On Mukai flops for Scorza varieties

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Abstract

I give three descriptions of the Mukai flop of type $E_{6,I}$, one in terms of Jordan algebras, one in terms of projective geometry over the octonions, and one in terms of $\mathbb{O}$-blow-ups. Each description shows that it is very similar to certain flops of type A. The Mukai flop of type $E_{6,II}$ is also described.

Introduction

In this article, I study a class of birational transformations called "Mukai flops". Let G/P be a flag variety. Recall [Ric 74] that the natural map $T^*G/P \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G, has image the closure of a single nilpotent orbit.

Sometimes, it happens that for two parabolic subgroups $P, Q \subset G$, the images in $\mathfrak{g}^*$ of T^*G/P and T^*G/Q are equal to the same orbit closure $\overline{\mathcal{O}}$, and that moreover, the above maps are birational isomorphisms. We therefore get a birational map

$$\begin{array}{ccccc} T^*G/P & & \dashrightarrow & & T^*G/Q, \\ & \searrow & & \swarrow & \\ & & \overline{\mathcal{O}} & & \end{array}$$

called a Mukai flop.

Since T^*G/P is a symplectic variety, nilpotent orbit closures provide a wide class of examples of symplectic singularities and were studied also for this reason. If $\overline{\mathcal{O}}$ is a nilpotent orbit closure, then B. Fu showed that any symplectic resolution of $\overline{\mathcal{O}}$ is given by a map $T^*G/P \to \overline{\mathcal{O}}$ [Fu 03]. On the other hand, in [Nam 04], it is proved that any Mukai flop can be described using fundamental ones, when P (and Q) is a maximal parabolic subgroup : G is then of type A, D_{2n+1} or E_6. In some sense, this provides a complete understanding of the different symplectic resolutions of $\overline{\mathcal{O}}$ and the relations between them.

In fact, the classical fundamental flops, when G is of type A_n or D_{2n+1}, are easy to describe. The only items which are not very well understood in this matter are the fundamental Mukai flops of type E_6, and the purpose of this article is to fill this gap.

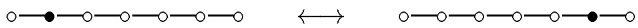
Along with this motivation in birational geometry, these flops are key ingredients for the definition of generalized dual varieties [Cha 06] for a subvariety of the homogeneous space G/P, when G is of type E_6 and P is the parabolic subgroup corresponding to the root α_1 or α_3, with Bourbaki's notations [Bou 68].

For example, an easy consequence of theorem 3.3 is theorem 2.1 in [Ch 06], which generalizes the fact that the dual variety of the smooth quadric in $\mathbb{P}V$ defined by an invertible symmetric map $f : V \to V^*$ is the quadric in $\mathbb{P}V^*$ defined by f^{-1}, when the usual projective space $\mathbb{P}V$ is replaced by any Scorza variety (see subsection 3.1 for the definition of Scorza varieties; for example, a grassmannian of 2-dimensional subspaces of an even-dimensional fixed space, and $E_6/P_1, P_1$ the parabolic subgroup of the adjoint group of type E_6 corresponding to the root α_1, are Scorza varieties).

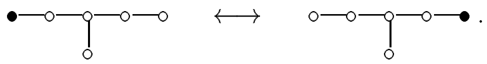
Finally, a third motivation is the study of the geometry of exceptional homogeneous spaces. For example, subsection 3.5 starts a study of the geometric properties of E_6/P_3, with a rather detailed description of its tangent bundle.

There are two flops of type E_6, denoted $E_{6,I}$ (then P corresponds to the root α_1 and Q to α_6) and $E_{6,II}$ ($P = P_2, Q = P_5$). I give three descriptions of the flop $E_{6,I}$: one via the geometry of the corresponding flag variety, one using Jordan algebras, and one using a new class of birational transformations that I call $\mathbb{O}$-blow-ups.

In fact, these three constructions work uniformly for G/P any Scorza variety. This gives for example a common description of the flop



and the flop



This allows to understand better the latter.

I now describe more precisely the contents of this article. Let k be a field and let $x \in \mathbb{P}^n_k$. Then a non-vanishing tangent vector $t \in T_xX$ defines a unique line l with the following properties :

- $x \in l$,
- $t \in T_x l$.

Moreover, the rational map $t \mapsto l$ is clearly the quotient map $T_xX \simeq \mathbb{A}^n_k \dashrightarrow \mathbb{P}^{n-1}_k$, where $\mathbb{P}^{n-1}_k$ denotes the variety of lines in $\mathbb{P}^n_k$ through x. Dually, we have a similar rational map $T^*_xX \dashrightarrow (\mathbb{P}^{n-1}_k)^{\vee}$.

Section 3 is devoted to proving the same kind of results when the variety $\mathbb{P}^n_k$ is replaced by a Scorza variety (see subsection 3.1), which after [Cha 05] is considered as a projective space $\mathbb{P}^n_{\mathcal{A}}$ over a composition algebra $\mathcal{A}$, so that when $\mathcal{A} = k$, we recover $\mathbb{P}^n_k$. So, in this section, I show theorems 3.2 and 3.3, which have the following interpretation in terms of projective geometry over $\mathcal{A}$: given a generalized projective space $\mathbb{P}^n_{\mathcal{A}}$ and a point $x \in \mathbb{P}^n_{\mathcal{A}}$, there is a rational quadratic map $\overline{\nu_x^+} : T_x\mathbb{P}^n_{\mathcal{A}} \dashrightarrow \mathbb{P}^{n-1}_{\mathcal{A}}$, which maps a tangent vector to the unique $\mathcal{A}$-line through it. Dually, there is a similar map $\overline{\nu_x^-} : T^*_x\mathbb{P}^n_{\mathcal{A}} \dashrightarrow (\mathbb{P}^{n-1}_{\mathcal{A}})^{\vee}$.

Propositions 3.4 and 3.5 show that polarizing ν_x^+ (resp. ν_x^-), one gets an isomorphism between the variety of lines in $\mathbb{P}^n_{\mathcal{A}}$ through x and the Fano variety of maximal linear subspaces included in $\mathbb{P}^{n-1}_{\mathcal{A}}$ (resp. $(\mathbb{P}^{n-1}_{\mathcal{A}})^\vee$). These two results don't have analogs when $\mathcal{A} = k$.

Note that this last $(\mathbb{P}^{n-1}_{\mathcal{A}})^\vee$ is the projective space of hyperplanes containing x; it is therefore included in $(\mathbb{P}^n_{\mathcal{A}})^\vee$. The connection with Mukai flops is as follows : assume that G/P is the Scorza variety $\mathbb{P}^n_{\mathcal{A}}$. Let $x \in G/P$. We will see that there is a Mukai flop $T^*\mathbb{P}^n_{\mathcal{A}} \dashrightarrow T^*(\mathbb{P}^n_{\mathcal{A}})^\vee$. The structure map $T^*G/Q \to G/Q$ and this Mukai flop yield a composition $T^*_x\mathbb{P}^n_{\mathcal{A}} = T^*_xG/P \dashrightarrow T^*G/Q \to G/Q = (\mathbb{P}^n_{\mathcal{A}})^\vee$. Theorem 3.3 shows that this composition is the map $\overline{\nu_x}$: $T^*_x\mathbb{P}^n_{\mathcal{A}} \dashrightarrow (\mathbb{P}^{n-1}_{\mathcal{A}})^\vee \subset (\mathbb{P}^n_{\mathcal{A}})^\vee$.

Then, I show a general canonical isomorphism of quotients of tangent spaces to homogeneous spaces (theorem 4.1). As a particular case, this theorem gives a way of computing a Mukai flop $T^*G/P \dashrightarrow T^*G/Q$ once we know the composition $T^*G/P \dashrightarrow G/Q$. I deduce a description of the flop of type $E_{6,I}$ (proposition 4.1), in terms of Jordan algebras.

In subsection 4.3, I give a maybe more geometric description of the flop $E_{6,I}$. Recall that the minimal resolution of the simplest Mukai flop $T^*\mathbb{P}^n \dashrightarrow T^*(\mathbb{P}^n)^\vee$ is the blow-up of $T^*\mathbb{P}^n$ along the zero section. I show that the same result holds for the $E_{6,I}$-flop $T^*\mathbb{P}^2_{\mathbb{O}} \dashrightarrow T^*(\mathbb{P}^2_{\mathbb{O}})^\vee$, if one replaces the usual notion of blow-up with an octonionic version of it (theorem 4.2).

Finally, concerning the Mukai flop of type $E_{6,II}$, I use the fact that the homogeneous space E_6/P_3 can be realized as the space of lines included in E_6/P_1. Theorem 4.3 uses this model and the study of the tangent bundle $T(E_6/P_3)$ performed in subsection 3.5 to give also a description of the Mukai flop of type $E_{6,II}$.

Sections 1 and 2 study the restriction of the flops to a cotangent space in the two cases when G is of type E_6. They are of course L-equivariant rational maps, if L is a Levi factor of P, and happen to be quite subtil. In each case, I show that they are the only P-equivariant rational map $T^*_xG/P \dashrightarrow (G/Q)_x$ (propositions 1.5 and 2.1), if $(G/Q)_x$ denotes the variety of y's in G/Q with stabilizor Q_y such that $P_x \cap Q_y$ is parabolic.

In the case of a flop of type $E_{6,I}$ for instance, we get a $Spin_{10}$-equivariant rational map; G/P is often called the Moufang plane $\mathbb{P}^2_{\mathbb{O}}$ (it is some kind of octonionic projective plane). As an example of the above discussion, the restriction of the flop to a cotangent space should interpret as the "quotient map" $\mathbb{A}^2_{\mathbb{O}} \dashrightarrow \mathbb{P}^1_{\mathbb{O}}$. In the first section, I show that this map has some properties of such a quotient; for example, its fibers carry a natural structure of algebra isomorphic with the octonions (corollary 1.12). I also study the projective geometry of the corresponding spinor variety.

Similarly, section 2 gives a model for the restriction of the Mukai flop of type $E_{6,II}$ to a cotangent space. In this case, a Levi factor contains $SL_2 \times SL_5$ and the relevant factor of T^*E_6/P_3 is $Hom(\mathbb{C}^2, (\wedge^2\mathbb{C}^5)^*)$. The given classification of $(GL_2 \times GL_5)$-orbits in $Hom(\mathbb{C}^2, (\wedge^2\mathbb{C}^5)^*)$ allows to understand the E_6-orbits in T^*E_6/P_3. Finally, corollary 4.3 states that the Mukai flop is defined only on the open orbit of T^*E_6/P_3, and describes the image of all orbits in T^*E_6/P_3 as nilpotent orbits in $\mathfrak{e}_6$.

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Contents

1 The group $Spin_{10}$ and $\mathbb{P}^1_{\mathbb{O}}$.

1.1 Geometric definition of composition algebras

For more details on composition algebras, the reader my consult [Cha 05]. I recall that if R is a ring, then $\mathbb{R}_R, \mathbb{C}_R, \mathbb{H}_R$ and $\mathbb{O}_R$ denote the four usual split composition algebras over R. Therefore, $\mathbb{R}_R = R, \mathbb{C}_R = R \oplus R$, $\mathbb{H}_R$ is the algebra of 2×2-matrices with coefficients in R, and $\mathbb{O}_R$ is obtained from $\mathbb{H}_R$ by Cayley-Dickson's process.

Their norms will be denoted N. If $\mathcal{A}$ is one of those and $z \in \mathcal{A}$, then L_z and R_z denote the endomorphisms of $\mathcal{A}$ of left and right multiplication by z, and $L(z), R(z) \subset \mathcal{A}$ their images.

In the following, we will have to define a composition algebra structure on a vector space by geometric means. This subsection explains how it is possible. In this section, k is an algebraically closed field of caracteristic different from 2.

Proposition 1.1. *Let V be a k-vector space of dimension a, with $a \in \{1, 2, 4\}$. Let $x_0 \in V - \{0\}$ and $\mathcal{N} \subset \mathbb{P}V$ be a smooth quadric such that the class of x_0 in $\mathbb{P}V$ does not belong to $\mathcal{N}$.*

Then if $a \in \{1, 2\}$, there exists a unique composition algebra structure on V with unit x_0 and such that $\mathcal{N}$ is the quadric of elements with vanishing norm. If $a = 4$, there are two such composition algebras.

Therefore, giving a composition algebra structure on V is equivalent with giving a smooth quadric in $\mathbb{P}V$ and a point out of its affine cone (if $a = 4$, we must moreover choose a component of the variety of maximal isotropic subspaces of the quadric).

Proof : The existence of the algebra is an immediate consequence of the fact that $Aut(\mathcal{N})$ acts transitively on $\mathbb{P}V - \mathcal{N}$.

The unicity in the cases when $a \in \{1, 2\}$ is easy. Assume $a = 4$ and let $(x, y) \mapsto xy$ be a composition product satisfying the conditions of the proposition. Let $\mathcal{L}, \mathcal{R} \simeq \mathbb{P}^1$ be the two families of isotropic lines. For $x \in \mathcal{Q}$, denote $l(x)$ (resp. $r(x)$) the isotropic line in $\mathcal{L}$ (resp. $\mathcal{R}$) containinig x. Up to changing the algebra structure $(x, y) \mapsto xy$ into $(x, y) \mapsto yx$, we may assume that $\forall x \in \mathcal{N}, L(x) \in \mathcal{L}$. Therefore, $L(x) = l(x)$ and $R(x) = r(x)$.

If $z \in V$, let $[z]$ denote its class in $\mathbb{P}V$. Then, for generic $x, y \in \mathcal{N}$, $[xy] = L(x) \cap R(y) = l(x) \cap r(y)$. Therefore, the product of two elements of $\mathcal{N}$ is fixed up to a scale once $\mathcal{N}$ is. One checks also that $xy = 0$ if and only if $r(x) \cap l(y)$ is orthogonal to the unit, with respect to the scalar product defined by $\mathcal{N}$. In view of lemma 1.1 applied to left multiplication by $x \in \mathcal{N}$, the proposition is proved. □

Lemma 1.1. *Let V and W be vector spaces and $f, g : V \to W$ linear maps. Let $X \subset \mathbb{P}V$ be an irreducible variety included in no hyperplane of $\mathbb{P}V$. Assume that the induced rational maps $[f]_{|X}, [g]_{|X} : X \dashrightarrow \mathbb{P}W$ are equal and that $\ker f = \ker g$. Then there exists $\lambda \in k - \{0\} : f = \lambda g$.*

Proof : f and g have the same image, spanned by $f(X) = g(X)$. They also have the same kernel by hypothesis. Therefore, there is a linear automorphism h of this common image, such that $g = hf$. Since $[f]_{|X} = [g]_{|X}$, any vector in $f(X)$ is an eigenvector for h, from which the lemma follows. □

We now consider the case of the octonions.

Proposition 1.2. *Let V be an 8-dimensional vector space and $\mathcal{N} \subset \mathbb{P}V$ a smooth quadric. Let G denote the grassmannian of maximal isotropic subspaces of $\mathcal{N}$, and let l be an isomorphism between $\mathcal{N}$ and an irreducible component of G. Assume $\forall x \in \mathcal{N}, x \in l(x)$ and let $x_0 \in V - \{0\}$ such that $[x_0] \notin \mathcal{N}$.*

Then there exists a unique composition algebra structure on V with unit x_0 and such that for all $x \in \mathcal{N}$, we have $l(x) = L(x)$.

Proof : Given an octonionic structure on V, it is known as "triality principle" [Che 97, chapter IV] that L is an isomorphism on its image, which is a connected component of G.

The unicity of the algebra structure follows the lines of the previous proposition. Let $(x, y) \mapsto xy$ be an algebra structure on V with unit x_0 and such that $L = l$. If $x \in \mathcal{N}$ is generic, then the line (x, x_0) meets $\mathcal{N}$ at x and $\overline{x}$. Therefore, x_0 determines the conjugation. By hypothesis, $L(x) = l(x)$, therefore we get $R(x)$ as $\overline{l(\overline{x})}$. Now, the class of the product xy in $\mathbb{P}V$ is again $L(x) \cap R(y)$, and $xy = 0$ if and only if $\dim L(x) \cap R(x) = 3$, as is well-known [Che 97, IV.4.2]. Therefore, lemma 1.1 proves the unicity of the algebra.

Let us prove its existence. Put on V an arbitrary structure of composition algebra $(x, y) \mapsto xy$ such that the quadric of elements of vanishing norm is $\mathcal{N}$. This induces isomorphisms L, R between $\mathcal{N}$ and the components of G. Set $r(x) = \overline{l(\overline{x})}$. We can assume that L and l have the same image. Therefore, there exist $f, g \in Aut(\mathcal{N})$ such that $l(x) = L(f(x))$ and $r(x) = R(g(x))$. The hypothesis $x \in l(x)$ implies $x \in r(x)$, and so $f(x) \in L(x)$ and $g(x) \in R(x)$ [Cha 05, proposition 1.1]. By the following lemma 1.2, there exist invertible α, β such that $f(x) = x\alpha$ and $g(y) = \beta y$. The composition algebra $x * y = (x\alpha)(\beta y)$, with unit $\beta^{-1}\alpha^{-1}$, satisfies the conditions of the proposition. □

Lemma 1.2. *Let $m : \mathbb{O}_k \to \mathbb{O}_k$ a linear map preserving $\mathcal{N}$ and such that $\forall x \in \mathcal{N}, m(x) \in L(x)$. Then there exists $\alpha \in \mathbb{O}_k$ such that $\forall x \in \mathbb{O}_k, m(x) = x\alpha$.*

Proof : Left to the reader [Cha 03, p.48]. □

1.2 The 8-dimensional quadric as $\mathbb{P}^1_{\mathbb{O}}$

I have juste recalled the triality principle, which implies that the three 8-dimensional fundamental representations of $Spin_8$ can be identified with the algebra of octonions. The goal of this subsection is to relate the group $Spin_{10}$ with the octonions, see proposition 1.4. To study the representations of $Spin_{10}$, my strategy is to restrict them to representations of $Spin_8$. Before proving proposition 1.4, I need to make a computation in Clifford algebras. My notations are those of [Che 97].

Let V be a k-vector space of even dimension and equipped with a non-degenerate quadratic form q. Let $V' \subset V$ be a codimension two subspace in V such that $q_{|V'}$ is non-degenerate. Let C, C' denote the Clifford algebra of V, V' (the Clifford algebra of V is the tensor algebra of V mod out by the relations $x \otimes x = q(x)$). Let α be the "main antiautomophism" of V, defined by $\alpha(v_1 \dots v_k) = v_k \dots v_1$.

