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A limit theorem for a random walk in a stationary scenery coming from a hyperbolic dynamical system

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Abstract. *In this paper, we extend a result of Kesten and Spitzer [13]. Let us consider an invertible probability dynamical system* $(M,\mathcal{F},\nu,T)$ *and* $f : M \to \mathbb{R}$ *some function with null expectation. We define the stationary sequence* $(\xi_k := f\circ T^k)_{k\in\mathbb{Z}}$*. Let* $(S_n)_{n\ge 0}$ *be a simple symmetric random walk on* $\mathbb{Z}$ *independent of* $(\xi_k)_{k\in\mathbb{Z}}$*. We are interested in the study of the sequence of random variables of the form* $(\sum_{k=1}^n \xi_{S_k})_{n\ge 1}$*. We give examples of partially hyperbolic dynamical systems* $(M,\mathcal{F},\nu,T)$ *and of functions* f *such that* $\left(\frac{1}{n^{\frac{3}{4}}}\sum_{k=1}^n \xi_{S_k}\right)_{n\ge 1}$ *converges in distribution.*

1 Introduction

In [13], Kesten and Spitzer prove that, if $(\xi_n)_{n\in\mathbb{Z}}$ is a sequence of independent identically distributed satisfying a central limit theorem and if $(S_n)_{n\ge 0}$ is the simple symmetric random walk on $\mathbb{Z}$ independent of $(\xi_k)_{k\in\mathbb{Z}}$, then $\left(\frac{1}{n^{\frac{3}{4}}}\sum_{i=1}^n \xi_{S_k}\right)_{n\ge 1}$ converges in distribution. In this paper, our goal is to establish such a result when $(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of random variables given by a dynamical system with some hyperbolic properties. More precisely, we study the cases when $(\xi_k = f\circ T^k)_{k\in\mathbb{Z}}$, with f a ν-centered Hölder continuous function and when $(M,\mathcal{F},\nu,T)$ is one of the following dynamical systems :

- the transformation T is an ergodic algebraic automorphism of the torus $M=\mathbb{T}^{d_0}$ endowed with its normalised Haar measure ν (for some $d_0\ge 2$);
- the transformation T is a diagonal transformation on a compact quotient M of $Sl_{d_0}(\mathbb{R})$ by a discrete subgroup, M being endowed with a natural T-invariant probability measure ν;
- the transformation T is the Sinai billiard transformation.

In these situations, we prove that $\left(\frac{1}{n^{\frac{3}{4}}}\sum_{i=1}^n \xi_{S_k}\right)_{n\ge 1}$ converges in distribution to the random variable $\sqrt{\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]}\Delta_1$, where Δ_1 has the limit distribution of $\left(\frac{1}{n^{\frac{3}{4}}}\sum_{i=1}^n \hat{\xi}_{S_k}\right)_{n\ge 1}$ obtained by Kesten and Spitzer when $(\hat{\xi}_m)_m$ is a sequence of independent identically distributed random variables with null expectation and with variance 1. Let us notice that, in our cases, $\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]$ is well defined and is nonnegative since it is the limit of the variance of $\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\xi_l$ as n goes ti infinity.

We also get the same result of convergence in distribution for the following sequence $(\xi_k)_{k\in\mathbb{Z}}$. Let us consider the same examples of dynamical systems $(M,\mathcal{F},\nu,T)$. Instead of taking $\xi_k = f\circ T^k$, we suppose that, conditionally to $\omega \in M$, $(\xi_k)_{k\in\mathbb{Z}}$ is an independent sequence of random variables with values in $\{-1;1\}$. We suppose that, conditionally to $\omega \in M$, $\xi_k(\omega,\cdot)$ is equal to 1 with probability $h\circ T^k(\omega)$, for some nonnegative Hölder continuous function h with expectation $\frac{1}{2}$. This model is envisaged by Guillotin-Plantard and Le Ny in [8] for other questions and with other hypotheses on $(M,\mathcal{F},\nu,T)$ and on f.

Moreover we generalize this to the case when ξ_k takes p values (conditionally to $\omega \in M$, $(\xi_k)_{k\in\mathbb{Z}}$ is an independent sequence of random variables, ξ_k being equal to θ_j with probability $f_j \circ T^k(\omega)$, with $f_1 + ... + f_p = 1$ and with $f_1, ..., f_p$ are nonnegative Hölder continuous functions).

In section 2, we state a general result under technical hypotheses of decorrelation (our theorem 1). Section 5 is devoted to the proof of this result (the idea of the proof is inspired by one step of an inductive method of Jan [9, 11] used in [14]).

In section 3, we give some applications of our abstract theorem 1. We apply our theorem 1 to the examples mentionned previously (ergodic algebraic automorphisms of the torus, diagonal transformation of a compact quotient of $Sl_{d_0}(\mathbb{R})$, billiard transformation). The proofs of the results of section 3 are done in sections 3 and 4.

2 A technical result

Theorem 1 *Let $(S_n)_{n\geq 1}$ and $(\xi_k)_{k\in\mathbb{Z}}$ be two sequences of random variables defined on the same probability space $(\Omega,\mathcal{T},\mathbb{P})$ such that :*

1. *$(S_n)_{n\geq 0}$ and $(\xi_k)_{k\in\mathbb{Z}}$ are independent one of the other;*
2. *$(S_n)_{n\geq 0}$ is a simple symmetric random walk on $\mathbb{Z}$;*
3. *$(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of centered random variables admitting moments of the fourth order;*
4. *we have :*
$$\sum_{p\geq 0}\sqrt{1+p}\,|\mathbb{E}[\xi_0\xi_p]| < +\infty$$
$$\text{and} \quad \sup_{N\geq 1} N^{-2}\sum_{k_1,k_2,k_3,k_4=0,\dots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]| < +\infty.$$
5. *There exists some $C>0$, some $(\varphi_{p,s})_{p,s\in\mathbb{N}}$ and some integer $r\geq 1$ such that :*
$$\forall (p,s)\in\mathbb{N}^2,\quad \varphi_{p+1,s}\leq \varphi_{p,s} \quad \text{and} \quad \lim_{s\to+\infty}\sqrt{s}\varphi_{rs,s}=0$$
and such that, for all integers n_1,n_2,n_3,n_4 with $0\leq n_1\leq n_2\leq n_3\leq n_4$, for all real numbers $\alpha_{n_1},...,\alpha_{n_2}$ and $\beta_{n_3},...,\beta_{n_4}$, we have :
$$\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k}, e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right| \leq C\left(1+\sum_{k=n_1}^{n_2}|\alpha_k|+\sum_{k=n_3}^{n_4}|\beta_k|\right)\varphi_{n_3-n_2,n_4-n_3}.$$

Then, the sequence of random variables $\left(\frac{1}{n^{\frac{3}{4}}}\sum_{i=1}^n \xi_{S_k}\right)_{n\geq 1}$ converges in distribution to $\sqrt{\sum_{p\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_p]}\Delta_1$, where $\Delta_1 := \int_{\mathbb{R}} L_1(x)\,dB_x$, where $(B_x)_{x\in\mathbb{R}}$ and $(b_t)_{t\geq 0}$ are two independent standard brownian motions and $(L_t(x))_{t\geq 0}$ is the local time at x of $(b_t)_{t\geq 0}$, i.e. $L_t(x) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{(x-\varepsilon,x+\varepsilon)}(b_s)\,ds$.

Let us notice that the point 5 of our theorem 1 is true if $(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of random variables satisfying the following α-mixing condition (cf. for example [10], lemma 1.2) :
$$\lim_{n\to+\infty}\sqrt{n}\alpha_n = 0, \quad \text{with} \quad \alpha_n := \sup_{p\geq 0;\ m\geq 0}\ \sup_{A\in\sigma(\xi_{-p},\dots,\xi_0)}\ \sup_{B\in\sigma(\xi_n,\dots,\xi_{n+m})}|\mathbb{P}(A\cap B)-\mathbb{P}(A)\mathbb{P}(B)|\,.$$

3 Applications

Now let us give some examples of stationary sequences $(\xi_k)_k$ satisfying the points 3, 4 and 5 of our theorem 1. We say that $(M,\mathcal{F},\nu,T)$ is an invertible dynamical system if $(M,\mathcal{F},\nu)$ is a probability space endowed with an invertible bi-measurable transformation $T:M\rightarrow M$.

