AN INDUCTION PRINCIPLE FOR THE WEIGHTED p-ENERGY MINIMALITY OF $x/\|x\|$

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Abstract. In this paper, we investigate minimizing properties of the map $x/\|x\|$ from the Euclidean unit ball $\mathbf{B}^n$ to its boundary $\mathbb{S}^{n-1}$, for the weighted energy functionals $E^n_{p,\alpha}(u) = \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u\|^p dx$. We establish the following induction principle: if the map $\frac{x}{\|x\|} : \mathbf{B}^{n+1} \to \mathbb{S}^n$ minimizes $E^{n+1}_{p,\alpha}$ among the maps $u : \mathbf{B}^{n+1} \to \mathbb{S}^n$ satisfying $u(x) = x$ on $\mathbb{S}^n$, then the map $\frac{y}{\|y\|} : \mathbf{B}^n \to \mathbb{S}^{n-1}$ minimizes $E^n_{p,\alpha+1}$ among the maps $v : \mathbf{B}^n \to \mathbb{S}^{n-1}$ satisfying $v(y) = y$ on $\mathbb{S}^{n-1}$.

This result enables us to enlarge the range of values of p and α for which $x/\|x\|$ minimizes $E^n_{p,\alpha}$.

1. Introduction and statement of main results

Let $\mathbf{B}^n$ be the unit Euclidean ball of $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ its bourdary. For any couple (α, p) of real numbers, with $\alpha \geq 0$ and $p \geq 1$, we define the r^α-weighted p-energy functional of a map $u : \mathbf{B}^n \to \mathbb{S}^{n-1}$ by

$$E^n_{p,\alpha}(u) = \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u\|^p dx.$$

This functional is nothing but the p-energy functional associated with the Riemannian metric $r^{\alpha(n-p)} g_{euc}$ on the ball $\mathbf{B}^n$, where g_{euc} is the Euclidean metric, the sphere $\mathbb{S}^{n-1}$ being endowed with its standard metric. The functional $E^n_{p,\alpha}$ is to be considered on the Sobolev space

$$W^{1,p}_\alpha(\mathbf{B}^n, \mathbb{S}^{n-1}) = \{u \in W^{1,p}((\mathbf{B}^n r^{\alpha(n-p)} g_{euc}), \mathbb{R}^n); \|u\| = 1 a.e\}.$$

The question is to know which map minimizes $E_{p,\alpha}$ among the maps in $W^{1,p}_\alpha(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $u(x) = x$ on $\mathbb{S}^{n-1}$.

Following the arguments of Hildebrant, Kaul and Widman [11], the map $x/\|x\|$ is a natural candidate to be the minimizer of $E^n_{p,\alpha}$, for $p \in [1, n+\alpha)$ (notice that $x/\|x\| \in W^{1,p}_\alpha(\mathbf{B}^n, \mathbb{S}^{n-1})$ if and only if $p < n + \alpha$).

The minimality of $x/\|x\|$ was first established for the 2-energy functional $E^n_2 := E^n_{2,0}$ by Jäger and Kaul [14] in dimension $n \geq 7$, then by Brezis, Coron and Lieb [2] in dimension 3. Coron and Gulliver [5] actually proved the minimality of $x/\|x\|$ for the p-energy functional $E^n_p := E^n_{p,0}$ for any integer $p \in \{1, \cdots, n-1\}$ and any dimension $n \geq 3$. An alternative proof using the null Lagrangian method (or calibration method) of this last result was obtained by Lin [15] and Avellaneda and Lin [1].

Hardt and Lin [7, 8] and Hardt, Lin and Wang [9, 10] developed the study of the singularities of p-harmonic and p-minimizing maps and obtained results extending those of Schoen and Uhlenbelck [16, 17]. A consequence of their results is the minimality of $x/\|x\|$ for E^n_p for any $p \in (n-1, n)$.