Let $V' = N' \oplus P'$ be a decomposition into isotropic subspaces. Let $x_0, y_0 \in V$ be orthogonal to V' and such that $q(x_0, y_0) = 1$. Denote $N = N' \oplus k.x_0$ and $P = P' \oplus k.y_0$.

Let $C_N \subset C$ (resp. $C'_N \subset C'$) be the subalgebra of C (resp. C') generated by N (resp. N'). Let $f' \in C'_N$ be the product of the elements of a basis of N' and $f = f' y_0$. Let $\mathbb{S}^{\pm}$ and $\mathbb{S}'^{\pm}$ be the spin representations of $Spin(V)$ and $Spin(V')$. We may choose $\mathbb{S}^+$ (resp. $\mathbb{S}^-$) be the subspace of even (resp. odd) elements of C_N, and similarly for $\mathbb{S}'^{\pm}$.

There are isomorphisms $\varphi^{\pm}$ between $\mathbb{S}'^+ \oplus \mathbb{S}'^- = C'_N$ and $\mathbb{S}^{\pm} = C_N^{\pm}$, given by $\varphi^+(u'_+ + u'_-) = u'_+ + u'_- x_0$ and $\varphi^-(u'_+ + u'_-) = u'_+ x_0 + u'_-$. Finally, there is a quadratic map $\beta : C_N \times C_N \to \wedge V$, where $\beta(u, v)$ is the image of $uf\alpha(v) \in C$ in $\wedge V$ under the canonical vector space isomorphism $C \simeq \wedge V$ [Che 97, p.102,103 and II 1.6]. Let $\beta' : C'_N \times C'_N \to \wedge V'$ be the similar map for V'.

Proposition 1.3. *Let $r' = \dim V'/2$. Let $u'_+, v'_+ \in \mathbb{S}'^+$ and let $u'_-, v'_- \in \mathbb{S}'^-$. We have*

$$\begin{aligned} & \beta[\varphi^+(u'_+ + u'_-), \varphi^+(v'_+ + v'_-)] \\ = \; & \beta'(u'_+, v'_+) \wedge y_0 - x_0 \wedge y_0 \wedge \beta'(u'_+, v'_-) + \beta'(u'_+, v'_-) \\ + \; & (-1)^{r'}(x_0 \wedge y_0 \wedge \beta'(u'_-, v'_+) + \beta'(u'_-, v'_+)) - x_0 \wedge \beta'(u'_-, v'_-), \end{aligned}$$

and

$$\begin{aligned} & \beta[\varphi^-(u'_+ + u'_-), \varphi^-(v'_+ + v'_-)] \\ = \; & x_0 \wedge \beta'(u'_+, v'_+) + (-1)^{r'}(x_0 \wedge y_0 \wedge \beta'(u'_+, v'_-) + \beta'(u'_+, v'_-)) \\ + \; & y_0 \wedge x_0 \wedge \beta'(u'_-, v'_+) + \beta'(u'_-, v'_+) - \beta'(u'_-, v'_-) \wedge y_0. \end{aligned}$$

Proof : We have, in the Clifford algebra C, $u'_+ f' y_0 \alpha(v'_+) = u'_+ f' \alpha(v'_+) y_0$, so $\beta[\varphi^+(u'_+), \varphi^+(v'_+)] = \beta'(u'_+, v'_+) \wedge y_0$. We can compute the other terms $\beta[\varphi^+(u'_\pm), \varphi^+(v'_\pm)]$ using the facts

$$u'_+ f' y_0 \alpha(v'_- x_0) = y_0 x_0 u'_+ f' \alpha(v'_-), \quad u'_- x_0 f' y_0 \alpha(v'_+) = (-1)^{r'} x_0 y_0 u'_- f' \alpha(v'_+),$$
$$\text{and} \quad u'_- x_0 f' y_0 \alpha(v'_-) = x_0 y_0 x_0 u'_- f' \alpha(v'_-) = x_0 u'_- f' \alpha(v'_-).$$

The computation of $\beta[\varphi^-(u'_+ + u'_-), \varphi^-(v'_+ + v'_-)]$ is similar. □

Our second task is to describe spinor representations using octonions. Let $V = H_2(\mathbb{O}_k)$ denote the 10-dimensional k-vector space of 2×2 hermitian matrices with entries in $\mathbb{O}_k$. Let det be the quadratic form on $H_2(\mathbb{O}_k)$ defined by $\det\left(\begin{pmatrix} t & z \\ \overline{z} & u \end{pmatrix}\right) = tu - N(z)$ ($t, u \in k$ and $z \in \mathbb{O}_k$). Recall [Che 97, III 1.2,III 1.4] that the variety of maximal isotropic subspaces of V has two components; they will be denoted $G_Q^+(5, V)$ and $G_Q^-(5, V)$. Moreover, there are natural projective embeddings $G_Q^\pm(5, V) \subset \mathbb{PS}^\pm$ in the projectivized spinor representations, the elements of $\mathbb{S}^\pm$ which class are in $G_Q^\pm(5, V)$ being called "pure spinors".

Let $\nu_2^+ : \mathbb{O}_k \times \mathbb{O}_k \to H_2(\mathbb{O}_k)$ the quadratic map defined by $\nu_2^+(a, b) = \begin{pmatrix} N(a) & a\overline{b} \\ b\overline{a} & N(b) \end{pmatrix}$ and μ^+ the polarization of ν^+ : $\mu^+((a, b), (c, d)) = \nu_2^+(a + c, b + d) - \nu_2^+(a, b) - \nu_2(c, d))$. Similarly, let $\nu_2^-(a, b) = \begin{pmatrix} N(b) & a\overline{b} \\ b\overline{a} & N(a) \end{pmatrix}$ and μ^- the polarization of ν_2^-.

Let $X^+ = X^- \subset \mathbb{P}(\mathbb{O}_k \oplus \mathbb{O}_k)$ be defined by $[(a, b)] \in X^\pm \iff \nu_2^\pm(a, b) = 0$.

Proposition 1.4. *The variety $X^\pm$ is isomorphic with $G_Q^\pm(5, V)$. An isomorphism $X^\pm \to G_Q^\pm(5, V)$ maps (u, v) on the image of $\mu^\pm((u, v), .)$.*

Proof : Let $q = -\det$, $V' = \left\{\begin{pmatrix} 0 & z \\ \overline{z} & 0 \end{pmatrix}\right\} \simeq \mathbb{O}_k$, $x_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ and $y_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let β_k denote the component in $\wedge^k V \subset \wedge V$ of β. Since q restricts to the norm of octonions on $V' \simeq \mathbb{O}_k$, by the triality principle [Che 97, Chapter IV], with the notations of proposition 1.3, there are linear isomorphisms $\mathbb{S}'^\pm \to \mathbb{O}_k$ such that the map $\beta'_1 : \mathbb{S}'^+ \times \mathbb{S}'^- \to V'$ identifies with the product of octonions, and $(\beta'_0)^+ : \mathbb{S}^+ \times \mathbb{S}^+ \to k, (\beta'_0)^- : \mathbb{S}^- \times \mathbb{S}^- \to k$ identify with the scalar product of octonions. Composing with the automorphism $b \mapsto \overline{b}$, of $\mathbb{S}'^- \simeq \mathbb{O}_k$, we may assume that β'_1 is in fact given by $(a, b) \mapsto a\overline{b}$.

By proposition 1.3, $\mathbb{S}^+$ and $\mathbb{S}^-$ therefore identify with $\mathbb{O}_k \oplus \mathbb{O}_k$ in such a way that $\beta_1^+((a, b), (a, b)) = 2N(a)y_0 - 2N(b)x_0 + 2a\overline{b}$ and $\beta_1^-((a, b), (a, b)) = -2N(a)x_0 + 2N(b)y_0 + 2a\overline{b}$, that is to say, $\beta_1^\pm = \mu^\pm$.

By proposition [Che 97, III 5.2] the spinor varieties $G_Q^\pm(5, V) \subset \mathbb{PS}^\pm$ are defined by the equations $N(a) = N(b) = 0, a\overline{b} = 0$, which is equivalent to $\nu_2^\pm = 0$. Therefore, they are isomorphic with $X^\pm$. Moreover, since the linear space corresponding to s is the image of $\mu^\pm(s, .)$ [Che 97, III 4.4], the proposition is proved. □

In the sequel, we will identify both $\mathbb{S}^+$ and $\mathbb{S}^-$ with $\mathbb{O}_k \oplus \mathbb{O}_k$, keeping however in mind the fact that $\mathbb{S}^+$ and $\mathbb{S}^-$ are non-equivalent $Spin_{10}$-representations. The projectivization $\mathbb{PS}^\pm$ of $\mathbb{S}^\pm$ have two $Spin_{10}$-orbits, by [Igu 70, prop. 2 p.1011]. The closed orbits are X^+ and X^-.

Now comes the explanation of the title of this subsection : the variety of classes of matrices $\left[\begin{pmatrix} t & z \\ \bar{z} & u \end{pmatrix}\right] \in \mathbb{P}V$ with $tu - N(z) = 0$ is a $Spin_8$-conformal compactification of the variety of classes of matrices of the form $\left[\begin{pmatrix} 1 & z \\ \bar{z} & N(z) \end{pmatrix}\right]$ which is isomorphic with $\mathbb{O}_k \simeq \mathbb{A}^1_{\mathbb{O}}$, therefore, it can be thought as $\mathbb{P}^1_{\mathbb{O}}$. Moreover, the projectivisations $\overline{\nu_2^{\pm}} : \mathbb{S}^{\pm} \dashrightarrow \mathbb{P}\{\det = 0\}$ of the maps $\nu_2^{\pm}$ are some kind of quotient maps $\mathbb{A}^2_{\mathbb{O}} \dashrightarrow \mathbb{P}^1_{\mathbb{O}}$. Proposition 1.8 and corollary 1.12 illustrate this viewpoint.

For the moment, we show that $\overline{\nu_2^+}$ and $\overline{\nu_2^-}$ are the only natural (ie $Spin_{10}$-equivariant) candidates for such a kind of quotient (proposition 1.5). Let $Q \subset \mathbb{P}V$ denote the quadric defined by det.

Lemma 1.3. *There is a unique 15-dimensional $Spin_{10}$-orbit in $(\mathbb{P}\mathbb{S}^+ - X^+) \times Q$.*

Proof : Let $(s_1, x_1), (s_2, x_2) \in (\mathbb{P}\mathbb{S}^+ - X^+) \times Q$. We may assume that $s_1 = s_2 = s$. Let $G_0 \subset Spin_{10}$ be the stabilizor of s. From the proof of [Igu 70, prop. 2 p.1011], it follows that $\overline{\nu_2^+}(s) \in Q$ is the only line in Q stabilized by G_0. Therefore, $x_1 = x_2 = \nu_2^+(s)$. □

Proposition 1.5. $\overline{\nu_2^+} : \mathbb{S}^+ \dashrightarrow Q$ *is the only $(k^* \times Spin_{10})$-equivariant rational map $\mathbb{S}^+ \dashrightarrow Q$.*

Proof : Let $\nu : \mathbb{S}^+ \dashrightarrow Q$ be any $(k^* \times Spin_{10})$-equivariant rational map $\mathbb{S}^+ \dashrightarrow Q$. Then $\overline{\nu_2^+}$ and ν induce rational maps $\mathbb{P}\mathbb{S}^+ \dashrightarrow Q$, which will be denoted with the same letter. Since ν is $Spin_{10}$-equivariant, it is defined on $\mathbb{P}\mathbb{S}^+ - X^+$. Therefore, the variety of $\{(s, \nu(s)) : s \in \mathbb{P}\mathbb{S}^+ - X^+\}$ is a 15-dimensional orbit in $(\mathbb{P}\mathbb{S}^+ - X^+) \times Q$. By lemma 1.3, it is equal to the orbit $\{(s, \overline{\nu_2^+}(s)) : s \in \mathbb{P}\mathbb{S}^+ - X^+\}$. □

1.3 Projective geometry of the spinor variety

We keep the notations of the previous subsection; namely, $V = H_2(\mathbb{O}_k)$, $\mathbb{S}^+ = \mathbb{S}^- = \mathbb{O}_k \oplus \mathbb{O}_k$ are the two spinor representations of $Spin_{10}$, and $\nu_2^{\pm} : \mathbb{S}^{\pm} \to V$ are the quadratic $Spin_{10}$-equivariant maps defined above. Their polarizations are denoted $\mu^{\pm}$. We denote $Q \subset \mathbb{P}V$ the smooth quadric defined by det. If $(a, b) \in \mathbb{O}_k \oplus \mathbb{O}_k$ we denote $[a, b]$ its class in $\mathbb{P}(\mathbb{O}_k \oplus \mathbb{O}_k)$. Finally, if $X \subset \mathbb{P}^n$ is a variety and $x \in X$, let $T_x X$ its tangent space and let $\widehat{X} \subset \mathbb{A}_k^{n+1}$ denote the affine cone over X.

Recall from [Che 97, III 2.3] that there is a $Spin_{10}$-equivariant perfect pairing $\mathbb{S}^+ \times \mathbb{S}^- \to k$. This allows identifying $\mathbb{S}^-$ with the dual of $\mathbb{S}^+$. Recall that the dual variety of a variety X is the closure of the set of tangent hyperplanes, where a tangent hyperplane is by definition a hyperplane containing a tangent space $T_x X$ at a smooth point $x \in X$.

Proposition 1.6. *The equivariant isomorphism $\mathbb{P}\mathbb{S}^{+^*} \simeq \mathbb{P}\mathbb{S}^-$ identifies the dual variety of X^+ with X^-.*

Proof : The dual variety of X^+ is a $Spin_{10}$-stable closed variety. Since in $\mathbb{P}S^-$ there are only two orbits, by [Igu 70, prop 2 p.1011], it is either the whole projective space $\mathbb{PS}^-$, which is absurd, or the variety X^-. $\square$

If $X \subset \mathbb{P}^n$ is a subvariety of projective space, and if $z \in \mathbb{P}^n - X$, the entry locus of z is classically defined as the closure of the set of points $x \in X$ such that the line joining x and z meets X at at least two distinct points.

If $s \in \mathbb{PS}^\pm - X^\pm$, denote $L_s^\pm$ the variety $\overline{(\nu_2^\pm)^{-1}(k^*.\nu_2^\pm(t))} \subset \mathbb{S}^\pm$, where $t \in \mathbb{S}^+$ is such that $[t] = s$. Let $L^\pm$ denote the variety $\{(s,v) \in (\mathbb{PS}^\pm - X^\pm) \times \mathbb{S}^\pm : v \in L_s^\pm\}$. Finally, let $\overline{\nu_2^\pm} : \mathbb{PS}^\pm - X^\pm \to Q$ denote the map induced by $\nu_2^\pm : \mathbb{S}^\pm \to V$.

Proposition 1.7. *Let $s \in \mathbb{PS}^+ - X^+$. Then the entry locus Q_s^+ of s in X^+ is a smooth 6-dimensional quadric in the 7-dimensional projective space $\mathbb{P}L_s^+$. Moreover, the fibration $L \to \mathbb{PS}^+ - X^+$ is locally trivial and is the push-back by $\overline{\nu_2^+}$ of a vector bundle on $Q \subset \mathbb{P}V$.*

Remark : The bundle over Q of the proposition is often called the spinor bundle.

Proof : Since $\mathbb{P}S^+ - X^+$ is a single $Spin_{10}$-orbit, it is enough to check the first claim of the proposition for $s = [1,0]$. Computing Q_s^+ is equivalent with solving the equation $(1,0) = (a,x) + (b,y)$ in $\mathbb{O}_k \oplus \mathbb{O}_k$, with (a,x) and (b,y) in the affine cone over X^+. Equivalently, (a,x) satisfies $N(a) = N(x) = 0$ and $a\overline{x} = 0$, and similarly for (b,y).

Now, the equality $a + b = 1$ implies $N(a,b) \neq 0$ ($N(.,.)$ denotes the polarization of N). This, in turn, implies $R(a) \cap R(b) = \{0\}$ [Che 97, IV 4.4]. Since $a\overline{x} = 0$, $x \in R(a)$ [Cha 05, proposition 1.1] and similarly $y \in R(b)$. Since $x = -y$, it follows that $x \in R(a) \cap R(b)$, so $x = 0$.

We thus have proved that the entry locus Q_s^+ is included in the variety of elements $[a,0]$ with $N(a) = 0$. Conversely, this smooth quadric is included in Q_s^+. Since if $N(a) \neq 0$, then left multiplication by a is invertible, and a direct computation shows that $L_s^+ = \mathbb{O}_k \oplus \{0\}$.

To show that L^+ is a vector bundle, let $s \in \mathbb{S}^+$ and $x = [\nu_2^+(s)] \in Q$. Let $L_x \subset V$ denote the line corresponding to x. First recall by definition that the image of the restriction of ν_2^+ to L_s^+ is the line of multiples of $\nu_2^+(s)$. Therefore, $\mu^+(s, L_s^+) = k.\nu_2^+(s)$. The linear space $\mu^+(s, \mathbb{S}^+)$ is $\widehat{T_x Q}$, thus the kernel of the composition $\mathbb{S}^+ \overset{\mu^+(s,.)}{\longrightarrow} \widehat{T_x X} \to \widehat{T_x X}/L_x$ is exactly L_s^+. Therefore, L^+ is the kernel of a morphism of vector bundles over $\mathbb{PS}^+ - X^+$ with constant rank; so, it is locally free.

Since L_s^+ is constant on a fiber $(\overline{\nu_2^+})^{-1}(x)$, L^+ is the push-back of a vector bundle on Q. $\square$

We now study the family of quadrics $\{Q_s^+\}$. Let $G(8, \mathbb{S}^+)$ denote the grassmannian of 8-dimensional linear spaces in $\mathbb{S}^+$ and consider the variety $\mathcal{Q} \subset G(8, \mathbb{S}^+)$ of 8-dimensional linear spaces L in $\mathbb{S}^+$ such that $X^+ \cap \mathbb{P}L$ is a smooth 6-dimensional quadric.

Proposition 1.8. *The variety $\mathcal{Q}$ is $Spin_{10}$-equivariantly isomorphic with Q and any element of $\mathcal{Q}$ is of the form Q_s. Moreover, let $s, t \in \mathbb{PS}^+ - X^+$; one of the following holds :*

1. $L_s^+ = L_t^+$ *and* $Q_s = Q_t$.
2. $Q_s^+ \cap Q_t^+ = \mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ \simeq \mathbb{P}^3$.
3. $\mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ = \emptyset$.