Hypothesis 2 *Let us consider an invertible dynamical system $(M,\mathcal{F},\nu,T)$ such that there exists $C_0>0$, there exist two real sequences $(\varphi_n)_{n\ge 0}$ and $(\kappa_m)_{m\ge 0}$ and, for any function $g:M\rightarrow\mathbb{C}$, there exist $K_g^{(1)}\in[0;+\infty]$ and $K_g^{(2)}\in[0;+\infty]$ such that, for all bounded functions $g,\tilde{g},h,\tilde{h}:M\rightarrow\mathbb{C}$:*

1. *for all integer $n\ge 0$, we have : $|Cov_\nu(g,h\circ T^n)|\le c_0\left(\|g\|_\infty\|h\|_\infty+\|h\|_\infty K_g^{(1)}+\|g\|_\infty K_h^{(2)}\right)\varphi_n$;*
2. *for all integer $m\ge 0$, we have : $K^{(1)}_{g\circ T^{-m}}\le c_0K_g^{(1)}$;*
3. *for all integer $m\ge 0$, and all $k=0,...,m$, we have : $K^{(2)}_{h\circ T^k}\le c_0K_h^{(2)}(1+\kappa_m)$;*
4. *we have : $K^{(1)}_{g\times\tilde{g}}\le\|g\|_\infty K^{(1)}_{\tilde{g}}+\|\tilde{g}\|_\infty K^{(1)}_g$;*
5. *we have : $K^{(2)}_{h\times\tilde{h}}\le\|h\|_\infty K^{(2)}_{\tilde{h}}+\|\tilde{h}\|_\infty K^{(2)}_h$;*
6. *the sequence $(\varphi_n)_{n\ge 0}$ is decreasing and there exists an integer $r\ge 1$ such that : $\sup_{n\ge 1}n^6(1+\kappa_n)\varphi_{nr}<+\infty$.*

For some hyperbolic or partially hyperbolic transformations, such properties are satisfied with $K_g^{(1)}$ some Hlder constant of g along the unstable manifolds and $K_h^{(2)}$ some Hlder constant of h along the stable-central manifolds, with $\varphi_n=\alpha^n$ for some $\alpha\in]0;1[$ and $\kappa_m=m^\beta$ for some $\beta\ge 0$. Let us mention, for example, the ergodic algebraic automorphisms of the torus as well as the diagonal transformation on compact quotient of $Sl_{d_0}(\mathbb{R})$ (cf. [12]). Moreover, in the case of the Sinai billiard transformation, these properties come from [6, 5]. Since the earliest work of Sinai [15], these billiard systems have been studied by many authors (let us mention [1, 2, 3, 4, 7]). More precisely, we state :

Proposition 3 *Let us consider an integer $d_0\ge 2$. Let $(M,\mathcal{F},\nu,T)$ be one of the following dynamical systems :*

(i) M is the d_0-dimensional torus $\mathbb{T}^{d_0}=\mathbb{R}^{d_0}/\mathbb{Z}^{d_0}$ endowed with its Borel σ-algebra $\mathcal{F}$ and with the normalised Haar measure ν on $\mathbb{T}^{d_0}$ and T is an algebraic automorphism of $\mathbb{T}^{d_0}$ given by a matrix $S\in Sl_{d_0}(\mathbb{Z})$ the eigenvalues of which are not root of the unity. We endow $\mathbb{T}^{d_0}$ with the metric d induced by the natural metric on $\mathbb{R}^{d_0}$.

(ii) M is a compact quotient of $Sl_{d_0}(\mathbb{R})$ by a discrete subgroup Γ of $Sl_{d_0}(\mathbb{R})$: $M:=\{x\Gamma;\ x\in Sl_{d_0}(\mathbb{R})\}$; endowed with the normalised measure ν induced by the Haar measure on $Sl_{d_0}(\mathbb{R})$. The transformation T corresponds to the multiplication on the left by a diagonal matrix $S=diag(T_1,...,T_{d_0})\in Sl_{d_0}(\mathbb{R})$ not equal to the identity and such that, for all $i=1,...,d_0-1$, $T_i\ge T_{i+1}>0$. We endow M with the metric d induced by a right-translations invariant riemanian metric on $SL_{d_0}(\mathbb{R})$.

(iii) $(M,\mathcal{F},\nu,T)$ is the time-discrete dynamical system given by the discrete Sinai billiard (corresponding to the reflection times on a scatterer). We suppose that the billiard domain is $\mathcal{D}:=\mathbb{T}^2\setminus\left(\bigcup_{i=1}^I O_i\right)$, where the scatterers O_i are open convex subsets of $\mathbb{T}^2$, the closures of which are pairwise disjoint and the boundaries of which are C^3 smooth with non-null curvature. We use the parametrisation by (r,φ) introduced by Sinai in [15] and we denote by d the natural corresponding metric.

Let $\eta>0$. We can define $g\mapsto K_g^{(1)}$ and $g\mapsto K_g^{(2)}$ such that hypothesis 2 is true and such that, for any bounded $g:M\rightarrow\mathbb{C}$, $K_g^{(1)}$ and $K_g^{(2)}$ are dominated by the Hölder constant $C_g^{(\eta)}$ of g of order η (eventually multiplied by some constant).

In the case (iii), this is still true if we replace $C_g^{(\eta)}$ by :

$$C_g^{(\eta,m)} := \sup_{C\in\mathcal{C}_m} \sup_{x,y\in C,\ x\neq y} \frac{|g(x)-g(y)|}{\max(d(T^k(x),T^k(y)); k=-m,..,m)^\eta},$$

for some integer $m \geq 0$, with $\mathcal{C}_m = \{A\cap B;\ A\in\xi_m^u,\ B\in\xi_m^s\}$ with ξ_m^u and ξ_m^s as in [5] (page 7). (We recall that, for any $k=-m,...,m$, the map T^k si C^1 on each atom of $\mathcal{C}_m$).

Proof. Let $\eta > 0$.

- In the cases *(i)* and *(ii)*, we denote by $\Gamma^{(s,e)}$ the set of stable-central manifolds and by Γ^u the set of unstable manifolds. In [12], each $\gamma^u \in \Gamma^u$ is endowed with some metric d^u and each $\gamma^{(s,e)} \in \Gamma^{(s,e)}$ is endowed with some metric $d^{(s,e)}$ such that there exist $\tilde{c}_0 > 0$, $\delta_0 \in]0;1[$ and $\beta > 0$ such that, for any integer $n \geq 0$, for any $\gamma^u \in \Gamma^u$ and any $\gamma^{(s,e)} \in \Gamma^{(s,e)}$, we have :
  - For any $y,z\in\gamma^u$, $d^u(y,z)\geq d(y,z)$ and for any $y',z'\in\gamma^{(s,e)}$, $d^{(s,e)}(y',z')\geq d(y',z')$.
  - For any $y,z\in\gamma^u$, there exists $\gamma^u_{(n)}$ such that $T^{-n}(y)$ and $T^{-n}(z)$ belong to $\gamma^u_{(n)}$ and we have : $d^u(T^{-n}(y),T^{-n}(z)) \leq \tilde{c}_0(\delta_0)^n d^u(y,z)$.
  - For any $y,z\in\gamma^{(s,e)}$, there exists $\gamma^{(s,e)}_{(n)}$ such that $T^n(y)$ and $T^n(z)$ belong to $\gamma^{(s,e)}_{(n)}$ and we have : $d^{(s,e)}(T^n(y),T^n(z)) \leq \tilde{c}_0(1+n^\beta)d^{(s,e)}(y,z)$.

  Let us define :

  $$K_f^{(1)} := \sup_{\gamma^u\in\Gamma^u} \sup_{y,z\in\gamma^u:y\neq z} \frac{|f(y)-f(z)|}{(d^u(y,z))^\eta} \quad\text{and}\quad K_f^{(2)} := \sup_{\gamma^{(s,e)}\in\Gamma^{(s,e)}} \sup_{y,z\in\gamma^{(s,e)}:y\neq z} \frac{|f(y)-f(z)|}{(d^{(s,e)}(y,z))^\eta}.$$

  Hence, the points 2, 3, 4 and 5 of hypothesis 2 are satisfied with $\kappa_n = n^\beta$. Moreover, these two quantities are less than the Hölder constant of order η of f.

  In [12], the point 1 of hypothesis 2 is proved in the particular case *(ii)*. The same proof can be used in the case *(i)*. We get a sequence $(\varphi_n)_n$ decreasing exponentially fast (cf. lemme 1.3.1 in [12]).

- Let us now consider the case *(iii)*. Let us consider an integer $m \geq 0$. Let us cconsider the set Γ^s of homogeneous stable curves and the set Γ^u of homogeneous unstable curves (see [5] page 7 for the definition of these curves). We recall that there exist two constants $c_1 > 0$ and $\delta_1 \in]0;1[$ such that :
  - let y and z belonging to the same homogeneous unstable curve. Then, for any integer $n \geq 0$, $T^{-n}(y)$ and $T^{-n}(z)$ belong to a same homogeneous unstable curve and we have : $d(T^{-n}(y),T^{-n}(z)) \leq c_1{\delta_1}^n$. Moreover, for any integer $p\geq 0$, y and z belong to the same atom of ξ_p^u. Moreover, if y and z belong to the same atom of ξ_m^s, then $T^m(y)$ and $T^m(z)$ belong to a same homogeneous unstable curve.
  - let y and z belonging to the same homogeneous stable curve. Then, for any integer $n\geq 0$, $T^n(y)$ and $T^n(z)$ belong to a same homogeneous stable curve and we have : $d(T^n(y),T^n(z)) \leq c_1{\delta_1}^n$. Moreover, for any integer $p \geq 0$, y and z belong to the same atom of ξ_p^s. Moreover, if y and z belong to the same atom of ξ_m^u, then $T^{-m}(y)$ and $T^{-m}(z)$ belong to a same homogeneous stable curve.