Finally, Hong [12] and Wang [18] have proved independently the minimality of $x/\|x\|$ for E_p^n in dimension $n \geq 7$ for any $p \leq n - 2\sqrt{n-1}$.

In order to close the question of the E_p^n-minimality of $x/\|x\|$ for the remaining values of p, Hong [13] suggested a new idea. Indeed, he observed that, for any $p \in (2,n)$, the minimality of $x/\|x\|$ for E_p^n follows from the minimality of $x/\|x\|$ for the r^{2-p}-weighted 2-energy $E_{2,2-p}^n$. Unfortunately, we have proved in a previous paper [3], the existence of an interval of values of $p \in (1, n-1)$ for which the map $x/\|x\|$ fails to be a minimizer of $E_{2,2-p}^n$. Nevertheless, we proved that $x/\|x\|$ minimizes the r^α-weighted p-energy $E_{p,\alpha}^n$ for any $\alpha \geq 0$ and any integer $p \leq n-1$. Actually, we obtained in [4] the minimality of $x/\|x\|$ for more general weighted p-energy functionals of the form $E_{p,f}^n(u) = \int_{\mathbf{B}^n} f(\|x\|)\|\nabla u\|^p dx$, where $p \leq n-1$ is an integer and $f : [0,1] \to \mathbb{R}^+$ is a continuous non-decreasing function.

Our aim in this paper is to show that the minimizing properties of $x/\|x\|$ in dimension n and $n+1$ are not independent. Indeed, we will prove that, for any $p \in [1, n+\alpha+1)$, if $x/\|x\|$ minimizes the r^α-weighted p-energy $E_{p,\alpha}^{n+1}$ in dimension $n+1$, then it also minimizes the $r^{\alpha+1}$-weighted p-energy $E_{p,\alpha+1}^n$ in dimension n.

Theorem 1. *Let $n \geq 2$ be an integer, $\alpha \geq 0$ be a real number and $p \in [1, n+\alpha+1)$. If the map $\frac{x}{\|x\|} : \mathbf{B}^{n+1} \mapsto \mathbb{S}^n$ minimizes the r^α-weighted p-energy $E_{p,\alpha}^{n+1}$ among the maps $u \in W_\alpha^{1,p}(\mathbf{B}^{n+1}, \mathbb{S}^n)$ satisfying $u(x) = x$ on $\mathbb{S}^n$, then the map $\frac{y}{\|y\|} : \mathbf{B}^n \mapsto \mathbb{S}^{n-1}$ minimizes the $r^{\alpha+1}$-weighted p-energy $E_{p,\alpha+1}^n$ among the maps $v \in W_{\alpha+1}^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $v(y) = y$ on $\mathbb{S}^{n-1}$.*

The proof of this theorem relies on a construction which associates to each map $u : \mathbf{B}^n \to \mathbb{S}^{n-1}$ such that $u(y) = y$ on $\mathbb{S}^{n-1}$, a map $\overline{u} : \mathbf{B}^{n+1} \to \mathbb{S}^n$ such that $\overline{u}(x) = u(x)$ in $\mathbf{B}^n \times \{0\}$ and $\overline{u}(x) = x$ on the unit sphere $\mathbb{S}^n$, in such a way that, if u_0 is the map defined in $\mathbf{B}^n$ by $u_0(y) = \frac{y}{\|y\|}$, then $\overline{u}_0$ is exactly the map defined on $\mathbf{B}^{n+1}$ by $\overline{u}_0(x) = \frac{x}{\|x\|}$. The energy $E_{p,\alpha}^{n+1}(\overline{u})$ of $\overline{u}$ is estimated above in terms of the energy $E_{p,\alpha+1}^n(u)$ of u and the equality holds in the estimate for the map $u_0 = \frac{y}{\|y\|}$.