Remark : Although this does not make sense due to the lack of associativity of the octonions, the maps $\nu_2^\pm : \mathbb{S}^\pm \dashrightarrow Q$ should be some kind of quotient maps $\mathbb{A}^2_{\mathbb{O}} \dashrightarrow \mathbb{P}^1_{\mathbb{O}}$. The linear space L_s^+ can be interpreted as the set of $\mathbb{O}_k$-multiples of s (in $\mathbb{O}_k \oplus \mathbb{O}_k$). With this point of view, the proposition says that for two non-degenerate (out of X^+) vectors in $\mathbb{O}_k \oplus \mathbb{O}_k$, there are three possibilities : either they are linked (1), either they are free (3), either they are "weakly linked" (2). This last case would not occur if we would consider non-split octonions, say for example over the field $\mathbb{R}$ of real numbers. The same situation holds when one considers two non-degenerate vectors $v, w \in \mathbb{H}_k \oplus \mathbb{H}_k$. In fact, it is easy to check that

$$\dim\left(\{\lambda.v : \lambda \in \mathbb{H}_k\} \cap \{\lambda.w : \lambda \in \mathbb{H}_k\}\right) \in \{0, 2, 4\}.$$

(see the remark after lemma 2.1 in [Cha 05]).

Proof : Proposition 1.7 and the fonctorial property of grassmannians show that there is a map $\psi : Q \to \mathcal{Q}$. In the other way, let $l \in \mathcal{Q}$. Let δ be a generic line in $\mathbb{P}l$; this line meets the quadric $X^+ \cap \mathbb{P}l$ in two points x and y. Since ν_2^+ vanishes on $\widehat{X}^+$, for any s in δ, we have $\overline{\nu_2}^+(s) = \overline{\mu}^+(x, y)$. Therefore, $\overline{\nu_2}^+$ is constant on the generic lines in $\mathbb{P}l$, so it is constant on $\mathbb{P}l$. This proves that there is a map $\varphi : \mathcal{Q} \to Q$, induced by ν_2^+.

It is obvious, by construction, that φ and ψ are inverse maps, so the first point of the proposition is proved.

The rest of the proposition follows. In fact, set $s = (1, 0)$, so that $L_s^+ = \mathbb{O}_k \oplus 0$. If $t = s$, then $L_s^+ = L_t^+$. If $t = (1, b)$, with $N(b) = 0$, then an easy computation shows that $\mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ = Q_s^+ \cap Q_t^+ = \{(c, 0) : c \in R(b)\}$. If $t = (0, 1)$, then $L_t^+ = 0 \oplus \mathbb{O}_k$ and so $\mathbb{P}L_s^+ \cap \mathbb{P}L_t^+ = \emptyset$.

Since there are three $Spin_{10}$-orbits in $Q \times Q$, these three examples exhaust all the possibilities for a couple $(L_s^+, L_t^+) \in \mathcal{Q} \times \mathcal{Q}$. □

Let $s \in \mathbb{P}\mathbb{S}^+ - X^+$. Define Q_s^- as the intersection of X^- with the orthogonal of L_s^+ (in other words, $Q_s^- \subset (X^+)^*$ is the variety of tangent hyperplanes which contain L_s^+).

Proposition 1.9. *With notations above, Q_s^- is a 6-dimensional smooth quadric in X^-. Moreover, its linear span in $\mathbb{S}^-$ is the closure of $(\nu_2^-)^{-1}(k^*.\nu_2^+(s)) =: L_s^-$.*

Proof : Arguing as in the proof of proposition 1.4 one can show that the equivariant duality between $\mathbb{S}^+$ and $\mathbb{S}^-$ is $\langle (a, b), (c, d)\rangle = N(a, c) + N(b, d)$. Therefore, if $s = [1, 0]$, then Q_s^- is the variety $\{[0, b] : N(b) = 0\}$. Its linear span is $0 \oplus \mathbb{O}_k$, which is sent by ν_2^- on $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \nu_2^+(s)$. □

Let $\varphi^\pm$ denote the isomorphisms between $X^\pm$ and the components of the grassmannian of maximal isotropic subspaces in Q. We have another caracterisation of the quadrics Q_s^+ and Q_s^- :

Proposition 1.10. *Let $x \in X^\pm$ and $s \in \mathbb{S}^+$. Then $x \in Q_s^\pm$ if and only if $\nu_2^+(s) \in \varphi^\pm(x)$.*

Proof : By proposition 1.7, there exists a quadratic form q_s on L_s^+, which zero locus is Q_s^+, and such that $\forall u \in L_s^+, \nu_2^+(u) = q_s(u)\nu_2^+(s)$. Let $x \in \widehat{Q}_s^+ \subset L_s^+$, then for $u \in L_s^+$, we have $\mu^+(x,u) = q_s(x,u)\nu_2^+(s)$. Therefore, $\nu_2^+(s) \in \{\mu^+(x,u) : u \in \mathbb{S}^+\} = \varphi^+(x)$.

The converse implication $\nu_2^+(s) \in \varphi^+(x) \Rightarrow x \in Q_s^+$ follows by a dimension count argument.

In view of proposition 1.9, the proof of the same result for Q_s^- is similar. □

1.4 Equivariant octonionic structure on the fibers on $\nu_2^\pm$

In a honest projective space $\mathbb{P}_k^n$, over a field k, the choice of an element $v \in \mathbb{A}_k^{n+1}$ identifies the closure of the fiber of the quotient map $\mathbb{A}_k^{n+1} \dashrightarrow \mathbb{P}_k^n$ with k, since any element in this fiber can uniquely be written as $\lambda.v$, with $\lambda \in k$. Therefore, this fiber carries the structure of a field, isomorphic with k.

We will see (corollary 1.12) in this subsection something analogous for $\nu_2^\pm$, which is interpreted as a quotient map. However, let $s \in \mathbb{S}^+$; the image of the stabilizer of s in $GL(L_s^+)$ contains $Spin_7$ by [Igu 70, prop 2 p.1011]. Therefore, there is no hope to give L_s^+ an equivariant octonionic structure.

I will show that given two generic spinors $s, t \in \mathbb{S}^+$, there are equivariant octonionic structures on L_s^+ and L_t^+ (and indeed the stabilizor of two elements has a quotient isomorphic with G_2). I don't know how to interpret the necessity of two spinors to define such a structure in terms of octonionic projective geometry.

Let $s, t \in \mathbb{PS}^+ - X^+$ such that $\langle \nu_2^+(s), \nu_2^+(t) \rangle \neq 0$. The idea of the geometric definition of an octonionic structure on L_s^+ is as follows : we have the two quadrics Q_s^+ and Q_s^-. Let Q_s denote variety of lines in Q containing $[\nu_2^+(s)]$. Then Q_s is isomorphic with a 6-dimensional quadric. By proposition 1.10, Q_s^+ and Q_s^- parametrize the maximal isotropic linear spaces of Q_s. The point is to show that s and t yield an isomorphism $Q_s^+ \xrightarrow{\sim} Q_s^-$. Then, proposition 1.2 gives the octonionic structure.

The next proposition yields the isomorphism $\psi : Q_s^+ \xrightarrow{\sim} Q_s^-$. Let $x \in Q_s^+$ be such that the line through x and s is not a tangent line to Q_s^+. Call $\overline{x}$ the other point of intersection of this line with Q_s. Moreover, set $r(x) = \langle T_x X^+, L_s^+ \rangle^\perp \subset \mathbb{PS}^{+*} = \mathbb{PS}^-$.

Proposition 1.11. *For all $x \in Q_s^+$, $r(x)$ is a maximal isotropic subspace of Q_s^-. Moreover, if (x,s) is not a tangent line to Q_s, then $r(x)$ and $r(\overline{x})$ are supplementary subspaces of L_s^-. Call $\psi(x)$ the image of t by the projection on $r(x)$ with center $r(\overline{x})$. Then $\psi : Q_s^+ \to Q_s^-$ is an isomorphism.*

Proof : Assume $s = [1,0]$ and $t = [0,1]$. Let $x = [a,0] \in Q_s^+$ (therefore $N(a) = 0$). Since X^+ is the variety of pairs $[a,b]$ with $N(a) = N(b) = 0$ and $a\overline{b} = 0$, $T_x X^+ = \{[c,d] : N(a,c) = 0 \text{ and } a\overline{d} = 0\}$. Therefore, its orthogonal is the set of $[c,d]$ with c colinear with a and $d \in R(c)$. So $r(x) = \{[0,d] : d \in R(a)\}$. This is endeed a maximal isotropic subspace of Q_s^-.

Moreover, we have $\overline{x} = [\overline{a}, 0]$, and so $r(\overline{x}) = \{[0,d] : d \in R(\overline{a})\}$. Therefore, $r(x)$ and $r(\overline{x})$ are supplementary.

Finally, since $t = [0,1] = [0, (a+\overline{a})/2]$, we deduce that $\psi(x) = [0,a]$. We have proved that $\psi([a,0]) = [0,a]$, so ψ is an isomorphism. □

Corollary 1.12. *Let $s,t \in \mathbb{PS}^+ - X^+$ such that $\langle \nu_2^+(s), \nu_2^+(t)\rangle \neq 0$. Then L_s^+ has a natural structure of algebra, isomorphic with $\mathbb{O}_k$.*
Moreover, when (s,t) vary, this octonionic structure on the vector bundle with fiber L_s^+ varies algebraically.

Proof : We have isomorphisms $\psi^\pm$ between $Q_s^\pm$ and the components of the variety of maximal isotropic subspaces in Q_s, as explained at the beginning of this paragraph.

If $s = (1,0)$ and $t = (0,1)$, it follows from the proof of proposition 1.11 that $\forall x \in Q_s^+, \dim\,(\psi^+(x) \cap \psi^-(\psi(x))) = 3$. This is analogous to the condition $x \in l(x)$ of proposition 1.2, and therefore s and the isomorphisms ψ, ψ^+, ψ^- define a unique octonionic structure on L_s^+.

It follows by general arguments that this octonionic structure varies algebraically. Alternatively, one can give another construction of this octonionic structure, where the algebraicity is clear.

Let $x = \nu_2^+(s)$ and $M = \widehat{T_x Q}/k.x$. Then, as one checks one the example $s = (1,0), t = (0,1)$, $\mu^+(s,.)$ restricts to an isomorphism ν_t between L_t^+ and M and $\mu^+(t,.)$ to an isomorphism ν_s between L_s^+ and M. We can therefore give an octonionic structure to L_s^+ by setting

$$\forall u,v \in L_s^+,\ uv = \nu_s^{-1}[\mu^+(u, \nu_t^{-1}(\nu_s(v)))].$$

A direct computation shows that this octonionic structure is the same as the previous one. □

2 Geometry associated with two skew-forms in k^5

In this section, we consider a model for the restriction of the Mukai flop of the second kind to a tangent space. In the first subsection, we prove lemmas which will suffice defining this restriction, in section 4. The second subsection will be used when classifying the orbits in T^*G/P, for G of type E_6 and P corresponding to α_3. The third subsection shows that the involved rational map is the unique equivariant rational map.

Let k denote an arbitrary field.

2.1 A rational map $Hom(k^2, (\wedge^2 k^5)^*) \dashrightarrow G(3, k^5)$

Let r be an integer and let F be a vector space of dimension $2r+1$. An element ω in $\wedge^2 F^*$ yields a skew-symmetric map $F \to F^*$ which will be denoted L_ω. The rank, image, and kernel of ω will be those of L_ω. If $f_1, f_2 \in F$, $\omega(f_1,f_2)$ will denote the number $L_\omega(f_1)(f_2)$.

Lemma 2.1. *Let $\omega \in \wedge^2 F^*$ of rank $2t$ and $U \subset F$ a linear subspace of dimension $2r+1-t$ and such that $\omega \perp \wedge^2 U$. Then*

- *If $u \in U$, then $\omega(u, U) \equiv 0$.*
- $\ker \omega \subset U$.

Proof : Taking a basis of F containing a basis of U and decomposing ω along this basis, one checks that the condition $\omega \perp \wedge^2 U$ is equivalent to $\forall u,v \in U, \omega(u,v) = 0$, proving the first point.

Therefore, we have $L_\omega(U) \subset U^\perp$, and since $2t = \mathtt{rg}(L_\omega) \leq \mathtt{rg}(L_{\omega|U}) + t$, we have $\mathtt{rg}(L_{\omega|U}) = t$ and so $L_\omega(U) = U^\perp$. Since moreover L_ω is skew-symmetric, it follows that $\ker L_\omega = (\mathtt{Im}\ L_\omega)^\perp \subset (U^\perp)^\perp = U$. □

Notation 2.1. *Let $\omega_1, \omega_2 \in \wedge^2 F^*$. We denote*

$$l(\omega_1, \omega_2) := \{f \in F : \forall u \in \ker \omega_1, \omega_2(u, f) = 0\}.$$

Lemma 2.2. *Assume $2(2r+1) = 5t$. Let $\omega_1, \omega_2 \in \wedge^2 F^*$ with rank $2t$ be such that*

1. *$\ker \omega_1 \cap \ker \omega_2 = \{0\}$, and*
2. *$L_{\omega_2}(\ker \omega_1) \cap L_{\omega_1}(\ker \omega_2) = \{0\}$.*

If a linear subspace $U \subset F$ of dimension $2r+1-t$ is such that $\wedge^2 U \perp \omega_i, i = 1, 2$, then $U = l(\omega_1, \omega_2) \cap l(\omega_2, \omega_1)$.

We will see in lemma 2.3 that for the minimal possible values of r, t, which are those of interest to describe Mukai's flop, $U = l(\omega_1, \omega_2) \cap l(\omega_2, \omega_1)$ satisfies indeed $\wedge^2 U \perp \omega_i, i = 1, 2$; this is not the case in general.

Proof : Let $u \in \ker \omega_1$. By the previous lemma, we have $\ker \omega_1 \subset U$, so $u \in U$. If $f \in U$, it follows that $\omega_2(u, f) = 0$, so $f \in l(\omega_1, \omega_2)$ and $U \subset l(\omega_1, \omega_2)$. By symmetry, we have also $U \subset l(\omega_2, \omega_1)$. By condition (1),

$$\dim \ker \omega_1 = \dim L_{\omega_2}(\ker \omega_1) = \dim L_{\omega_1}(\ker \omega_2) = 2r + 1 - 2t,$$

and by condition (2),

$$\dim(L_{\omega_2}(\ker \omega_1) + \dim L_{\omega_1}(\ker \omega_2)) = 2(2r + 1 - 2t).$$

Since we know that U is orthogonal to this space, and since by hypothesis $2r + 1 - 2(2r + 1 - 2t) = 4t - (2r + 1) = 2r + 1 - t = \dim U$, U is exactly the orthogonal of this space, proving the lemma. □

Notation 2.2. *Denote $U(\omega_1, \omega_2) := l(\omega_1, \omega_2) \cap l(\omega_2, \omega_1)$.*

Lemma 2.3. *Assume $r = t = 2$ and let $\omega_1, \omega_2 \in \wedge^2 F^*$ be arbitrary. Then there exists $U \subset F$ of dimension 3 such that $\wedge^2 U \perp \omega_1, \omega_2$. Therefore, if the two conditions of lemma 2.2 are satisfied, then $\wedge^2 U(\omega_1, \omega_2) \perp \omega_1, \omega_2$. If moreover ω_1', ω_2' are linear combinations of ω_1, ω_2 which also satisfy the two conditions, then $U(\omega_1', \omega_2') = U(\omega_1, \omega_2)$.*

Proof : The second claim is a consequence of the first and the lemma 2.2. The third claim follows from the second since $\wedge^2 U(\omega_1, \omega_2) \perp \omega_1', \omega_2'$. It is therefore enough to prove the first claim.

Let $G = \wedge^2 F^* \oplus \wedge^2 F^* \simeq Hom(k^2, \wedge^2 F^*)$. There is a natural $GL_2 \times GL(F)$ action on G. Let $G(3, F)$ denote the grassmannian of 3-spaces in F and consider the incidence variety $I \subset G(3, F) \times \mathbb{P}G$ defined by $(U, [\omega_1, \omega_2]) \in I$ if and only if $\wedge^2 U \perp \omega_1, \omega_2$. It is a closed projective $GL_2 \times GL(F)$-stable variety. Therefore, its projection $p_2(I) \subset \mathbb{P}G$ also.

Now, let $f_1, \ldots, f_5$ be a basis of F and $f_1^*, \ldots, f_5^*$ be the dual basis of F^*. Set $\omega_1 = f_4^* \wedge f_1^* + f_5^* \wedge f_2^*$ and $\omega_2 = f_4^* \wedge f_2^* + f_5^* \wedge f_3^*$. It is clear that if $U = \mathtt{Vect}(f_1, f_2, f_3)$, then $\wedge^2 U \perp \omega_1, \omega_2$; therefore $[\omega_1, \omega_2] \in p_2(I)$. It is proved in [KS 77, proof of proposition 13 p.94] that the $GL_2 \times GL(F)$-orbit through $[\omega_1, \omega_2]$ is dense (it also follows from lemma 2.5); therefore $p_2(I) = G$ and the existence claim of the lemma is proved. □

2.2 $GL_2 \times GL_5$-orbits

Let as above F a 5-dimensional vector space over k. In this subsection, I describe the $GL_2 \times GL(F)$-orbits in $Hom(k^2, \wedge^2 F^*)$, and prove where the previous rational map $U : \mathbb{P}Hom(k^2, \wedge^2 F^*) \dashrightarrow G(3, F)$, defined on the open orbit by notation 2.2, extends.

We start with a result of co-diagonalisation of 2-forms of maximal rank :

Lemma 2.4. *Let $\omega_1, \omega_2 \in \wedge^2 F^*$ be forms such that $\forall(\alpha_1, \alpha_2) \in k^2 - \{(0,0)\}$, $\alpha_1\omega_1 + \alpha_2\omega_2$ has rank 4. Then there exists a basis $f_1^*, \dots, f_5^*$ of F^* such that*

$$\begin{aligned} \omega_1 &= f_2^* \wedge f_4^* + f_3^* \wedge f_5^* \\ \omega_2 &= f_1^* \wedge f_5^* + f_3^* \wedge f_4^*. \end{aligned}$$

Proof : For $i \in \{1,2\}$ and $u, v \in F$, we denote $\langle u, v\rangle_i := L_{\omega_i}(u)(v)$.

Assume that $\ker\omega_1 = \ker\omega_2$. Denote K this 1-dimensional vector space. Then ω_1, ω_2 belong to $\wedge^2(F/K)^*$. The variety of degenerate 2-forms in $(F/K)^*$ is a hypersurface, so there exists $(\alpha_1, \alpha_2) \in k^2 - \{(0,0)\}$ such that $\alpha_1\omega_1 + \alpha_2\omega_2$ is degenerate, contradicting the hypothesis of the lemma.

