On the algebra of viscoelastic responses

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We study the class of functions which are responses to the unit impulse for linear viscoelastic systems and the algebraic operations which preserve this class. This allows to yield and to structure a large number of explicit analytical expressions which can be used in symbolic computer systems for the construction of various rheological models.

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The restrictions imposed by the thermodynamics of irreversible phenomena on viscoelastic systems were discovered and studied by several authors sometimes independently (cf Eckart (1948), Meixmer (1953), Biot (1954), Coleman (1964), Fung (1965), Mandel (1966)). We consider here the hypotheses of the theory of M. Biot (1954) and, as this theory, the following study applies as well to other physical (electrical, chimical) phenomena provided that the assumptions (Onsager principle, existence of normal variables, linearity) be valid. The conclusion of this theory, whose reasoning is recalled in part I below for completeness, is that every viscoelastic system can be approximated (cf Fung (1965), Mandel (1966), or more recently Tschoegl (1989), Pipkin (1989)) by a grouping in parallel or in series of a finite number of dashpots and of springs. Nevertheless it would be a shame to limit this theory to this single conclusion, on one hand because the grouping in parallel of an infinite sequence of models of Maxwell and a dashpot and a spring **does not yield** the general response (as does not the grouping in series of Kelvin-Voigt models and a dashpot and a spring, cf II.3 below) on the other hand because this theory yields the explicit class of functions which are possible responses and because this class, related to several important mathematical questions, possesses remarkable algebraic properties which allow to construct and to link together a large variety of examples with explicit analytical forms.

The increasing use by engineers of symbolic computer systems make the building up of such a library of response functions a tool for modelisation which is complementary to the usual numerical computations.

I. Thermodynamics and viscoelasticity

I.1 The classical argument about restrictions to the viscoelasticity due to thermodynamics is the following :

Let a system be acted on by generalized forces Q_i, $i = 1, \ldots, n$ and described by associated geometrical parameters q_i, $i = 1, \ldots, n$ such that the work of external forces can be written $\sum_{i=1}^{n} Q_i dq_i$. In the neighbourhood of a stable equilibrium state, where the q_i's are taken to be zero, the thermodynamical potential of the system writes

$$W = \frac{1}{2} \sum_{i,j} a_{ij} q_i q_j$$

where the matrix (a_{ij}) is symmetric positive semi-definite (the word stable is taken here in the wide sense). Computing the entropy variation during a short time interval, neglecting the inertial forces, and assuming small velocities give for the dissipated power :

$$D = \frac{1}{2} \sum_{i,j} b_{ij} \dot{q}_i \dot{q}_j$$

where the matrix (b_{ij}) is symmetric by Onsager's principle and positive semi-definite by the second principle of thermodynamics. The evolution equation is then

$$\frac{\partial D}{\partial \dot{q}_i} + \frac{\partial W}{\partial q_i} = Q_i$$

thus

(1) $$\sum_j a_{ij} q_j + \sum_j b_{ij} \dot{q}_j = Q_i.$$

In the case of linear viscoelasticity free from aging effects, coefficients (a_{ij}) and (b_j) are constants, and the linear relation between the history of forces $(Q_i(t))$ and that of parameters $(q_i(t))$ commutes with time translations, it expresses therefore by a convolution product and is known by the response $f_{ij}(t)$ of the parameter q_i to the unit jump of the force Q_j. Let us write equation (1) under the form

$$Aq + B\dot{q} = Q \tag{2}$$

where A and B are $n \times n$ symmetric positive semi-definite matrices and let us assume first that B be positive definite.

Let us consider on $\mathbb{R}^n$ the Euclidean structure associated with B whose scalar product is

$$(u, v)_B = < u, Bv > = {}^t uBv$$

(denoting by $< ., . >$ the usual scalar product on $\mathbb{R}^n$). For this new Euclidean structure the operator $B^{-1}A$ is self-adjoint :

$$(u, B^{-1}Av)_B = < u, Av > = < Au, v > = < v, Au > = (v, B^{-1}Au)_B$$

Hence there exists a B-orthonormal basis $(T_k)_{k=1,\dots,n}$ on which $B^{-1}A$ is diagonal, in other words

$$\begin{aligned} (\psi_j, B^{-1}A\psi_k)_B &= < \psi_j, A\psi_k > = \begin{cases} 0 & \text{if } j \neq k \\ \lambda_k \geq 0 & \text{if } j = k \end{cases} \\ (\psi_j, \psi_k)_B &= < \psi_j, B\psi_k > = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \end{aligned}$$

in particular

$$A\psi_k - \lambda_k B\psi_k = 0 \qquad \forall k.$$

Let $\hat{q}(\theta)$ [resp. $\hat{Q}(\theta)$] be the Laplace transform of $q(t)$ [resp. $Q(t)$] $(\hat{q}(\theta) = \int_0^\infty e^{-\theta t} q(t)dt)$.