Thanks to Theorem 1 and the minimality results mentioned above, we deduce the following :

Corollary 1. *The map $y/\|y\|$ minimizes $E_{p,\alpha}^n$ among the maps in $W_\alpha^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ which coincide with $y/\|y\|$ on $\mathbb{S}^{n-1}$ in the following cases :*

i) $\alpha \in \mathbb{N}$ and $p \in (n+\alpha-1, n+\alpha)$,

ii) $\alpha \geq \beta$ and p is an integer in $\{1, \cdots, n+\beta-1\}$, for any real number $\beta \geq 0$.

iii) $\alpha \in \mathbb{N}$, $n+\alpha \geq 7$ and $p \leq n+\alpha-2\sqrt{n+\alpha-1}$,

1.1. Construction of $\overline{u}$ and proof of Theorem 1.1.

Let $\mathbf{B}^{n+1}$ and $\mathbb{S}^n$ be the unit open ball and the unit sphere of $\mathbb{R}^{n+1}$ and let $\mathbf{B}^n$ and $\mathbb{S}^{n-1}$ be the unit open ball and the unit sphere of $\mathbb{R}^n$ that we identify with the subspace $\mathbb{R}^n = \mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. Moreover, we write $x = (x_1, \cdots, x_n, x_{n+1})$ a vector of $\mathbb{R}^{n+1}$, $y = (y_1, \cdots, y_n)$ a vector of $\mathbb{R}^n$, $\langle ., . \rangle$ the standard metric of $\mathbb{R}^{n+1}$ and $(e_1, \cdots, e_n, e_{n+1})$ the standard basis of $\mathbb{R}^{n+1}$.

Let Π_n be the projection defined by :

$$\begin{array}{rcl}\Pi_n : \mathbf{B}^{n+1} & \longrightarrow & \mathbf{B}^n \\ (x_1, \cdots, x_{n+1}) & \longrightarrow & (x_1, \cdots, x_n, 0) = (x_1, \cdots, x_n).\end{array}$$

Consider the map φ_n defined on $B^n \backslash \mathbb{R}e_{n+1}$ by $\varphi_n(x) = \frac{\Pi_n(x)}{\|\Pi_n(x)\|}$.
We define the map $\overline{u}$ defined by

$$\begin{array}{rcl}\overline{u} : \mathbf{B}^{n+1} & \longrightarrow & \mathbf{B}^n \\ x & \longrightarrow & \langle e_{n+1}, \frac{x}{\|x\|}\rangle e_{n+1} + \langle \varphi_n(x), \frac{x}{\|x\|}\rangle u(\|x\|\varphi_n(x)).\end{array}$$

Lemma 1. *For any $x \in \mathbf{B}^{n+1}$, $\|\nabla\overline{u}(x)\|^2 = \frac{1}{\|x\|^2} + \|\nabla u(\|x\|\varphi_n(x))\|^2$.*

Proof. For any $i \in \{1, \cdots, n\}$, we have,

$$\begin{aligned} d\overline{u}(x).e_i &= \frac{\langle \Pi_n(x), e_i\rangle}{\|x\|}\left(\frac{1}{\|\Pi_n(x)\|} - \frac{\|\Pi_n(x)\|}{\|x\|^2}\right) u(\|x\|\varphi_n(x)) \\ &\quad - \frac{\langle e_{n+1}, x\rangle\langle x, e_i\rangle}{\|x\|^3} e_{n+1} \\ &\quad + du(\|x\|\varphi_n(x)).\left(\frac{\langle \Pi_n(x), e_i\rangle \Pi_n(x)}{\|x\|^2} + e_i - \frac{\langle \Pi_n(x), e_i\rangle \Pi_n(x)}{\|\Pi_n(x)\|^2}\right)\end{aligned}$$

and,

$$\begin{aligned} d\overline{u}(x).e_{n+1} &= \frac{e_{n+1}}{\|x\|} - \frac{\langle e_{n+1}, x\rangle^2}{\|x\|^3} e_{n+1} - \frac{\langle e_{n+1}, x\rangle}{\|x\|^3}\|\Pi_n(x)\| u(\|x\|\varphi_n(x)) \\ &\quad + \frac{\langle e_{n+1}, x\rangle}{\|x\|^2} du(\|x\|\varphi_n(x)).\Pi_n(x).\end{aligned}$$