We consider $0 \neq f_1 \in \ker\omega_1$ and $0 \neq f_2 \in \ker\omega_2$; f_1 and f_2 are therefore not colinear.

Assume now that $L_{\omega_1}(f_2)$ and $L_{\omega_2}(f_1)$ are colinear. Denote I this common image. The map $F^* \to F^*/I$ induces a map $\wedge^2 F^* \to \wedge^2(F^*/I)$ Let $\overline{\omega}_i \in \wedge^2(I^\perp)^*$ denote the image of ω_i under this map. Both $\overline{\omega}_1$ and $\overline{\omega}_2$ vanish on f_1, f_2. Therefore, they are proportional 2-forms : let $\alpha_1\overline{\omega}_1 + \alpha_2\overline{\omega}_2$ be a non-trivial relation. Since $I^\perp$ is an isotropic subspace for $\alpha_1\omega_1 + \alpha_2\omega_2$, this form does not have rank 4, contradicting the hypothesis.

We set $f_5^* = L_{\omega_2}(f_1)$ and $f_4^* = L_{\omega_1}(f_2)$; f_4^* and f_5^* are therefore not colinear. Note that $\langle f_4^*, f_1\rangle = \langle f_2, f_1\rangle_1 = 0$ because $L_{\omega_1}(f_1) = 0$, and that $\langle f_4^*, f_2\rangle = \langle f_2, f_2\rangle_1 = 0$; therefore, $f_4^* \in \langle f_1, f_2\rangle^\perp$, and similarly $f_5^* \in \langle f_1, f_2\rangle^\perp$.

We now let $[\omega_i]$ be the composition $F \stackrel{L_{\omega_i}}{\to} F^* \to F^*/\langle f_4^*, f_5^*\rangle$. I claim that $\ker[\omega_i] = \langle f_4^*, f_5^*\rangle^\perp$.

Note that $\mathtt{Im}L_{\omega_i} = f_i^\perp \supset \langle f_4^*, f_5^*\rangle$. I will prove the claim when $i = 1$. Both spaces are 3-dimensional and contain $\langle f_1, f_2\rangle$. So let f such that $L_{\omega_1}(f) = f_5^*$, and let us see that $f \in \langle f_4^*, f_5^*\rangle^\perp$. Since $L_{\omega_1}(f) = f_5^*$ by assumption, $\langle f_5^*, f\rangle = 0$. Similarly, $\langle f_4^*, f\rangle = \langle f_2, f\rangle_1 = -\langle f_5^*, f_2\rangle = -\langle f_1, f_2\rangle_2 = 0$, since $L_{\omega_2}(f_2) = 0$. So the claim is proved.

Looking at the surjective maps $\langle f_4^*, f_5^*\rangle^\perp \stackrel{L_{\omega_1}, L_{\omega_2}}{\longrightarrow} \langle f_4^*, f_5^*\rangle$, one proves that there exists $f_3 \in \langle f_4^*, f_5^*\rangle^\perp$ such that $L_{\omega_1}(f_3) \in \langle f_5^*\rangle - \{0\}$ and $L_{\omega_2}(f_3) \in \langle f_4^*\rangle - \{0\}$. Up to scaling f_1 (and so $f_5^* = L_{\omega_1}(f_1)$) and f_2 (and so f_4^*), we may assume that $L_{\omega_1}(f_3) = f_5^*$ and $L_{\omega_2}(f_3) = f_4^*$.

Up to now, the vectors $f_1, f_2, f_3, f_4^*, f_5^*$ were determined, up to a scale, by ω_1 and ω_2. We now make a more significant choice for f_4 : let $f_4 \in (f_5^*)^\perp$ such that $\langle f_4^*, f_4\rangle = 1$. Note that this implies $\langle f_3, f_4\rangle_2 = \langle f_2, f_4\rangle_1 = 1$, by definition of f_3 and f_4^*. We moreover choose $f_5 \in (f_4^*)^\perp$ such that $\langle f_5^*, f_5\rangle = 1$ and $\langle f_4, f_5\rangle_1 = \langle f_4, f_5\rangle_2 = 0$. Note that this implies $\langle f_5, f_3\rangle_1 = \langle f_5, f_1\rangle_2 = -1$.

It is then easy to check that for $i \in \{1, \ldots, 5\}$, we have $f_5^*(f_i) = \delta_{i,5}$ and $f_4^*(f_i) = \delta_{i,4}$. So it will not conflict notations to consider the dual basis $(f_1^*, \ldots, f_5^*)$ of the basis $(f_1, \ldots, f_5)$ of F. In this dual basis, ω_1 and ω_2 are as in the proposition. □

Let $(f_1^*, \ldots, f_5^*)$ be a basis of F^*. Let ω_1, ω_2 denote the forms $f_2^* \wedge f_4^* + f_3^* \wedge f_5^*, f_1^* \wedge f_5^* + f_3^* \wedge f_4^*$. We now classify the $GL_2 \times GL(F)$-orbits in $Hom(k^2, \wedge^2 F^*)$.

Lemma 2.5. *There are eight $GL_2 \times GL_5$-orbits in $Hom(k^2, \wedge^2 F^*)$. The following array gives elements in each orbit, its dimension and a label.*

label	$f((1,0))$	$f((0,1))$	dim
$A_2 + 2A_1$	ω_1	ω_2	20
$A_2 + A_1$	ω_1	$f_1^* \wedge f_2^*$	18
A_2	ω_1	$f_2^* \wedge f_4^*$	16
$3A_{1,a}$	ω_1	$f_2^* \wedge f_3^*$	15
$3A_{1,b}$	$f_1^* \wedge f_2^*$	$f_1^* \wedge f_3^*$	12
$3A_{1,c}$	ω_1	ω_1	11
$2A_1$	$f_1^* \wedge f_2^*$	$f_1^* \wedge f_2^*$	8
A_1	0	0	0

Finally, the closure of an orbit $\mathcal{O}$ contains the orbit $\mathcal{O}'$ if and only if $\mathcal{O}$ lies above $\mathcal{O}'$ in this array, except that the closure of the orbit labelled $3A_{1,b}$ does not contain the orbit labelled $3A_{1,c}$.

Proof : Granting the classification of the orbits, I leave it to the reader to check the dimensions of the orbits and the decomposition of their closures.

So let again $F = k^5$ and $f \in Hom(k^2, \wedge^2 F^*)$. If the rank of f, as a morphism of vector spaces, is one, then there are three cases (labelled $A_1, 2A_1, 3A_{1,c}$), according to the rank (as an element of $\wedge^2 F^*$) of a generic element of its image.

Assume f has rank two. If all non-vanishing elements of the image of f have rank 4, then, by lemma 2.4, we are in case $A_2 + 2A_1$. If all these elements are degenerate, then it is well-known that we are in case $3A_{1,b}$.

Otherwise, we may assume that $f((1,0))$ has rank 4 and $\omega := f((0,1))$ has rank 2. There is a basis $f_1, \ldots, f_5$ of F such that in terms of the dual basis $f_1^*, \ldots, f_5^*$, $f((1,0)) = \omega_1$. The kernel of L_{ω_1} is generated by f_1. Consider the 4-dimensional subspace $F' := \langle f_2^*, f_3^*, f_4^*, f_5^* \rangle \subset F^*$ and the 5-dimensional projective space $\mathbb{P} \wedge^2 F'$ containing the 4-dimensional quadric $G(2, F')$ of classes of elements of rank 2. The generic element $\omega_1 \in \wedge^2 F'$ defines a polar hyperplane (with respect to the quadric) in $\mathbb{P} \wedge^2 F'$, which will be denoted H. Note that $H \cap G(2, F')$ is a smooth 3-dimensional quadric.

- Assume first that L_ω does not vanish on f_1 and let g_1^* be an element in $\mathtt{Im}\ L_\omega$ not vanishing on f_1. Let $g_2^* \neq 0$ be an element in $\mathtt{Im}\ L_\omega \cap \mathtt{Im}\ L_{\omega_1}$. The variety of classes of elements of the form $[g_2^* \wedge g^*], g^* \in F'$ is a $\mathbb{P}^2$ in the quadric $G(2, F')$. Therefore, it can't be included in H. Let $g_4^* \in F'$ such that $[g_2^* \wedge g_4^*] \notin H$; the projective line through $[g_2^* \wedge g_4^*]$ and $[\omega_1]$ is therefore a secant line : let $[g_3^* \wedge g_5^*]$ be an element in the intersection of this line and $G(2, F')$. We can assume $\omega_1 = g_2^* \wedge g_4^* + g_3^* \wedge g_5^*$ and $\omega = g_1^* \wedge g_2^*$; therefore, we are in the case labelled $A_2 + A_1$.

- Assume now that $L_\omega(f_1)=0$. In this case, both ω and ω_1 are in $\wedge^2 F'$. There are two $GL(F')$-orbits for the projective line through ω and ω_1 : either it is a secant line to the quadric $G(2,F')$, either it is a tangent line; this corresponds to the cases A_2 and $3A_{1,a}$. □

Recall the rational map U of notation 2.2. It is a model for the restriction of the Mukai flop of the second kind to a tangent space, so it is interesting to know where it is defined.

Lemma 2.6. *The open orbit is the locus where U is defined.*

Proof : Let as before $\omega_1 = f_2^*\wedge f_4^* + f_3^*\wedge f_5^*$, and let $\omega_2(t) = f_1^*\wedge f_5^* + t.f_2^*\wedge f_3^*$. Let $f : k^2 \to \wedge^2 F^*$ be defined by $f((1,0)) = \omega_1$ and $f((0,1)) = \omega_2(t)$; we have $U(f) = \langle f_5^*, f_2^*\rangle$.

The same construction with $\omega_2'(t) = f_1^*\wedge f_5^* + t.f_3^*\wedge f_4^*$ yields $U(f') = \langle f_5^*, f_4^*\rangle$. Now, since $\omega_2(t)$ and $\omega_2'(t)$ converge to $f_1^*\wedge f_5^*$, this proves that U is not defined at the point f_0 defined by $f_0((1,0)) = f_2^*\wedge f_4^* + f_3^*\wedge f_5^*$ and $f_0((0,1)) = f_1^*\wedge f_5^*$.

Therefore, the indeterminacy locus of U contains the orbit labelled $A_2 + A_1$; since it is closed, it contains all the orbits but the open one. □

2.3 Unicity of the equivariant rational map

Recall that F is a 5-dimensional vector space over k. In this subsection, I show that the rational map $U : Hom(k^2, \wedge^2 F^*) \dashrightarrow G(3,F)$ of notation 2.2 is the unique $(GL_2\times GL(F))$-equivariant rational map $Hom(k^2,\wedge^2F^*) \dashrightarrow G(3,F)$. This is a result analogous to proposition 1.5, and the strategy of proof is the same : we caracterize its graph as an orbit of minimal dimension.

Let O denote the open $(GL_2\times GL(F))$-orbit in $Hom(k^2,\wedge^2F^*)$.

Lemma 2.7. *In $O\times G(3,F)$, there is a unique 20-dimensional orbit.*

Proof : The set of (f,α) with $\alpha = U(f)$ is such an orbit. Let (f,α) be in an orbit of dimension 20. Since O is homogeneous, we can assume that $f((1,0)) = \omega_1 = f_2^*\wedge f_4^* + f_3^*\wedge f_5^*$ and $f((0,1)) = \omega_2 = f_1^*\wedge f_5^* + f_3^*\wedge f_4^*$. Let G_0 denote the stabilizor of f in $GL_2\times GL(F)$; we have $G_0 = \{1\}\times G_1$, with $G_1 = \left\{ \begin{pmatrix} t & 0 & 0 & 0 & 0\\ 0 & t & 0 & 0 & 0\\ 0 & 0 & t & 0 & 0\\ a & c & d & t^{-1} & 0\\ b & d & a & 0 & t^{-1}\end{pmatrix} : t\in k^*; a,b,c,d\in k\right\}$ (these matrices express the action of G_1 on F^* in the dual basis $f_1^*,\dots,f_5^*$). This fact can be proved by a direct computation; we will only use the obvious fact that G_1 stabilizes f.

Let $F_\alpha\subset F$ denote the 3-dimensional subspace corresopnding to α, and let $F_\alpha^\perp \subset F^*$ denote its orthogonal. Since (f,α) belongs to an orbit of dimension 20, G_1 must stabilize α. Assume there exists $f^*\in F_\alpha^\perp$, with $f^* = \sum x_i f_i^*$ and $x_1\neq 0$. The action of G_1 implies that f_5^* belongs to $F_\alpha^\perp$, and so also f_4^*. Therefore we have a contradiction. The same contradiction arises if $F_\alpha^\perp$ contains a form with non-vanishing coefficient along f_2^*. From this it follows easily that $F_\alpha^\perp$ is generated by f_4^* and f_5^*, so $\alpha = U(f)$. □

Proposition 2.1. *There is a unique $(GL_2\times GL(F))$-equivariant rational map $Hom(k^2,\wedge^2F^*) \dashrightarrow G(3,F)$.*

Proof : U is such a rational map. Let u denote any $(GL_2 \times GL(F))$-equivariant rational map $Hom(k^2, \wedge^2 F^*) \dashrightarrow G(3, F)$.

Recall that $O \subset Hom(k^2, \wedge^2 F^*)$ denotes the open orbit; since u is equivariant, it is defined on O and surjective. The variety $\{(f, u(f)) : f \in O\}$ is a 20-dimensional orbit in $O \times G(3, F)$; therefore, by lemma 2.7, it is equal to the variety $\{(f, U(f)) : f \in O\}$. □

3 Tangency in Scorza varieties

In a projective space, given a point x and a non-vanishing tangent vector $t \in T_x X$, there is a unique line l through x and such that $t \in T_x X$. Similarly, given a non-vanishing cotangent form $f \in T_x^* X$, there is a unique hyperplane h such that f vanishes on $T_x h$. Therefore, a tangent vector defines a line and a cotangent form a hyperplane. This will be extended to a projective space over a composition algebra in this section. Both of these maps will be also defined using Jordan algebras.

3.1 Notations for Scorza varieties

Let $\mathcal{A}$ be a composition algebra over $\mathbb{C}$, of dimension a. If n is an integer, let $H_n(\mathcal{A})$ denote the space of $(n \times n)$ hermitian matrices with coeficients in $\mathcal{A}$.

Let

$$\begin{array}{cccc} \nu_2 & : & \mathcal{A}^n & \to & H_n(\mathcal{A}) \\ & & (z_1, \ldots, z_n) & \mapsto & (z_i \overline{z}_j)_{1 \leq i,j \leq n} \end{array}$$

be the map generalizing that of section 1. Recall from [Cha 05] that in $H_n(\mathcal{A})$ there is a notion of rank. The variety of rank $n-1$-elements is a hypersurface; let det denote a reduced equation of this hypersurface. The variety of rank one matrices may be described, by [Cha 05, theorem 3.1 (4) and proposition 4.2], as the closure $X = \overline{\{[\nu_2(1, z_2, \ldots, z_n)] \ : \ z_i \in \mathcal{A}\}}$.

Scorza varieties were defined and classified by F. Zak as varieties having some extremal properties with respect to their secant varieties [Zak 93, Cha 03]. For our purpose, the following theorem will serve as a definition :

Theorem 3.1 (Zak). *Let $a \in \{1, 2, 4, 8\}$ and n be integers. A Scorza variety of type (n, a) is a pair (V, X), where V is a $\mathbb{C}$-vector space, and $X \subset \mathbb{P}V$ is a projective variety projectively isomorphic to the variety of classes of rank one matrices in the projectivisation of the space $H_n(\mathcal{A})$ (with $\dim \mathcal{A} = a$).*

X is a kind of projective space; moreover, one can define a dual "projective space" $X^\vee \subset \mathbb{P}V^*$, non-canonically isomorphic with X, and an incidence relation for $(x, h) \in X \times X^\vee$ denoted $x \vdash h$. In fact, $X^\vee$ is the variety of hyperplanes containing $n-1$ general tangent spaces to X and $x \vdash h$ if and only if $T_x X \subset h$. For $h \in X^\vee$, the Schubert cell of x's incident to h will be denoted C_h. The quadratic representation corresponding to the Scorza variety $(V^*, X^\vee)$ will be denoted $U^\vee$; therefore, $U^\vee$ is a quadratic map $V^* \to Hom(V, V^*)$.

For the convenience of the reader, I recall, given a and n, the corresponding Scorza varieties and their automorphism group ($G(n_1, n_2)$ denotes the grassmannian of n_1-dimensional subspaces in $\mathbb{C}^{n_2}$.

a	$\mathcal{A}$	V	X	$X^\vee$	$Aut(X)$
1	$\mathbb{C}$	$S^2\mathbb{C}^n$	$\mathbb{P}^{n-1}$	$(\mathbb{P}^{n-1})^\vee$	PGL_n
2	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}^n\otimes\mathbb{C}^n$	$\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$	$(\mathbb{P}^{n-1})^\vee\times(\mathbb{P}^{n-1})^\vee$	$PGL_n\times PGL_n$
4	$M_2(\mathbb{C})$	$\wedge^2\mathbb{C}^{2n}$	$G(2,2n)$	$G(2n-2,2n)$	PGL_{2n}
8	$\mathbb{O}_\mathbb{C}$	$\dim 27$	E_6/P_1	E_6/P_6	E_6

In each case, it is well-known that there is a Mukai flop $T^*X \dashrightarrow T^*X^\vee$. One aim of the rest of the article is to describe this flop.

Let (X,V) be a Scorza variety of type (n,a). Recall from [Ch 06, section 1] the "quadratic representation" : it is a quadratic map $V\to Hom(V^*,V)$, canonically defined using only X. If $A\in V$, we will denote $U_A\in Hom(V^*,V)$ the image of A under the quadratic representation.

In concrete terms, when we will have to compute a quadratic representation in V, we will allways do the following. First, we will identify V with $H_n(\mathcal{A})$. Second, we will choose the scalar product $(A,B)=\mathrm{tr}(AB)$, which identifies V and V^*. These two choices will not affect the final result. Then, to compute $U_A(B)$, for $A\in V$ and $B\in V^*\simeq V$, we will allways manage to be in the situation when all the coefficients of A and B belong to an associative subalgebra of $\mathcal{A}$ (this holds, for example, if $\mathcal{A}$ itself is associative). Then we use the fact that $U_A(B)$ is ABA, where juxtaposition stands for the usual product of matrices [Ch 06].