  In [5], for any y,z, Chernov defines : $s_+(x,y) := \min\{n\geq 0\ :\ y\notin\xi_n^s(x)\}$ and $s_-(x,y) := \min\{n\geq 0\ :\ y\notin\xi_n^u(x)\}$, where $\xi_n^s(x)$ (resp. $\xi_n^u(x)$) is the atom of ξ_n^s (resp. ξ_n^u) containing the point x.

  Following Chernov in [5] (page 15), let us introduce the following quantities :

  $$\tilde{K}_f^{(1)} := \sup_{\gamma^u\in\Gamma^u} \sup_{y,z\in\gamma^u y\neq z} \frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_+(y,z)}}$$

  and

  $$\tilde{K}_f^{(2)} := \sup_{\gamma^s\in\Gamma^s} \sup_{y,z\in\gamma^s y\neq z} \frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

In the definition of [5], the suprema are taken over all unstable and stable curves instead of homogeneous unstable and stable curves. However, in the proofs of theorems 4.1, 4.2 and 4.3 of [5], Chernov only uses Hölder continuity on homogeneous stable and unstable curves. We observe that we have : $\tilde{K}_f^{(i)} \leq 2\|f\|_\infty {\delta_1}^{-\eta m} + K_f^{(i)}$, with :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y,z\in\gamma^u; y\neq z; s_+(y,z)\geq m+1} \frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_+(y,z)}}$$

and

$$\tilde{K}_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{y,z\in\gamma^s; y\neq z; s_-(y,z)\geq m+1} \frac{|f(y)-f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

With these definitions, we have :

$$K_f^{(1)} \leq (\delta_1)^{-\eta(m+1)}(c_1)^\eta C_f^{(\eta,m)} \text{ and } K_f^{(2)} \leq (\delta_1)^{-\eta(m+1)}(c_1)^\eta C_f^{(\eta,m)}.$$

Let us prove the first inequality. Let two points y and z belonging to the same homogeneous unstable curve such that $s^+(y,z) \geq m+1$. Then $y' := T^{s_+(y,z)-1}(y)$ and $z' := T^{s_+(y,z)-1}(z)$ belong to the same homogeneous unstable curve. Therefore, for any $k = -m, ..., m$, we have :

$$\begin{aligned} d(T^k(y),T^k(z)) &= d(T^{-(s_+(y,z)-1-k)}(y'), T^{-(s_+(y,z)-1-k)}(z')) \\ &\leq c_1 {\delta_1}^{s_+(y,z)-1-k} \\ &\leq c_1 {\delta_1}^{s_+(y,z)-(m+1)}. \end{aligned}$$

Hence, since y and z belong to the same atom of $\mathcal{C}_m$, we have :

$$|f(y)-f(z)| \leq C_f^{(\eta,m)}(c_1)^\eta {\delta_1}^{\eta s_+(y,z)} {\delta_1}^{-\eta(m+1)}.$$

The proof of the second inequality is analogous.

Let two points y and z. If y and z belong to the same homogeneous unstable curve, then, for any integer $n \geq 0$, we have $s_+(T^{-n}(y),T^{-n}(z)) = s_+(y,z)+n$. In the same way, if y and z belong to the same homogeneous stable curve, then for any integer $n \geq 0$, we have $s_-(T^n(y),T^n(z)) = s_-(y,z)+n$.

Hence, we get points 2, 3, 4 and 5 of hypothesis 2 with $\kappa_n = 1$.

Moreover, Chernov establishes the existence of $c_3 > 0$ and of $\alpha_3 \in]0;1[$ such that, for any integer $n \geq 0$, for any bounded $\mathbb{C}$-valued functions f and g, we have :

$$|Cov(f, g\circ T^n)| \leq c_3 \left(\|f\|_\infty \|g\|_\infty + \|f\|_\infty K_g^{(2)} + \|g\|_\infty K_f^{(1)}\right)(\alpha_3)^n$$

(cf. theorem 4.3 in [5] and the remark after theorem 4.3 in [5]). This gives the points 1 and 5 of our hypothesis 2,

qed.

Theorem 4 *Let us suppose hypothesis 2. Let $f : M \to \mathbb{R}$ be a bounded function.*

(a) Let us suppose that f is ν-centered, that $K_f^{(1)} < +\infty$ and $K_f^{(2)} < +\infty$. We suppose that there exists some real number $c_1 > 0$ such that, for any real number α, we have : $K^{(1)}_{\exp(i\alpha f)} \leq c_1|\alpha|$ and $K^{(2)}_{\exp(i\alpha f)} \leq c_1|\alpha|$. Then $(\xi_k := f\circ T^k)_{k\in\mathbb{Z}}$ satisfies the points 3, 4 and 5 of our theorem.

(b) Let us suppose that f takes its values in $[0;1]$. Moreover let us suppose that there exists some $c_1 > 0$ such that, for any $a,b \in \mathbb{C}$, we have $K^{(1)}_{af+b} \leq c_1|a|$ and $K^{(2)}_{af+b} \leq c_1|a|$.

Let $\left(\Omega_1 :=]0;1[^{\mathbb{Z}}, \mathcal{F}_1 := (\mathcal{B}(]0;1[))^{\times\mathbb{Z}}, \nu_1 := \lambda^{\otimes\mathbb{Z}}\right)$ where λ is the Lebesgue measure on $]0;1[$. We define $(\xi_k)_{k\in\mathbb{Z}}$ on the product $(\Omega_2 := M\times\Omega_1, \mathcal{F}_2 := \mathcal{F}\otimes\mathcal{F}_1, \nu_2 := \nu\otimes\nu_1)$ as follows :

$$\xi_k(\omega,(z_m)_{m\in\mathbb{Z}}) := 2.\mathbf{1}_{\{z_k \leq f\circ T^k(\omega)\}} - 1.$$

Then $(\xi_k)_{k\in\mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

(c) *Let us fix an integer $p \geq 2$. Let us fix p real numbers $\theta_1, ..., \theta_p$ (and $\theta_0 := 0$) and p non-negative functions $f_1, ..., f_p : M \to [0;1]$ such that $\int_M (\theta_1 f_1 + ... + \theta_p f_p)\, d\nu = 0$ and $f_1 + ... + f_p = 1$ and such that there exists $c_2 > 0$ such that, for all complex numbers $a_1, ..., a_{p-1}, b$, we have*

$$\max(K^{(1)}_{a_1 f_1 + ... + a_{p-1} f_{p-1} + b}, K^{(2)}_{a_1 f_1 + ... + a_{p-1} f_{p-1} + b}) \leq c_2(|a_1| + ... + |a_{p-1}|).$$

Let $\left(\Omega_1 :=]0;1[^{\mathbb{Z}}, \mathcal{F}_1 := (\mathcal{B}(]0;1[))^{\otimes \mathbb{Z}}, \nu_1 := \lambda^{\otimes \mathbb{Z}}\right)$ where λ is the Lebesgue measure on $]0;1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) = \sum_{l=1}^{p} (\theta_l - \theta_{l-1})\, \mathbf{1}_{\{z_k \leq \sum_{j=1}^{l} f_j(T^k(\omega))\}},$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

Let us make some comments on the point *(b)*. Conditionally to $\omega \in M$, $(\tilde{\xi}_k(\omega, \cdot))_{k \in \mathbb{Z}}$ is a sequence of independent random variables with values in $\{-1;1\}$ and $\tilde{\xi}_k(\omega, \cdot)$ is equal to 1 with probability $f \circ T^k(\omega)$. This model is envisaged by Guillotin-Plantard and Le Ny in [8].

The case *(c)* is a generalization of the case *(b)* to the case when $\tilde{\xi}_k$ takes p values (conditionally to $\omega \in M$, $\tilde{\xi}_k(\omega, \cdot)$ is equal to θ_j with probability $f_j \circ T^k(\omega)$).

A direct consequence of proposition 3 and of theorem 4 is :

Theorem 5 *Let $(M, \mathcal{F}, \nu, T)$ be as in proposition 3. Let $\eta > 0$. Let $p \geq 2$. Let $f, f_1, ..., f_p : M \to \mathbb{R}$ be $(p+1)$ bounded Hlder continuous function of order η (or, in the case (iii) of proposition 3, we suppose that these functions are bounded and such that $C_f^{(\eta,m)} < +\infty$ and $\sup_{i=1,...,p} C_{f_i}^{(\eta,m)} < +\infty$ for some integer $m \geq 0$).*

We suppose that $f_1, ..., f_p$ are non-negative functions satisfying $f_1 + ... + f_p = 1$.