Since $\|u(x)\|^2 = 1$, one has $\langle du(x).h, u(x)\rangle = 0$ for any $h \in \mathbb{R}^n$. Hence, for any $i \in \{1, \cdots, n\}$, we have,

$$\begin{aligned} \|d\overline{u}(x).e_i\|^2 &= \frac{\langle \Pi_n(x), e_i\rangle^2}{\|x\|^2}\left(\frac{1}{\|\Pi_n(x)\|} - \frac{\|\Pi_n(x)\|}{\|x\|^2}\right)^2 + \frac{\langle e_{n+1}, x\rangle^2\langle x, e_i\rangle^2}{\|x\|^6} \\ &+ \left\|du(\|x\|\varphi_n(x)).\left(\langle \Pi_n(x), e_i\rangle\left(\frac{1}{\|x\|^2} - \frac{1}{\|\Pi_n(x)\|^2}\right)\Pi_n(x) + e_i\right)\right\|^2.\end{aligned}$$

$$\begin{aligned} \|d\overline{u}(x).e_{n+1}\|^2 &= \frac{1}{\|x\|^2}\left(1 - \frac{\langle e_{n+1}, x\rangle^2}{\|x\|^2}\right)^2 + \frac{\langle e_{n+1}, x\rangle^2}{\|x\|^6}\|\Pi_n(x)\|^2 \\ &\quad + \frac{\langle e_{n+1}, x\rangle^2}{\|x\|^4}\|du(\|x\|\varphi_n(x)).\Pi_n(x)\|^2.\end{aligned}$$

Finally, we have,

$$\|\nabla\overline{u}(x)\|^2 = \frac{1}{\|x\|^2} + \|\nabla u(\|x\|\varphi_n(x))\|^2. \qquad \blacksquare$$

Let x_{n+1} be a real number in $(0, 1)$, consider the set $A_{x_{n+1}} = (x_{n+1}e_{n+1} + e_{n+1}^{\perp}) \cap \mathbf{B}^{n+1} \backslash \mathbb{R}e_{n+1}$, where $e_{n+1}^{\perp}$ is the orthogonal subspace to $\mathbb{R}e_{n+1}$ for

$\langle .,. \rangle$. Let θ be the map :

$$\begin{array}{rcl} \theta : A_{x_{n+1}} & \longrightarrow & C_{x_{n+1}} = \{y \in \mathbb{R}^n ; \|y\| > |x_{n+1}|\} \\ x = (x_1, \cdots, x_{n+1}) & \longrightarrow & \|x\| \varphi_n(x) = y = (y_1, \cdots, y_n). \end{array}$$

Lemma 2. *For any $y \in \mathbb{R}^n$, the Jacobian determinant of θ^{-1} is :*

$$Jac(\theta^{-1})(y) = \frac{\left(\|y\|^2 - x_{n+1}^2\right)^{\frac{n-2}{2}}}{\|y\|^{n-2}}.$$

Proof For any $i \in \{1, \cdots, n\}$ and for any $x \in A_{x_{n+1}}$,

$$d\theta(x).e_i = \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2}\right) \frac{\langle \Pi_n(x), e_i \rangle}{\|\Pi_n(x)\|} \Pi_n(x) + \frac{\|x\|}{\|\Pi_n(x)\|} e_i.$$

Let us set, for any $i \in \{1, \cdots, n\}$,

$$\lambda_i = \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2}\right) \frac{\langle \Pi_n(x), e_i \rangle}{\|\Pi_n\|} \text{ and } \alpha = \frac{\|x\|}{\|\Pi_n(x)\|}.$$