Recall also that for any integer $r<n$ there is a well-defined variety $G_\mathcal{A}(r,X)$ parametrizing Scorza subvarieties of type (r,a) in X. To an element $A\in\mathbb{P}V$ of rank r is associated a subvariety $X_A\in G_\mathcal{A}(r,X)$ and its linear span in $\mathbb{P}V$ is denoted Σ_A [Ch 06, proposition 1.3].

As explained in [Cha 05] and [Ch 06], the Scorza varieties admit a model over $\mathbb{Z}$, and the quadratic representation is defined over $\mathbb{Z}$. Therefore, all the following constructions are valid on this base, and we get a description of Mukai flops over $\mathbb{Z}$. For the clarity of redaction, I will work over $\mathbb{C}$, since it is the usual context of Mukai flops.

In the following, (V,X) will be a Scorza variety of type (n,a), and G denotes the automorphism group of X.

3.2 A generic tangent vector defines a line

Let $x\in X$ and let $L_x\subset V$ be the line it represents. We have $T_xX = Hom(L_x,\widehat{T_xX}/L_x)$. Let $t\in T_xX$; in the next proposition, I say that $T\in\widehat{T_xX}$ represents t if the morphism $t\in Hom(L_x,\widehat{T_xX}/L_x)$ has image the line generated by the class modulo L_x of T.

By [Ch 06, proposition 1.5], the $\mathcal{A}$-lines through a point $x\in X$ are naturally parametrized by a subvariety of $\mathbb{P}(V/\widehat{T_xX})$. I say that a representative of an $\mathcal{A}$-line through x is L (with $L\in V/\widehat{T_xX}$) if the class of L in $\mathbb{P}(V/\widehat{T_xX})$ corresponds to l.

Theorem 3.2. *Let $x\in X$ and $t\in T_xX$ generic. There exists a unique $\mathcal{A}$-line $l\in G_\mathcal{A}(2,X)$ such that $x\in l$ and $t\in T_xl$. A representative for l in $V/\widehat{T_xX}$ is $L=[U_T(A)]$, if $T\in\widehat{T_xX}$ represents t and A is a generic element in V^*.*

Notation 3.1. *Let ν_x^+ denote the quadratic map $T_xX \to V/\widehat{T_xX}, T \mapsto U_T(A)$ of this theorem.*

Proof : Let $x \in X$ and $t \in T_xX$ be generic. Let T represent t. Then T has rank two, so by [Ch 06, proposition 1.4], T defines the $\mathcal{A}$-line X_T. We will prove that X_T is the unique $\mathcal{A}$-line with the properties of the proposition. To this end, we assume that $n = 3$ to simplify notations, since larger values of n would not change the argument.

We assume that $V = H_n(\mathcal{A})$ and X is the variety of rank one matrices. By [Ch 06, proposition 1.3], we can assume $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then X_T is the set of rank one matrices of the form $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We therefore check that $x \in X_T$ and $T \in \widehat{T_xX_T} = \begin{pmatrix} * & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Conversely, let $l \in G_{\mathcal{A}}(2, X)$ such that $x \in l$ and $t \in T_xl$. Let $B \in \mathbb{P}V$ of rank 2 such that $l = X_B$. Then since T represents t and $t \in T_xl$, we have $T \in \widehat{T_xl} \subset \Sigma_B$. By [Ch 06, proposition 1.4], $\Sigma_B = \Sigma_T$ and $l = X_T$.

The fact that $L = [U_T(A)] \in V/\widehat{T_xX}$ is a representative for l follows from the fact that by [Ch 06, proposition 1.4] again, Σ_T is the image of U_T, and the fact that the isomorphism of [Ch 06, proposition 1.5] maps the $\mathcal{A}$-line X_T on the line $\mathtt{Im}\ U_T/\widehat{T_xX} \subset V/\widehat{T_xX}$. □

Let d be an integer and $\mathcal{A}$ a composition algebra; recall the map $\nu_2 : \mathcal{A}^d \to H_d(\mathcal{A})$ defined in subsection 3.1. Its projectivisation $\overline{\nu}_2 : \mathcal{A}^d \dashrightarrow \mathbb{P}H_d(\mathcal{A})$ may be considered as a kind of quotient map $\mathcal{A}^d \dashrightarrow \mathbb{P}^{d-1}_{\mathcal{A}}$ [Cha 05, subsection 3.4].

Corollary 3.1. *There are identifications of T_xX with $\mathcal{A}^{n-1}$ and $V/\widehat{T_xX}$ with $H_{n-1}(\mathcal{A})$ such that ν_x^+ identifies with $\nu_2 : \mathcal{A}^{n-1} \to H_{n-1}(\mathcal{A})$.*

Proof : With the notations of the previous proof, to see that ν_x^+ identifies with ν_2, we choose the scalar product $(A, B) \mapsto \mathrm{tr}(AB)$ on $V = H_n(\mathcal{A})$, which identifies V and V^*, and moreover we choose $A \in V^*$ to be the linear form corresponding to the identity matrix in $V = H_n(\mathcal{A})$. Then, by subsection 3.1, if $T = \begin{pmatrix} t & \overline{z}_1 & \overline{z}_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix}$, then $U_T(A) = T^2 = \begin{pmatrix} * & * & * \\ * & N(z_1) & z_1\overline{z}_2 \\ * & z_2\overline{z}_1 & N(z_2) \end{pmatrix}$. Therefore, ν_x^+ identifies with ν_2. □

3.3 A generic cotangent form defines a hyperplane

A cotangent form $f \in T_xX^*$ is an element $f \in Hom(L_x^*, (\widehat{T_xX}/L_x)^*)$. I say that $\tilde{f} \in V^*$ represents f if $\tilde{f}_{|\widehat{T_xX}}$ generates the image of f. Recall (subsection 3.1) that for $h \in X^\vee$, C_h denotes the Schubert cell in X defined by h. Let $\mu : T^*X \dashrightarrow T^*X^\vee$ denote the Mukai flop and $\pi : T^*X^\vee \to X^\vee$ the projection.

Theorem 3.3. *Let $x \in X$, $x_0 \in L_x - \{0\}$, and $f \in T_x^*X$ generic. There exists a unique $h \in X^\vee$ such that f vanishes on T_xC_h. If $\tilde{f} \in V^*$ represents f, then a representative of h is $U_{\tilde{f}}^\vee(x_0) \in (V/\widehat{T_xX})^*$. Finally, $\pi \circ \mu(x, f) = h$.*

Notation 3.2. *Let ν_x^- denote the quadratic map $T_x^*X \to (V/\widehat{T_xX})^*, \tilde{f} \mapsto U_{\tilde{f}}^\vee(x_0)$ of this theorem.*

Proof : The last claim follows from the first and [Cha 06], where it is proved that $\pi \circ \mu(x,f)$ is the only $h \in X^\vee$ such that f vanishes on T_xC_h.

To simplify notations, we assume in the proof that $V = H_3(\mathcal{A})$ and we identify V and V^* via the scalar product $(A,B) = \mathrm{tr}(AB)$. Assume as before that $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and that $f\left(\begin{pmatrix} t & \overline{z}_1 & \overline{z}_2 \\ z_1 & 0 & 0 \\ z_2 & 0 & 0 \end{pmatrix}\right) = \mathtt{Re}(z_2)$.

Let $h_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $y = \begin{pmatrix} t & z & 0 \\ \overline{z} & u & 0 \\ 0 & 0 & 0 \end{pmatrix}$, with $t,u \in \mathbb{C}$ and $z \in \mathcal{A}$ and $tu - N(a) = 0$, be an element of $\widehat{X}$. By [Cha 05, theorem 3.1 (4) and proposition 4.2], $\widehat{X} = \overline{\{[\nu_2(1,z_1,z_2)] : z_i \in \mathcal{A}\}}$. We deduce that if $(m_{i,j}) \in \widehat{X}$, then the minors $m_{i,i}m_{j,j} - N(m_{i,j})$ vanish. It follows that if $t \neq 0$, $T_y\widehat{Y}$ is orthogonal to h_0.

Therefore, by continuity, the intersection of $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with X lies in the Schubert cell C_{h_0}, and for dimension reasons we have equality. This shows that f vanishes on $T_xC_{h_0}$. Therefore $h = h_0$.

Finally, let $\tilde{f} = \begin{pmatrix} 0 & \overline{z}_1 & \overline{z}_2 \\ z_1 & t & \overline{z} \\ z_2 & z & u \end{pmatrix}$ be a linear form ($t,u \in \mathbb{C}$ and $z, z_1, z_2 \in \mathcal{A}$ are arbitrary); then $U^\vee(\tilde{f}).x = \tilde{f}.x.\tilde{f} = \tilde{f}\begin{pmatrix} 0 & \overline{z}_1 & \overline{z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N(z_1) & z_1\overline{z}_2 \\ 0 & z_2\overline{z}_1 & N(z_2) \end{pmatrix}$, so that ν_x^- identifies with ν_2. Moreover, if $z_1 = 0$ and $z_2 = 1$, then $\tilde{f}$ represents f and we have $[\nu_x(\tilde{f})] = h_0$, as claimed. □

In the proof of the theorem, we showed :

Corollary 3.2. *There are identifications of T_xX^* and $(V/\widehat{T_xX})^*$ with $\mathcal{A}^{n-1}$ and $H_{n-1}(\mathcal{A})$ such that ν_x^- identifies with $\nu_2 : \mathcal{A}^{n-1} \to H_{n-1}(\mathcal{A})$.*

3.4 The variety of lines through a point in $\mathbb{P}^n_{\mathcal{A}}$ as Fano variety of maximal linear subspaces of $\mathbb{P}^{n-1}_{\mathcal{A}}$

The goals of this subsection are propositions 3.4 and 3.5.

The normal bundle to X in $\mathbb{P}V$ twisted by (-1) will be denoted N and let $\pi : N \to X$ (resp. $\overline{\pi} : \mathbb{P}N \to X$) be the structure map of this vector bundle (resp. its projectivisation). Similarly, let ψ and $\overline{\psi}$ denote the natural maps $TX(-1) \to X$ and $\mathbb{P}TX(-1) \to X$. Let $x \in X$; the quotient map $V \to V/\widehat{T_xX} = N_x$ will be denoted π_x. The normal bundle N admits an interesting subvariety : the image of $\widehat{X}$. This variety will be denoted $N(X)$: by definition, the fiber $N(X)_x := \pi^{-1}(x) \cap N(X)$ is $\pi_x(\widehat{X})$. Recall [Ch 06] that $(N_x, \mathbb{P}N(X)_x)$ is a Scorza variety of type $(n-1, a)$.

Assume $a > 1$. Let $F(0,1,X)$ denote the variety of couples (x,l) where $x \in \mathbb{P}V$, $l \subset \mathbb{P}V$ is a projective line, and $x \in l \subset X$. The map which sends a pair $(x,l)\in F(0,1,X)$ to $(x,t)\in \mathbb{P}TX$, where $t\in \mathbb{P}T_xX$ is the projectivisation of the tangent vector of l at x shows that $F(0,1,X)$ can be considered as a subvariety of $\mathbb{P}TX$. By [Ch 06, lemma 1.2 and proposition 1.3], $F(0,1,X)$ is homogeneous.

The first interesting point is that $(\overline{\psi}^* N)_{|F(0,1,X)}$ admits a subbundle included in $\overline{\psi}^{-1}(N(X))$. For $(x,l)\in F(0,1,X)$ and $x\neq y\in l$, define $T_y := \pi_x(\widehat{T_yX})$.

Proposition 3.3. *T_y does not depend on $y\in l$ and $T_y\to F(0,1,X)$ defines a rank $(ra/2+1)$-subbundle of $(\overline{\psi}^* N)_{F(0,1,X)}$, entirely included in $\overline{\psi}^{-1}(N(X))$.*

Proof : Assume for the simplicity of notations that $n=3$. I use the fact that if z_1,z_2,z_3 generate an associative subalgebra of $\mathcal{A}$, then $\nu_2(z_1,z_2,z_3)\in\widehat{X}$ [Cha 05, proposition 4.2]. The condition on z_1,z_2,z_3 holds for example if $\mathcal{A}$ itself is associative or if $z_1=1$, since in $\mathbb{O}_\mathbb{C}$, the subalgebra generated by two elements is allways associative.

Let $x=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ and $y=\begin{pmatrix}1&\overline{z}&0\\z&0&0\\0&0&0\end{pmatrix}$ in $H_3(\mathcal{A})$, with $z\in\mathcal{A}$ and $N(z)=0$. Then the line through $[x]$ and $[y]$ in $\mathbb{P}H_3(\mathcal{A})$ lies in X, because $x+ty=\nu_2(1,tz,0)$. Moreover, differentiating ν_2, we have $\widehat{T_xX}=\left\{\begin{pmatrix}*&*&*\\ *&0&0\\ *&0&0\end{pmatrix}\right\}$ and $\widehat{T_yX}=\left\{\begin{pmatrix}\mathtt{Re}(u)&u\overline{z}+\overline{v}&\overline{w}\\ z\overline{u}+v&\mathtt{Re}(zv)&z\overline{w}\\ w&w\overline{z}&0\end{pmatrix}: u,v,w\in\mathcal{A}\right\}$. It follows that

$$\widehat{T_yX}/\widehat{T_xX}\simeq\left\{\begin{pmatrix}*&z\overline{w}\\ w\overline{z}&0\end{pmatrix}: w\in\mathcal{A}\right\}.$$

Therefore, this space does not change if y is replaced by a point of the line through x and y. Since $F(0,1,X)$ is homogeneous, this holds for any of its elements. Therefore, $T_{(x,l)}$ is allways a $(a/2+1)$-linear subspace of $V/\widehat{T_xX}$. It follows that it is a subbundle of $(\overline{\psi}^* N)_{|F(0,1,X)}$, as it is locally the image of the bundle $\widehat{T_yX}$ (y a local section of l different from x) under a constant rank vector bundle map.

Moreover, since $\begin{pmatrix}0&0&0\\0&*&z\overline{w}\\0&w\overline{z}&0\end{pmatrix}$ belongs to $\widehat{X}$, $\begin{pmatrix}*&z\overline{w}\\ w\overline{z}&0\end{pmatrix}$ belongs to $N(X)_x$, and so the subbundle T is included in $\overline{\psi}^{-1}(N(X)_x)$. □

Let $x\in X$ and denote $F(x,1,X)$ the variety of lines through x and included in X. Theorem 3.2 yields a quadratic map $\nu_x^+:\widehat{T_xX}/L_x\to N_x$, well-defined up to a scale. Let $\mu_x^+(.,.)$ denote its polarization.

Proposition 3.4. *The map $l\mapsto T_{(x,l)}$ defines a map between $F(x,1,X)$ and some components of the Fano variety of maximal linear subspaces in $\mathbb{P}N(X)_x$. This map is an isomorphism when $a\neq 4$, and is surjective with fibers isomorphic to $\mathbb{P}^1$ when $a=4$. Moreover, let $0\neq t\in T_xl$; $T_{(x,l)}$ is the image of $\mu_x^+(t,.)$.*

In particular, in the case $\mathcal{A}=\mathbb{O}_\mathbb{C}$ this proposition proves that the variety of lines in X through a fixed point $x\in X$ is isomorphic with a 10-dimensional spinor variety; this fact is proved in [LM 03, prop 3.4 p.77], but I give here a direct proof which makes it clear which isotropic spaces this spinor variety parametrizes.

Proof : First of all, by [Ch 06, proposition 1.4], there are two P-orbits in $\mathbb{P}T_xX$, if P denotes the stabilizor of x in G. Therefore, $F(x,1,X)=\mathbb{P}\{\nu_x=0\}\subset\mathbb{P}T_xX$. We know that ν_x identifies with ν_2 and in [Cha 05, section 3.4], I described the locus where ν_2 vanishes; therefore, we get the following array ($G_Q^+(5,10)$ denotes a 10-dimensional spinor variety and Q an 8-dimensional projective quadric; $G(2,2n-2)$ is the grassmannian of 2-dimensional spaces in $\mathbb{C}^{2n-2}$) :

V	$F(x,1,X)$	$\mathbb{P}N(X)_x$	$\frac{a}{2}(n-2)+1$
$H_n(\mathbb{C}_\mathbb{C})$	$\mathbb{P}^{n-2}\amalg\mathbb{P}^{n-2}$	$\mathbb{P}^{n-2}\times\mathbb{P}^{n-2}$	$n-1$
$H_n(\mathbb{H}_\mathbb{C})$	$\mathbb{P}^1\times\mathbb{P}^{2n-3}$	$G(2,2n-2)$	$2n-3$
$H_3(\mathbb{O}_\mathbb{C})$	S^+	Q	5

Let now $z\in\mathcal{A}$ such that $N(z)=0$; the image of $\mu_x^+((z,0,\dots,0),.)$ is

$$\left\{\begin{pmatrix} * & z\overline{u}_1,\dots,z\overline{u}_{n-2} \\ u_1\overline{z} & \\ \vdots & 0 \\ u_{n-2}\overline{z} & \end{pmatrix} : u_i\in\mathcal{A}\right\} \quad ;$$

it is of dimension $1+\frac{a}{2}(n-2)$, so it is a maximal linear subspace of $N(X)_x$. If $l\in F(x,1,X)$, the fact that the image of $\mu_x^+(l,.)$ is $T_{(x,l)}$ is a consequence of the formula for ν_x^+ and the computation of $T_{(x,l)}$ made in the proof of proposition 3.3.

In the case when $\mathcal{A}=\mathbb{O}_\mathbb{C}$, proposition 1.4 shows that the map of the proposition is an isomorphism. I leave it to the reader to check that in case $\mathcal{A}=\mathbb{C}_\mathbb{C}$, it is an isomorphism, and in case $\mathcal{A}=\mathbb{H}_\mathbb{C}$, it has fibers isomorphic with $\mathbb{P}^1$. □

Let $\nu_x^-:(\widehat{T_xX/L_x})^*\to N_xX^*\subset V^*$ be the quadratic map of theorem 3.3 and μ_x^- its polarization. We know that $(N_x,\mathbb{P}N(X)_x)$ is a Scora variety of type $(n-1,a)$; let $\mathbb{P}N(X)_x^\vee\subset\mathbb{P}N_x^*$ denote its dual Scorza variety.

