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CROSSING AND ALIGNMENTS OF PERMUTATIONS

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Abstract. We introduce the notion of crossings and nestings of a permutation. We compute the generating function of permutations with a fixed number of weak exceedances, crossings and nestings. We link alignments and permutation patterns to these statistics. We generalize to the case of decorated permutations. Finally we show how this is related to the stationary distribution of the Partially ASymmetric Exclusion Process (PASEP) model.

1. Introduction

We introduce the notion of crossings and nestings of a permutation. The purpose of this paper is to link permutation patterns and these statistics. Our main result is the following:

Theorem 1. *The number $B(n,k,\ell,m)$ of permutations σ of $[n]$ with k weak exceedances, ℓ crossings and m nestings is equal to the number $D(n,k,\ell,m)$ of permutations of $[n]$ with $n-k$ descents, ℓ occurrences of the pattern $31-2$ and m occurrences of the pattern $2-31$.*

We prove this theorem by exhibiting their generating function. Let

$$F(q,p,x,y) = \cfrac{1}{1-b_0x-\cfrac{\lambda_1x^2}{1-b_1x-\cfrac{\lambda_2x^2}{1-b_2x-\cfrac{\lambda_3x^2}{\ddots}}}} \tag{1}$$

with $[n]_{p,q} = p^{n-1}+p^{n-2}q+\cdots+pq^{n-2}+q^{n-1}$, $b_n = y[n+1]_{p,q}+[n]_{p,q}$ and $\lambda_n = y[n]_{p,q}^2$.

Proposition 1. *The coefficient of $x^ny^kx^\ell y^m$ in $F(q,p,x,y)$ is equal to $B(n,k,\ell,m)$.*

Proposition 2. *The coefficient of $x^ny^kx^\ell y^m$ in $F(q,p,x,y)$ is equal to $D(n,k,\ell,m)$.*

We also propose a bijective proof of Theorem 1. To prove this we use a bijection of Foata and Zeilberger [7, 10], a bijection due to Françon and Viennot [11] and results from [4, 5].

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The notions of crossings and nestings are closely related to the notion of alignments, defined in [18], which come from the enumeration of totally positive Grassmann cells. The link between alignments and patterns was conjectured by Steingrímsson and Williams [17].

These alignments define some new q-analogs of the Eulerian numbers $\hat{E}_{k,n}(q)$ were introduced by Williams [18] building on work of Postnikov [14]. Let $[k]_q$ be $1+q+\cdots+q^{k-1}$, then

Proposition 3. [18] *The number of permutations of $[n]$ with k weak exceedances and ℓ alignments is the coefficient of $q^{(k-1)(n-k)-\ell}$ in*

$$\hat{E}_{k,n}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{k(i-1)} \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).$$

These numbers have the property that if $q=1$ they are the Eulerian numbers, if $q=0$ they are the Narayana numbers and if $q=-1$ they are the binomial coefficients. See [18] for details. Williams proved in [18] using permutation tableaux that $E(q,x,y) = \sum_{n,k} q^{n-k} \hat{E}_{k,n}(q) y^k x^n$ is equal to

$$\sum_{i=0}^{\infty} \frac{y^i (q^{2i+1} - y)}{q^{i^2+i+1}(q^i - q^{i+1}[i]_q x + [i]_q xy)}.$$

Here we exhibit the generating function $\hat{E}(q,x,y) = \sum_{n,k} \hat{E}_{k,n}(q) y^k x^n$ and show that

Theorem 2.

$$\hat{E}(q,x,y) = F(q,1,x,y).$$

This proves the conjecture of Steingrímsson and Williams [17].

We define the combinatorial statistics in Section 2. We then prove Proposition 1 and Theorem 2 in Section 3. In Section 4 we prove Proposition 2. This also gives a proof of Theorem 1 which is implied by Propositions 1 and 2. In Section 5 we propose a direct bijective proof of Theorem 1. In Section 6 we generalize to the case of decorated permutations. In Section 7 we show how these numbers appear naturally in the stationary distribution of the PASEP model [2].

2. Definitions

Let $\sigma = (\sigma(1), \ldots, \sigma(n))$ be a permutation of $[n] = \{1, 2, \ldots, n\}$. The number of weak exceedances of a permutation σ is the cardinality of the set $\{j \mid \sigma(j) \geq j\}$. We denote this number by $WEX(\sigma)$.