Hence, for any $i \in \{1, \cdots, n\}$,

$$d\theta(x).e_i = \lambda_i \Pi_n(x) + \alpha e_i.$$

Then, for any $x \in A_{x_{n+1}}$,

$$\begin{aligned} Jac(\theta)(x) &= \det(\lambda_1 \Pi_n(x) + \alpha e_1, \lambda_2 \Pi_n(x) + \alpha e_2, \cdots, \lambda_n \Pi_n(x) + \alpha e_n) \\ &= \sum_{i=1}^n \det(\alpha e_1, \cdots, \alpha e_{i-1}, \lambda_i \Pi_n(x), \alpha e_{i+1}, \cdots, \alpha e_n) + \alpha^n \det(e_1, \cdots, e_n) \\ &= \sum_{i=1}^n \alpha^{n-1} \lambda_i \det(e_1, \cdots, e_{i-1}, \Pi_n(x), e_{i+1}, \cdots, e_n) + \alpha^n, \end{aligned}$$

where det is the determinant in the basis $(e_1, \cdots, e_{n+1})$.

$$\begin{aligned} Jac(\theta)(x) &= \sum_{i=1}^n \alpha^{n-1} \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2}\right) \frac{\langle \Pi_n(x), e_i \rangle^2}{\|\Pi_n(x)\|} + \alpha^n \\ &= \alpha^{n-1} \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2}\right) \frac{\|\Pi_n(x)\|^2}{\|\Pi_n(x)\|} + \alpha^n = \alpha^{n-2}. \end{aligned}$$

Therefore, we have

$$Jac(\theta)(x) = \frac{\|x\|^{n-2}}{\|\Pi_n(x)\|^{n-2}} = \frac{\|y\|^{n-2}}{\left(\|y\|^2 - x_{n+1}^2\right)^{\frac{n-2}{2}}}.$$

We deduce that,

$$Jac(\theta^{-1})(y) = \frac{\left(\|y\|^2 - x_{n+1}^2\right)^{\frac{n-2}{2}}}{\|y\|^{n-2}}.$$

Lemma 3. *For any $x\in \mathbf{B}^{n+1}$, we have,*

$$\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\|\nabla\overline{u}(x)\|^{p}dx_{1}\cdots dx_{n+1} \;\leq\; n^{p/2-1}\int_{\mathbf{B}^{n+1}}\frac{1}{\|x\|^{p-\alpha}}dx_{1}\cdots dx_{n+1} +2(1-1/n)^{1-p/2}\,W_{n-1}\int_{\mathbf{B}^{n}}\|y\|^{\alpha+1}\|\nabla u(y)\|^{p}dy,$$

where $W_{n-1}=\int_0^{\frac{\pi}{2}}(cos(\gamma))^{n-1}d\gamma$.

Proof. Writing $\frac{1}{\|x\|^2}+\|\nabla u(\|x\|\varphi_n(x))\|^2=\frac{1}{n}(n\frac{1}{\|x\|^2})+(1-\frac{1}{n})(\frac{1}{1-\frac{1}{n}}\|\nabla u(\|x\|\varphi_n(x))\|^2)$ and using that $x\to x^{\frac{p}{2}}$ is a convex map, for any $p\in[1,n+\alpha+1)$, we have,

$$\begin{aligned}\|\nabla\overline{u}(x)\|^{p} &= \left(\frac{1}{\|x\|^{2}}+\|\nabla u(\|x\|\varphi_n(x))\|^{2}\right)^{p/2}\\ &\leq n^{p/2-1}\frac{1}{\|x\|^{p}}+(1-1/n)^{1-p/2}\|\nabla u(\|x\|\varphi_n(x))\|^{p}.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\|\nabla\overline{u}(x)\|^{p}dx_{1}\cdots dx_{n+1} &\leq n^{p/2-1}\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\frac{1}{\|x\|^{p}}dx_{1}\cdots dx_{n+1}\\ &\quad+(1-1/n)^{1-p/2}\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\|\nabla u(\|x\|\varphi_n(x))\|^{p}dx_{1}\cdots dx_{n+1}\\ &\leq n^{p/2-1}\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\frac{1}{\|x\|^{p}}dx_{1}\cdots dx_{n+1}\\ &\quad+2(1-1/n)^{1-p/2}\int_0^1 dx_{n+1}\int_{\mathbf{B}^{n}}\|x\|^{\alpha}\|\nabla u(\|x\|\varphi_n(x))\|^{p}dx_{1}\cdots dx_{n}.\end{aligned}$$