(a) *Let us suppose that f is ν-centered. Then $(\xi_k := f \circ T^k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.*

(b) *Let us suppose that f takes its values in $[0;1]$ and that we have $\int_M f\, d\nu = \frac{1}{2}$.*

Let $\left(\Omega_1 :=]0;1[^{\mathbb{Z}}, \mathcal{F}_1 := (\mathcal{B}(]0;1[))^{\otimes \mathbb{Z}}, \nu_1 := \lambda^{\otimes \mathbb{Z}}\right)$ where λ is the Lebesgue measure on $]0;1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :

$$\tilde{\xi}_k(\omega, (z_m)_{m \in \mathbb{Z}}) := 2.\mathbf{1}_{\{z_k \leq f \circ T^k(\omega)\}} - 1.$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

(c) *Let us fix p real numbers $\theta_1, ..., \theta_p$ (and $\theta_0 = 0$) such that $\int_M (\theta_1 f_1 + ... + \theta_p f_p)\, d\nu = 0$ and Let $\left(\Omega_1 :=]0;1[^{\mathbb{Z}}, \mathcal{F}_1 := (\mathcal{B}(]0;1[))^{\times \mathbb{Z}}, \nu_1 := \lambda^{\otimes \mathbb{Z}}\right)$ where λ is the Lebesgue measure on $]0;1[$. We define $(\xi_k)_{k \in \mathbb{Z}}$ on the product $(\Omega_2 := M \times \Omega_1, \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}_1, \nu_2 := \nu \otimes \nu_1)$ as follows :*

$$\xi_k(\omega, (z_m)_{m \in \mathbb{Z}}) = \sum_{l=1}^{p} (\theta_l - \theta_{l-1})\, \mathbf{1}_{\{z_k \leq \sum_{j=1}^{l} f_j(T^k(\omega))\}},$$

Then $(\xi_k)_{k \in \mathbb{Z}}$ satisfies points 3, 4 and 5 of our theorem.

Let us observe that, in the case *(iii)* of proposition 3, we can take the function f constant on each atom of $\mathcal{C}_m$ for some integer $m \geq 0$. For example $f = \mathbf{1}_{\bigcup_{k \geq k_0} \mathbb{H}_k} - \mathbf{1}_{\bigcup_{k \geq k_0} \mathbb{H}_{-k}}$ satisfies the case *(a)* of theorem 5 for the Sinai billiard (with the notations k_0 and $\mathbb{H}_k$ of [5] page 5). In the case *(c)* of theorem 5, we can take $p = 3$, $\theta_1 = 1$, $\theta_2 = -1$, θ_3=0, $f_1 = \mathbf{1}_{\bigcup_{k \geq k_0} \mathbb{H}_k}$, $f_2 = \mathbf{1}_{\bigcup_{k \geq k_0} \mathbb{H}_{-k}}$, $f_3 = \mathbf{1} - f_1 - f_2$ in the case of the Sinai billiard (with again the notations of [5] page 5).

4 Proof of theorem 4

In the cases (a), (b) and (c), it is easy to see that $(\xi_k)_k$ is a stationary sequence of bounded random variables

4.1 Proof of *(a)*

We have :

$$\begin{aligned}\sum_{p\geq 0}\sqrt{1+p}|\mathbb{E}[\xi_0\xi_p]| &= \sum_{p\geq 0}\sqrt{1+p}|\mathbb{E}_\nu[f.f\circ T^p]|\\ &\leq c_0\|f\|_\infty\left(\|f\|_\infty+K_f^{(1)}+K_f^{(2)}\right)\sum_{p\geq 0}\sqrt{1+p}\varphi_p<+\infty.\end{aligned}$$

Let us consider an integer $N\geq 1$. We have :

$$\frac{1}{N^2}\sum_{k_1,k_2,k_3,k_4=0,\dots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|\leq\frac{24}{N^2}\sum_{0\leq k_1\leq k_2\leq k_3\leq k_4\leq N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|\,.$$

Let us consider the set $E_N^{(1)}$ of (k_1,k_2,k_3,k_4) such that $0\leq k_1\leq k_2\leq k_3\leq k_4\leq N-1$ and $k_4-k_3\geq N^{\frac{1}{3}}$ We have :

$$\begin{aligned}\sum_{(k_1,k_2,k_3,k_4)\in E_N^{(1)}}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]| &= \sum_{(k_1,k_2,k_3,k_4)\in E_N^{(1)}}\left|Cov_\nu\left(f\circ T^{k_1-k_3}f\circ T^{k_2-k_3}f,f\circ T^{k_4-k_3}\right)\right|\\ &\leq c_0N^4\left(\|f\|_\infty^4+\|f\|_\infty^3(K_f^{(2)}+3c_0K_f^{(1)})\right)\varphi_{\lceil N^{\frac{1}{3}}\rceil}\\ &\leq c_0N^2\left(\|f\|_\infty^4+\|f\|_\infty^3(K_f^{(2)}+3c_0K_f^{(1)})\right)\sup_{n\geq 1}n^6\varphi_n.\end{aligned}$$

Let us consider the set $E_N^{(2)}$ of (k_1,k_2,k_3,k_4) such that $0\leq k_1\leq k_2\leq k_3\leq k_4\leq N-1$ and $k_4-k_3<N^{\frac{1}{3}}$ and $k_3-k_2\geq rN^{\frac{1}{3}}$. We have :

$$\begin{aligned}\sum_{(k_1,k_2,k_3,k_4)\in E_N^{(2)}}|Cov\left(\xi_{k_1}\xi_{k_2},\xi_{k_3}\xi_{k_4}\right)| &= \sum_{(k_1,k_2,k_3,k_4)\in E_N^{(2)}}\left|Cov_\nu\left(f\circ T^{k_1-k_2}f,(f.f\circ T^{k_4-k_3})\circ T^{k_3-k_2}\right)\right|\\ &\leq c_0N^4\left(\|f\|_\infty^4+2c_0\|f\|_\infty^3(K_f^{(2)}+K_f^{(1)})\right)(1+\kappa_{\frac{1}{r}\lceil rN^{\frac{1}{3}}\rceil})\varphi_{\lceil rN^{\frac{1}{3}}\rceil}\\ &\leq c_0N^2\left(\|f\|_\infty^4+2c_0\|f\|_\infty^3(K_f^{(2)}+K_f^{(1)})\right)\sup_{n\geq 1}n^6(1+\kappa_n)\varphi_{rn}.\end{aligned}$$

Moreover, we have :

$$\begin{aligned}\sum_{(k_1,k_2,k_3,k_4)\in E_N^{(2)}}|\mathbb{E}[\xi_{k_1}\xi_{k_2}]\mathbb{E}[\xi_{k_3}\xi_{k_4}]| &\leq \left(\sum_{0\leq k_1\leq k_2\leq N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}]|\right)^2\\ &\leq \left(N\sum_{k\geq 0}\left|\mathbb{E}_\nu[f.f\circ T^k]\right|\right)^2\\ &\leq N^2\left(c_0\left(\|f\|_\infty^2+\|f\|_\infty(K_f^{(1)}+K_f^{(2)})\right)\sum_{k\geq 0}\varphi_k\right)^2.\end{aligned}$$

Let us consider the set $E_N^{(3)}$ of (k_1,k_2,k_3,k_4) such that $0\le k_1\le k_2\le k_3\le k_4\le N-1$ and $k_4-k_3<N^{\frac13}$ and $k_3-k_2<rN^{\frac13}$ and $k_2-k_1\ge r(1+r)N^{\frac13}$. By the same method, we get :

$$\sum_{(k_1,k_2,k_3,k_4)\in E_N^{(3)}}|\mathbb{E}\left[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}\right]| \quad\le\quad N^2\frac{c_0}{(1+r)^6}\left(\|f\|_\infty^4+3c_0\|f\|_\infty^3(K_f^{(2)}+K_f^{(1)})\right)\sup_{n\ge1}n^6(1+\kappa_n)\varphi_{rn}.$$

Since the number of (k_1,k_2,k_3,k_4) such that $0\le k_1\le k_2\le k_3\le k_4\le N-1$ and that do not belong to $E_N^{(1)}\cup E_N^{(2)}\cup E_N^{(3)}$ is bounded by $N^2 2(r+1)^3$, we get :

$$\sup_{N\ge1}\frac{1}{N^2}\sum_{k_1,k_2,k_3,k_4=0,\dots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|<+\infty.$$

Now, let us prove the point 5. Let n_1, n_2, n_3 and n_4 be four integers such that $0\le n_1\le n_2\le n_3\le n_4$. Let us consider any real numbers $\alpha_{n_1},..,\alpha_{n_2}$ and $\beta_{n_3},...,\beta_{n_4}$. We have :

$$\begin{aligned}\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k},e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right| &= \left|Cov_\nu\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k f\circ T^{-(n_2-k)}},\left(e^{i\sum_{k=n_3}^{n_4}\beta_k f\circ T^{k-n_3}}\right)\circ T^{n_3-n_2}\right)\right|\\
&\le c_0\left(1+K^{(1)}_{\exp\left(i\sum_{k=n_1}^{n_2}\alpha_k f\circ T^{-(n_2-k)}\right)}+K^{(2)}_{\exp\left(i\sum_{k=n_3}^{n_4}\beta_k f\circ T^{k-n_3}\right)}\right)\varphi_{n_3-n_2}\\
&\le c_0\left(1+\sum_{k=n_1}^{n_2}K^{(1)}_{\exp(i\alpha_k f\circ T^{-(n_2-k)})}+\sum_{k=n_3}^{n_4}K^{(2)}_{\exp(i\beta_k f\circ T^{k-n_3})}\right)\varphi_{n_3-n_2}\\
&\le c_0\left(1+\sum_{k=n_1}^{n_2}c_0c_1|\alpha_k|+\sum_{k=n_3}^{n_4}c_0c_1|\beta_k|(1+\kappa_{n_4-n_3})\right)\varphi_{n_3-n_2}.\end{aligned}$$

We conclude by taking $\varphi_{p,s}:=(1+\kappa_s)\varphi_p$.