Let

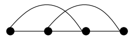
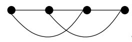
- $C_+(i) = \{j \mid j < i \leq \sigma(j) < \sigma(i)\}$,
- $C_-(i) = \{j \mid j > i > \sigma(j) > \sigma(i)\}$,

and $C_+(\sigma) = \sum_{i=1}^{n} |C_+(i)|$.

Definition 1. *The number of crossings of a permutation σ is equal to*

$$C_+(\sigma) + C_-(\sigma).$$

We can also define those parameters using the permutation diagram. We draw a line and put the numbers from 1 to n and we draw an edge from i to $\sigma(i)$ above the line if $i \leq \sigma(i)$ and under the line otherwise. For example, the permutation diagram of $\sigma = (4, 7, 3, 6, 2, 1, 5)$ is on Figure 1.

$C_+(\sigma)$ is the number of pairs of edges above the line that cross or touch  and $C_-(\sigma)$ is the number of pairs of edges under the line that cross .

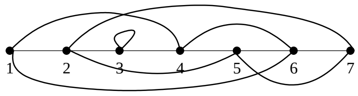


FIGURE 1. The permutation diagram of $\sigma = (4, 7, 3, 6, 2, 1, 5)$

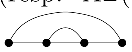
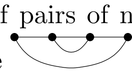
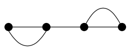
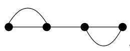
For $\sigma = (4, 7, 3, 6, 2, 1, 5)$ on Figure 1, we have $C_+(\sigma) = 2$ and $C_-(\sigma) = 1$. The pairs of edges contributing to $C_+(\sigma)$ are $\{(1, 4), (4, 6)\}$ and $\{(1, 4), (2, 7)\}$. The pair of edges contributing to $C_-(\sigma)$ is $\{(7, 5), (6, 1)\}$.

Then let

- $A_+(i) = \{j \mid j < i \leq \sigma(i) < \sigma(j)\}$;
- $A_-(i) = \{j \mid j > i > \sigma(i) > \sigma(j)\}$;
- $A_{+,-}(i) = \{j \mid j \leq \sigma(j) < \sigma(i) < i\} \cup \{j \mid \sigma(i) < i < j \leq \sigma(j)\}$.

We then set $A_+(\sigma) = \sum_{i=1}^{n} |A_+(i)|$.

For $\sigma = (4, 7, 3, 6, 2, 1, 5)$, $A_+(\sigma) = 3$, $A_-(\sigma) = 1$, and $A_{+,-}(\sigma) = 2$.

We can again define these parameters using the permutation diagram of permutation σ of $[n]$. Then $A_+(\sigma)$ (resp. $A_-(\sigma)$) is the number of pairs of nested edges above (resp. under) the line  (resp. under the line ) and $A_{+,-}(\sigma)$ is the number of pairs of edges such that one is above the line and the other under and such that their supports do not intersect  .

For $\sigma = (4, 7, 3, 6, 2, 1, 5)$, the pairs of edges contributing to $A_+(\sigma)$ are $\{(1, 4), (3, 3)\}$, $\{(2, 7), (3, 3)\}$ and $\{(2, 7), (4, 6)\}$. The pair of edges contributing to $A_-(\sigma)$ is $\{(5, 2), (6, 1)\}$. The pairs of edges contributing to $A_{+,-}(\sigma)$ are $\{(7, 5), (1, 4)\}$ and $\{(7, 5), (3, 3)\}$.

Definition 2. *The number of nestings of a permutation σ is equal to*

$$A_+(\sigma) + A_-(\sigma).$$

Definition 3. *The number of alignments of a permutation σ is equal to*

$$A_+(\sigma) + A_-(\sigma) + A_{+,-}(\sigma).$$

This definition looks a bit different but is equivalent to the definition in [18].

A descent in a permutation is an index i such that $\sigma(i) > \sigma(i+1)$. An ascent in a permutation is an index i such that $\sigma(i) < \sigma(i+1)$. We denote by $DES(\sigma)$ the number of descents of σ.

The pattern 31−2 (resp. 2−31, 13−2) occurs in σ if there exist $i < j$ such that $\sigma(i) > \sigma(j) > \sigma(i+1)$ (resp. $\sigma(j+1) < \sigma(i) < \sigma(j)$, $\sigma(i+1) > \sigma(j) > \sigma(i)$). We denote by $(31{-}2)(\sigma)$ the number of occurrences of the pattern 31−2. For example,

$(31{-}2)(4,7,3,6,2,1,5)=2$, as $73{-}6$ and $73{-}5$ are the two occurrences of the pattern $31{-}2$.