Using the change of variables $y=\theta(x)$ and Lemma 1.2 we get,

$$\begin{aligned}\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\|\nabla\overline{u}(x)\|^{p}dx_{1}\cdots dx_{n+1} &\leq n^{p/2-1}\int_{\mathbf{B}^{n+1}}\frac{1}{\|x\|^{p-\alpha}}dx_{1}\cdots dx_{n+1}\\ &\quad+2(1-1/n)^{1-p/2}\int_0^1 dx_{n+1}\int_{C^{n+1}}\frac{\left(\|y\|^2-x_{n+1}^2\right)^{\frac{n-2}{2}}}{\|y\|^{n-2}}\|y\|^{\alpha}\|\nabla u(y)\|^{p}dy_1\cdots dy_n,\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{B}^{n+1}}\|x\|^{\alpha}\|\nabla\overline{u}(x)\|^{p}dx_{1}\cdots dx_{n+1} &\leq n^{p/2-1}\int_{\mathbf{B}^{n+1}}\frac{1}{\|x\|^{p-\alpha}}dx_{1}\cdots dx_{n+1}\\ &\quad+2(1-1/n)^{1-p/2}\int_{\mathbf{B}^{n}}\|y\|^{\alpha}\|\nabla u(y)\|^{p}\left(\int_0^{\|y\|}\frac{\left(\|y\|^2-x_{n+1}^2\right)^{\frac{n-2}{2}}}{\|y\|^{n-2}}dx_{n+1}\right)dy.\end{aligned}$$

But we have,

$$\begin{aligned}\int_0^{\|y\|}\left(\frac{\|y\|^2-x_{n+1}^2}{\|y\|^2}\right)^{\frac{n-2}{2}}dx_{n+1} &= \int_0^{\|y\|}\left(1-\left(\frac{x_{n+1}}{\|y\|}\right)^2\right)^{\frac{n-2}{2}}dx_{n+1}\\ &= \int_0^1\|y\|(1-t^2)^{\frac{n-2}{2}}dt=\|y\|\int_0^{\pi/2}(cos\gamma)^{n-1}d\gamma.\end{aligned}$$

Let us set $W_{n-1} = \int_0^{\pi/2} (cos\gamma)^{n-1} d\gamma$. Then, we have the inequality,

$$\int_{\mathbf{B}^{n+1}} \|x\|^{\alpha} \|\nabla \overline{u}(x)\|^p dx_1 \cdots dx_{n+1} \leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1 \cdots dx_{n+1} + 2(1-1/n)^{1-p/2} W_{n-1} \int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy. \quad \blacksquare$$

Lemma 4. *Let* Γ *be the Gamma function defined by,*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

We have,

$$W_{n-1} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} = \frac{\sqrt{\pi}}{2}.$$

Proof From the equality $\Gamma(x+1) = x\Gamma(x)$ for any $x \in (0, +\infty)$, we have, if $n = 2l$, where $l \in \mathbb{N}$,

$$\Gamma(\frac{2l+1}{2}) = \frac{2l-1}{2} \frac{2l-3}{2} \cdots \frac{3}{2} \frac{1}{2} \Gamma(\frac{1}{2})$$

and

$$\Gamma(\frac{2l}{2}) = \frac{2l-2}{2} \frac{2l-4}{2} \cdots \frac{2}{2} \Gamma(1).$$