We have a similar result for the cotangent space :

Proposition 3.5. *The map $l\mapsto \mathtt{Im}\ \mu_x^-(t,.)$, where $0\neq t\in T_xl$ defines a morphism between $F(x,1,X)$ and some components of the Fano variety of maximal linear subspaces in $\mathbb{P}N(X)_x^\vee$. It is an isomorphism if $a\neq 4$, and is surjective with fibers isomorphic to $\mathbb{P}^1$ if $a=4$.*

From the array in the proof of proposition 3.4, we see that in case $n=3$, $\mathbb{P}N(X)_x$ is a smooth quadric of dimension a. So there are two families of maximal linear subspaces in $N(X)_x$. In case $\mathcal{A}=\mathbb{C}_\mathbb{C}$, the two families are described by proposition 3.4. But in case $a\geq 4$, we only get one family. The other family comes with proposition 3.5, because we can use the canonical isomorphism $\mathbb{P}N(X)_x^\vee=\mathbb{P}N(X)_x$ which holds since $\mathbb{P}N(X)_x$ is a smooth quadric. One can check that we indeed find two different families with the two dual constructions of propositions 3.4 and 3.5.

3.5 The tangent bundle to the variety of lines in a Severi variety

In this subsection, we prepare the description of the Mukai flop of the second kind. Let $X \subset \mathbb{P}V$ be a Scorza scheme of type $(3, a)$ (these schemes are called Severi varieties in [Zak 93]) and assume $a \geq 2$. Note [Cha 05] that if $a = 2$, then X is isomorphic with $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and that if $a = 4$, then X is isomorphic with the grassmannian $G(2,6) \subset \mathbb{P}^{14}$ of 2-dimensional subspaces of $\mathbb{C}^6$.

Let Y denote an irreducible component of the variety of projective lines in $\mathbb{P}V$ which are included in X. If $a = 2$, then $Y \simeq (\mathbb{P}^2)^{\vee} \times (\mathbb{P}^2)$ and if $a = 4$, then Y is isomorphic with the flag variety $F(1,3,6)$ of 1-dimensional subspaces included in a 3-dimensional subspace included in a fixed $\mathbb{C}^6$. If $a = 8$, then it follows from [LM 03, theorem 4.3 p.82] that Y is the quotient G/P_3, where G is a simply-connected group of type E_6 and P_3 is the parabolic subgroup corresponding to the simple root α_3. Therefore, the Mukai flop of the second kind is a rational map $T^*Y \dashrightarrow T^*Y^{\vee}$, where $Y^{\vee} = G/P_5$.

The aim of this subsection is to describe the tangent bundle TY. As before, this will be done in a unified way for all Severi varieties with $a \geq 2$ (if $a = 1$, the variety Y is empty).

Let us start with an easy lemma. Let $\det(.,.,.)$ be the polarization of the degree 3 polynomial det (that is, the unique trilinear symmetric form such that $\forall v \in V, \det(v,v,v) = 6\det(v)$).

Lemma 3.1. *Let X be a Severi variety and $x \in X$. Then we have*

$$\widehat{T_x X} = \{v : \forall w \in V, \det(x,v,w) = 0\}.$$

Proof : By [Cha 05, propositions 3.5 and 4.2], the ideal of X is generated by the quadratic equations $\det(x,x,.) = 0$. Therefore, by differentiation, we get the given equations for the tangent space at x. □

Now, let $\alpha \in Y$. The 2-dimensional linear space it represents will be denoted L_α. We set

$$\begin{aligned} S_\alpha &:= \langle T_x \widehat{X} \rangle_{x \in L_\alpha - \{0\}} \\ I_\alpha &:= \textstyle\bigcap_{x \in L_\alpha - \{0\}} T_x \widehat{X}. \end{aligned}$$

It is clear that S and I are G-homogeneous subbundles of the trivial bundle $V \otimes \mathcal{O}_Y$ over Y. We moreover consider the quotient bundles defined by $A_\alpha := I_\alpha / L_\alpha, B_\alpha := S_\alpha / I_\alpha, C_\alpha := V/S_\alpha$.

Proposition 3.6. *The ranks of the bundles A, B, C are, respectively, $3a/2 - 2, a+2, a/2+1$. There is a G-equivariant short exact sequence of bundles*

$$0 \to Hom(L, A) \to TY \to \wedge^2 L^* \otimes C^* \to 0.$$

Remark : The image of $Hom(L,A)_\alpha$ in $T_\alpha Y$ may be described geometrically, by [LM 03, theorem 4.3 p.82], as the linear subspace generated by the tangent vectors to lines through α included in Y.

Proof : Let $u \in \mathcal{A}$ such that $N(u) = 0$. Let $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & \overline{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}$

be vectors in $\widehat{X}$. The tangent spaces $T_x\widehat{X}$ and $T_y\widehat{X}$ were computed during the proof of proposition 3.3; it follows from this computation that $T_x\widehat{X} \cap T_y\widehat{X} = \begin{pmatrix} * & . & . \\ R(u) & 0 & 0 \\ u^\perp & 0 & 0 \end{pmatrix}$ (the dots replace coefficients above the diagonal which are conjugates of elements under it) and that $\langle T_x\widehat{X}, T_y\widehat{X}\rangle = \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & L(u) & * \end{pmatrix}$. Note that these spaces do not change when x is replaced by $\lambda.x$, and y by $\nu.y$, $\lambda, \nu \in \mathbb{C}$. Therefore, if α represents the subspace generated by x and y, we have $I_\alpha = T_x\widehat{X} \cap T_y\widehat{X}$ and $S_\alpha = \langle T_x\widehat{X}, T_y\widehat{X}\rangle$. We see that $\dim I_\alpha = 3a/2$ and $\dim S_\alpha = 5a/2 + 2$. The first result on the ranks of the vector bundles therefore follows.

Let $\alpha \in Y$; I now define a map $T_\alpha Y \to \wedge^2 L_\alpha^* \otimes C_\alpha^*$. Let $G(2,V)$ denote the grassmannian of 2-dimensional linear subspaces of V. We use the fact that $T_\alpha Y$, as a subspace of $T_\alpha G(2,V)$, may be described as the set of $\varphi : L_\alpha \to V/L_\alpha$ such that $\forall x \in L_\alpha, \varphi(x) \in T_x\widehat{X}/L_\alpha$. So an element $\varphi \in T_\alpha Y \subset Hom(L_\alpha, V/L_\alpha)$ defines a linear map

$$\begin{array}{cccc} \varphi_0 & : & L_\alpha \otimes L_\alpha & \to & V^* \\ & & x \otimes y & \mapsto & (w \mapsto \det(x, \varphi(y), w)). \end{array}$$

Now, if $y = \lambda.x$, with $\lambda \in \mathbb{C}$, then $\varphi(y) \in T_y\widehat{X} = T_x\widehat{X}$, and so by lemma 3.1, $\det(x, \varphi(y), w) = 0$ for all $w \in V$. Therefore, φ_0 induces a linear map $\varphi_1 : \wedge^2 L_\alpha \to V^*$.

Moreover, assume there exists $x \in L_\alpha - \{0\}$ such that $w \in T_x\widehat{X}$. Then we have $\det(x, \varphi(y), w) = \det(x, w, \varphi(y)) = 0$ because $w \in T_x\widehat{X}$. Choosing $y \in L_\alpha$ not colinear with x, this proves that $\varphi_1(\wedge^2 L_\alpha) \subset S_\alpha^\perp$. Since $S_\alpha^\perp = C_\alpha^*$, we therefore get an element $\varphi_2 \in \wedge^2 L_\alpha^* \otimes C_\alpha^*$. The map $\varphi \mapsto \varphi_2$ is the map $T_\alpha Y \to \wedge^2 L_\alpha^* \otimes C_\alpha^*$ of the proposition.

From the realization of $T_\alpha Y$ as a subspace of $Hom(L_\alpha, V/L_\alpha)$, it is moreover clear that $Hom(L_\alpha, A_\alpha)$ is a subspace of $T_\alpha Y$. Assume now that $\varphi_2 = 0$. This implies that if $x, y \in L_\alpha$ and $w \in V$, then $\det(x, \varphi(y), w) = 0$. By lemma 3.1 again, this implies that $\varphi(y) \in T_x\widehat{X}$. It follows that $\mathtt{Im}\ \varphi \subset A_\alpha$ and $\varphi \in Hom(L_\alpha, A_\alpha)$. Since $\dim Y = 25$, the above map $\varphi \mapsto \varphi_2$ is surjective and the sequence of the proposition is exact. □

We will see (proposition 3.8) that the projectivised bundle $\mathbb{P}A$ contains a subvariety which is isomorphic to the relative grassmannian $G(2,C)$ of 2-dimensional subspaces in C. Here is a first result in this direction.

Proposition 3.7. *There is a G-equivariant injective map of bundles $\psi : \wedge^2 C \otimes \wedge^2 L \to A$. The cokernel bundle is trivial except when $a = 4$, in which case it is a line bundle.*

Proof : Assume first that $a = 4$. Let E be a 6-dimensional vector space; we have already seen that $Y = F(1,3,E)$. So a point α in Y defines a 1-dimensional subspace E_1 of E and a 3-dimensional subspace E_3 of E; moreover, $E_1 \subset E_3$. Consider now E_1, E_3 as bundles over Y.

We have $V \otimes \mathcal{O}_Y = \wedge^2 E$, $L = E_1 \wedge E_3 = E_1 \otimes (E_3/E_1)$, $A = E_1 \otimes (E/E_3) \oplus \wedge^2(E_3/E_1)$ and $C = \wedge^2(E/E_3)$. Set $A' = E_1 \otimes (E/E_3)$. Recall that if Z is a 3-dimensional vector space, then $\wedge^2(\wedge^2 Z)$ is canonically isomorphic with $Z \otimes \wedge^3 Z$.

Therefore,

$$\begin{aligned}
\wedge^2 C \otimes \wedge^2 L &= (E/E_3) \otimes \wedge^3(E/E_3) \otimes E_1 \otimes E_1 \otimes \wedge^2(E_3/E_1) \\
&= (E/E_3) \otimes \wedge^3(E/E_3) \otimes E_1 \otimes \wedge^3 E_3 \\
&= E_1 \otimes (E/E_3) \otimes \wedge^6 E \\
&= A'.
\end{aligned}$$

The last equality follows from the fact that $\wedge^6 E$ is the trivial line bundle on Y. We therefore get the map $\wedge^2 C \otimes \wedge^2 L \to A$, which is injective and has 1-dimensional cokernel.

The case when $a = 2$ is similar.

Assume now that $a = 8$. In this case, I don't know any better proof than checking the weights. Recall from [Bou 68] the following : the highest weight of V is $\lambda = \frac{1}{3}\left[\begin{smallmatrix}4&5&6&4&2\\&&3&&\end{smallmatrix}\right]$ and the lowest is $\frac{1}{3}\left[\begin{smallmatrix}-2&-4&-6&-5&-4\\&&-3&&\end{smallmatrix}\right]$. Let x be a vector of weight λ and y a vector of weight $s_{\alpha_1}(\lambda) = \frac{1}{3}\left[\begin{smallmatrix}1&5&6&4&2\\&&3&&\end{smallmatrix}\right]$. We may assume that L_α is the space generated by x and y. I claim that the weights of C_α are $\frac{1}{3}\left[\begin{smallmatrix}-2&-4&-6&-5&-4\\&&-3&&\end{smallmatrix}\right]$, $\frac{1}{3}\left[\begin{smallmatrix}-2&-4&-6&-5&-1\\&&-3&&\end{smallmatrix}\right]$, $\frac{1}{3}\left[\begin{smallmatrix}-2&-4&-6&-2&-1\\&&-3&&\end{smallmatrix}\right]$, $\frac{1}{3}\left[\begin{smallmatrix}-2&-4&-3&-2&-1\\&&-3&&\end{smallmatrix}\right]$ and $\frac{1}{3}\left[\begin{smallmatrix}-2&-4&-3&-2&-1\\&&0&&\end{smallmatrix}\right]$. In fact, first, we see that these weights are obtained from the lowest adding successively $\alpha_6, \alpha_5, \alpha_4, \alpha_2$ (this proves by the way that if $L \simeq SL_2 \times SL_5$ is included in a Levi factor of P_3, then C_α is an irreducible SL_5-module). Second, the corresponding weight lines are not in $T_x\widehat{X}$ (resp. neither in $T_y\widehat{X}$) since the weights of this linear subspace are the sum of λ (resp. $s_{\alpha_1}(\lambda)$) and a root. Since no root has a coefficient -3 in α_4, the claim follows.

Adding the two highest weights of C_α and the two weights of L_α, one gets $\frac{1}{3}\left[\begin{smallmatrix}1&2&6&4&2\\&&3&&\end{smallmatrix}\right]$. This is exactly the highest weight of A. Therefore, there is an L-equivariant map $\wedge^2 C_\alpha \otimes \wedge^2 L_\alpha \to A_\alpha$. Since this is a map between irreducible L-representations, it is also a P_3-equivariant map, proving the proposition. $\square$

Lemma 3.2. *Let $\alpha \in Y$ and $x, y \in \mathbb{P}I_\alpha - \mathbb{P}L_\alpha$ such that $x \equiv y \mod L_\alpha$. Then $x \in X$ if and only if $y \in X$.*

<u>Proof :</u> Let $z_1 \neq z_2 \in \mathbb{P}L_\alpha$ and $i \in \{1, 2\}$. By definition of I_α, $x \in T_{z_i}X$. If $x \in X$, then the projective line (xz_i) through x and z_i meets X at the points z_i and x, and with multiplicity at least two at z_i. Since X is defined by quadratic equations, $(xz_i) \subset X$. Therefore, the plane (xz_1z_2) meets X along the three lines $(z_1z_2), (xz_1), (xz_2)$; so this plane is included in X. Therefore, $y \in X$. $\square$

Notation 3.3. *Let $A' \subset A$ denote the image of $\wedge^2 C \otimes \wedge^2 L$ under the map of proposition 3.7. Let $X(\alpha) \subset \mathbb{P}A'_\alpha$ denote the intersection of the image of X under the rational projection $\mathbb{P}I_\alpha \dashrightarrow \mathbb{P}A_\alpha$ and $\mathbb{P}A'_\alpha$.*

Proposition 3.8. *Assume $a \geq 4$. Let $\alpha \in Y$ and $x \in X(\alpha)$. The projectivisation of the inverse of the isomorphism $\psi_\alpha : \wedge^2 C_\alpha \otimes \wedge^2 L_\alpha \to A'_\alpha$ maps x on the element in $G(2, C_\alpha)$ representing the 2-dimensional space $T_y\widehat{X}/S_\alpha \subset V/S_\alpha$, if $y \in I_\alpha$ is any vector with class x in $\mathbb{P}A_\alpha$.*

Proof : Assume first that $a = 8$. Let $\alpha \in Y$. Since A_α is an irreducible SL_5-representation isomorphic with $\wedge^2 \mathbb{C}^5$, there is a unique non-trivial invariant subvariety in $\mathbb{P}A_\alpha$, and therefore it is $X(\alpha)$. If $a = 4$, then obviously we also have $X(\alpha) = \mathbb{P}A'_\alpha$.

Let $a \in \{4, 8\}$ and assume as in the proof of proposition 3.6 that $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 & \overline{u} \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}$ and L_α is spanned by x and y. If $z = \begin{pmatrix} 1 & 0 & \overline{v} \\ 0 & 0 & 0 \\ v & 0 & 0 \end{pmatrix}$, with $\langle u, v \rangle = 0$, then the 3-dimensional space generated by x, y, z lies in X and $T_z \widehat{X}/S_\alpha \subset C_\alpha$ is 2-dimensional and does not change if z is replaced by a linear combination of x, y and z. By homogeneity of $X(\alpha)$, this fact holds for any $[z] \in X(\alpha)$ and so we have a well-defined map $X(\alpha) \to G(2, C_\alpha)$. Since there is only one such P_3-equivariant map, this map also coincides with the restriction of the projectivisation of ψ^{-1}. $\square$

We now assume $a = 8$, and conclude this subsection classifying the E_6-orbits in T^*Y. By propositions 3.6 and 3.7, there is a vector bundle map $T^*Y \to Hom(L^* \otimes \wedge^2 L, \wedge^2 C^*) = Hom(L, \wedge^2 C^*)$; I denote it h.

Proposition 3.9. *Let $\alpha \in Y$ and $f, g \in T^*Y$, and assume f and g both don't vanish. Then f, g lay in the same E_6-orbit if and only if the two elements $h(f), h(g) \in Hom(L_\alpha, \wedge^2 C^*_\alpha)$ lay in the same $(GL(L_\alpha) \times GL(C_\alpha))$-orbit.*

In view of lemma 2.5, this gives a complete understanding of the E_6-orbits in T^*Y.

Proof : Let $P \subset E_6$ be the stabilizor of α and $L(P)$ a Levi factor of P. We know that the image of $L(P)$ in $End(Hom(L_\alpha, \wedge^2 C_\alpha))$ is the same as that of $GL(L_\alpha) \times GL(C_\alpha)$. If $h(f) = h(g) = 0$, then, by proposition 3.6, f and g are elements in $(\wedge^2 L_\alpha \otimes C_\alpha) - \{0\}$, which is obviously homogeneous under $L(P)$, and so lay in the same P-orbit.

Assume $h(f) \neq 0$ and $h(g) \neq 0$. Since by hypothesis $h(f)$ and $h(g)$ lay in the same $L(P)$-orbit, we may assume that $h(f) = h(g)$. Let $R_u(P)$ denote the unipotent radical of P; $R_u(P)$ acts trivially on the irreducible P-representation $Hom(L_\alpha, \wedge^2 C^*_\alpha)$. Therefore, it is enough to prove that the $R_u(P)$-orbit of f is dense in $h^{-1}(h(f))$. Equivalently, we will prove that the image of the action of the Lie algebra of $R_u(P)$ on f contains $\wedge^2 L_\alpha \otimes C_\alpha$.

It is enough to prove this when $h(f)$ is in the minimal non-zero orbit of $GL(L_\alpha) \times GL(C_\alpha)$ in $Hom(L_\alpha, \wedge^2 C^*_\alpha)$. This, in turn, can be verified at the level of weights. In fact, we assume that $h(f)$ is a highest weight vector of $Hom(L_\alpha, \wedge^2 C^*_\alpha)$. Therefore, $h(f)$ has weight $\left[\begin{smallmatrix} 1 & 1 & 2 & 2 & 1 \\ & & 2 & & \end{smallmatrix}\right]$. In fact, as we saw in the proof of proposition 3.7, the highest weight of L^*_α is $\frac{1}{3}\left[\begin{smallmatrix} -1 & -5 & -6 & -4 & -2 \\ & & -3 & & \end{smallmatrix}\right]$, and the two highest weights of C^*_α are $\frac{1}{3}\left[\begin{smallmatrix} 2 & 4 & 6 & 5 & 4 \\ & & 3 & & \end{smallmatrix}\right]$ and $\frac{1}{3}\left[\begin{smallmatrix} 2 & 4 & 6 & 5 & 1 \\ & & 3 & & \end{smallmatrix}\right]$.