4.2 Proof of *(b)* and of *(c)*

Let us consider *(c)* which is an extension of the case *(b)* (by taking $p=2$, $\theta_1=1$, $\theta_2=-1$, $f_1=f$ and $f_2=1-f$). Let us define the function $g:=\sum_{j=1}^p\theta_j f_j$ (in the case *(b)*, we have : $g=2f-1$). This function is ν-centered and satisfies $K_g^{(1)}+K_g^{(2)}<+\infty$. We observe that, conditionally to $\omega\in M$, the expectation of $\xi_k(\omega,\cdot)$ is equal to $g\circ T^k(\omega)$. Using the Fubini theorem and starting by integrating over Ω_1, we observe that, for any integers k and l, we have : $\mathbb{E}[\xi_k\xi_l]=\mathbb{E}_\nu[g\circ T^k.g\circ T^l]$ and that, for any integers k_1,k_2,k_3,k_4, we have : $\mathbb{E}\left[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}\right]=\mathbb{E}_\nu\left[\prod_{j=1}^4 g\circ T^{k_j}\right]$. Hence, we can prove the point 4 of theorem 1 as we proved it for *(a)*.

Now, let us prove the point 5 of theorem 1. We observe that, conditionally to $\omega\in M$, the expectation of $\exp(iu\xi_k(\omega,\cdot))$ is $h_u\circ T^k$ with $(h_u:=\sum_{l=1}^p e^{i\theta_l u}f_l)$. This function can be rewritten : $h_u=e^{i\theta_p u}+\sum_{l=1}^{p-1}\left(e^{i\theta_l u}-e^{i\theta_p u}\right)f_l$. The modulus of this function is bounded by 1 and we have :

$$\max\left(K_{h_u}^{(1)},K_{h_u}^{(2)}\right)\le c_2 2p\max_{j=0,\dots,p}|\theta_j|\ |u|.$$

Let n_1, n_2, n_3 and n_4 be four integers such that $0\le n_1\le n_2\le n_3\le n_4$. Let us consider any real numbers $\alpha_{n_1},..,\alpha_{n_2}$ and $\beta_{n_3},...,\beta_{n_4}$. We have :

$$\begin{aligned}&\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k},e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right|=\\
&\quad=\left|Cov_\nu\left(\prod_{k=n_1}^{n_2}h_{\alpha_k}\circ T^k,\prod_{k=n_3}^{n_4}h_{\beta_k}\circ T^k\right)\right|\\
&\quad\le c_0\left(1+c_0c_2 2p\max_{j=0,\dots,p}|\theta_j|\left(\sum_{k=n_1}^{n_2}|\alpha_k|+\sum_{k=n_3}^{n_4}|\beta_k|\right)\right)(1+\kappa_{n_4-n_3})\varphi_{n_3-n_2}.\end{aligned}$$

5 Proof of theorem 1

To prove our result of convergence in distribution, we use characteristic functions. Let us fix some real number t. We will show that :

$$\lim_{n\to+\infty} \mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{k=1}^{n}\xi_{S_k}\right)\right] = \mathbb{E}\left[\exp\left(it\sqrt{\sum_{p\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_p]}\Delta_1\right)\right].$$

Let us notice that we have (cf [13] lemma 5, for example) :

$$\mathbb{E}\left[\exp\left(iu\Delta_1\right)\right] = \mathbb{E}\left[\exp\left(-\frac{u^2}{2}\int_{\mathbb{R}}(L_1(x))^2\,dx\right)\right].$$

Hence, it is enough to prove that :

$$\lim_{n\to+\infty} \mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{k=1}^{n}\xi_{S_k}\right)\right] = \mathbb{E}\left[\exp\left(-\frac{t^2}{2}\sum_{p\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_p]\int_{\mathbb{R}}(L_1(x))^2\,dx\right)\right].$$

In the following, for any integer $m\geq 1$ and any integer k, we define :

$$N_m(k) := Card\{j=1,...,m \ : \ S_j = k\}.$$

We notice that, for any integer $n\geq 1$, we have :

$$\sum_{j=1}^{n}\xi_{S_j} = \sum_{k\in\mathbb{Z}}\xi_k N_n(k).$$

In the step 1 of our proof, we will use the following facts :

$$C_0 := \sup_{n\geq 1}\sup_{K>0} K^2 n^{-1}\mathbb{P}\left(\max_{m=1,\dots,n}|S_m|\geq K\right) < +\infty,$$

$$C_1 := \sup_{n\geq 1}\sup_{k\in\mathbb{Z}} n^{-\frac{1}{2}}\|N_n(k)\|_6 < +\infty,$$

$$C_2 := \sup_{n\geq 1}\sup_{k,\ell\in\mathbb{Z}}\frac{\|N_n(\ell)-N_n(k)\|_2}{\sqrt{1+|\ell-k|}n^{\frac{1}{4}}} < +\infty.$$

The first fact comes from the Kolmogorov inequality. We refer to [13] lemmas 1, 2, 3 and 4 for the proof of the other facts.

5.1 Step 1 : Technical part

This is the big part of our proof. In this part, we prove that the following quantity goes to zero as n goes to $+\infty$:

$$\left|\mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell\in\mathbb{Z}}\xi_\ell N_n(\ell)\right)\right] - \mathbb{E}\left[\exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell,k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)\right]\right|.$$

Let us fix $\varepsilon>0$. We will prove that, if n is large enough, this quantity is less than ε.

Our proof is inspired by a method used by Jan to establish central limit theorem with rate of convergence (cf. [11], [9], method also used in [14]). More precisely, we adapt the idea of the first step of the inductive method of Jan.

- For any $K \geq 1$ and any integer $n \geq 1$, we have :

$$\mathbb{P}\left(\max_{m=1,\dots,n} |S_m| \geq K\sqrt{n}\right) \leq \frac{C_0 n}{K^2 n} = \frac{C_0}{K^2}.$$

Let us fix $K \geq 1$ such that $2\frac{C_0}{K^2} < \frac{\varepsilon}{10}$. Then, we have

$$\left|\mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell\in\mathbb{Z}}\xi_\ell N_n(\ell)\right)\right] - \mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil}^{\lceil K\sqrt{n}\rceil}\xi_\ell N_n(\ell)\right)\right]\right| \leq 2\frac{C_0}{K^2} < \frac{\varepsilon}{10} \quad (1)$$

and :

$$\left|\mathbb{E}\left[\exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell,k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)\right] - \mathbb{E}\left[\exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil}^{\lceil K\sqrt{n}\rceil}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)\right]\right| < \frac{\varepsilon}{10}. \quad (2)$$

Hence we have to estimate :

$$A_n := \left|\mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil}^{\lceil K\sqrt{n}\rceil}\xi_\ell N_n(\ell)\right)\right] - \mathbb{E}\left[\exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil}^{\lceil K\sqrt{n}\rceil}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)\right]\right|. \quad (3)$$

- In the following, L will be some real number bigger than 8 and large enough and n any integer bigger than 1 and large enough such that : $\frac{2K\sqrt{n}}{L} \geq L$. We will have : $\frac{K\sqrt{n}}{L} \leq \left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor \leq \frac{5K\sqrt{n}}{L}$.

- We split our sums $\sum_{\ell=-\lceil K\sqrt{n}\rceil}^{\lceil K\sqrt{n}\rceil}$ in L sums over $\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor$ terms and one sum over less than L terms and so over less than $\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor$ terms.