Equation (2) writes

$$(A + \theta B)\hat{q} = \hat{Q}. \tag{3}$$

Then, if $\hat{q}(\theta)$ is expanded on the basis (ψ_k),

$$\hat{q}(\theta) = \sum_{k=1}^{n} \xi_k(\theta)\psi_k \tag{4}$$

it follows

$$(\lambda_k + \theta)\xi_k = < \psi_k, \hat{Q} > .$$

If, among the n parameters only m $(m < n)$ are observable, i.e. if $\hat{Q} = (\hat{Q}_1, \dots, \hat{Q}_m, 0, \dots, 0)$ it comes

$$\xi_k = \frac{1}{\lambda_k + \theta} \sum_{j=1}^{m} \psi_{k_j} \hat{Q}_j$$

and by (4)

$$\hat{q}_i = \sum_{j=1}^{m} \hat{Q}_j [\sum_{k=1}^{n} \frac{1}{\lambda_k + \theta} \psi_{k_j} \psi_{k_j}].$$

Therefore, denoting by $f_{ij}(t)$ the response of q_i to the unit jump of Q_j, one has

$$\hat{f'_{ij}} = \sum_{k=1}^{n} \frac{1}{\lambda_k + \theta} \psi_{k_j} \psi_{k_j}$$

and hence

$$f_{ij}(t) = \sum_{k=1}^{n} \int_0^t e^{-\lambda_k s} ds J_{ij}^{(k)}.$$

It is easily seen that if larger assumptions are taken on matrix B allowing zero and infinite eigenvalues, one gets finally the following form

$$f_{ij}(t) = \sum_{k=1}^{n} (1 - e^{-\lambda_k t}) J_{ij}^{(k)} + tL_{ij} + K_{ij} \tag{5}$$

where the matrices $(J^{(k)})k = 1, \ldots, n, L, K$ are symmetric positive semi-definite.

I.2 Starting from equation (5) and passing to the limit give that the responses of viscoelastic systems to unit jump are,
in the case of a single observable parameter, of the form

$$f(t) = \int_{\mathbb{R}_+^*} (1 - e^{-\lambda t}) \nu(d\lambda) + bt + c \tag{6}$$

where ν is a positive σ-finite measure on $\mathbb{R}_+^* =]0, \infty[$ such that $\int_0^\infty \frac{x}{1+x} d\nu(x) < +\infty$, and $b \geq 0$, $c \geq 0$;
and in the case of m observable parameters, of the form

$$f_{ij}(t) = \int_{\mathbb{R}_+^*} (1 - e^{-\lambda t}) \nu_{ij}(d\lambda) + tL_{ij} + K_{ij}, \tag{7}$$

where $\nu = (\nu_{ij})$ is a symmetric positive semi-definite matrix of σ-finite measures on $\mathbb{R}_+^*$ which satisfy

$$\int_{\mathbb{R}_+^*} \frac{x}{1+x} d|\nu_{ij}|(x) < +\infty \qquad \forall i, j = 1, \ldots, n$$

and where matrices L and K are symmetric positive semi-definite.

This passing to the limit can be performed either, using physical arguments, by taking the set of pointwise limits of functions of the form (5), or by considering a viscoelastic continuum, with matrices A and B replaced by non bounded self-adjoint operators in a Hilbert space. Then the argument of part I above extends and by the spectral representation of self -adjoint operators, gives directly the forms (6) and (7). A detailed mathematical proof of this derivation would bring nothing more to the present article which uses henceforth only formulae (6) and (7).

II. Operations on Bernstein functions

The functions of the form (6) are called Bernstein functions. They occur essentially in potential theory and especially in the theory of convolution semi-groups (cf Berg and Forst (1975)), we recall first their fundamental properties.

II.1 Fundamental properties of Bernstein functions

Property 1 *A function* $f : \mathbb{R}_+^* \to \mathbb{R}$ *is Bernstein if and only if,* f *is* C^∞, $f \geq 0$, *and* $(-1)^p D^p f \leq 0 \quad \forall p \geq 1$.