We will use bijections between permutations and weighted bicolored Motzkin paths. A bicolored Motzkin path of length n is a sequence $c=(c_1,\ldots,c_n)$ such that $c_i \in \{N,S,E,\bar{E}\}$ for $1\leq i\leq n$ and such that if $h_i=\{j<i \mid c_j=N\}-\{j<i \mid c_j=S\}$ then $h_1=0$, $h_i\geq 0$ for $2\leq i\leq n$ and $h_{n+1}=0$.

3. Crossings, nestings and alignments

3.1. **Crossings and nestings.** We use a bijection of Foata and Zeilberger [10] between permutations and weighted bicolored Motzkin paths. We could also use the bijection of Biane [1]. We refer to [5] for a compact definition of these bijections.

To any permutation σ we associate a pair (c,w) made of a bicolored Motzkin path $c=(c_1,c_2,\ldots,c_n)$ and a weight $w=(w_1,\ldots,w_n)$. The path is created using the following rules :

- $c_i=N$ if $i<\sigma(i)$ and $i<\sigma^{-1}(i)$
- $c_i=E$ if $i\leq\sigma(i)$ and $i\geq\sigma^{-1}(i)$
- $c_i=\bar{E}$ if $i>\sigma(i)$ and $i<\sigma^{-1}(i)$
- $c_i=S$ if $i>\sigma(i)$ and $i>\sigma^{-1}(i)$

The weight is created using the following rules :

- $w_i=yp^{|A_+(i)|}q^{|C_+(i)|}$ if $c_i=N,E$.
- $w_i=p^{|A_-(i)|}q^{|C_-(i)|}$ if $c_i=S,\bar{E}$.

This implies that $\prod_{i=1}^n w_i$ is equal to $y^{WEX(\sigma)}q^{C_+(\sigma)+C_-(\sigma)}p^{A_+(\sigma)+A_-(\sigma)}$. Let $\mathcal{P}_n$ be the set of pairs (c,w) obtained from permutations of $[n]$. Then the coefficient of $y^kq^\ell p^m$ in

$$\sum_{(c,w)\in\mathcal{P}_n}\prod_{i=1}^n w_i$$

is $B(n,k,\ell,m)$ the number of permutations of $[n]$ with k weak exceedances, ℓ crossings and m nestings.

For example $\sigma=(4,1,5,6,2,3)$ gives the path $(N,\bar{E},N,E,S,S)$ and the weight $(y,1,yq,yq^2,q,1)$.

We now compute the generating function $\sum_n x^n\sum_{(c,w)\in\mathcal{P}_n}\prod_{i=1}^n w_i$.

Lemma 3. *If $i\leq\sigma(i)$ then*

$$|C_+(i)|=h_i-|A_+(i)|;$$

and if $i>\sigma(i)$ then

$$|C_-(i)|=h_i-1-|A_-(i)|.$$

Proof. It is easy to prove by induction that

$$h_i=|\{j<i \mid \sigma(j)\geq i\}|=|\{j\geq i \mid \sigma(j)<i\}|.$$

Using the definitions given in Section 1, if $i\leq\sigma(i)$ then

$$A_+(i)\cup C_+(i)=\{j<i \mid \sigma(j)\geq i\}$$

and if $i>\sigma(i)$ then

$$A_-(i)\cup C_-(i)=\{j>i \mid \sigma(j)<i\}=\{j\geq i \mid \sigma(j)<i\}\backslash\{i\}.$$

This implies the result. □

Thanks to that Lemma, using the machinery developed in [9, 16], we get directly that the generating function

$$\sum_n x^n \sum_{(c,w)\in\mathcal{P}_n} \prod_{i=1}^{n} w_i.$$

is

$$\cfrac{1}{1-b_0x-\cfrac{\lambda_1x^2}{1-b_1x-\cfrac{\lambda_2x^2}{1-b_2x-\cfrac{\lambda_3x^2}{\ddots}}}}$$

with $b_n = y[n+1]_{p,q} + [n]_{p,q}$ and $\lambda_n = y[n]_{p,q}^2$. This is $F(q,p,x,y)$ and proves Proposition 1.

Note that this bijection implies that

Proposition 4. *The number of permutations with k weak exceedances and ℓ crossings and m nestings is equal to the number of permutations with k weak exceedances and ℓ nestings and m crossings.*

Similar results are known for set partitions and matchings [3, 12, 13].

3.2. **Link with alignments.** From the preceding result, we know that the coefficient of $x^ny^kq^\ell$ in $F(q,1,x,y)$ is the number of permutations of $[n]$ with k weak exceedances and ℓ crossings.