For any $k = 0, ..., L-1$, we define :

$$a_{k,n,L} = \exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil+k\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor}^{-\lceil K\sqrt{n}\rceil+(k+1)\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor-1}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)$$

and

$$b_{k,n,L} = \exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil+k\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor}^{-\lceil K\sqrt{n}\rceil+(k+1)\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor-1}\xi_\ell N_n(\ell)\right).$$

Moreover, we define :

$$a_{L,n,L} = \exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil+L\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor}^{\lceil K\sqrt{n}\rceil}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)$$

and

$$b_{L,n,L} = \exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil+L\left\lfloor \frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor}^{\lceil K\sqrt{n}\rceil}\xi_\ell N_n(\ell)\right).$$

Let us notice that, for any $k = 0, ..., L$, we have :

$$|a_{k,n,L}| \leq 1 \quad \text{and} \quad |b_{k,n,L}| \leq 1.$$

We have :

$$
\begin{aligned}
|A_n| &= \left|\mathbb{E}\left[\prod_{k=0}^{L} b_{k,n,L} - \prod_{k=0}^{L} a_{k,n,L}\right]\right| \\
&= \left|\sum_{k=0}^{L}\mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_{m,n,L}\right)(b_{k,n,L}-a_{k,n,L})\prod_{m'=k+1}^{L} a_{m',n,L}\right]\right|. \qquad (4)
\end{aligned}
$$

- Now we explain how we can restrict our study to the sum over the k such that $(r+1)^3 \le k \le L-1$. Indeed, the number of k that do not satisfy this is equal to $(r+1)^3+1$. Let us consider any $k = 0, ..., L$. We have :

$$
\mathbb{E}\left[|b_{k,n,L}-1|\right] \le \frac{|t|}{n^{\frac{3}{4}}}\mathbb{E}\left[\left|\sum_{\ell=\cdots}^{\cdots}\xi_\ell N_n(\ell)\right|\right]
$$

and :

$$
\mathbb{E}\left[|a_{k,n,L}-1|\right] \le \frac{t^2}{2n^{\frac{3}{2}}}\mathbb{E}\left[\left|\sum_{\ell=\cdots}^{\cdots}\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_m]N_n(\ell)^2\right|\right].
$$

But, for any integers α and β with $\beta \ge 1$, we have :

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{\ell=\alpha+1}^{\alpha+\beta}\xi_\ell N_n(\ell)\right)^2\right] &\le \sum_{\ell=\alpha+1}^{\alpha+\beta}\sum_{m=\alpha+1}^{\alpha+\beta}|\mathbb{E}[\xi_\ell\xi_m]|\;|\mathbb{E}[N_n(\ell)N_n(m)]| \\
&\le \sum_{\ell=\alpha+1}^{\alpha+\beta}\sum_{m=\alpha+1}^{\alpha+\beta}|\mathbb{E}[\xi_\ell\xi_m]|\;\|N_n(\ell)\|_2\|N_n(m)\|_2 \\
&\le (C_1)^2 n\beta\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|.
\end{aligned}
$$

From which, we get :

$$
\begin{aligned}
\mathbb{E}\left[|b_{k,n,L}-1|\right] &\le \frac{|t|}{n^{\frac{3}{4}}}\sqrt{(C_1)^2 n\frac{5K\sqrt{n}}{L}\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|} \\
&\le \frac{|t|}{\sqrt{L}}C_1\sqrt{5K\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|}.
\end{aligned}
$$

Moreover we have :

$$
\begin{aligned}
\mathbb{E}\left[|a_{k,n,L}-1|\right] &\le \frac{t^2}{2n^{\frac{3}{2}}}\mathbb{E}\left[\sum_{\ell=\cdots}^{\cdots}\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_m]N_n(\ell)^2\right] \\
&\le \frac{t^2}{2n^{\frac{3}{2}}}\frac{5K\sqrt{n}}{L}\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m](C_1)^2 n \\
&\le \frac{5t^2}{2}\frac{K(C_1)^2}{L}\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m].
\end{aligned}
$$

Let $L_1 \ge 8$ be such that for all $L \ge L_1$, we have :

$$
((r+1)^3+1)\frac{|t|}{\sqrt{L}}C_1\sqrt{5K\sum_{m\in\mathbb{Z}}|\mathbb{E}[\xi_0\xi_m]|} < \frac{\varepsilon}{20}
$$

and

$$
((r+1)^3+1)\frac{5t^2}{2}\frac{K(C_1)^2}{L}\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m] < \frac{\varepsilon}{20}.
$$

Then, if we have $L \geq L_1$ and $n \geq 1$ such that $\frac{2K\sqrt{n}}{L} \geq L$, we have :

$$\mathbb{E}\left[|b_{L,n,L} - a_{L,n,L}|\right] + \sum_{k=0}^{(r+1)^3-1} \mathbb{E}\left[|b_{k,n,L} - a_{k,n,L}|\right] < \frac{\varepsilon}{10}. \tag{5}$$

It remains to estimate :

$$\sum_{k=(r+1)^3}^{L-1} \left| \mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_{m,n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^{L} a_{m',n,L} \right] \right|. \tag{6}$$

- We estimate :

$$B_{n,L} := \sum_{k=(r+1)^3}^{L-1} \left| \mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^3} b_{m,n,L} \right) \left(\left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right) \times \right.\right.$$

$$\left.\left. \times \left(\left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right) \left(\prod_{m'=k-r}^{k-1} b_{m',n,L} \right) (b_{k,n,L} - a_{k,n,L}) \prod_{m'=k+1}^{L} a_{m',n,L} \right] \right|.$$

We have :

$$B_{n,L} \leq \sum_{k=(r+1)^3}^{L-1} \left\| \left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L} \right) - 1 \right\|_3 \left\| \left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L} \right) - 1 \right\|_3 \|b_{k,n,L} - a_{k,n,L}\|_3 .$$

  - We have :

$$\|b_{k,n,L} - 1\|_3 \leq \frac{|t|}{n^{\frac{3}{4}}} \left\| \sum_{\ell=\cdots}^{\cdots} \xi_\ell N_n(\ell) \right\|_3 .$$

  For any integers α and β with $\beta \geq 1$, we have :

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{\ell=\alpha+1}^{\alpha+\beta} \xi_\ell N_n(\ell) \right)^4 \right] &\leq \sum_{\ell_1,\ell_2,\ell_3,\ell_4=\alpha+1}^{\alpha+\beta} \left|\mathbb{E}\left[\xi_{\ell_1}\xi_{\ell_2}\xi_{\ell_3}\xi_{\ell_4}\right]\right| (C_1)^4 n^2 \\ &\leq (C_1)^4 n^2 C_2' \beta^2. \end{aligned} \tag{7}$$

  with $C_2' := \sup_{N\geq 1} N^{-2} \sum_{k_1,k_2,k_3,k_4=0,\dots,N-1} |\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|$. Hence, we have :

$$\begin{aligned} \|b_{k,n,L} - 1\|_3 &\leq \frac{|t|}{n^{\frac{3}{4}}} \left((C_1)^4 n^2 C_2' \left(\frac{5K\sqrt{n}}{L} \right)^2 \right)^{\frac{1}{4}} \\ &\leq |t| C_1 (C_2')^{\frac{1}{4}} \sqrt{\frac{5K}{L}}. \end{aligned}$$

  - We have :

$$\begin{aligned} \|a_{k,n,L} - 1\|_3 &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k\in\mathbb{Z}} \mathbb{E}[\xi_0\xi_k] \left\| \sum_{\ell=\cdots}^{\cdots} N_n(\ell)^2 \right\|_3 \\ &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k\in\mathbb{Z}} \mathbb{E}[\xi_0\xi_k] \sum_{\ell=\cdots}^{\cdots} \|N_n(\ell)\|_6^2 \\ &\leq \frac{t^2}{2n^{\frac{3}{2}}} \sum_{k\in\mathbb{Z}} \mathbb{E}[\xi_0\xi_k] \frac{5K\sqrt{n}}{L} (C_1)^2 n \\ &\leq \frac{5t^2}{2} \sum_{k\in\mathbb{Z}} \mathbb{E}[\xi_0\xi_k] \frac{K}{L} (C_1)^2. \end{aligned}$$

– Using formula (7),we get :

$$\begin{aligned}\left\|\left(\prod_{m=k-(r+1)^3+1}^{k-(r+1)^2} b_{m,n,L}\right)-1\right\|_3 &\le \frac{|t|}{n^{\frac{3}{4}}}\left\|\sum_{\ell=-\lceil K\sqrt{n}\rceil+(k-(r+1)^3+1)\left\lfloor\frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor}^{-\lceil K\sqrt{n}\rceil+(k-(r+1)^2+1)\left\lfloor\frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor-1}\xi_\ell N_n(\ell)\right\|_3\\ &\le \frac{|t|}{n^{\frac{3}{4}}}\left((C_1)^4 n^2 C_2'(r(r+1)^2)^2\left(\frac{5K\sqrt{n}}{L}\right)^2\right)^{\frac{1}{4}}\\ &\le |t|C_1(C_2')^{\frac{1}{4}}\sqrt{r}(r+1)\sqrt{\frac{5K}{L}}.\end{aligned}$$

– Analogously, we get :

$$\left\|\left(\prod_{m=k-(r+1)^2+1}^{k-r-1} b_{m,n,L}\right)-1\right\|_3 \le |t|C_1(C_2')^{\frac{1}{4}}\sqrt{r}(r+1)\sqrt{\frac{5K}{L}}.$$

Hence, we have :

$$\begin{aligned}B_{n,L} &\le L\left(|t|C_1(C_2')^{\frac{1}{4}}\sqrt{r}(r+1)\sqrt{\frac{5K}{L}}\right)^2\left(|t|C_1\,(C_2')^{\frac{1}{4}}\sqrt{\frac{5K}{L}}+\frac{5t^2}{2}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_k]\frac{K}{L}(C_1)^2\right)\\ &\le |t|^2(C_1)^2(C_2')^{\frac{1}{2}}r(r+1)^2 5K\left(|t|C_1\,(C_2')^{\frac{1}{4}}\sqrt{\frac{5K}{L}}+\frac{5t^2}{2}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_k]\frac{K}{L}(C_1)^2\right).\end{aligned}$$

Let $L_1' \ge L_1$ be such that, for all $L \ge L_1$, the right term of this last inequality is less than $\frac{\varepsilon}{10}$. Then, for any $L \ge L_1'$ and any $n \ge 1$ such that $\frac{2K\sqrt{n}}{L} \ge L$, we have : $B_{n,L} \le \frac{\varepsilon}{10}$.