Since the weight of $\wedge^2 L$ is $\frac{1}{3}\left[\begin{smallmatrix} 5 & 10 & 12 & 8 & 4 \\ & & 6 & & \end{smallmatrix}\right]$ and the heighest weight of C is $\frac{1}{3}\left[\begin{smallmatrix} -2 & -4 & -3 & -2 & -1 \\ & & 0 & & \end{smallmatrix}\right]$, the heighest weight of $\wedge^2 L_\alpha \otimes C$ is $\left[\begin{smallmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 1 & & \end{smallmatrix}\right] = \omega_2$. Since this is the highest weight of $Hom(L_\alpha, \wedge^2 C^*_\alpha)$ plus $\alpha_3 + \alpha_4$, which is a root of $R_u(P)$, we are done. $\square$

4 Mukai flops of type E_6

Let (V, X) be a Scorza variety and $(V^*, X^\vee)$ the dual Scorza variety. An element in T^*X will be denoted (x, α), where $x \in X$ and α is a linear form on T_xX. The flop is a map $T^*X \dashrightarrow X^\vee, (x, \alpha) \mapsto (h, \kappa) = (h(x, \alpha), \kappa(x, \alpha))$.

For flops of type $E_{6,I}$, the element $h(x, \alpha)$ was described in the preceeding section. The complete description of Mukai flops should also include a formula for $\kappa(x, \alpha)$. However, it is not easy to follow the identification of $T_x^*E_6/P_1$, seen more or less as a subspace of V^*, with a subspace of the Lie algebra of G. Instead, given $h(x, \alpha)$, I explain in the next subsection a general geometric way to put our hands on $\kappa(x, \alpha)$.

4.1 Canonical isomorphism of quotients of tangent spaces to flag varieties

Let G be a reductive algebraic group and let $\mathcal{P}, \mathcal{Q}$ denote two flag varieties parametrizing two classes of parabolic subgroups of G. Let $\mathcal{R}$ denote the flag variety of parabolic subgroups which are intersections of a parabolic subgroup in $\mathcal{P}$ and a parabolic subgroup in $\mathcal{Q}$. Since a parabolic subgroup in $\mathcal{R}$ is contained in exactly one subgroup in $\mathcal{P}$ (resp. $\mathcal{Q}$), $\mathcal{R}$ is canonically isomorphic with a subvariety of $\mathcal{P} \times \mathcal{Q}$; an element in $\mathcal{R}$ will threrefore be denoted (x, y), with $x \in \mathcal{P}$ and $y \in \mathcal{Q}$.

If $x \in \mathcal{P}$, let $\mathcal{Q}_x$ denote the variety of parabolic subgroups y such that $(x, y) \in \mathcal{R}$, and define similarly $\mathcal{P}_y$. The following quite general theorem allows, as a special case, describing Mukai flops $T^*X \dashrightarrow T^*X^\vee$ of type $E_{6,I}$ as soon as we know the composition $T^*X \dashrightarrow T^*X^\vee \to X^\vee$. Since it is the case by theorem 3.3, it will be easy to deduce a Jordan-theoretic formula for this Mukai flop, in proposition 4.1.

In the next theorem, $(C, 0, t)$ will denote a pointed curve $(C, 0)$ which is smooth at 0, together with a tangent vector t at the point 0. Moreover, if Y is an algebraic variety and $f : C \to Y$ is a map, then $f'(0) \in T_{f(0)}Y$ will denote the derivative $df_0(t)$. I say that there is a Mukai flop $T^*\mathcal{P} \dashrightarrow T^*\mathcal{Q}$ if the natural maps $T^*\mathcal{P} \to \mathfrak{g}, T^*\mathcal{Q} \to \mathfrak{g}$ are birational and have the same image.

Theorem 4.1. *Let $(x, y) \in \mathcal{R}$. Then there is a canonical isomorphism $\mu(x, y)$: $\frac{T_x\mathcal{P}}{T_x\mathcal{P}_y} \to \frac{T_y\mathcal{Q}}{T_y\mathcal{Q}_x}$. If $(C, 0, t)$ is as above, and if $\gamma : (C, 0) \to (\mathcal{P}, x)$ is any map, such an isomorphism maps the class of $\gamma'(0)$ on the class of $\delta'(0)$, if $\delta : (C, 0) \to (\mathcal{Q}, y)$ is any map such that $(\gamma, \delta)(C) \subset \mathcal{R}$.*
If, moreover, there is a Mukai flop $T^*\mathcal{P} \dashrightarrow T^*\mathcal{Q}$, then this flop maps a generic form $f \in (T_x\mathcal{P}/T_x\mathcal{P}_y)^*$ to $(y, {}^t\mu(x, y)^{-1}(f))$.

Proof : Let $\pi_P : \mathcal{R} \to \mathcal{P}$ and $\pi_Q : \mathcal{R} \to \mathcal{Q}$ denote the natural projections. Consider the diagram

$$\begin{array}{ccccc} \frac{T_x\mathcal{P}}{T_x\mathcal{P}_y} & \stackrel{\varphi_P}{\longleftarrow} & \frac{T_{(x,y)}\mathcal{R}}{\langle T_{(x,y)}\pi_P^{-1}(x), T_{(x,y)}\pi_Q^{-1}(y)\rangle} & \stackrel{\varphi_Q}{\longrightarrow} & \frac{T_y\mathcal{Q}}{T_y\mathcal{Q}_x} \\ \| & & \| & & \| \\ \frac{\mathfrak{g}/\mathfrak{p}}{\mathfrak{q}/\mathfrak{p}} & \simeq & \frac{\mathfrak{g}/(\mathfrak{p}\cap\mathfrak{q})}{\langle\mathfrak{p},\mathfrak{q}\rangle/(\mathfrak{p}\cap\mathfrak{q})} & \simeq & \frac{\mathfrak{g}/\mathfrak{q}}{\mathfrak{p}/\mathfrak{q}} \end{array},$$

where φ_P (resp. φ_Q) is induced by the differential $d_{(x,y)}\pi_P$ (resp. $d_{(x,y)}\pi_Q$). All the terms on the second line are canonically isomorphic with $\mathfrak{g}/\langle\mathfrak{p},\mathfrak{q}\rangle$. Obviously, the diagram commutes, so φ_P and φ_Q are isomorphisms. Let $\mu(x,y):\frac{T_x\mathcal{P}}{T_x\mathcal{P}_y}\to\frac{T_y\mathcal{Q}}{T_y\mathcal{Q}_x}$ denote the canonical isomorphism $\varphi_Q\circ\varphi_P^{-1}$.

Let $f\in(T_x\mathcal{P}/T_x\mathcal{P}_y)^*$ be generic. Let $(y',f')\in T^*\mathcal{Q}$ denote the image of (x,f) by the flop $T^*\mathcal{P}\dashrightarrow T^*\mathcal{Q}$. By [Cha 06], y' is the only element in $\mathcal{Q}$ such that f vanishes on $T_x\mathcal{P}_y$; since by assumption, f vanishes on $T_x\mathcal{P}_y$, $y'=y$. Moreover, we have canonical isomorphisms $(T_x\mathcal{P}/T_x\mathcal{P}_y)^*\simeq\mathfrak{u}(\mathfrak{p})\cap\mathfrak{u}(\mathfrak{q})\simeq(T_y\mathcal{Q}/T_y\mathcal{Q}_x)^*$, and under this isomorphism, f is mapped to f' by definition of the Mukai flop. It is clear that this isomorphism is the transpose of $\mu(x,y)^{-1}$, so the last claim of the proposition is proved.

If (γ,δ) are as in the proposition, then $(\gamma'(0),\delta'(0))\in T_{(x,y)}\mathcal{R}$; denote by $[\gamma'(0),\delta'(0)]$ its class in $\frac{T_{(x,y)}\mathcal{R}}{\langle T_{(x,y)}\pi_P^{-1}(x),T_{(x,y)}\pi_Q^{-1}(y)\rangle}$. By definition of φ_P and φ_Q, we have $\varphi_P([\gamma'(0),\delta'(0)])=[\gamma'(0)]$ and $\varphi_Q([\gamma'(0),\delta'(0)])=[\delta'(0)]$. We therefore have, as expected, $\varphi_Q\circ\varphi_P^{-1}([\gamma'(0)]=[\delta'(0)]$.

□

4.2 Mukai flop of type $E_{6,I}$ in terms of Jordan algebras

Let $\mu(x,y)$ denote the isomorphism of proposition 4.1. In this subsection, I give an expression of $\mu(x,y)$ in the case of Scorza varieties, in terms of Jordan algebras. Therefore, this gives also a formula for the Mukai flop.

More precisely, let (V,X) be a Scorza variety of type (n,a) and let $(V^*,X^\vee)$ be the dual Scorza variety. Let $(x,h)\in X\times X^\vee$ such that $x\vdash h$. Let us choose $(\tilde{x},\tilde{h})\in V\times V^*$ such that $[\tilde{x}]=x$ and $[\tilde{h}]=h$. This identifies T_xX (resp. $T_hX^\vee$) with $T_{\tilde{x}}\widehat{X}/\mathbb{C}.\tilde{x}$ (resp. $T_{\tilde{h}}\widehat{X^\vee}/\mathbb{C}.\tilde{h}$). The previous isomorphism $\mu(x,h)$: $T_xX/T_xC_h\simeq T_hX^\vee/T_hC_x$ induces an isomorphism $T_{\tilde{x}}\widehat{X}/T_{\tilde{x}}\widehat{C_h}\simeq T_{\tilde{h}}\widehat{X^\vee}/T_{\tilde{h}}\widehat{C_x}$.

The goal of this section is to give a formula for this isomorphism in Jordan terms.

For $A,B\in V$, let $\sigma_A(B)\in V^*$ denote the linear form $U\mapsto D_A^2\det(B,U)$. Note that this is equal, modulo $D_A\det$, to $S_A(B)$ [Cha 06]. For $h\in X^\vee$, let $V(h):=(T_hX^\vee)^\perp\subset V$.

Lemma 4.1. *Let $A\in V(h)$. Then $\sigma_A(\tilde{x})$ is proportional to $\tilde{h}$.*

Proof : We can assume that $V=H_3(\mathcal{A})$ and $X\subset\mathbb{P}V$ is the variety of rank one elements. Identify V and V^* as usually. Since $X^\vee$ is homogeneous under G, we may assume that $h=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$. Then $V(h)=\left\{\begin{pmatrix}*&*&0\\ *&*&0\\0&0&0\end{pmatrix}\right\}$. It is enough to prove the lemma for generic A in $V(h)$, so we may assume that A has rank 2. Moreover, let G_h denote the stabilizor in $Aut(X)$ of h. It is clear that G_h acts transitively on the set of rank 2 elements of $V(h)$, and on the set of its rank 1 elements. So we may assume $\tilde{x}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$. Moreover, for the stabilizor of x in

G_h, the set of A's of rank 2 is made of two orbits and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in the open orbit, as it is easily checked case by case.

Therefore, it is enough to compute $\sigma_A(\tilde{x})$ for these choices of x and A. Let $m = (m_{i,j}) \in V$. Then one computes

$$D_A^2 \det(m, m) = \sum_{i<n} m_{i,i} m_{n,n} - \sum_{i<n} N(m_{n,i}). \tag{1}$$

The lemma immediately follows. □

Let as before $x \in X, h \in C_x, A \in V(h)$, and let $\tilde{x} \in V, \tilde{h} \in V^*$ represent x and h.

Proposition 4.1. *Let $v \in T_{\tilde{x}}\widehat{X}$, and let $[v]$ denote its class in $T_{\tilde{x}}\widehat{X}/T_{\tilde{x}}\widehat{C_h} \simeq T_x X/T_x C_h$. If $\sigma_A(\tilde{x}) = \tilde{h}$, then the vector*

$$[\sigma_A(v)] \in T_{\tilde{h}}\widehat{X^{\mathsf{v}}}/T_{\tilde{h}}\widehat{C_x} \simeq T_h X^{\mathsf{v}}/T_h C_x$$

identifies with $\mu(x,h)([v])$.

The isomorphism $T_{\tilde{x}}\widehat{X}/T_{\tilde{x}}\widehat{C_h} \simeq T_x X/T_x C_h$ depends on the choice of $\tilde{x}$, and the isomorphism $T_{\tilde{h}}\widehat{C_x} \simeq T_h X^{\mathsf{v}}/T_h C_x$ depends on the choice of $\tilde{h}$. However, the proposition says that the corresponding map $T_x X/T_x C_h \to T_h X^{\mathsf{v}}/T_h C_x$ does not depend on these choices, neither on the choice of A, as long as $\sigma_A(\tilde{x}) = \tilde{h}$.

Proof : As in the previous lemma, we assume that $V = H_n(\mathcal{A})$. Let X_r denote the variety of rank r matrices. If $B \in X_{n-1}$, then $D_B \det$ belongs to X^{v}. Since $D_A \det = h$ and $\widehat{T_x X} \subset \widehat{T_A X_{n-1}}$, we have the implication $u \in \widehat{T_x X} \Longrightarrow \sigma_A(u) \in \widehat{T_h X^{\mathsf{v}}}$.

Now, let $v \in \widehat{T_x X}$ and let u be the class of v in $\widehat{T_x X}/\mathbb{C}.\tilde{x}$. Let $\varphi(\tilde{x}, \tilde{h}, A)(u)$ denote the element of $T_h X^{\mathsf{v}}$ corresponding to the class of $\sigma_A(v)$ in $\widehat{T_h X^{\mathsf{v}}}/\mathbb{C}.\tilde{h}$ (by lemma 4.1, this class depends only on u).

We first show that if $\tilde{x}, \tilde{h}, A$ are multiplied by a scalor, then $\varphi(\tilde{x}, \tilde{h}, A)$ does not vary. So let $\lambda, \mu, \nu \in \mathbb{C}$, and assume $\sigma_{\nu.A}(\lambda.\tilde{x}) = \mu.\tilde{h}$. Since by assumption $\sigma_A(\tilde{x}) = \tilde{h}$, this means that $\nu^{n-2}.\lambda = \mu$. Now, $\lambda.\tilde{x}$ will identify u with $\lambda.v$. Then, $\sigma_{\nu.A}(\lambda.v) = \nu^{n-2}.\lambda.\sigma_A(v) = \mu.\sigma_A(v)$, and the class of this vector will identify with $\varphi(\tilde{x}, \tilde{h}, A)(u) \in T_h X^{\mathsf{v}}$ with the choice $\mu.\tilde{h}$ instead of $\tilde{h}$.

Therefore, the claim is proved, and one can choose the same elements $\tilde{h}, \tilde{x}, A$ as in the proof of the lemma. By formula (1), if $v = \begin{pmatrix} t & \overline{w} & \overline{z} \\ w & 0 & 0 \\ z & 0 & 0 \end{pmatrix} \in \widehat{T_x X}$, then $\sigma_A(v) = \begin{pmatrix} 0 & 0 & -\overline{z} \\ 0 & 0 & 0 \\ -z & 0 & t \end{pmatrix}$, if one identifies V and V^* via the usual scalar product. Note that $T_{\tilde{h}}\widehat{C}_x = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix} \right\}$, so that the class of $\sigma_A(v)$ does not depend on w, neither on t.

Let us now compute $\mu(x,h)([v])$ using theorem 4.1, and assuming $w = 0$. So let $z \in \mathcal{A}$ and $t \in \mathbb{C}$. Recall that generic elements of $\widehat{X}$ can be written as $\nu_2(\alpha, z_1, z_2)$, with $\alpha \in \mathbb{C}, z_1, z_2 \in \mathcal{A}$, and $\nu_2(\alpha, z_1, z_2) = \begin{pmatrix} \alpha^2 & \alpha\overline{z}_1 & \alpha\overline{z}_2 \\ \alpha z_1 & N(z_1) & z_1\overline{z}_2 \\ \alpha z_2 & z_2\overline{z}_1 & N(z_2) \end{pmatrix}$.

Denote $x(t) = \nu_2(1,0,tz) = \begin{pmatrix} 1 & 0 & \overline{z}t \\ 0 & 0 & 0 \\ zt & 0 & N(z)t^2 \end{pmatrix}$; we have $x(t) \in \widehat{X}$ and $x'(0) = \begin{pmatrix} 0 & 0 & \overline{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}$.

Differetiating ν_2 we get

$$\widehat{T_{x(t)}X} = \left\{ \begin{pmatrix} 2\alpha & \overline{z}_1 & \overline{z}_2 + \alpha t\overline{z} \\ z_1 & 0 & tz_1\overline{z} \\ \alpha tz + z_2 & tz\overline{z}_1 & t(z\overline{z}_2 + z_2\overline{z}) \end{pmatrix} : \alpha \in \mathbb{C}, z_1, z_2 \in \mathcal{A} \right\}$$

Recall that the incidence relation between X and $X^\vee$ is : $x \vdash h$ if $h \supset \widehat{T_xX}$; therefore, if we set $h(t) = \begin{pmatrix} t^2N(z)/2 & 0 & -t\overline{z} \\ 0 & 0 & 0 \\ -tz & 0 & 1 \end{pmatrix}$, we have $x(t) \vdash h(t)$, and since $h'(0) = -\begin{pmatrix} 0 & 0 & \overline{z} \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix}$, the proposition follows. □

4.3 Mukai flops for Scorza varieties in terms of $\mathcal{A}$-blow-up

The simplest Mukai flop $T^*\mathbb{P}^n \dashrightarrow T^*(\mathbb{P}^n)^\vee$ can be resolved blowing-up the zero section. Let's recall this construction. Let $Z \subset T^*\mathbb{P}^n$ be the zero section, and let B be the blow-up of $T^*\mathbb{P}^n$ along Z. It is known that there is a map $B \to T^*(\mathbb{P}^n)^\vee$ such that the following triangle commutes :

$$\begin{array}{ccccc} & & B & & \\ & \swarrow & & \searrow & \\ T^*\mathbb{P}^n & & \dashrightarrow & & T^*(\mathbb{P}^n)^\vee. \end{array} \qquad (2)$$

Moreover, this is the minimal resolution, in the sense that for any other B' with the same property, there is a map $B' \to B$ and an obvious commutative diagram.