Moreover we have,

$$W_{2l-1} = \frac{(2l-2)(2l-4)\cdots 2}{(2l-1)(2l-3)\cdots 3.1}$$

then,

$$W_{2l-1} \frac{\Gamma(\frac{2l+1}{2})}{\Gamma(\frac{2l}{2})} = \frac{\sqrt{\pi}}{2}.$$

If $n = 2l+1$ where $l \in \mathbb{N}$, we obtain,

$$\Gamma(\frac{2l+2}{2}) = \frac{2l}{2} \frac{2l-2}{2} \cdots \frac{2}{2} \Gamma(1),$$

$$\Gamma(\frac{2l+1}{2}) = \frac{2l-1}{2} \cdots \frac{3}{2} \frac{1}{2} \Gamma(\frac{1}{2})$$

and

$$W_{2l} = \frac{(2l-1)(2l-3)\cdots 3.1}{(2l)(2l-2)\cdots 2} . \frac{\pi}{2}.$$

We deduce that

$$W_{2l} \frac{\Gamma(\frac{2l+2}{2})}{\Gamma(\frac{2l+1}{2})} = \frac{\sqrt{\pi}}{2}. \quad \blacksquare$$

Proof of Theorem 1.1. Suppose that $\overline{u}_0$ is a minimizer map of $E_{p,\alpha}$. Since the measure of $\mathbb{S}^n$ is $|\mathbb{S}^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ and since $\|\nabla \overline{u}_0\|(x) = \frac{n^{\frac{p}{2}}}{\|x\|}$, by

Lemma 1.3, for any map $u \in W^{1,p}_{\alpha+1}(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $u(y) = y$ on $\mathbb{S}^{n-1}$, we have,

$$\begin{aligned}
\frac{n^{p/2}}{n+1+\alpha-p}\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} &= \int_{\mathbf{B}^{n+1}} \|x\|^{\alpha}\|\nabla\overline{u_0}(x)\|^p dx_1\cdots dx_{n+1} \\
&\leq \int_{\mathbf{B}^{n+1}} \|x\|^{\alpha}\|\nabla\overline{u}(x)\|^p dx_1\cdots dx_{n+1} \\
&\leq n^{p/2-1}\int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1\cdots dx_{n+1} \\
&\quad +2(1\text{–}1/n)^{1-p/2}\, W_{n-1}\int_{\mathbf{B}^n} \|y\|^{\alpha+1}\nabla u(y)\|^p dy \\
&\leq \frac{1}{n}\frac{n^{p/2}}{n+1+\alpha-p}\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \\
&\quad + 2W_{n-1}(1-\frac{1}{n})(\frac{n}{n-1})^{p/2}\int_{\mathbf{B}^n} \|y\|^{\alpha+1}\|\nabla u(y)\|^p dy.
\end{aligned}$$

Then we have,

$$\begin{aligned}
2W_{n-1}\left(1-\frac{1}{n}\right)\left(\frac{n}{n-1}\right)^{p/2}\frac{\int_{\mathbf{B}^n}\|y\|^{\alpha+1}\|\nabla u(y)\|^p dy}{\int_{\mathbf{B}^n}\|y\|^{\alpha+1}\|\nabla \frac{y}{\|y)\|}\|^p dy} &\geq \frac{n^{p/2}}{n+1+\alpha-p}\frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \\
&\quad \times\frac{n+1+\alpha-p}{(n-1)^{p/2}}\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}\left(1-\frac{1}{n}\right)
\end{aligned}$$

By Lemma 1.4 we finally get,

$$\int_{\mathbf{B}^n}\|y\|^{\alpha+1}\|\nabla u(y)\|^p dy \geq \int_{\mathbf{B}^n}\|y\|^{\alpha+1}\|\nabla\frac{y}{\|y\|}\|^p dy \qquad \blacksquare$$

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