In particular, it is seen that Bernstein functions are concave and constitute a convex cone. Property 1 is often taken as definition.

Property 2 *The Bernstein functions can be represented under the form (6) and the triplet* (ν, b, c) *is uniquely determined by* f. *They have an extension to a continuous function in the half plane* $Rez \geq 0$ *which is holomorphic in the open half plane* $Rez > 0$. *The limit of every pointwise convergent (on* $\mathbb{R}_+^*$*) sequence of Bernstein functions is a Bernstein function.*

The most fundamental property is that they are in one to one correspondance with vaguely continuous semi-group of sub-probability measures on $\mathbb{R}_+$ and are therefore related to the probabilistic theory of processes with stationary independent increments (cf Bouleau (1991)) :

Property 3 *Let* f *be a Bernstein function. Then there exists a family of positive measures* $(\mu_\tau)_{\tau>0}$ *on* $\mathbb{R}_+$ *such that*

i) $\mu_\tau(\mathbb{R}_+) \leq 1$

ii) $\mu_\tau * \mu_\sigma = \mu_{\tau+\sigma} \qquad \forall \sigma, \tau > 0$ *(convolution semi-group)*

iii) $\lim_{\tau\to 0} \mu_\tau = \delta_0 \qquad$ *vaguely*

iv) $\int_{\mathbb{R}_+} e^{-tx} \mu_\tau(dx) = e^{-\tau f(t)} \qquad \forall \tau > 0$

such a family is unique, and conversely for a family satisfying i), ii), iii) the function f *given by iv) is Bernstein.*

The measures μ_τ are probability measures ($\mu_\tau(\mathbb{R}_+) = 1$) if and only if $f(0) = 0$.

The Bernstein functions are also related to the family of completely monotone functions (functions $g : \mathbb{R}_+^* \to \mathbb{R}_+$ such that $(-1)^P D^P g \geq 0 \quad \forall p \geq 0$) :

Property 4 f *is Bernstein if and only if, for every* $\tau > 0$, $g = e^{\tau f}$ *is completely monotone.*

II.2 Subordination in Bochner sense

A remarkable property of the cone of Bernstein functions is that it is stable by composition in the following sense :

Property 5 *If* φ *and* ψ *are Bernstein,* $\psi(0) = 0$, *then* $\varphi \circ \psi : x \to \varphi(\psi(x))$ *is Bernstein.*

This property possesses a semi-group interpretation as shown by Bochner and also a probabilistic one known as **subordination** (cf for instance Bouleau and Chateau (1989), Chateau (1990)). It allows from known Bernstein functions to obtain new ones and it will be used in part III below.

This property, when interpretating a viscoelastic material as a clock, has the following physical meaning :

Let us consider :

a) a body A with rectilinear uniform movement, at zero at time 0.

b) two indices B and C bound respectively to the responses to the unit jump of two viscoelastic materials and moving parallely to A.

If we note the successive positions $a_0 = 0, a, \ldots, a_n, \ldots$ of A when B reaches regular distances $0, h, 2h, \ldots, nh, \ldots$, then the successive positions $\alpha_0 = 0, \alpha_1, \ldots, \alpha_n, \ldots$ of A when C reaches the a_n's are also the ones reached by A when a third viscoelastic body D has its index on $0, h, 2h, \ldots, nh, \ldots,$.

II.3 Parallel and serial grouping : conjugate materials

When two rheological models with the same numbers of observables parameters are put in series, the response obtained is the sum of the responses :

$$f_{ij}(t) = f_{ij}^{(1)}(t) + f_{ij}^{(2)}(t)$$

where as the putting in parallel corresponds to the sum of relaxation functions

$$r_{ij}(t) = r_{ij}^{(1)}(t) + r_{ij}^{(2)}(t).$$

By the fact that the responses to the unit jump are related to the relaxation functions by the relation

$$(\sum_{k=1}^{m} f'_{ik} * r'_{kj})_{ij} = \delta_0.I \tag{8}$$

where I is the identity matrix, and where the derivative are taken in the sense of distributions, it follows from the results of Hirsch (1972) on the Stieltjes transform that the relaxation functions are the distributions of the form

$$r_{ij}(t) = A_{ij} + B_{ij}\delta_0(t) + \int_{\mathbb{R}_+^*} e^{-tx} d\varphi_{ij}(x) \tag{9}$$

where the matrices A, B $\varphi = (\varphi_{ij})$ are symmetric positive semi-definite, and the measures φ_{ij} are such that

$$\int_0^\infty \frac{1}{1+x} d|\varphi_{ij}(x)| > +\infty \qquad i, j = 1, \ldots, m.$$

It follows that the primitive functions R_{ij} vanishing on the negative real axis of the functions $r_{ij}(t)$ are exactly the functions of the form (7). There exists therefore a material where responses to unit jump are the $R_{ij}(t)$'s. This material can be called **conjugate** to the initial material. The conjugation relation is involutive and the serial grouping of the initial materials corresponds to the parallel grouping of the conjugate materials and conversely.