In this subsection, I give a similar resolution of the rational map $T^*X \dashrightarrow T^*X^\vee$, if X is a Scorza variety and $X^\vee$ the corresponding dual Scorza variety. In fact, the main idea is that since X behaves like a projective space $\mathbb{P}^n_{\mathcal{A}}$ over $\mathcal{A}$, one should consider an "$\mathcal{A}$-blow-up".

Let me make a heuristic comment. Given a composition algebra $\mathcal{A}$, I believe in the existence of a category $\mathcal{A}-Var$ of $\mathcal{A}$-varieties, containing projective spaces and grassmanians over $\mathcal{A}$. Moreover, if $Y \subset X$ is a closed immersion in this category, then there should be an object $Bl_X(Y)$ over X defined by a universal property analogous to that defining usual blow-ups, but in the category $\mathcal{A}-Var$. Since for the moment I don't know how to define $\mathcal{A} - Var$, I will not give this construction here. In the following we will only have very simple $\mathcal{A}$-blow-ups to do, and in these simple cases we can guess what the blow-up should be.

So let $\mathcal{A}$ be a composition algebra over $\mathbb{C}$ of dimension a and $n \geq 2$ an integer, with $n = 2$ if $\mathcal{A} = \mathbb{O}_{\mathbb{C}}$. Let the affine space $\mathbb{A}^n_{\mathcal{A}}$ be just $\mathcal{A}^n$, the affine (an)-dimensional space over $\mathbb{C}$.

Recall that in subsection 1.2, I introduced a map $\overline{\nu}_2 : \mathbb{A}^2_{\mathbb{O}} \dashrightarrow \mathbb{P}^1_{\mathbb{O}}$, where by definition $\mathbb{P}^1_{\mathbb{O}}$ is an 8-dimensional smooth quadric. Recall also the rank 8 vector bundle L over $\mathbb{P}^1_{\mathbb{O}}$ of proposition 1.7. By definition, L is a subbundle of the trivial bundle $\mathbb{A}^2_{\mathbb{O}} \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{O}}}$ of rank 16 over $\mathbb{P}^1_{\mathbb{O}}$. Therefore, if $\mathcal{L}$ denotes the total space of the vector bundle L, there is an inclusion $\mathcal{L} \subset \mathbb{A}^2_{\mathbb{O}} \times \mathbb{P}^1_{\mathbb{O}}$. Therefore we have a map $\mathcal{L} \to \mathbb{A}^2_{\mathbb{O}}$.

The case of associative algebras is simpler and was studied in [Cha 05] : recall that there is a rational map $\overline{\nu}_2 : \mathbb{A}^n_{\mathcal{A}} \dashrightarrow \mathbb{P}^{n-1}_{\mathcal{A}}$ and a rank a subbundle $\mathcal{L}$ of the trivial bundle $\mathbb{A}^n_{\mathcal{A}} \otimes \mathcal{O}_{\mathbb{P}^{n-1}_{\mathcal{A}}}$ with fiber $\mathbb{A}^n_{\mathcal{A}}$ over $\mathbb{P}^{n-1}_{\mathcal{A}}$. This subbundle is also defined by $\mathcal{L}_x = \overline{\{v \in \mathbb{A}^n_{\mathcal{A}} : \overline{\nu}_2(v) \text{ is defined and } \overline{\nu}_2(v) = x\}}$. Let $\mathcal{L}$ denote its total space.

Definition 4.1. *The $\mathcal{A}$-blow-up $Bl_{\mathbb{A}^n_{\mathcal{A}}}(0)$ of the affine space $\mathbb{A}^n_{\mathcal{A}}$ at the origin is the map $\mathcal{L} \to \mathbb{A}^n_{\mathcal{A}}$.*

Recall that in $\mathbb{A}^2_{\mathbb{O}}$ there are three $Spin_{10}$-orbits : the open orbit, the point 0, and the affine cone $\widehat{\mathbb{S}}$ over a spinor variety $\mathbb{S} \subset \mathbb{P}\mathbb{A}^2_{\mathbb{O}}$. The map $Bl_{\mathbb{A}^2_{\mathbb{O}}}(0) \to \mathbb{A}^2_{\mathbb{O}}$ is an isomorphism above the open orbit, and the fiber over 0 is isomorphic with $\mathbb{P}^1_{\mathbb{O}}$. Except for the existence of the intermediate orbit $\widehat{\mathbb{S}} - \{0\}$ in $\mathbb{A}^2_{\mathbb{O}}$, the situation is therefore very similar to that of the usual blow-up of the origin in $\mathbb{A}^2_{\mathbb{C}}$. A similar statement holds in general for the blow-up of $\mathbb{A}^n_{\mathcal{A}}$. The following result gives another analogy with usual blow-ups :

Proposition 4.2. *This $\mathcal{A}$-blow-up is the minimal resolution of the rational map $\overline{\nu}_2 : \mathbb{A}^n_{\mathcal{A}} \dashrightarrow \mathbb{P}^{n-1}_{\mathcal{A}}$.*

Proof : Let $\pi : \mathcal{L} \to \mathbb{A}^n_{\mathcal{A}}$ denote this $\mathcal{A}$-blow-up. The restriction of π to the regular locus of $\overline{\nu}_2$ is an isomorphism. By definition, there are maps $Bl_{\mathbb{A}^n_{\mathcal{A}}}(0) \to \mathbb{A}^n_{\mathcal{A}}$ and $Bl_{\mathbb{A}^n_{\mathcal{A}}}(0) \to \mathbb{P}^{n-1}_{\mathcal{A}}$ such that the diagram

$$\begin{array}{ccccc} & & Bl_{\mathbb{A}^n_{\mathcal{A}}}(0) & & \\ & \swarrow & & \searrow & \\ \overline{\nu}_2 : \mathbb{A}^n_{\mathcal{A}} & & \dashrightarrow & & \mathbb{P}^{n-1}_{\mathcal{A}} \end{array}$$

commutes.

Let

$$\begin{array}{ccccc} & & B' & & \\ & \swarrow & & \searrow & \\ \overline{\nu}_2 : \mathbb{A}^n_{\mathcal{A}} & & \dashrightarrow & & \mathbb{P}^{n-1}_{\mathcal{A}} \end{array}$$

be another resolution. Then we have a map $B' \to \mathbb{A}^n_{\mathcal{A}} \times \mathbb{P}^{n-1}_{\mathcal{A}}$. Since the above diagram is commutative, the image of this map is $Bl_{\mathbb{A}^n_{\mathcal{A}}}(0)$, and we get the desired map $B' \to Bl_{\mathbb{A}^n_{\mathcal{A}}}(0)$. □

Let (V,X) be a Scorza variety of type (n,a) with $n \geq 3$. The above construction of the blow-up of a point in the fixed vector-space $\mathbb{A}^{n-1}_{\mathcal{A}}$ extends readily to a blow-up of the zero section in the vector bundle T^*X. In fact, let $x \in X$; recall (theorem 3.3) that we have a quadratic map $\nu_x^- : T_x^*X \to N_x^*X$, where $N_xX = V/\widehat{T_xX}$, and so a rational map $\overline{\nu_x^-} : T_x^*X \dashrightarrow \mathbb{P}N_x^*X$. This map is isomorphic with our model map $\overline{\nu}_2 : \mathcal{A}^{n-1} \dashrightarrow \mathbb{P}^{n-2}_{\mathcal{A}}$. Letting x vary, we get an algebraic map $\nu^- : T^*X \to N^*X$ over X, and so a rational map $\overline{\nu^-} : T^*X \dashrightarrow \mathbb{P}N^*X$.

In subsection 3.4, the projectivisation of the image of ν^- was denoted $\mathbb{P}N(X)^{\vee}$; $\mathbb{P}N(X)^{\vee}$ is a locally trivial fibration over X with fibers Scorza varieties of type $(n-1,a)$. Let $p_X : \mathbb{P}N(X)^{\vee} \to X$ denote the natural projection.

Consider the bundle $p_X^*T^*X$ above $\mathbb{P}N(X)^{\vee}$. An element of this bundle will be denoted (x,h,f), with $x \in X, h \in \mathbb{P}N(X)^{\vee}_x$ and $f \in T_x^*X$. Globalizing the above construction, let $\mathcal{L} \subset p_X^*T^*X$ be defined as the closure of the set of $(x,h,f) \in p_X^*T^*X$ such that $\overline{\nu_x^-}(f)$ is defined and equals h.

Lemma 4.2. *$\mathcal{L} \subset p_X^*T^*X$ is a subbundle.*

Proof : Assume first that $a = 8$. Then it is simply a global version of proposition 1.7. By theorem 3.3, We know that ν^- is a global algebraic map, which on each fiber T^*X is isomorphic with the map $\nu_2^- : \mathbb{O}_k \oplus \mathbb{O}_k \dashrightarrow \mathbb{P}^1_{\mathbb{O}}$ defined in subsection 1.2. Therefore, the argument of proposition 1.7 works in this situation. The case of associative composition algebras $\mathcal{A}$ is similar and left to the reader. □

Let $Z \subset T^*X$ denote the zero section.

Definition 4.2. *The $\mathcal{A}$-blow-up $Bl_{T^*X}(Z)$ of T^*X along Z is the map $\mathcal{L} \to T^*X$.*

Theorem 4.2. *This $\mathcal{A}$-blow-up is the minimal resolution of the Mukai flop $\mu : T^*X \dashrightarrow T^*X^{\vee}$.*

Proof : Globalizing the proof of proposition 4.2, we see that $Bl_{T^*X}(Z)$ is the minimal resolution of the rational map $\overline{\nu^-} : T^*X \dashrightarrow \mathbb{P}N(X)^{\vee}$. In view of theorem 3.3, it is also the minimal resolution of the composition $T^*X \overset{\mu}{\dashrightarrow} T^*X^{\vee} \to X^{\vee}$. Now, by theorem 4.1, resolving the Mukai flop $T^*X \dashrightarrow T^*X^{\vee}$ is equivalent with resolving its projection to $X^{\vee}$, so the theorem follows. □

4.4 Mukai flop of type $E_{6,II}$

Let $Y = E_6/P_3$ be the homogeneous space considered in subsection 3.5, $Y^{\vee} = E_6/P_5$ the "dual" homogeneous space and A, B, C the homogeneous vector bundles over Y defined there. Let also $X = E_6/P_1$ and $X^{\vee} = E_6/P_6$.

We already used the fact that Y is isomorphic with the Fano variety of projective lines included in X. Similarly, $Y^{\vee}$ is the variety of lines included in $X^{\vee}$. Denote as before $\mathbb{P}V$ the ambient space of X. As we have already seen, $X^{\vee}$ identifies with the set of hyperplanes in V which contain two tangent spaces to X.

Therefore, given a point $\alpha \in Y$, which represents a projective line l_α contained in X, and given two points $x \neq y \in X$, any hyperplane $h \subset \mathbb{P}V$ which contains the span of T_xX and T_yX can be considered as an element of $X^{\vee}$. A codimension two subspace $V_\beta \subset V$ containing this span defines a pencil of hyperplanes belonging to $X^{\vee}$, or a point in $Y^{\vee}$.

Let $\alpha \in Y$. Recall that the linear space $V/\langle \widehat{T_x X}, \widehat{T_y X}\rangle$ (where x and y are different points of the line l_α) was denoted C_α in subsection 3.5. Let $\mathbb{P}C^*$ denote the projective bundle over Y and $G(3,C)$ the relative grassmannian of 3-spaces in C. The preceeding remarks show that there are natural maps $\mathbb{P}C^* \to X^\vee$ and $G(3,C) \to Y^\vee$. Let $f : G(3,C) \to Y^\vee$ be this map.

For any $\alpha \in Y$, let $g_\alpha : Hom(L_\alpha, \wedge^2 C_\alpha^*) \dashrightarrow G(3, C_\alpha)$ be the map defined by lemmas 2.2 and 2.3 using $F = C_\alpha$ (namely, $g_\alpha(\varphi) = U(\varphi(l_1), \varphi(l_2))$ for $\varphi \in Hom(L_\alpha, \wedge^2 C_\alpha^*)$ and any non-colinear $l_1, l_2 \in L_\alpha$). By propositions 3.6 and 3.7, there is a natural vector bundle map $T^*Y \to Hom(L^* \otimes \wedge^2 L, \wedge^2 C^*) = Hom(L, \wedge^2 C^*)$, which I denote h.

Let finally $\mu : T^*Y \dashrightarrow T^*Y^\vee$ be the Mukai flop and $\pi : T^*Y^\vee \to Y^\vee$ the structure map.

Theorem 4.3. *The composition*

$$T^*Y \xrightarrow{h} Hom(L, \wedge^2 C^*) \overset{g}{\dashrightarrow} G(3,C) \xrightarrow{f} Y^\vee$$

equals the composition

$$T^*Y \overset{\mu}{\dashrightarrow} T^*Y^\vee \xrightarrow{\pi} Y^\vee.$$

Remark : This describes the rational map $\pi \circ \mu$. The rational map μ itself is then described using proposition 4.1.

Proof : Let $\alpha \in Y$ and generic $\eta \in T^*_\alpha Y$. We know that $\pi \circ \mu(\eta)$ is the unique $\beta \in f(G(3, C_\alpha))$ such that η vanishes on the tangent space $T_\alpha SC_\beta$ at α of the Schubert cell $SC_\beta \subset Y$ defined by β. So first, we compute $T_\alpha SC_\beta$.

If $\beta = f(\beta_0)$, with $\beta_0 \in G(3, C_\alpha)$, let $c_\beta \subset C_\alpha$ be the 3-dimensional subspace corresponding to β_0. Let a_β denote the image of $\wedge^2 c_\beta \otimes \wedge^2 L_\alpha \subset \wedge^2 C_\alpha \otimes \wedge^2 L_\alpha$ in A_α under the isomorphism of proposition 3.7. By proposition 3.8, if the class modulo L_α of $x \in X$ is in $\mathbb{P}a_\beta$, then $\widehat{T_x X}/S_\alpha \subset c_\beta$.

Let $v_\beta \subset V$ denote the inverse image of a_β under the projection $I_\alpha \to A_\alpha = I_\alpha / L_\alpha$. Since SC_β is the variety of lines $l \subset X$ such that $\forall x \in l, \widehat{T_x X}/S_\alpha \subset c_\beta$, we deduce $G(2, v_\beta) \subset SC_\beta$.

Now, given $\alpha \in Y$, the cell $SC_\alpha \in Y^\vee$ identifies with $G(3, C_\alpha)$. By symmetry, SC_β is also isomorphic with a six-dimensional grassmannian, and so $G(2, v_\beta) = SC_\beta$. Therefore, $T_\alpha SC_\beta = Hom(L_\alpha, a_\beta)$.

Now, we complete the proof. We already saw that a cotangent form $\eta \in T^*_\alpha Y$ defines an element $h(\eta) \in Hom(L_\alpha, \wedge^2 C_\alpha^*)$. Given the previous computation of $T_\alpha SC_\beta$, η vanishes on $T_\alpha SC_\beta$ if and only if $\wedge^2 c_\beta \perp \mathtt{Im}\ h(\eta)$. Therefore, we can conclude thanks to lemmas 2.2 and 2.3. □

Recall that if V_2 and V_5 are vector spaces of respective dimensions 2 and 5, there are 8 $(GL(V_2) \times GL(V_5))$-orbits in $Hom(V_2, \wedge^2 V_5)$, which were given a label in lemma 2.5. We also use the standart labels for nilpotent orbits, as in [McG 02, p.202].

Corollary 4.3. *Let $\alpha \in Y$ and $0 \neq f \in T^*_\alpha Y$. Under the natural map $T^*Y \to \mathfrak{e}_6$, f is mapped to the nilpotent orbit with the same label as that of the $(GL(L) \times GL(C))$-orbit $h(f) \in Hom(L, \wedge^2 C^*)$ belongs. The Mukai flop is defined exactly on the open orbit of T^*Y.*

In this corollary, I mean that if $h(f)$ is the orbit labelled $3A_{1,a}, 3A_{1,b}$ or $3A_{1,c}$, then it is mapped on the nilpotent orbit labelled $3A_1$.

Proof : This corollary follows from dimension arguments, which are not so illuminating on the geometry of the resolution. The map $T^*Y \to \mathfrak{e}_6$ being birational and proper, it has 50-dimensional closed image; so it is the closure of the unique 50-dimensional orbit in $\mathfrak{e}_6$, labelled $A_2 + 2A_1$. By the given graph of orbit closures [McG 02, p.212], the image of T^*Y is the union of the orbits labelled $0, A_1, 2A_1, 3A_1, A_2, A_2 + A_1, A_2 + 2A_1$.

Let $\alpha \in Y$ and $f, g \in T^*_\alpha Y$, with $f \neq 0$ and $g \neq 0$. Then f and g lay in the same E_6-orbit if and only if $h(f), h(g) \in Hom(L_\alpha, \wedge^2 C^*_\alpha)$ lay in the same $(GL(L_\alpha) \times GL(C_\alpha))$-orbit, by proposition 3.9. It is clear that the zero section in T^*Y is mapped to the 0-orbit in $\mathfrak{e}_6$. Let us label the other E_6-orbits in T^*Y by the labels of their images in $Hom(L, \wedge^2 C^*)$.

We first begin with a trivial remark : the image of an orbit in T^*Y is an orbit in $\mathfrak{e}_6$ of non-greater dimension. From this it follows that the orbits in T^*Y labelled $A_2 + 2A_1, A_2 + A_1, A_1$ map to the orbits in $\mathfrak{e}_6$ with the same label.

Suppose the orbit in T^*Y labelled $3A_{1,a}$ maps to the orbit in $\mathfrak{e}_6$ labelled A_2. The the fibers above the nilpotent orbit A_2 would have dimension 3, and the preimage of the nilpotent orbit labelled $3A_1$ would be included in the orbits labelled $3A_{1,b}$ and $3A_{1,c}$. So the fibers above this orbit would have dimension 1 or 2, contradicting the semi-continuity of the dimensions of the fibers of a morphism. Therefore, $3A_{1,a}$ maps to $3A_1$.

We know that the resolution $T^*Y \to \mathfrak{e}_6$ is semi-small, so the 38-dimensional orbit in T^*Y labelled $2A_1$ cannot contract to the 22-dimensional orbit labelled A_1; therefore, it maps to $2A_1$.

We deduce that the fibers above the A_1-orbit are 8-dimensional; by semi-continuity again, the orbits labelled $3A_{1,b}, 3A_{1,c}$ map to the orbit $3A_1$.

Since by lemma 2.5 the map U of notation 2.2 is defined only on the open orbit of $Hom(L_\alpha, \wedge^2 C^*_\alpha)$, the Mukai flop is also defined only on the open orbit of T^*Y. □

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