From Section 1 we also know that the number of permutations of $[n]$ with k weak exceedances and ℓ alignments is the coefficient of $q^{(k-1)(n-k)-\ell}x^ny^k$ in $\hat{E}(q,x,y)$.

We want to prove Theorem 2 which states that $\hat{E}(q,x,y) = F(q,1,x,y)$. It follows from the following proposition:

Proposition 5. *For any permutation with k weak exceedances the number of crossings plus the number of alignments is $(k-1)(n-k)$.*

Proof. We suppose that σ is a permutation of $[n]$ with k weak exceedances. For any i with $1 \le i < n$, we first define :

- $B_+(i) = \{j \mid j < i \le \sigma(j)\}$
- $B_-(i) = \{j \mid \sigma(j) < i \le j\}$

Note that $h_i = |B_+(i)| = |B_-(i)|$ and that for $i > \sigma(i)$, $A_-(i) \cup C_-(i) \cup \{i\} = B_-(i)$. Therefore

$$A_-(\sigma) + C_-(\sigma) = \sum_{i>\sigma(i)} (|B_-(i)| - 1) = k - n + \sum_{i>\sigma(i)} |B_+(i)| \tag{2}$$

For $i \le \sigma(i)$, let

$$E_+(i) = \{j \mid i \in C_+(j)\} = \{j \mid i < j \le \sigma(i) < \sigma(j)\}.$$

It is easy to see that :

$$E_+(i) \cup A_+(i) = B_+(\sigma(i)).$$

Therefore

$$\begin{aligned} A_+(\sigma)+C_+(\sigma) &= \sum_{i\le\sigma(i)} |C_+(i)|+|A_+(i)| \\ &= \sum_{i\le\sigma(i)} |E_+(i)|+|A_+(i)| \\ &= \sum_{i\le\sigma(i)} |B_+(\sigma(i))| \\ &= \sum_{i>\sigma(i)} |D_+(i)|, \end{aligned}$$

where

$$D_+(i)=\{j \mid j\le\sigma(j) \text{ and } i\in B_+(\sigma(j))\}.$$

It is easy to see that for $i>\sigma(i)$, $D_+(i)=\{j \mid j\le\sigma(j) \text{ and } \sigma(i)<\sigma(j)<i\}$ and therefore that $B_+(i)\cup D_+(i)\cup A_{+,-}(i)=\{j \mid j\ge\sigma(j)\}$. As they are pairwise disjoint then

$$|B_+(i)|+|D_+(i)|+|A_{+,-}(i)|=k.$$

Combining equations (2) and (3) we get that $A_+(\sigma)+C_+(\sigma)+A_-(\sigma)+C_-(\sigma)+A_{+,-}(\sigma)=k-n+\sum_{i>\sigma(i)}|D_+(i)|+|B_+(i)|+|A_{+,-}(i)|$. This concludes the proof of Proposition 5. □

4. Permutation patterns

Continued fractions like the one presented in Equation (1) of Theorem 1 were studied combinatorially in [4, 5]. Theorem 10 in [5] associated with Theorem 22 in [4] tells us that the coefficient of $x^n y^k q^\ell p^m$ in $F(x,y,q,p)$ is the number $D(n,k,\ell,m)$ of permutations σ of $[n]$ with $n-k$ descents, ℓ occurrences of the patterns 31−2 and m occurrences of the pattern 2−31. This is Proposition 2.

This can also be proved bijectively thanks to a bijection of Françon and Viennot [11]. We present now that bijection. See also [5].

Given a permutation $\sigma=(\sigma(1),\ldots,\sigma(n))$, we set $\sigma(0)=0$ and $\sigma(n+1)=n+1$. Let $\sigma(j)=i$. Then i is

- a valley if $\sigma(j-1)>\sigma(j)<\sigma(j+1)$
- a double ascent if $\sigma(j-1)<\sigma(j)<\sigma(j+1)$
- a double descent if $\sigma(j-1)>\sigma(j)>\sigma(j+1)$
- a peak if $\sigma(j-1)<\sigma(j)>\sigma(j+1)$
- the beginning (resp. end) of a descent if $\sigma(j)>\sigma(j+1)$ (resp. $\sigma(j-1)>\sigma(j)$)
- the beginning (resp. end) of a ascent if $\sigma(j)<\sigma(j+1)$ (resp. $\sigma(j-1)<\sigma(j)$).

If $\sigma(j)=i$ then we define $(31{-}2)(i)$ (resp. $(2{-}31)(i)$) to be the number of indices $k<j$ (resp. $k>j$) such that $\sigma(k-1)>\sigma(j)>\sigma(k)$.

