since inverse images of recognizable languages are recognizable, $\pi^{-1}(X)$ is recognizable. Furthermore, since A^* is finitely generated and free, the notions of recognizability and rationality coincide, hence $\pi^{-1}(X) \cap S$ is rational in A^*. Since it is contained in S, its entropy is zero and its stars can thus be decomposed according to Lemma **??**. Hence, $\pi^{-1}(X) \cap S$ can be written as a finite union of finite products of words and stars of words. But then its image X is of this form as well since $\pi(B \cup C) = \pi(B) \cup \pi(C)$, $\pi(BC) = \pi(B)\pi(C)$ and $\pi(C^*) = \pi(C)^*$. □

2.3 The Plactic Monoid

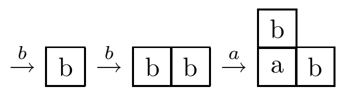
As mentioned in the introduction, the plactic monoid is closely related to Young tableaux via Schensted's algorithm. So a few explanations with regard to that subject are in order here. Let $(\lambda_1, \ldots, \lambda_r)$ be a partition of a positive integer n, that is a sequence of positive integers such that $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_r$ and $\sum_i \lambda_i = n$. Then the Ferrers diagram associated with this sequence consists of rows of boxes with λ_i boxes in the $r - i + 1$-st row for $1 \leq i \leq r$. A Young tableau is a Ferrers diagram where the boxes are filled by the elements of a totally ordered alphabet in such a way that the elements increase from left to right in each row and strictly decrease from top to bottom in each column. Schensted's algorithm assigns to each word w of a language A^* a Young tableau, called its insertion tableau $P(w)$, as follows.

1. The first letter a_1 of w is put in a box.

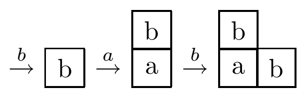
2. If the n-th letter a_n of w is greater or equal to every letter in the last row of the insertion tableau of $a_1 a_2 \dots a_{n-1}$ then put it in a box and adjoin it to the right of the last row of this last tableau.

3. Otherwise find the leftmost letter a_l in the last row of $P(a_1 \dots a_{n-1})$ which is strictly greater than a_n, put a_n into a_l's box and do step 2 with a_l starting in the row above the last row.

Thus the insertion tableau of bba with $a < b$ is constructed in the following way.



The insertion tableau for bab is obtained as follows.



Thus, different words may lead to the same insertion tableau, suggesting the introduction of a relation on the words of A^* that takes account of this phenomenon. It is clear that the relation on A^* defined by $w \cong u$ if and only if $P(w) = P(u)$ is an equivalence relation. Knuth proved that this equivalence relation is, in fact, the congruence generated by

$$\begin{cases} aba \equiv baa, \quad bba \equiv bab & \text{for every pair } a, b \in A \text{ with } a < b \\ acb \equiv cab, \quad bca \equiv bac & \text{for every triple } a, b, c \in A \text{ with } a < b < c. \end{cases} \tag{2}$$

Thus, the plactic monoid $Pla(A)$ on the alphabet A is defined to be the quotient of A^* by this congruence relation.

Since the map P from A^* to Young tableaux is onto, there exist right inverses of P from the Young tableaux to A^* whose images will yield cross sections of the plactic monoid $Pla(A)$. The two most obvious ones are obtained by concatenating the rows or the columns, respectively, of a Young tableau. The cross section obtained from concatenating the columns yields a rational cross section of the plactic monoid (cf [**?**]). In the case of the plactic monoid $Pla(a, b)$ this cross section has the form $S = (ba)^* a^* b^*$. The action $S \times A^* \to S$ related to this cross section by Lemma **??** is given by

$$(ba)^i a^j b^k . a = \begin{cases} (ba)^{i+1} a^j b^{k-1} & \text{if} \quad k \geq 1 \\ (ba)^i a^{j+1} & \text{if} \quad k = 0 \end{cases}$$

and

$$(ba)^i a^j b^k . b = (ba)^i a^j b^{k+1}$$

on the generators of A^*. It is easy to check that

$$(ba)^i a^j b^k.bba = (ba)^i a^j b^k.bab = (ba)^{i+1} a^j b^{k+1}$$

and

$$(ba)^i a^j b^k.aba = (ba)^i a^j b^k.baa = \begin{cases} (ba)^{i+2} a^j b^{k-1} & \text{if} \quad k \geq 1 \\ (ba)^{i+1} a^{j+1} & \text{if} \quad k = 0 \end{cases}$$

Note that the restriction of this action to $S \times S \to S$ defines a monoid structure on S which turns S into a monoid isomorphic to $Pla(a,b)$. Thus $Pla(a,b)$ will be identified with $(S,.)$ in the sequel. Note that

$$(ba)^i a^j b^k.b^n = (ba)^i a^j b^{k+n},$$
$$(ba)^m a^n.(ba)^i a^j b^k = (ba)^{i+m} a^{j+n} b^k,$$

and

$$b^n.(ba)^i a^j b^k = \begin{cases} (ba)^{i+j} b^{n-j+k} & if \quad n \geq j \\ (ba)^{i+n} a^{j-n} b^k & if \quad j < n \end{cases} .$$

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3 Recognizable Subsets of $Pla(a,b)$

Before starting the search for the recognizable subsets of the plactic monoid, note the following relationship between the recognizable subsets of the free monoid A^* and a cross section $(S,.)$.

Lemma 3.1 *Let L be a subset of $S \subseteq A^*$. Then L is a recognizable subset of $(S,.)$ if and only if*

$$[L] = \{u \in A^* | \textit{there is a } v \in L \textit{ such that } u \equiv v\}$$

is recognizable in A^*.

Thus, if S is recognizable in A^* then the recognizability of $L \subseteq S$ in S implies the recognizability of L in A^*. The converse of this statement, however, is not true. For example, in the case of the two letter plactic monoid, $(ba)^*$ is recognizable in A^*, while it is not recognizable in S for $[(ba)^*]$ is the semi Dyck language.

In order to study the recognizable subsets of the plactic monoid on two letters, it is worth noticing that every subset $L \subseteq Pla(a,b)$ can be written in the form

$$L = \bigcup_{i,j \geq 0} (ba)^i \, a^j L_{i,j} \tag{3}$$

where $L_{i,j}$ is a subset of b^*. The reason for this is that $(ba)^* \, a^* b^*$ is a cross section of $Pla(a,b)$. This representation is unique and the residuals of a subset L of $Pla(a,b)$ represented in this form look as follows.