- In the following, we suppose $L \ge L_1'$ and $\frac{2K\sqrt{n}}{L} \ge L$. It remains to estimate :

$$\sum_{k=(r+1)^3+1}^{L-1} C_{n,k,L,1,3}+C_{n,k,L,1,2}+C_{n,k,L,2,3}$$

where C_{n,k,L,j_0,j_1} is the following quantity :

$$\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}\right)\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L}\right)(b_{k,n,L}-a_{k,n,L})\prod_{m'=k+1}^{L} a_{m',n,L}\right]\right|$$

- Let j_0, j_1 be fixed. We estimate C_{n,k,L,j_0,j_1}. We have :

$$C_{n,k,L,j_0,j_1} \le D_{n,k,L,j_0,j_1}+E_{n,k,L,j_0,j_1},$$

where :

$$D_{n,k,L,j_0,j_1} := \left|\mathbb{E}\left[Cov_{|(S_p)_p}\left(\Delta_{n,k,L,j_1},\Gamma_{n,k,L,j_0}\right)\prod_{m'=k+1}^{L} a_{m',n,L}\right]\right|$$

and

$$E_{n,k,L,j_0,j_1} : \left|\mathbb{E}\left[\mathbb{E}\left[\Delta_{n,k,L,j_1}|\,(S_p)_p\right]\mathbb{E}\left[\Gamma_{n,k,L,j_0}|\,(S_p)_p\right]\prod_{m'=k+1}^{L} a_{m',n,L}\right]\right|.$$

with $\Delta_{n,k,L,j_1} := \prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}$ and $\Gamma_{n,k,L,j_0} := \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L}\right)(b_{k,n,L}-a_{k,n,L})$.

- Control of the terms with the product of the expectations.

  Let j_0, j_1 be fixed. Let $k = (r+1)^3, ..., L-1$. We can notice that E_{n,k,L,j_0,j_1} is bounded from away by the following quantity :

$$F_{n,k,L,j_0,j_1} := \mathbb{E}\left[\left|\mathbb{E}\left[\prod_{m=k-(r+1)^{j_0}+1}^{k} b_{m,n,L} - \left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L}\right) a_{k,n,L} \middle| (S_p)_p\right]\right|\right].$$

  We use the Taylor expansions of the exponential function. To simplify expressions, we will use the following notation :

$$\forall m \geq 0, \quad \alpha_{(m)} := -\lceil K\sqrt{n}\rceil + m \left\lfloor \frac{2\lceil K\sqrt{n}\rceil + 1}{L} \right\rfloor .$$

  – Let us show that, in F_{n,k,L,j_0,j_1}, we can replace

$$\prod_{m=k-(r+1)^{j_0}+1}^{k} b_{m,n,L} = \exp\left(\frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell)\right)$$

  by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{it}{n^{\frac{3}{4}}} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell) - \frac{t^2}{2n^{\frac{3}{2}}} \left(\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell)\right)^2 . \tag{8}$$

  Indeed the L^1-norm of the error between these two quantities is less than :

$$\frac{|t|^3}{6n^{\frac{9}{4}}} \mathbb{E}\left[\left|\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_n(\ell)\right|^3\right]$$

  which, according to formula (7), is less than :

$$\frac{|t|^3}{6n^{\frac{9}{4}}} \left((C_1)^4 n^2 C_2' \left(((r+1)^{j_0}) \frac{5K\sqrt{n}}{L}\right)^2\right)^{\frac{3}{4}} = \frac{|t|^3}{6} (C_1)^3 (C_2')^{\frac{3}{4}} \left((r+1)^{j_0} \frac{5K}{L}\right)^{\frac{3}{2}} .$$

  Hence, the sum over $k = (r+1)^3, ..., L-1$ of the L^1-norm of these errors is less than :

$$\frac{1}{\sqrt{L}} \frac{|t|^3}{6} (C_2')^{\frac{3}{4}} (C_1)^3 \left((r+1)^{j_0} 5K\right)^{\frac{3}{2}} .$$

  Let us consider $L_2 \geq L_1''$ such that, for all $L \geq L_2$, this last quantity is less than $\frac{\varepsilon}{10}$.

  – Let us introduce $Y_k := \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k)}-1} \xi_\ell N_n(\ell)$ and $Z_k := \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sum_{m\in\mathbb{Z}} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell)^2$. We show that, in F_{n,k,L,j_0,j_1}, we can replace

$$\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L}\right) a_{k,n,L} = \exp\left(\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k\right)$$

  by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k + \frac{1}{2}\left(\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k\right)^2 , \tag{9}$$

  Indeed, the L^1-norm of the error between these two quantities is less than :

$$\frac{1}{6} \mathbb{E}\left[\left|\frac{it}{n^{\frac{3}{4}}} Y_k - \frac{t^2}{2n^{\frac{3}{2}}} Z_k\right|^3\right] \leq \frac{4}{3} \mathbb{E}\left[\left|\frac{it}{n^{\frac{3}{4}}} Y_k\right|^3 + \left|\frac{t^2}{2n^{\frac{3}{2}}} Z_k\right|^3\right] .$$

According to formula (7), we have :

$$\frac{4}{3}\mathbb{E}\left[\left|\frac{it}{n^{\frac{3}{4}}}Y_k\right|^3\right] \leq \frac{4|t|^3}{3}(C_1)^3(C_2')^{\frac{3}{4}}\left((r+1)^{j_0}\frac{5K}{L}\right)^{\frac{3}{2}}.$$

Moreover, we have :

$$\begin{aligned}\frac{4}{3}\mathbb{E}\left[\left|\frac{t^2}{2n^{\frac{3}{2}}}Z_k\right|^3\right] &= \frac{t^6}{6n^{\frac{9}{2}}}\left(\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]\right)^3\mathbb{E}\left[\sum_{\ell_1\ell_2,\ell_3=\alpha_{(k)}}^{\alpha_{(k+1)}-1}N_n(\ell_1)^2N_n(\ell_2)^2N_n(\ell_3)^2\right]\\ &\leq \frac{t^6}{6n^{\frac{9}{2}}}\left(\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]\right)^3\left(\frac{5K\sqrt{n}}{L}\right)^3(C_1)^6n^3.\end{aligned}$$

The sum over $k=(r+1)^3,...,L-1$ of the L^1-norm of these errors is less than :

$$\frac{1}{\sqrt{L}}\frac{4|t|^3}{3}(C_2')^{\frac{3}{4}}(C_1)^3\left((r+1)^{j_0}5K\right)^{\frac{3}{2}}+\frac{1}{L^2}\frac{t^6}{6}\left(\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]\right)^3(5K)^3(C_1)^6.$$

Let us consider $L_2'\geq L_2$ such that, for all $L\geq L_2'$, this last quantity is less than $\frac{\varepsilon}{10}$.

– Now we show that, in formula (9), we can ommit the term with $(Z_k)^2$. Indeed, we have :

$$\frac{1}{2}\mathbb{E}\left[\left(\frac{t^2}{2n^{\frac{3}{2}}}Z_k\right)^2\right] \leq \frac{t^4}{8n^3}\left(\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]\right)^2\left(\frac{5K\sqrt{n}}{L}\right)^2(C_1)^4n^2.$$

The sum over $k=(r+1)^3,...,L-1$ of the L^1-norm of these errors is less than :

$$\frac{1}{L}\frac{t^4}{8}\left(\sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]\right)^2(5K)^2(C_1)^4.$$

Let us consider $L_2''\geq L_2'$ such that, for all $L\geq L_2''$, this last quantity is less than $\frac{\varepsilon}{10}$.

– From now, we fix $L:=L_2''$ and we consider an integer $n\geq\frac{L^4}{4K^2}$.