To any permutation, we associate a pair (c,w) made of a bicolored Motzkin path $c=(c_1,c_2,\ldots,c_n)$ and a weight $w=(w_1,\ldots,w_n)$. The path is created using the following rules :

- $c_i=N$ if i is a valley
- $c_i=E$ if i is a double ascent

- $c_i = \bar{E}$ if i is a double descent
- $c_i = S$ if i is a peak

The weight is created using the following rules :

- $w_i = yp^{(31-2)(i)}q^{(2-31)(i)}$ if $c_i = N, E$.
- $w_i = p^{(31-2)(i)}q^{(2-31)(i)}$ if $c_i = S, \bar{E}$.

This implies that $\prod_{i=1}^{n} w_i$ is equal to $y^{DES(\sigma)}p^{(31-2)(\sigma)}q^{(2-31)(\sigma)}$.

For example $\sigma = (6,2,1,5,3,4)$ gives the path $(N, \bar{E}, N, E, S, S)$ and the weight $(y, 1, yp, yp, y, 1)$.

Now we prove the following Lemma

Lemma 4. *For any i*

$$(31-2)(i) + (2-31)(i) = \begin{cases} h_i & \text{if } i \text{ is the beginning of an ascent} \\ h_i - 1 & \text{if } i \text{ is the beginning of a descent} \end{cases}$$

Proof. We prove this lemma by induction. If i is equal to 1 then i is the beginning of an ascent and $(31-2)(1) + (2-31)(1) = 0 = h_1$. If $i > 1$ then $(31-2)(i) + (2-31)(i) = (31-2)(i-1) + (2-31)(i-1) + v$ where v is zero, one or minus one. It is easy to see that v is one if $i-1$ is the end of a descent and i is the beginning of an ascent, v is minus one if $i-1$ is the end of an ascent and i is the beginning of a descent and 0 otherwise. That gives exactly the lemma. □

Let $\mathcal{P}_n$ be the set of pairs (c, w) obtained from permutations of $[n]$. Using the machinery developed in [9, 16], we get directly that $\sum_n x^n \sum_{(c,w)\in\mathcal{P}_n} \prod_{i=1}^{n} w_i$ equals $F(q, p, x, y)$ and proves Proposition 2.

5. Bijective proof of Theorem 1

Combining the bijection of Françon and Viennot and the inverse of bijection of Foata and Zeilberger [10], we get Theorem 1. We propose the direct mapping very similar to a bijection proposed in [5]. Starting from a permutation σ with k descents, ℓ occurrences of the patterns $31-2$ and m occurrences of the pattern $2-31$, we form a permutation τ with $n-k$ weak exceedances and ℓ crossings and m nestings.

We first form two two-rowed arrays τ_- and τ_+. We create the permutation τ from the tableaux of τ_- and τ_+. For i from 1 to n, if i is in the j^{th} entry of the first row of τ_- (resp. τ_+) then $\tau(i)$ is the j^{th} entry of the second row of τ_- (resp. τ_+).

The first row of τ_- contains all the entries of σ that are the beginning of a descent. They are sorted in increasing order. The second row of τ_- contains all the entries of σ that are the end of a descent. They are sorted such that if i is in first row then $\tau(i) < i$ and $(31-2)(i)$ in σ is equal to $A_-(i)$ in τ. One can easily check that this is always possible in a unique way. For each i in the first row starting from the smallest, $\tau(i)$ is the $((31-2)(i) + 1)^{th}$ smallest entry of the second row that is not yet chosen. Note that this implies that $(2-31)(i)$ in σ is equal to $C_-(i)$ in τ.

The first line of τ_+ contains all the entries of σ that are the beginning of an ascent and that are sorted in increasing order. The second line of τ_+ contains all the entries of σ that are not the end of a descent. They are sorted such that if i is in first row then $\tau(i) \geq i$ and $(2{-}31)(i)$ in σ is equal to $C_+(i)$ in τ. One can again easily check that this is always possible in a unique way and that this implies that $(31{-}2)(i)$ in σ is equal to $A_+(i)$ in τ.