Remark

In particular, we see that a countable combination of dashpots and springs yields a rheological model which, in the case of a single observable parameter, is of the forme (6) with a discrete measure ν (countable sum of Dirac masses). This is far from being all possible models as mentioned in the introduction.

III. Main analytical families

Henceforth we shall write only "response" for "response to the unit jump".

Because of the probabilistic interpretation of the underlying convolution semi-groups (cf Bouleau (1991)), the measure ν of formula (6) will be called the **Lévy measure**.

III.1 Discrete Lévy measures

a) Let us quote first for completeness the finite sums of Dirac masses which correspond to the usual springs-dashpots models :

Elastic material

$$f_1(t) = \frac{1}{a} 1_{t \geq 0}$$

damping (conjugate of the preceding one)

$$\tilde{f}_1(t) = at1_{\{t \geq 0\}}.$$

Maxwell material

$$f_2(t) = (at + b)1_{\{t\geq 0\}}.$$

Kelvin-Voigt material (conjugate of the preceding one)

$$\tilde{f}_2(t) = \frac{1}{a}(1 - e^{-\frac{a}{b}t})1_{t\geq 0}.$$

Finite combination of springs and dashopts

$$f_3(t) = (\sum_i \alpha_i(1 - e^{-\lambda_i t}) + at + b)1_{t\geq 0}$$

and its conjugate material

$$\tilde{f}_3(t) = (\sum_i \beta_i(1 - e^{-\mu_i t}) + ct + d)1_{t\geq 0}$$

with $ac = 0$ and

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \dots \qquad \text{if } c \neq 0$$

and

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \dots \qquad \text{if } a \neq 0.$$

b) Let X be an integer integrable random variable, then $\mathbb{E}[1 - e^{-tX}]$ is a response. Taking $X = aY$, $a > 0$, where Y has a geometrical law on $\mathbf{N}^*$, gives

$$f_4(t) = \frac{1 - e^{-ta}}{1 - pe^{-ta}} \qquad p \in]0, 1[,\ a > 0$$

and if Y follows a Poisson law with parameter θ on $\mathbf{N}^*$, one gets

$$f_5(t) = 1 - \exp[\theta e^{-ta} - \theta - ta], \qquad \theta > 0,\ a > 0.$$

Finally with $\mathbb{P}(Y = n) = (1-p)^\alpha \frac{\alpha(\alpha+1)\cdots(\alpha+n-2)}{(n-1)!} p^{n-1}, n \geq 1$ one obtains

$$f_6(t) = 1 - \frac{(1-p)^\alpha e^{-ta}}{(1 - pe^{-ta})^\alpha}, \qquad \alpha > 0, p \in]0, 1[.$$

c) It follows from the probabilistic interpretation of the Lévy measure that if f is of the form $f(t) = \int(1 - e^{-ty})d\nu(y)$ with ν of the form $\nu(dy) = \sum_{n=0}^\infty a_n\delta_{y_n}$, $a_n > 0$ (and satisfying $\int \frac{y}{1+y} d\nu(y) < +\infty$), then $g(f(t))$ is still of this form with a Lévy measure $\sum_{n=0}^\infty b_n\delta_{y_n}$ (the same y_n's) whatever be the Bernstein function g vanishing at zero.

Therefore, composing on the left $f_1, \dots, f_6$ with the other responses below yields responses in the class of discrete Lévy measures again.

III.2 Stable family of order α

Thus are called, because of the probabilistic interpretation, the responses of the form

$$f_7(t) = h_\alpha(t) = a.t^\alpha 1_{\{t\geq 0\}}, \qquad \alpha \in]0, 1[, a > 0.$$

From the formula

$$\int_0^\infty (1-e^{-\theta y})\frac{dy}{y^{\alpha+1}} = \frac{\Gamma(1-\alpha)}{\alpha}\theta^\alpha, \qquad \theta > 0,$$

it follows that the Lévy measure of f_7 is

$$\nu_7(dy) = \frac{a\alpha}{\Gamma(1-\alpha)}\frac{dy}{y^{\alpha+1}} \qquad \text{on } \mathbb{R}_+^*.$$

The response of the conjugate material is

$$\tilde{f}_7(t) = \frac{\sin\pi\alpha}{a\alpha(1-\alpha)\pi}t^{1-\alpha}1_{\{t\geq 0\}}$$

(by use of the relation (8) and Euler formula $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}$).