– Hence, it remains to estimate the following quantity called G_{n,k,L,j_0,j_1} :

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg|\mathbb{E}\Bigg[&\frac{it}{n^{\frac{3}{4}}}(Y_k+W_k)-\frac{t^2}{2n^{\frac{3}{2}}}(Y_k+W_k)^2-\frac{it}{n^{\frac{3}{4}}}Y_k+\frac{t^2}{2n^{\frac{3}{2}}}Z_k+\\ &+\frac{t^2}{2n^{\frac{3}{2}}}(Y_k)^2+\frac{it}{n^{\frac{3}{4}}}Y_k\frac{t^2}{2n^{\frac{3}{2}}}Z_k\Bigg|(S_p)_p\Bigg]\Bigg|\Bigg]\end{aligned}$$

with $W_k:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\xi_\ell N_n(\ell)$. Using the fact that the ξ_k are centered and independent of $(S_p)_p$, we get :

$$\begin{aligned}G_{n,k,L,j_0,j_1} &= \mathbb{E}\left[\left|\mathbb{E}\left[-\frac{t^2}{2n^{\frac{3}{2}}}(Y_k+W_k)^2+\frac{t^2}{2n^{\frac{3}{2}}}Z_k+\frac{t^2}{2n^{\frac{3}{2}}}(Y_k)^2\middle|(S_p)_p\right]\right|\right]\\ &= \frac{t^2}{2n^{\frac{3}{2}}}\mathbb{E}\left[\left|\mathbb{E}\left[(W_k)^2+2W_kY_k-Z_k\middle|(S_p)_p\right]\right|\right].\end{aligned}$$

Let us notice that we have :

$$Z_k:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2]N_n(\ell)^2+2\sum_{m\leq\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_n(\ell)^2\right).$$

– Let us show that, in the last expression of G_{n,k,L,j_0,j_1}, we can replace Z_k by :

$$\tilde{Z}_k := \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \left(\mathbb{E}[(\xi_\ell)^2] N_n(\ell)^2 + 2 \sum_{m \le \ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

Indeed, we have :

$$\begin{aligned} \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E}\left[\left| Z_k - \tilde{Z}_k \right| \right] &\le \frac{t^2}{n^{\frac{3}{2}}} \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sum_{m \le \ell-1} |\mathbb{E}[\xi_\ell \xi_m]| \, \|N_n(\ell)\|_2 \|N_n(m) - N_n(\ell)\|_2 \\ &\le \frac{t^2}{n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{p \ge 1} |\mathbb{E}[\xi_0 \xi_p]| C_1 \sqrt{n} C_2 n^{\frac{1}{4}} \sqrt{1+p}. \end{aligned}$$

The sum over $k = (r+1)^3, ..., L-1$ of these quantities is less than :

$$\frac{t^2}{n^{\frac{1}{4}}} 5K C_1 C_2 \sum_{p \ge 1} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]|,$$

which goes to zero when n goes to infinity. Hence, there exists some $n_0 \ge \frac{L^4}{4K^2}$ such that, for any integer $n \ge n_0$, this sum is less than $\frac{\varepsilon}{10}$.

– Hence we have to estimate :

$$\tilde{G}_{n,k,L,j_0,j_1} = \frac{t^2}{2n^{\frac{3}{2}}} \mathbb{E}\left[\left| \mathbb{E}\left[(W_k)^2 + 2W_k Y_k \middle| (S_p)_p \right] - \tilde{Z}_k \right| \right].$$

We have :

$$\mathbb{E}\left[(W_k)^2 \middle| (S_p)_p \right] = \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \left(\mathbb{E}[(\xi_\ell)^2] (N_n(\ell))^2 + 2 \sum_{m=\alpha_{(k)}}^{\ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

Hence we have :

$$\mathbb{E}\left[(W_k)^2 + 2W_k Y_k \middle| (S_p)_p \right] = \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \left(\mathbb{E}[(\xi_\ell)^2] (N_n(\ell))^2 + 2 \sum_{m=\alpha_{(k-(r+1)^{j_0}+1)}}^{\ell-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right).$$

We get :

$$\begin{aligned} \tilde{G}_{n,k,L,j_0,j_1} &= \frac{t^2}{n^{\frac{3}{2}}} \mathbb{E}\left[\left| \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sum_{m \le \alpha_{(k-(r+1)^{j_0}+1)}-1} \mathbb{E}[\xi_\ell \xi_m] N_n(\ell) N_n(m) \right| \right] \\ &\le \frac{t^2}{n^{\frac{3}{2}}} \frac{5K\sqrt{n}}{L} \sum_{m \ge (r+1)^{j_0} \frac{K\sqrt{n}}{L}} |\mathbb{E}[\xi_0 \xi_m]| (C_1)^2 n. \end{aligned}$$

The sum over $k = (r+1)^3, ..., L-1$ of these quantities is less than :

$$t^2 5K \sum_{m \ge (r+1)^{j_0} \frac{K\sqrt{n}}{L}} |\mathbb{E}[\xi_0 \xi_m]| (C_1)^2,$$

which goes to zero when n goes to infinity. Hence, there exists some $n'_0 \ge n_0$ such that, for any integer $n \ge n_0$, this sum is less than $\frac{\varepsilon}{10}$.

- Control of the covariance terms.
  Let j_0, j_1 be fixed. Let $k=(r+1)^3, ..., L-1$. We have :

$$D_{n,k,L,j_0,j_1} \leq \left|\mathbb{E}\left[Cov_{|(S_p)_p}\left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}, \prod_{m=k-(r+1)^{j_0}+1}^{k} b_{m,n,L}\right)\prod_{m'=k+1}^{L} a_{m',n,L}\right]\right| + \\ + \left|\mathbb{E}\left[Cov_{|(S_p)_p}\left(\prod_{m=0}^{k-(r+1)^{j_1}} b_{m,n,L}, \prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_{m,n,L}\right)\prod_{m'=k}^{L} a_{m',n,L}\right]\right|.$$

  But we have :

$$\prod_{m=\alpha}^{\alpha+\beta} b_{m,n,L} = \exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell=-\lceil K\sqrt{n}\rceil+\alpha\left\lfloor\frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor}^{-\lceil K\sqrt{n}\rceil+(\alpha+\beta+1)\left\lfloor\frac{2\lceil K\sqrt{n}\rceil+1}{L}\right\rfloor-1} \xi_\ell N_n(\ell)\right).$$

  Therefore, according to point 4 of the hypothesis of our theorem, we have :

$$D_{n,k,L,j_0,j_1} \leq 2\mathbb{E}\left[C\left(1+\frac{|t|}{n^{\frac{3}{4}}}\sum_{\ell\in\mathbb{Z}} N_n(\ell)\right)\left(\frac{rK\sqrt{n}}{2L}\right)^{-\frac{1}{2}}\sup_{s\geq r\frac{K\sqrt{n}}{2L}}\sqrt{s}\varphi_{rs,s}\right].$$

  Hence, we have :

$$\sum_{k=(r+1)^3}^{L-1} D_{n,k,L,j_0,j_1} \leq 2CL\sqrt{L}C(1+|t|n^{\frac{1}{4}})\frac{n^{-\frac{1}{4}}\sqrt{2}}{\sqrt{rK}}\sup_{s\geq r\frac{K\sqrt{n}}{2L}}\sqrt{s}\varphi_{rs,s}.$$

  which goes to zero as n goes to infinity. Hence, there exists some $N_0 \geq n_0'$ such that, for any integer $n \geq n_0$, this sum is less than $\frac{\varepsilon}{10}$.

Therefore, there exists N_0 (depending on t and on ε) such that, for any integer $n \geq N_0$, we have :

$$\left|\mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell\in\mathbb{Z}}\xi_\ell N_n(\ell)\right)\right] - \mathbb{E}\left[\exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell,k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)\right]\right| < \varepsilon.$$

This ends the step 1 of our proof.

5.2 Step 2 : Conclusion

In the previous section we proved that :

$$\lim_{n\to+\infty}\left|\mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell\in\mathbb{Z}}\xi_\ell N_n(\ell)\right)\right] - \mathbb{E}\left[\exp\left(-\frac{t^2}{2n^{\frac{3}{2}}}\sum_{\ell,k\in\mathbb{Z}}\mathbb{E}[\xi_\ell\xi_k]N_n(\ell)^2\right)\right]\right| = 0.$$

According to [13] lemma 6, we know that : $\left(\frac{1}{n^{\frac{3}{2}}}\sum_{\ell\in\mathbb{Z}} N_n(\ell)^2\right)_{n\geq 1}$ converges in distribution to $Z_1 := \int_{\mathbb{R}}(L_1(x))^2\,dx$. Hence, we get :

$$\lim_{n\to+\infty}\mathbb{E}\left[\exp\left(\frac{it}{n^{\frac{3}{4}}}\sum_{\ell\in\mathbb{Z}}\xi_\ell N_n(\ell)\right)\right] = \mathbb{E}\left[\exp\left(-\frac{t^2}{2}\sum_{k\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_k]\int_{\mathbb{R}}(L_1(x))^2\,dx\right)\right].$$

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