For example if $\sigma = (5, 1, 7, 4, 3, 6, 8, 2)$ then the $2{-}31$ sequence is

$$((31{-}2)(1), \ldots, (31{-}2)(8)) = (0, 1, 1, 1, 0, 1, 0, 0)$$

$$((2{-}31)(1), \ldots, (2{-}31)(8)) = (0, 0, 1, 1, 2, 1, 1, 0).$$

Then

$$\tau_- = \begin{pmatrix} 4, 5, 7, 8 \\ 2, 1, 3, 4 \end{pmatrix}$$

and

$$\tau_+ = \begin{pmatrix} 1, 2, 3, 6 \\ 8, 5, 6, 7 \end{pmatrix}$$

Then $\tau = (8, 5, 6, 2, 1, 7, 3, 4)$. One can check that $C_+(1) = 0$, $C_+(2) = 0$, $C_+(3) = 1$, $C_-(4) = 1$, $C_-(5) = 2$, $C_+(6) = 1$, $C_-(7) = 1$, $C_-(8) = 0$.

If σ is the image of τ, it is easy to see that

Lemma 5.

$$\begin{aligned} WEX(\tau) &= n - DES(\sigma) \\ C_+(\tau) + C_-(\tau) &= (2{-}31)(\sigma) \\ A_+(\tau) + A_-(\tau) &= (31{-}2)(\sigma). \end{aligned}$$

Therefore if σ has k descents, ℓ occurrences of $(2{-}31)$ and m occurrences of $(31{-}2)$ then σ has $n{-}k$ weak exceedances, ℓ crossings and m nestings.

This map is easily reversible.

We conclude this Section by proving a similar result. Given a permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$. Let $\pi = (\sigma(n), \ldots, \sigma(1))$. If σ has $k - 1$ ascents (or $n{-}k$ descents) and ℓ occurrences of the pattern $31{-}2$ (resp. $2{-}31$), then π has $k - 1$ descents and ℓ occurrences of the pattern $2 - 13$ (resp. $13{-}2$). Therefore

Proposition 6. *There is a one-to-one correspondance between permutations with k weak exceedances and ℓ crossings and m nestings and permutations with $k - 1$ descents and ℓ occurrences of the pattern $13{-}2$ and m occurrences of the pattern $2 - 13$.*

6. Generalization for decorated permutations

We can also derive the generating function of

$$A_{k,n}(q) = \sum_{i=0}^{k-1} \binom{n}{i} E_{k,n-i}.$$

These were introduced in [18].

The following corollary is an easy consequence of Theorem 2. Let $A(q, x, y) = \sum_{n,k} A_{k,n}(q) x^n y^k$.

Corollary 6.

$$A(q,x,y) = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{1 - b_2 x - \cfrac{\lambda_3 x^2}{\ddots}}}}$$

with $b_n = (1+y)[n+1]_q$ *and* $\lambda_n = yq[n]_q^2$.

Lauren Williams [17] proved that

$$A(q,x,y) = \frac{-y}{1-q} + \sum_{i\geq 1} \frac{y^i(q^{2i+1}-y)}{q^{i^2+i+1}(q^i - q^i[i+1]_q x + [i]_q xy)}$$

It would be interesting to have a direct proof of the identity of the formal series form and the continued fraction form of these generating functions [17].

We can also interpret these results combinatorially. In [18] the coefficient of $q^{(n-k)k-\ell}$ in $A_{k,n}(q)$ is interpreted in terms of decorated permutations with k weak exceedances and ℓ alignments . Let us define these notions. Decorated permutations are permutations where the fixed points are bicolored [14]. We color these fixed points by colors $\{+,-\}$. We say that $i \leq_+ \sigma(i)$ if $i < \sigma(i)$ or $i = \sigma(i)$ and i is colored with color $+$. We say that $i \geq_- \sigma(i)$ if $i > \sigma(i)$ or $i = \sigma(i)$ and i is colored with color $-$.

For a decorated permutation σ and i, let

- $A_+(i) = \{j \mid j < i \leq_+ \sigma(i) < \sigma(j)\}$
- $A_-(i) = \{j \mid j > i \geq_- \sigma(i) > \sigma(j)\}$
- $A_{+,-}(i) = \{j \mid i \geq_- \sigma(i) > \sigma(j) \geq j\} \cup \{j \mid \sigma(j) \geq j > i \geq_- \sigma(i)\}$
- $C_+(i) = \{j \mid i < j \leq \sigma(i) < \sigma(j)\}$
- $C_-(i) = \{j \mid j > i > \sigma(j) > \sigma(i)\}$

As for permutations we define $\mathrm{A}_+(\sigma) = \sum_i \mathrm{A}_+(i)$.