Moreover, it is possible to show that the only material which is its own conjugate material has for response

$$2\sqrt{\frac{t}{\pi}}1_{\{t\geq 0\}}$$

The stable family is of course closed by composition.

III.3 Homographic family

For $a > 0$ and $b > 0$, the function

$$f_8(t) = \frac{at}{1+bt}1_{\{t\geq 0\}}$$

is a response. This family is also closed by composition. The corresponding Lévy measure is

$$\nu_8(dy) = \frac{a}{b^2}e^{-y/b}dy \qquad \text{on } \mathbb{R}_+^*.$$

III.4 Logarithmic family For $a > 0$ and $b > 0$ the function

$$f_9 = a\,Log(1+bt)$$

is a response which corresponds to the Lévy measure

$$\nu_9(dy) = ae^{-y/b}\frac{dy}{y} \qquad \text{on } \mathbb{R}_+^*.$$

III.5 Some other responses

Easy computations yields also

$$f_{10}(t) = a(b - \frac{1-e^{-tb}}{t}) \qquad a > 0, b > 0$$

associated with $\nu_{10}(dy) = a1_{[0,b]}(y)dy$,

$$f_{11}(t) = a\{\frac{b^{n+1}}{n+1} - \frac{n!}{t^{n+1}}[1 - e^{-bt}(\frac{(bt)^n}{n!} + \frac{(bt)^{n-1}}{(n-1)!} + \cdots + 1)]\} \qquad a > 0, b > 0$$

associated with

$$\nu_{11}(dy) = 1_{[0,b]}(y) y^n dy, \qquad n \in \mathbf{N}$$

IV. Comments

IV.1 By combination of the preceding responses several behaviour at infinity can be obtained. For example, putting $f \prec g$ for $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$, gives

$$f_9 \circ f_9 \prec f_9 \prec f_7 \circ f_9 \prec f_7 \prec t.$$

Similarly, for the functions which possess a finite limit, the speed of the nearing to the asymptote can be varied.

In the same manner, the behaviour at zero can be chosen with a non vertical tangent (elastic behaviour) or with a vertical tangent (viscous behaviour) and with more or less curvature.

IV.2 There exists a relationship between the Logarithmic family and the homographic one :

If

$$f(t) = \frac{Log(1+bt)}{Log(1+b)} \qquad g(t) = \frac{1}{b}[1-(1+b)^{-t}]$$

one has

$$g \circ f(t) = \frac{t}{1+bt}$$

IV.3 The set $B_{0,1}$ of the responses vanishing at zero and with value 1 at $t=1$, is a monoid for composition.

The relation

$$f \sqsubset g \quad \Longleftrightarrow \exists h \in B_{0,1}, \quad g = f \circ h$$

defines an order on $B_{0,1}$. Let we call **branch** a totally ordered subset of $B_{0,1}$, each point of $B_{0,1}$ is the starting point of an uncountable family of branches. Nevertheless, $B_{0,1}$ does not possess the structure of a tree (cf Chateau (1990)).

IV.4 For building, with these families a multivariate viscoelastic response, it suffices to chose the f_{ij}'s among the differences of the preceding functions, and to check that the associated Lévy measure ν_{ij} define a positive semi-definite matrix (which means that the matrix $\nu_{ij}(A)$ is symmetric positive semi-definite for every Borel set $A \subset \mathbb{R}_+^*$).

For example, it can be taken

$$\begin{pmatrix} 1_{[0,1]}(y)dy & -1_{[0,1]}(y)ydy \\ 1_{[0,1]}(y)ydy & 1_{[0,1]}(y)dy \end{pmatrix}$$

which corresponds to the matrix of responses

$$\begin{pmatrix} 1 - \frac{1-e^{-t}}{t} & \frac{1-e^{-t}(t+1)}{t^2} - \frac{1}{2} \\ \frac{1-e^{-t}(t+1)}{t^2} - \frac{1}{2} & 1 - \frac{1-e^{-t}}{t}. \end{pmatrix}$$

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