With these notions, we can again define the number of alignments (resp. nestings, crossings) of a decorated permutation σ as $A_+(\sigma)+A_-(\sigma)+A_{+,-}(\sigma)$ (resp. $A_+(\sigma)+A_-(\sigma)$, $C_+(\sigma)+C_-(\sigma)$). The number of weak exceedances (resp. non-exceedances) of a decorated permutation is the cardinality of the set $\{i \mid i \geq_+ \sigma(i)\}$ (resp. $\{i \mid i < \sigma(i)\}$).

Let

$$A(q,p,x,y) = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{1 - b_2 x - \cfrac{\lambda_3 x^2}{\ddots}}}}$$

with $b_n = (1+y)[n+1]_{p,q}$ and $\lambda_n = yq[n]_{p,q}^2$.

A direct generalization of the bijection à la Foata-Zeilberger on decorated permutations gives :

Proposition 7. *The coefficient of $x^n y^k q^\ell p^m$ in $A(q,p,x,y)$ is the number of decorated permutations σ of $[n]$ with k weak exceedances, ℓ is the sum of the number of crossings and the number of non-exceedances and m nestings.*

We can also make a direct link with the alignments :

Proposition 8. *For any decorated permutation σ with k weak exceedances*

$$A_+(\sigma) + A_-(\sigma) + C_+(\sigma) + C_-(\sigma) + A_{+,-}(\sigma) + |\{j \mid j > \sigma(j)\}| = (n-k)k.$$

Proof. The proof is omitted as it follows exactly the same steps as the proof of Proposition 5. □

We could also derive a bijection à la Françon-Viennot on decorated permutations to interpret the $A_{k,n}(q)$ in terms of descents and permutations patterns. One way would be to define new decorated permutations where i is bicolored if and only if i is a double ascent and $(2{-}31)(i) = 0$.

7. Link with the PASEP model

The PASEP model [6] consists of black particles entering a row of n cells, each of which is occupied by a black particle or vacant. A particle may enter the system from the left hand side, hop to the right or to the left and leave the system from the right hand side, with the constraint that a cell contains at most one particle. We will say that the empty cells are filled with white particles $\circ$. A basic configuration is a row of n cells, each containing either a black $\bullet$ or a white $\circ$ particle. Let $\mathcal{B}_n$ be the set of basic configurations of n particles. We write these configurations as though they are words of length n in the language $\{\circ, \bullet\}^*$.

The PASEP defines a Markov chain P defined on $\mathcal{B}_n$ with the transition probabilities α, β, and q. The probability $P_{X,Y}$, of finding the system in state Y at time $t+1$ given that the system is in state X at time t is defined by:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then

$$P_{X,Y} = 1/(n+1); \qquad P_{Y,X} = q/(n+1) \tag{3a}$$

- If $X = \circ B$ and $Y = \bullet B$ then

$$P_{X,Y} = \alpha/(n+1). \tag{3b}$$

- If $X = B\bullet$ and $Y = B\circ$ then

$$P_{X,Y} = \beta/(n+1). \tag{3c}$$

- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

See an example for $n = 2$ in Figure 2.

This Markov chain has a unique stationary distribution [6]. The case $q = 0$ was studied combinatorially by Duchi and Schaeffer [8]. In [2] another combinatorial approach was taken to treat the general case.

Definition 4. [2] *Let $\mathcal{P}(n)$ be the set of bicolored Motzkin paths of length n. The weight of the path in $\mathcal{P}(0)$ is 1. The weight of any path p denoted by $w(p)$ is the product of the weights of its steps. The weight of a step, p_i, starting at height h is*

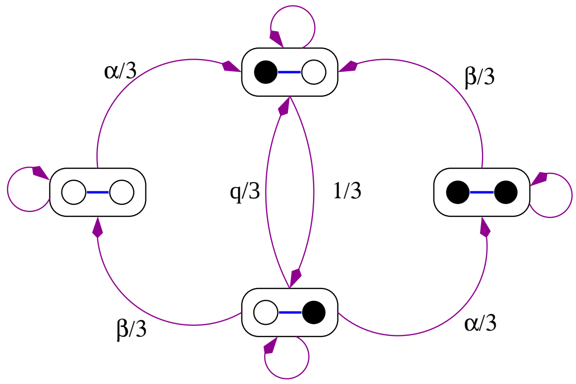


FIGURE 2. The chain P for $n = 2$.

given by:

$$\begin{array}{lll} \text{if } p_i = N & \text{then} & w(p_i) = [h+1]_q \\ \text{if } p_i = \bar{E} & \text{then} & w(p_i) = [h]_q + q^h/\alpha \\ \text{if } p_i = E & \text{then} & w(p_i) = [h]_q + q^h/\beta \\ \text{if } p_i = S & \text{then} & w(p_i) = [h]_q + q^h/(\alpha\beta) - q^{h-1}(1/\alpha - 1)(1/\beta - 1). \end{array}$$

Given a path p, $\theta(p)$ is the basic configuration such that each $\bar{E}$ and S step is changed to $\circ$ and each E and N step is changed to $\bullet$. Let

$$W(X) = \sum_{p \in \theta^{-1}(X)} w(p) \tag{5}$$

and

$$Z_n = \sum_{X \in B_n} W(X) \tag{6}$$

Theorem 7. [2] *At the steady state, the probability that the chain is in the basic configuration X is*

$$\frac{W(X)}{Z_n}$$

We can use these results and observations from the previous sections to get :

Theorem 8. *If $\alpha = \beta = 1$, at the steady state, the probability that the chain is in a basic configuration with k particles is*

$$\frac{\hat{E}_{k+1,n+1}(q)}{Z_n}.$$

Before proving that theorem, we need a Lemma

Lemma 9. *There is a weight preserving bijection between $\mathcal{P}(n,k)$ the set of weighted bicolored Motzkin paths of length n where the weight of any step starting at height h is $[h+1]_q$ and where k is the number of steps N plus the number of steps E and $\mathcal{P}'(n+1,k+1)$ the set of weighted bicolored Motzkin paths of length $n+1$ where the weight of any step starting at height h is $[h+1]_q$ if the step is N or E and $[h]_q$ otherwise and where $k+1$ is the number of steps N plus the number of steps E.*

Proof.

Starting with a weighted path (c,w) in $\mathcal{P}(n,k)$ with $c=(c_1,\ldots,c_n)$ and $w=(w_1,\ldots,w_n)$, we create a path (c',w') with $c'=(c'_1,\ldots,c'_{n+1})$ and $w'=(w'_1,\ldots,w'_{n+1})$.

We first set $c_0=N$ and $c_{n+1}=S$ and we construct the path c'. For $1\le i\le n+1$

- $c'_i=N$ if and only if $c_{i-1}=N$ or E and $c_i=N$ or $\bar{E}$
- $c'_i=E$ if and only if $c_{i-1}=N$ or E and $c_i=S$ or E
- $c'_i=\bar{E}$ if and only if $c_{i-1}=S$ or $\bar{E}$ and $c_i=N$ or $\bar{E}$
- $c'_i=S$ if and only if $c_{i-1}=S$ or $\bar{E}$ and $c_i=S$ or E

It is easy to see that the path is a bicolored Motzkin path that it is of length $n+1$ and that $k+1$ is the number of steps N plus the number of steps E. Moreover for $1\le i\le n$, if the starting height of c_i is j and $c'_i=N$ or E (resp. $c'_i=S$ or $\bar{E}$) then the starting height of c'_i is j (resp. $j+1$). Therefore if we set $w'_i=w_i$ for $1\le i\le n$ and $w'_{n+1}=1$, then $(c',w')\in\mathcal{P}'(n+1,k+1)$. This map is easily reversible. Remark : In this map the paths in $\mathcal{P}'$ do not have any steps $\bar{E}$ at height 0 and in particular do not start or end with $\bar{E}$. This because a path in $\mathcal{P}'$ having a $\bar(E)$ step at height 0 has weight zero and hence can be ignored. □

Proof of the Theorem. We want to compute for $\alpha=\beta=1$

$$W(k,n)=\sum_{\substack{X\in B_n\\ X\text{ has }k\text{ particles}}} W(X)=\sum_{p\in\mathcal{P}(n,k)} w(p).$$

Now we use the previous lemma and get

$$W(k,n)=\sum_{p\in\mathcal{P}'(n+1,k+1)} w(p).$$

Using [9, 16] and the definition of the weight of the steps of $\mathcal{P}'(n+1,k+1)$, we conclude that :

$$1+\sum_{n\ge 0} x^{n+1}\sum_{k=0}^{n} y^{k+1}W(k,n)=\hat{E}(q,x,y)$$

and therefore that $W(k,n)=\hat{E}_{k+1,n+1}(q)$. □

8. Conclusion

Several open problems naturally arise from this work :

- Can we define patterns for decorated permutations?
- Can we generalize these q-Eulerian numbers to understand the PASEP when $\alpha\neq 1$ or $\beta\neq 1$? Can we use the permutations tableaux [15, 17, 19]?
- Can we extend the definition of k-crossing and k-nesting that were defined for matchings and set partitions [3] ?

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